Path Integral Methods in Physics & Finance

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Lecture – 09
Brownian Motion

Welcome back, a recap of what we did last time; I introduced the concept of stochastic processes as a collection, a sequence of random variables. And these random variables would be indexed usually with reference to time; we can have the index set as a discrete set or a continuous set depending on the application and research. And then similarly the codomain of the random variable can either be a discrete set or a continuous set.

In a, we have different nomenclatures for the relevant type of random variables, a discrete time random and discrete time stochastic process and discontinuous time stochastic process and so on. And then I talked about certain equations that govern the behavior of stochastic processes; I talked about the Chapman Kolmogorov equation, the Fokker Planck equation and the master equation, we discussed all of that in the last lecture.

And now we move onto the concept of Brownian Motion, which in a sense is an application of the concept of stochastic processes. Let us say, let us start with the fundamental building block of Brownian motion which is a one step unbiased random walk, one step unbiased random walk.

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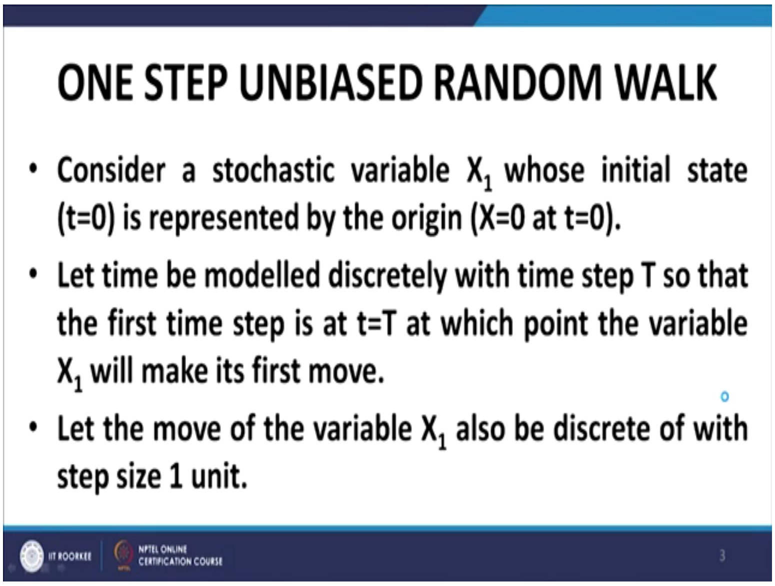

What is the random walk? Let us discuss with an example. Suppose you program your computer in such a way that, it picks out an integer randomly and depending on whether that choice happens to be an odd integer or an even integer. If it is an even integer say, it moves one step upwards and if it is an odd integer it moves one step downwards. I repeat you conduct a random experiment; let us say, let us make it even simpler, let us say you conduct a coin toss experiment.

You conduct a coin toss experiment; if you get a head, you move one step upwards or in the pointer moves one step upwards and if you get a tail, the pointer moves one step downwards. The probability of getting a head or a tail is equal that, that is why we use the word unbiased. So, we have a single step that is you start at t equal to 0, nothing happens up to a certain point in time; let us say we call it a capital T.

So, 0 to capital T is one time step; as soon as the time reaches capital T, you make the toss and immediately the system reacts by moving one step upwards or moving one step downward, this is the example of a one step random walk.

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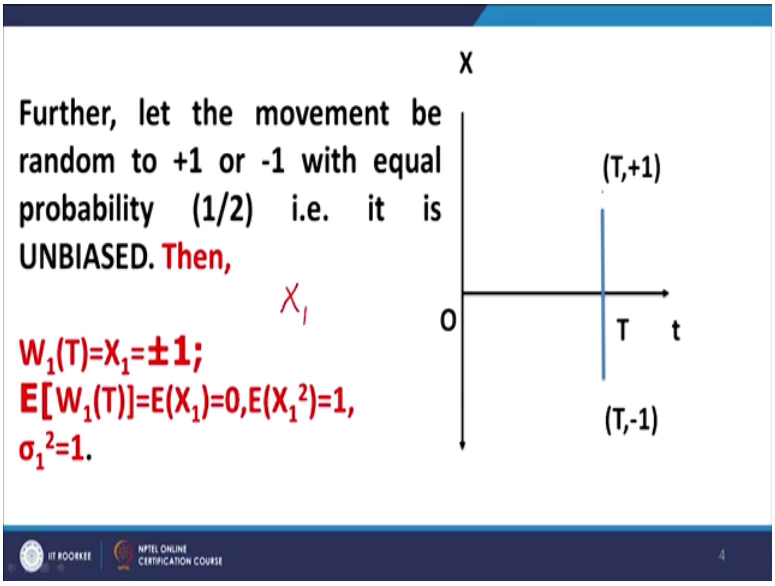

Let me show it by a diagram. So, here is the diagram, you are at the origin of the coordinate system; as soon as the clock strikes T which is the length of one time step, it need not necessarily be 1 second or 1 hour, it can be 1 unit of time. What howsoever you define that unit of time; not necessarily it has to be coincident with the conventional unit of time, it can be chosen as per the desired application.

So, 0 to capital T at point t the experiment is done, the coin is tossed; if it ends up with the head, the a particle or the pointer whatever moves up one step and if it turns out to be a tail, the pointer moves one step downwards. So, that means what? That means, at t equal to capital

T, we can have two possible positions of the random walker; either it can be at the coordinate T comma plus 1 depending on whether the coin gets a head or it can be at the point T minus 1 depending on whether you gets a tail.

Clearly from the symmetry it is quite obvious that, the mean of that of this random variable is 0. In fact, let us make it more a bit more formal; let us define a random variable X, let us call it X 1. Let us define this random variable X 1; this random variable X 1 can take the value plus 1 or minus 1 with equal probability. Random variable, I repeat random variable X 1 is can take the values plus 1 or minus 1 with equal probability.

Therefore the two values or the possible values that the system can take at T equal to capital T after the first step, it can be either plus 1 or minus 1 which is written usually in the form and W 1 of capital T; W 1 represents the possible values of the system of the random walker at time T. And this can be represented by that the variable that we chose that is X 1, it can take the value plus 1, it can take the value minus 1 right, but the probabilities are equal.

Clearly the expectation of W 1 is equal to the expectation of X 1; that is 1 by 2 into plus 1 plus 1 by 2 into minus 1 that gives you 0. And if you work out the expectation of X 1 squared; what do we get? We get X the expectation of X 1 squared is equal to 1 by 2 into plus 1 squared plus 1 by 2 into minus 1 squared that turns out to be 1.

So, the variance turns out to be 1. So, what are the critical parameters? The critical parameters of W 1 T; W 1 T is the spectrum of values, the various possible values that the system can take at time T and 1 represents the number of steps, which in this case is 1. So, W 1; 1 represents number of steps, T represent the point in time and that is equal to the expectation of this is equal to 0 and the expectation of W 1 square is equal to 1.

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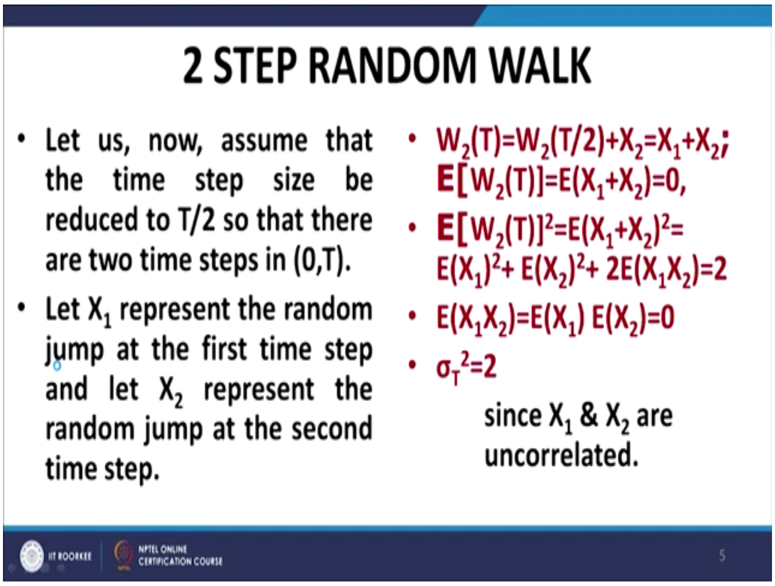

Now, we extend this to a 2 step model; how do we do it? Let us look at the diagram first before coming to the parameters.

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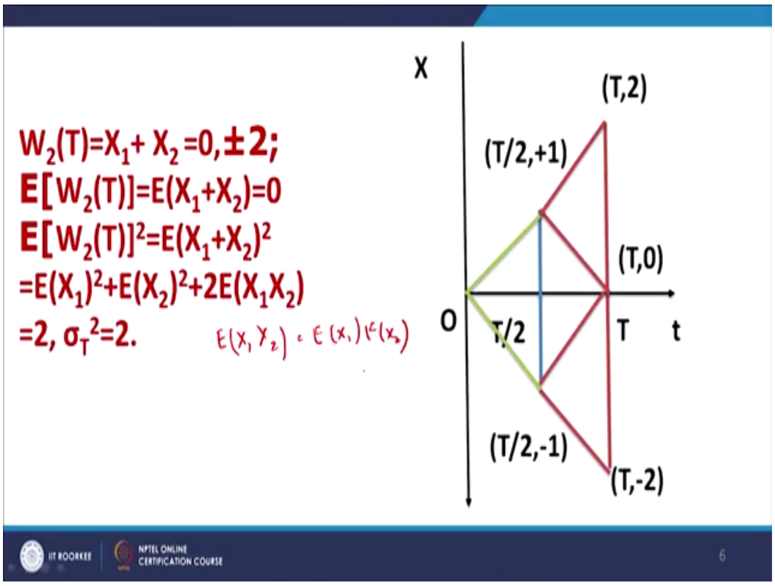

Now, what we do is, we split the interval 0 to t; remember we had an interval of 0 to t and in the previous case we made at, the system made a jump at t equal to capital T. Now we the system makes two jumps within this interval 0 to capital T.

But these two jumps are equally spaced in time. So, it makes a jump at t equal to capital T by 2 of plus 1 unit or one minus 1 unit; in other words the coin is tossed at t equal to T by 2. And whether it depending on whether you get a head or a tail, the system jumps to plus 1 or the system jumps to minus 1. So, the possible positions at t equal to capital T by 2 for the system to be is plus 1 or minus 1.

Thereafter when the clock strikes capital T, the system again tosses a coin or you toss a coin and depending on whether you get a head or tail from its previous position whatever that previous position was; whether it was plus 1 or minus 1, it makes a jump a second jump either to plus 1 plus 1 unit from where it was or minus 1 unit from where it was.

Clearly from the diagram you can see that, the possible values that the system can have at t equal to capital T; after two jumps the first jump at t equal to capital T by 2 and the second jump at t equal to capital 2, the spectrum of values that the system can possibly have are 2, 0 and minus 2 depending on where the system was at t equal to T by 2 when it made the second jump.

It can also be written in this form W 2; now 2 means there are two jumps, there are the number of jumps are equal to 2 and capital T means we are investigating the spectrum of values at time t equal to capital T. So, W 2 capital T is equal to X 1 plus X 2; remember X 1 and X 2 are defined similarly they are binary variables, which can take the value plus 1 and minus 1 each with probability 1 by 2.

But they are mutually independent, this is very important. How the system evolves in the second step is independent of how the system at evolved in the first step; in other words that the result of the second coin toss experiment does not in any way depend on the result of the first coin toss experiment. So, we have E W 2 T is equal to E of X 1 plus X 2; X 1 representing the first jump and X 2 representing the second jump.

Because the expectation of both X 1 and X 2 is 0; the expectation of X 1 plus X 2 also become 0. So, expectation of W 2 T is equal to 0. Let us look at the expectation of W 2 T squared; this is equal to E of X 1 plus X 2 square, which gives us E of X 1 square plus E of X 2 square plus 2 E X 1 X 2. Now because the X 1 and X 2 are independent events; so I can write this E X 1 X 2 as E of X 1 E of X 2, because X 1 and X 2 are independent event, because the two coin toss experiments are independent.

And we know that E of X 1 is 0, E of X 2 is also 0. So, this cross term X 1, X 2 term vanishes and we also know that E of X 1 is equal to, E of X 1 square is equal to 1 and E of X 2 square is equal to 1. So, what we end up with is that, E of W 2 squared is equal to 2 and the variance of this is, variance of the possible values that the system can take at time t equal to t capital T; if it makes 2 jumps at equidistant points in time 0 T by 2 and T is equal to 2.

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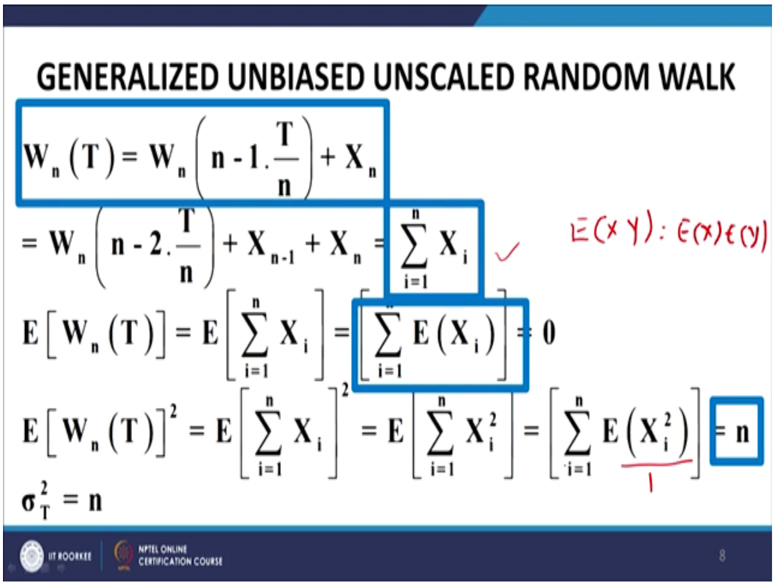

Continuing in this manner, you can arrive at the recursive relation which is given in the blue box here; W n of capital T, where n is now the number of steps, n is the number of steps. We have explicitly discussed the case of one step; we have discussed the case of two steps, now we generalized it to n steps. So, W n T is equal to where the system was at T n minus 1th step and then it makes a jump, either upwards by 1 unit or downwards by 1 unit in the nth step and that is represent by X n.

In other words you get the recursive relation, W n T is equal to W n minus 1 into T upon n; because each step is now of length T upon n plus each time step is of length T upon n and of course, in the in terms of variable it is plus 1 minus 1 always. So, it becomes plus whatever jump it makes in the nth step, which is a X n; which is represented by X n, where X n is again I repeat a random variable binary random variable which takes the values plus 1 and minus 1 with equal probabilities.

So, now you can continue this iteratively and you get this expression which is shown in the second box. In other words, W n T can be represented as a sum of binary random variables which can take values 0 and 1 with equal probability and most importantly which are mutually independent.

W X 1, X 2, X n all are mutually independent, they do not have any correlation between themselves and W n T, the spectrum of possible values that you get at time T; if the system evolves n times during 0 and T is given by the summation of X n, the summation being made over the number of steps.

Now, let us look at the what we get for the expectation values; the expectation value of W n T clearly is 0, because the expectation if you take expectation of a sum is the sum of the expectations and each of the expectation vanishes. So, it is clearly 0.

And similarly when we work out the expectation of W n T squared slightly more care is required; but we again note that the cross terms would again vanish, because the expectation the again the relation that I mentioned in the previous slide E x y is equal to E x E y and each of these two is 0.

So, the cross terms will all vanish and we will have expectation of summation of X i square is equal to expectation of X i square sum is summed over 1 to n each of these is 1, each of these expressions is 1, this is 1; and therefore because you have n ones, so total becomes n. In other words for an n step random walk; for an n step random walk, the expectation and variance of the spectrum of values at the end of the walk at time capital T is given by the expectation as 0 and the variance as n, where n is the number of steps.

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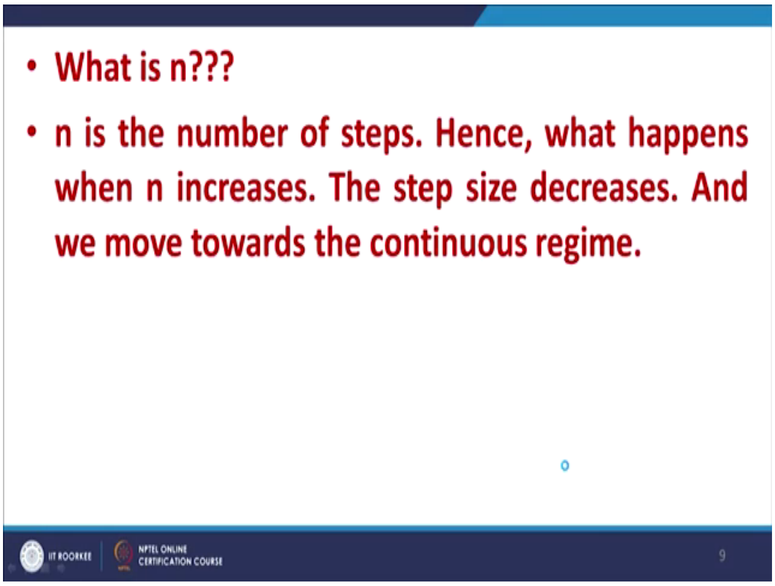

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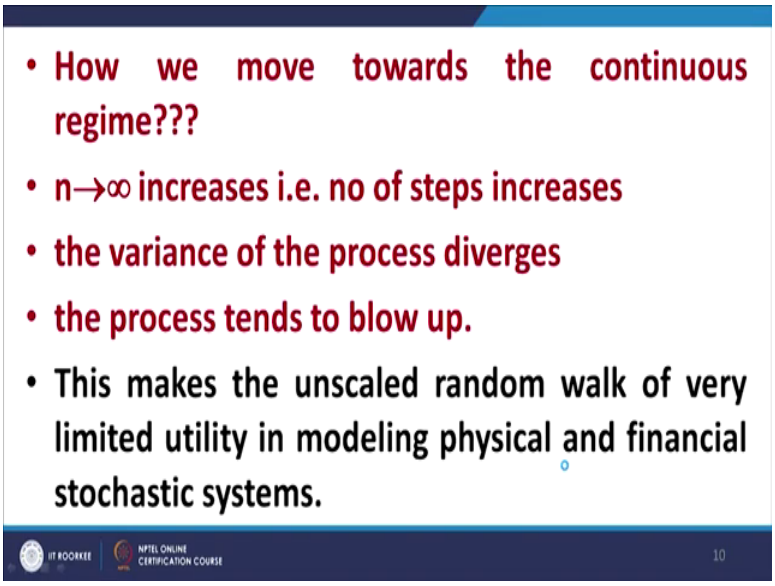

Now, this is very important, the n is the number of steps. Now let us look at the situation more carefully, more analytically. What we want at the end of the day is a framework, is a design that enables us to do continuous operations, mathematical operations that involves continuity and differentiation. Now in an effort to move towards that from this discrete framework, this is so far what we have done is that totally discrete framework.

In order to move from here towards the continuous framework, the obvious approaches to reduce the length of the time step; in other words to increase the number of time steps between 0 and capital T. As you increase the number of time steps, you move from discrete time to continuous time, right. So, let us see what happens when we move from discrete time to continuous time.

Now, recall, recall that for an n step binomial tree over time t capital T, the mean is given by 0 and the variance is given by n. Now as n tends to infinity, in other words how do we, how do we implement this program of moving from discreteness to continuity; we implement it by taking n tending to infinity limit. So, as you move from n tending to infinity, the number of steps increases; what happens to the variance?

Clearly the variance also increases; greater the number of steps, greater the variance and as you approach infinity, the variance also becomes unbounded, the process diverges. Now because the process diverges, it becomes incompatible to a number of applications in physics and finance; and therefore we have to evolve a mechanism whereby we can curtail this blow up of variance to a convergent situation, where the variance converges to a certain to a certain number, so that the applications can be, the spectrum of application can be increased particularly in physics and finance.

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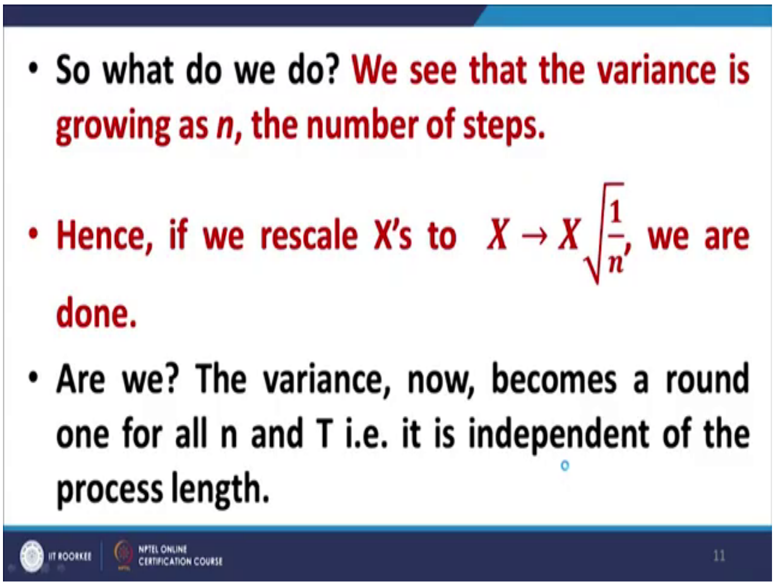

Now, how can we do that? Clearly the variance as we go from the discrete framework to the continuous framework, the variance goes up with the number of steps. So, if we scale, if we scale down the time instead of instead of or if we scale X; the jump in terms of X under root X upon under root n or X into under root 1 upon n, then what happens? Then if you work out the entire thing again, the various now turns out to be 1.

The variance has now become finite, the variance has now become convergent; and by decreasing the step size, by decreasing the step size or rescaling the step size in such a way that instead of having a steps as a plus minus 1, now I have a step size of plus minus 1 upon root n, then I find that the variance converges to 1.

But then again we have a problem, because the variances simply remains 1 irrespective of the number of steps, irrespective of the of the dynamics of the system. How the system involves is, does not impute itself or does not imprint in self itself on the variance and therefore, we have to look for something better.

Very often we find that in applications or in some of the applications, the variance tends to increase with time, it tends to increase linearly with time. To incorporate that we make a slight modification to our program of rescaling; instead of rescaling X to X into under root 1 upon n, instead of instead of this rescaling, I introduce a factor of T here.

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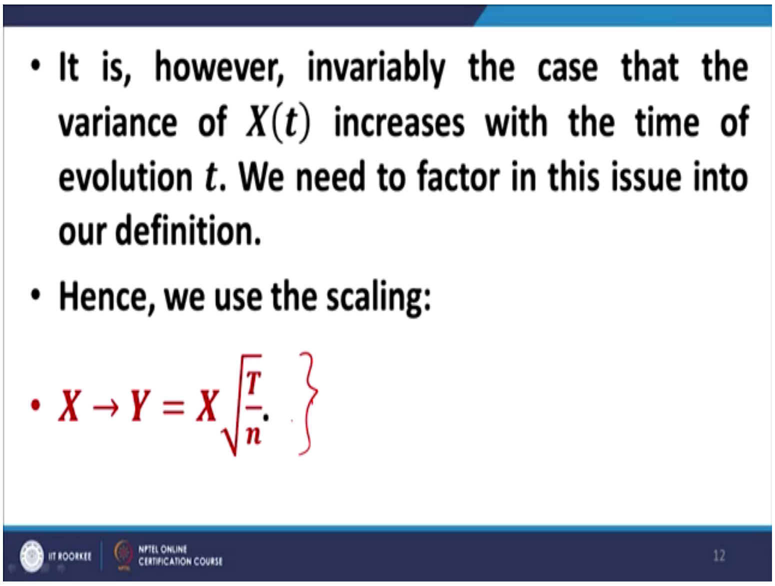

How do I do it? I do scaling in this form; I move from X to Y and write Y as under root T upon n.

What does it mean? It means that, is the step the jump size instead of being plus minus 1, is now plus minus under root T upon n. I repeat instead of the jump size being plus minus 1 which we started with; we now are looking at a jump size of plus minus under root T upon n, remember the time step continues to remain at T upon n. So, a quick summary of what we have at the moment, W and 0 is equal to 0; that means, the system starts or the particle starts at the origin.

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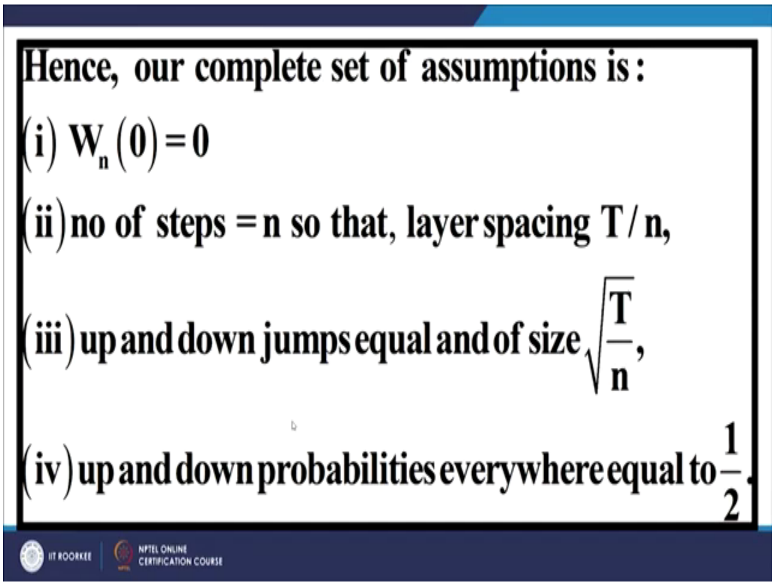

Number two, the number of steps is n, the time horizon is T. So, the length of each step is T upon n; up and down jumps are now rescaled instead of plus minus 1 to under root T upon n plus T upon n and minus T upon n. The probabilities remain unchanged; the probability of the up jump and the probability of the down jump remain 1 by 2 and to that extent we can, we are continuing with the system being unbiased.

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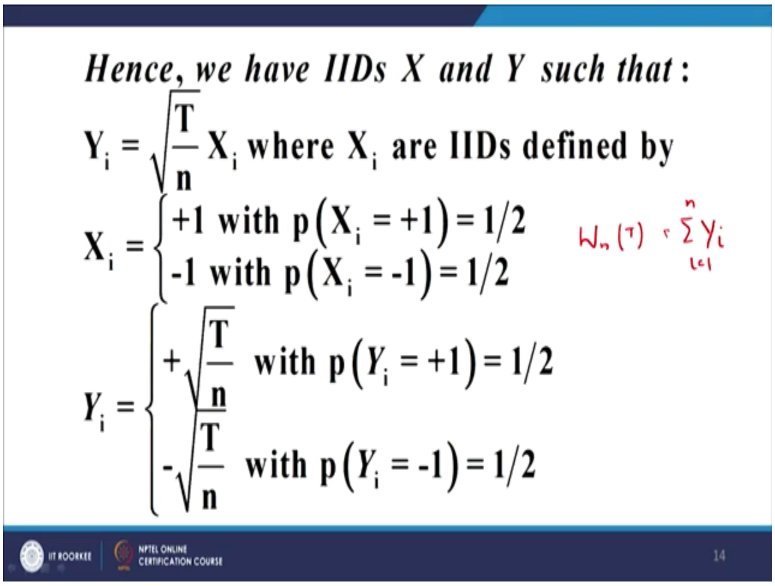

Let us now look at this. So, what do we have? With the W n T, W n T can now be described by summation of W i, Y i; I am sorry W n T can be described by summation up to n over of i Y i, where Y i is given by under root T upon n X i. X i was binary, we are taking values plus minus 1 with probabilities 1 by 2 each; therefore Y i takes the values plus under root T by n and minus under root T by n with respective probabilities 1 by 2.

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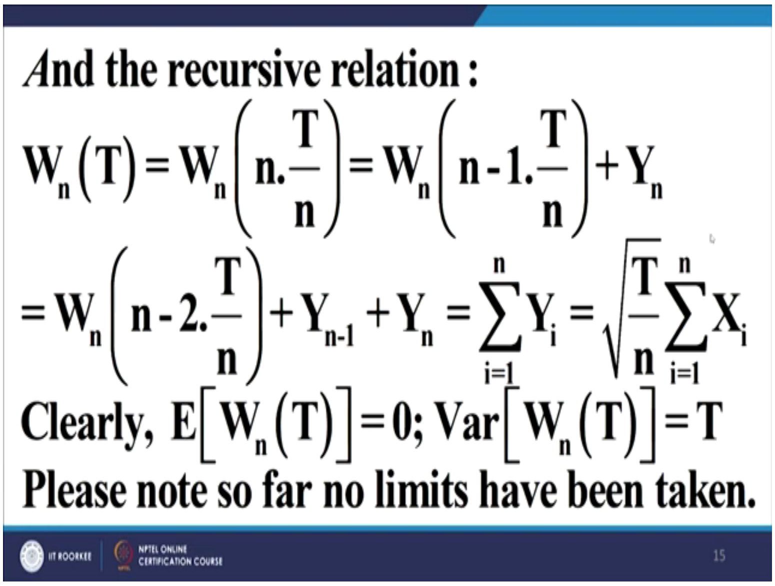

And the recursive relation now gets modified slightly; instead of W n T, instead of X n T being here, we now have Y n T. Otherwise it is more or less the same, the substitution is instead of; because the step size has changed, the step size has changed. So, when the object moves over the last step; when it moves over the last step, its jump is going to be either plus root T by n or minus root T by n instead of it being plus 1 or minus 1, this is reflected in this term Y n, right

So, and the critical parameters, the mean value will continue to be 0, no problem with that. The variance now, the variance now will become; re recall that the variance earlier when the size of the jump was n, now the variance. If the entire process is relooked into the variance will become; because you see you have rescaled the rescale the variable by an a factor of T by n. So, the under root T by n I am sorry. So, the variance will get rescaled by T by n.

So, earlier the variance was n, now the variance will become n into T by n and that is equal to T that is the time horizon of the evolution of the process. Just to repeat, since you have rescaled each random variable that comprise the entire stochastic process by under root T by n; the variance also gets scaled not by this factor, but the square of this factor that is T upon n. Earlier the variance was n, so it now becomes T upon n into n and that becomes T, that is the length of evolution of the system.

Please note we have not taken any limits tending to infinity so far, which we will do right now.

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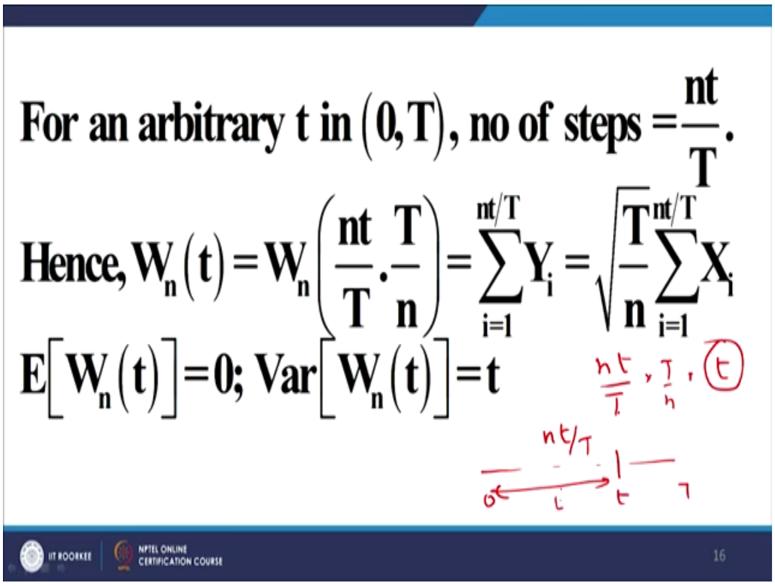

But before we do that, but before we do that, this was for the complete evolution of the system. What about an arbitrary time during the evolution of the system? Let us say this is my system, this is my 0, this is capital T, here is the some arbitrary point small t. What will be the number of steps between 0 and small t? They will naturally be n into t upon capital T. Why is that?

Well, the length of each step is capital T upon n; now this distance is small t, so the number of steps will be small t divided by capital T upon n that comes to n into small t divided by capital T. And therefore, when we work out the mean obviously, it remains 0 and the variance now becomes summation of Y and the number of variables involved will be n t n small t upon capital T. And therefore, when we do the simplification it will be, the variance will be n t upon capital T into T upon n that is equal to small t.

This is the variance of W n of small t; again it is equal to the length of evaluation, it is equal to this length, the variance is equal to small t. These two things are important; a mean is 0, the variance is equal to small t, where now t is any arbitrary point. Please note this t is an arbitrary point; if t is not the total length of evolution of the system, t is simply an arbitrary point, so that is important. Now we move to the next step.

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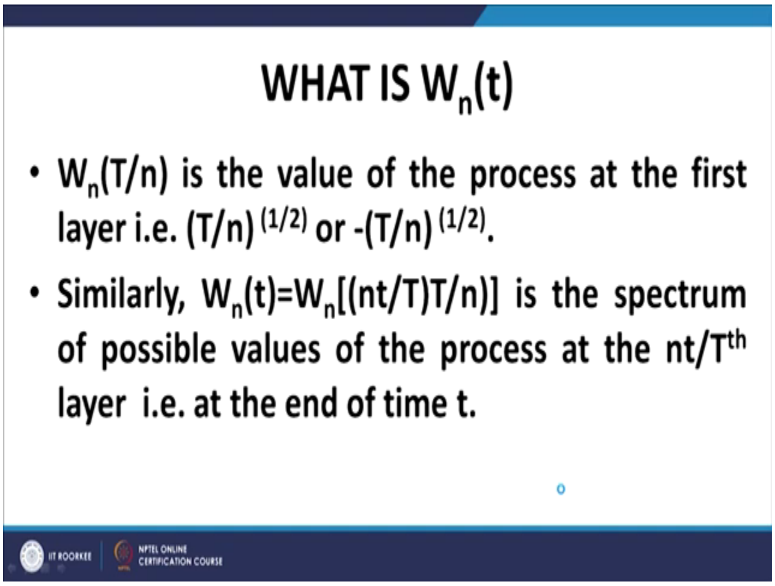

As far as W n t I have explained, I have a reiterated also W n t is the spectrum of values of the process, the possible values that the process can take at time t.

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These are realizations of the random walk, the first case is when there are 16 steps, the second up of 64 step; you can see the zigzagging increasing and the third is 256 steps and the fourth is 1024 steps, you can see the amount of zigzagging in the four step.

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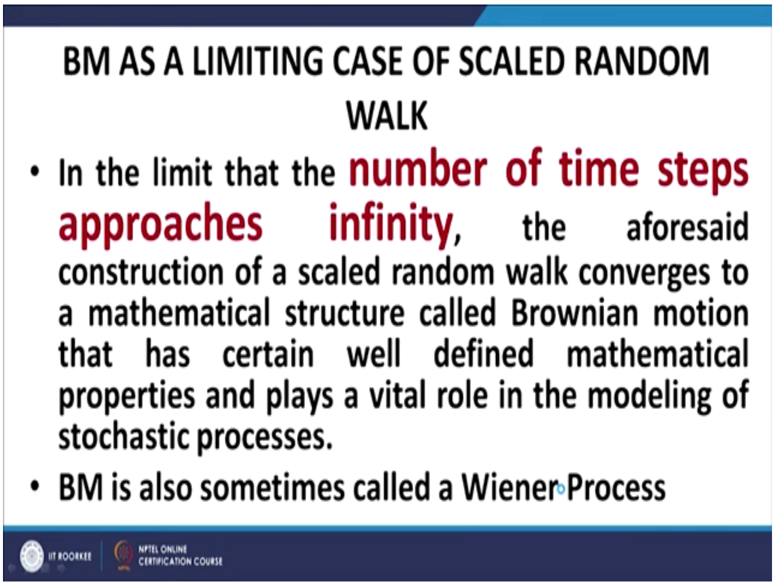

Now, what happens, so when we take the limit, we arrive at a certain mathematical structure; this mathematical structure has a very significant role in the entire gamut of stochastic calculus, it is known as Brownian motion. And we now arrive at or we now derive the expression for the or the salient feature or the defining property of Brownian motion and then we talk about its various properties.

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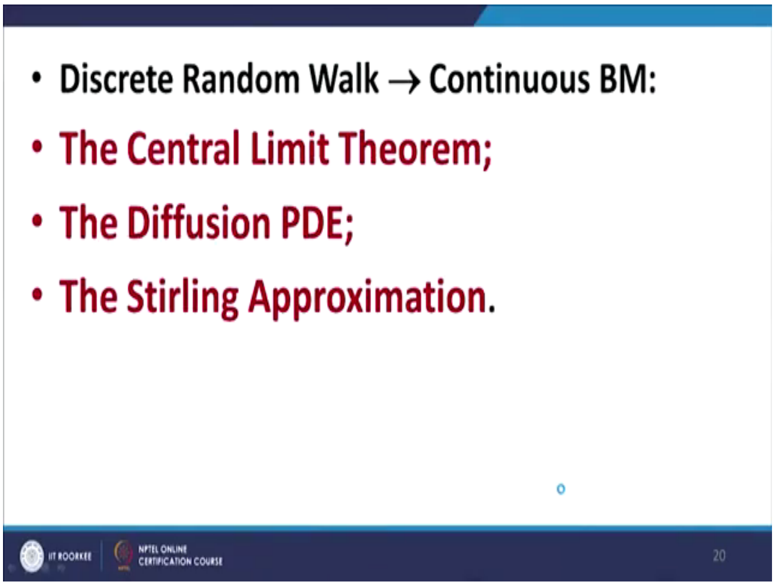

So, remember so far we have not taken any limits and we can move from this to the limiting structure to the continuous framework using any of these three approximations. We can use the central limit theorem, we can use the diffusion PDE or we can move with the stirling approximation.

I have already given an introduction to this stirling approximation, when we talked about the Gaussian distribution as being a limiting case of binomial distribution. So, that gives you a feel of what, how that this particular case could be manipulated to arrive at the continuous limit.

I shall start with the central limit theorem and then I will also discuss the diffusion partial differential equation.

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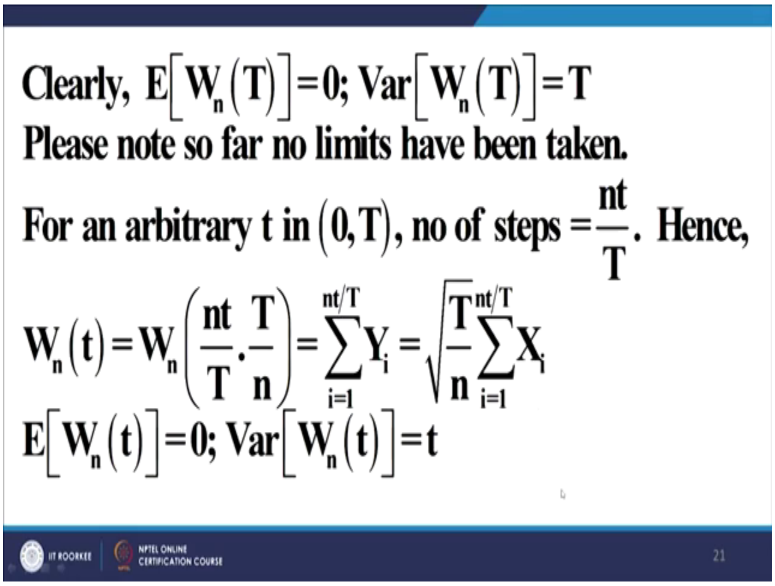

So, this is where we stood at the last slide. We had defined the random walk and we had defined the random walk with the property with three fundamental properties; the length of each step is capital T upon n time step, the jump is equal to plus minus under root T upon n, and the probabilities of the jumper 1 by 2, and the system starts at the origin.

These were the four fundamental property that we had just till now; we had not taken any limit so far.

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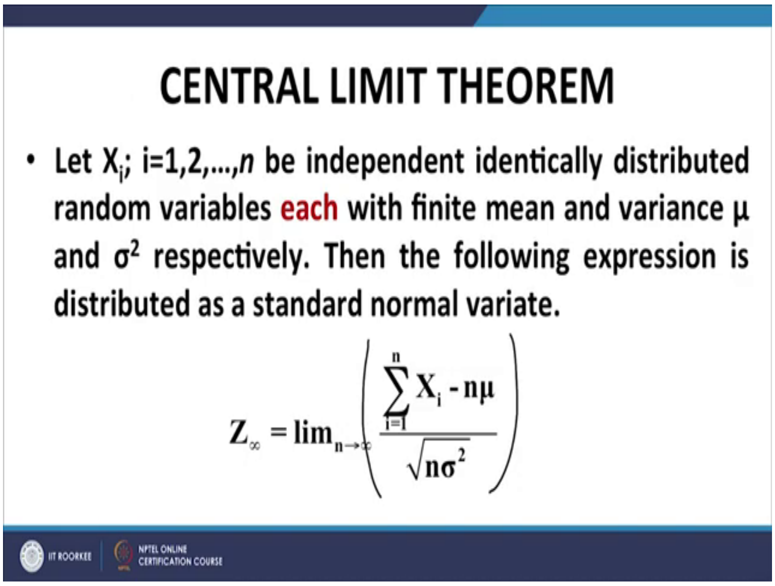

Now, what happens when we do the when we take the limits. Remember what does the central limit theorem say? The central limit theorem says that, if I define a variable, if suppose we have a set of variables X 1, X 2, X 3, X n; all of them are i ideals, that is independent identically distributed variables with a mean of mu and a variance of sigma square.

Then I define this variable sigma X i, summation of all these variables X 1, X 2, X 3, X n minus n of mu, mu is the mean; so n into mu and divide the whole thing by under root n sigma square, under root n sigma square and this expression will be a standard normal variate, in the limit variable standard normal variable in the limit that n tends to infinity. This is what we will invoke in moving from the discrete framework that we have with us to the continuous framework.

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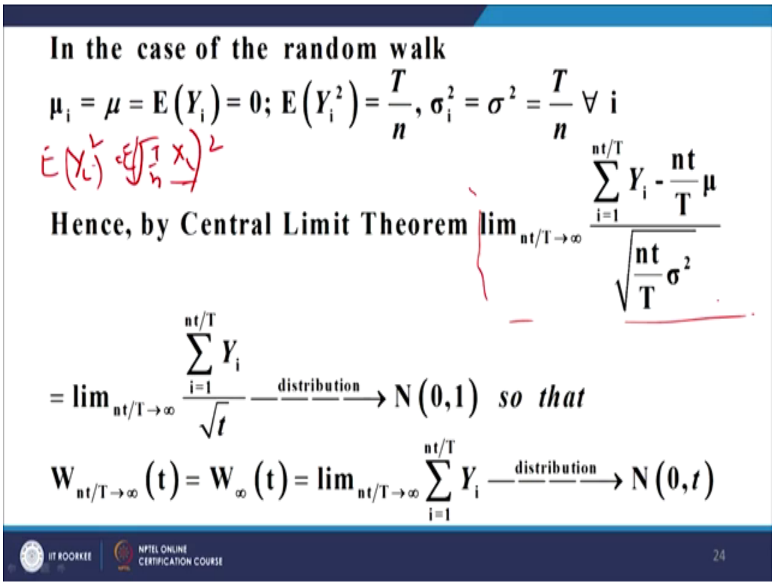

Recall, in our case mu was equal to E of Y i which was 0 and as far as E of Y i square was concerned, it was equal to T upon n; because we multiply Y i was nothing, but just to recap Y i was nothing, but under root T upon n X i.

Now, the stochastic part is X i, under root T upon X is not stochastic. So, when you take the expectation of both sides and square this; this under root T upon n can be taken outside the expectation operator, you get T upon n E of X i square, E of X i square we have already discussed it is 1. So, E of Y i square is equal to T upon n; therefore the variance of Y i square is also T upon n and this holds for each and every Y, because they are identically distributed.

Hence by the central limit theorem, what do we get? Summation of Y i minus mean, mean is in our case is 0 and under root n t upon T; because the number of steps is n t upon T. Recall that, we are talking about an arbitrary length in the Brownian motion evolution or in the in the evolution of the random walk; arbitrary length corresponds to the number of steps in n t upon capital T. And therefore, instead of n we are using n t upon capital T; we are simply now going to substitute the values.

So, this, this expression is 0, the mean is 0 and sigma square is equal to capital T upon n; so when you put capital T upon n here, and we are left with under root t, n cancels with this n, T cancels with this T, we are left with simply under root t in the denominator. So, the this whole thing simplifies to, this whole thing simplifies to summation n t upon capital T tending to infinity; sigma of Y i upon root t and this is normally distributed as a standard normal variable.

And therefore we find that, summation of Y i which is nothing, but W n t, which is nothing, but W n t; as now in the limit that n tending's to infinity, number of steps moving to infinity or n t upon capital T tending to infinity if you are talking about the of a particular arbitrary length. Then in that limit a summation of Y i or summation and W of t is equal to, it tends to the normal distribution with a mean of 0 and a variance of small t which is the arbitrary point time length at which you are working out this analysis.

And obviously, if you work out the analysis or you work out these parameters over the entire evolution of the system; then this becomes n of 0 capital T. So, just a quick recap, we have the mean of, we introduced the variables Y i here; Y i is equals to under root T upon n X i, therefore because the mean of X i is 0, the mean of X i square is 1.

We simply get mean of Y i is equal to 0, mean of Y i square is equal to T by n; and therefore variance of Y i square is equal to T by n. Substitute in the expression for the central limit theorem, we get summation of Y i upon root t is equal is the is distributed as a standard normal variate.

And therefore, but summation of Y i is nothing, but the, but W n or w infinity or at the moment when we taking the limit W infinity of t and that is normally distributed as 0, t. Because summation Y i is distributed as summation Y i upon root t is distributed as N 0, 1;

therefore, summation Y i is distributed as N 0, t. And summation Y i is nothing, but W n or W infinity of t and that is what.

In other words what have we concluded? What have we concluded? We have concluded that the spectrum of values at any arbitrary point in the in a process that follows Brownian motion is normally distributed with a mean of 0 and a variance equal to the time, since its initiation time since it the origin of the Brownian motion to the point at which it is being assessed. We will continue after the break.

Thank you.