Path Integral Methods in Physics & Finance

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Lecture- 08
Evolutionary Equations of Stochastic Processes

Right let us continue. Now, if you look at the Chapman Kolmogorov equation, it contains probability as the non-linear term; it is not a linear equation in probability and therefore, to solve the equation is a tedious task. And the equation can be simplified in some cases by introducing the concept of a transition rate.

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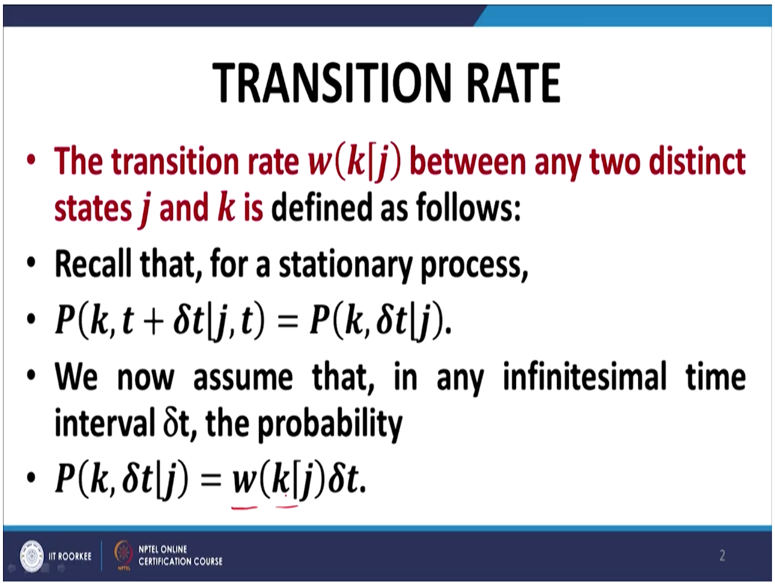

So, let us now look at what we mean by a transition rate. Now, this concept of transition rate is particularly relevant we are talk when we are talking about a stationary stochastic process a process where time where it is time translation invariant. So, in that case we can write the probability of k t plus delta t subject to j, t the conditional 2 point probability as k delta t subject to j 0 which is simply written as j.

We express this term this probability of the system moving from the state j to the state k in an infinitesimal time interval delta t in terms of an expression which is independent of time which is w k subject to j into delta t. So, this is a simplification that we introduce and this expression w k subject to j is called the transition rate.

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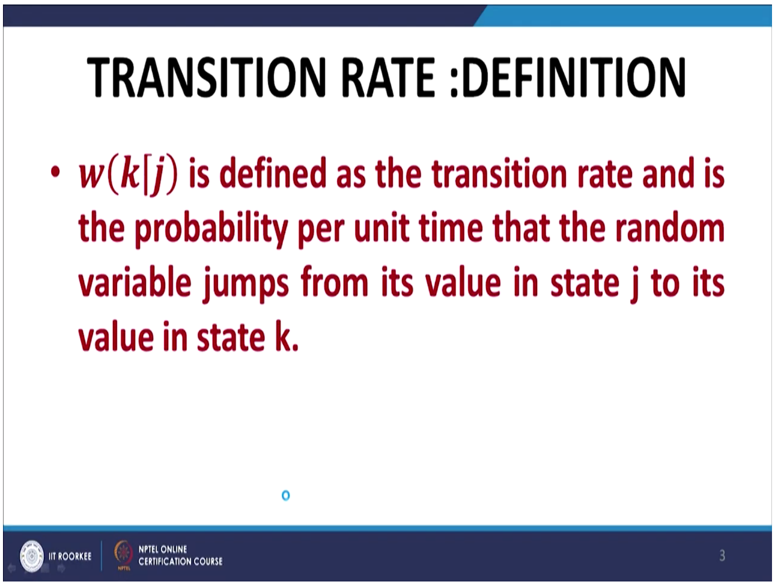

The transition rate if you look at it can be defined as a as the probability per unit time; probability per unit time of the system moving from the state j to the state k.

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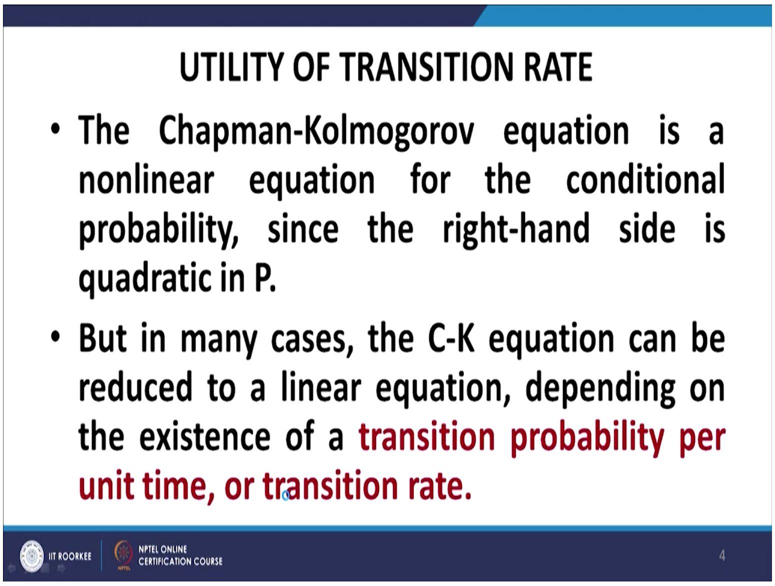

As I mentioned the utility of the transition rate comes into play; because the Chapman Kolmogorov equation is non-linear. And therefore, to simplify it further particularly in the context of stationary processes we can make use of the concept of transition rate which leads us to in fact the master equation.

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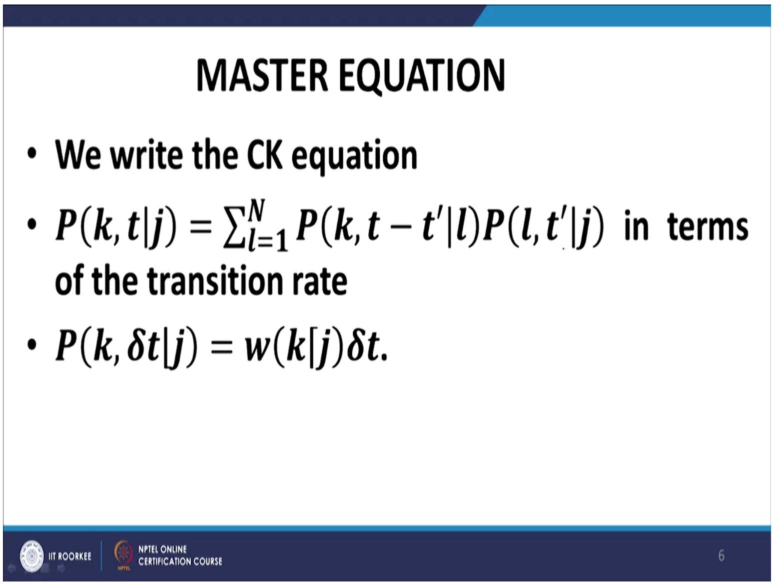

Let us look at the master equation. We start with the Chapman Kolmogorov equation C-K question. I will use C-K for brevity we have this particular equation as the Chapman Kolmogorov equation. We make use of the transition rate P k delta t j as w k j delta t.

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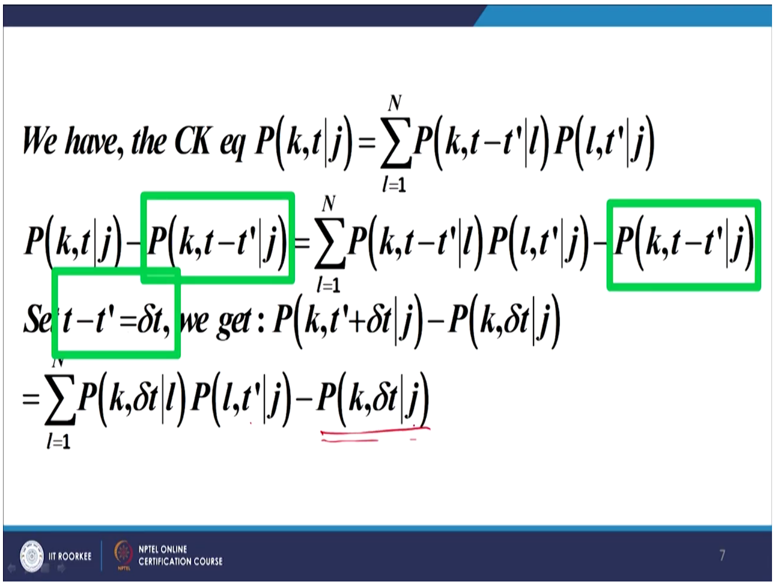

And we write it as P k t j is equal to this expression. This has come from the earlier slide P k t j. This is as it is the first term is as it is this is the original C-K equation and in the C-K equation what I do is I subtract this expression in the green box P k comma t minus t dash subject to j from both sides that is the first step. Then I replace t minus t dash by delta t.

When I replace t minus t dash by delta t, the left hand side becomes P k t dash plus delta t subject to the as say subject to j I am sorry minus P k delta t subject to j. And the right hand side become summation of P k delta t l P l t dash j minus this term we had subtracted from both sides.

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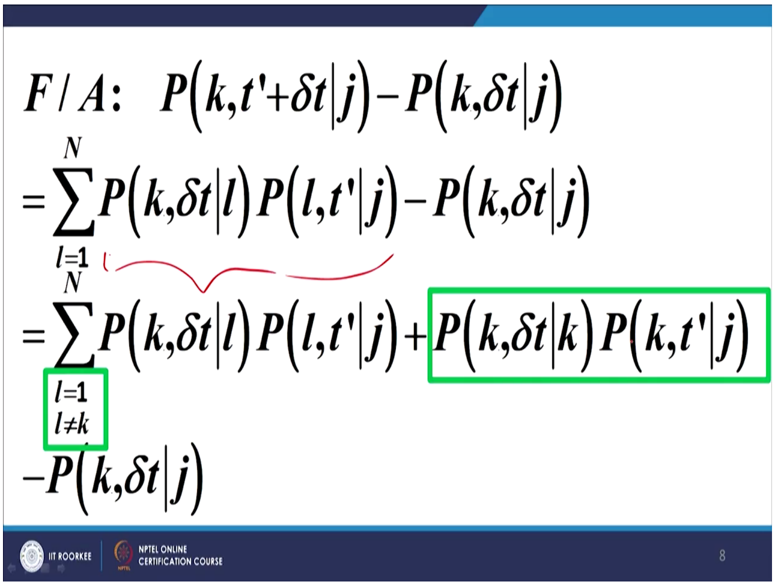

Now, this is what we had from the earlier slide the first equation is what we had from the earlier slide. What we do here is, we now let us look at the summation this particular summation P k delta t subject to l P l comma t dash subject to j at the summation is from l equal to 1 2 l equal to N that is over all states. We write the same summation over all state except the state l equal to k.

So, we introduced a constraint l equal to l unequal to k in the summation; that means, we have to separate out from the summation we have to add additional term which is which corresponds to l equal to k. Because now we the summation covers only all those terms where l is unequal to k whereas, the original equation was valid even for l equal to k. Therefore, we have to separately add on l equal to k and that is this particular term P k delta t subject to k P k t dash subject to j the rest is as earlier.

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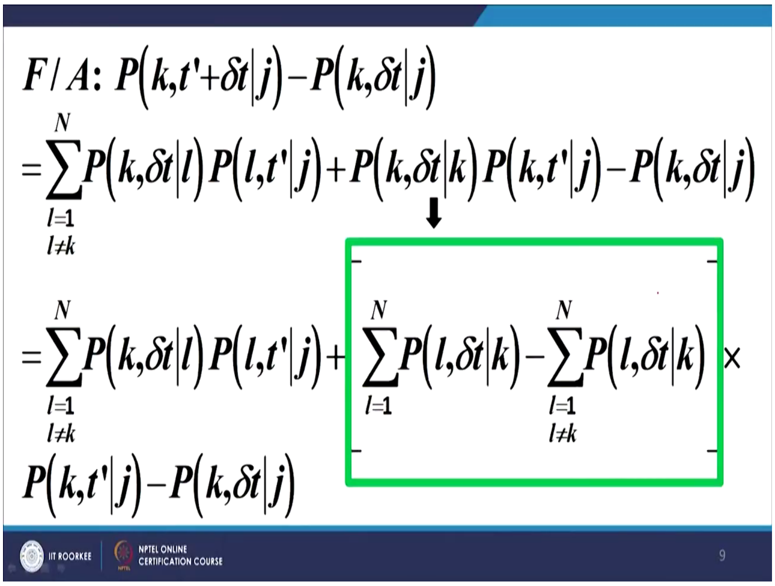

Now, this is what we have from above. Now, if you look at this expression P k delta t subject to k, now P k delta t subject to k can be written as l equal to 1 to N P l delta t k l equal to 1 to N l delta t k minus l equal to 1 to N P l delta t k l unequal to k because l equal to k if you put l equal to k you get this expression. So, I write the total summation over all states and I write the summation over all states except l equal to k. When you subtract these two expression you get simply this expression and the rest is written as it is.

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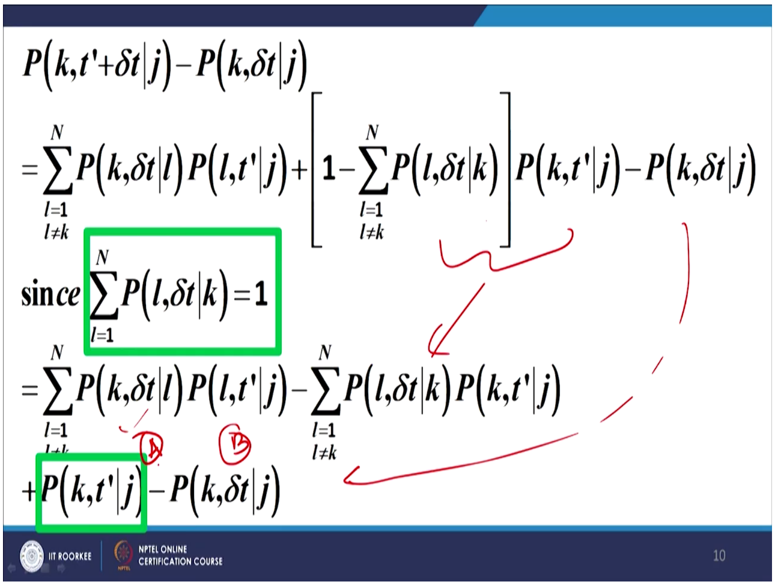

Now, if you look at this particular term which we have from the last slide. If you look at this particular term k P l delta t subject to this is what is this? This is the probability of the system moving to state l in time delta t when it was originally in state k and this is being summed over all possible state cell. So, this must be equal to 1. The probabilities being summed over all possible state cell all possible state cell. Therefore, it covers the entire samples present.

Therefore, it this expression is equal to 1, that is what we have substituted here and the rest is as it is. So, on simplification what we get? This expression as it is and this expression is this particular expression. This expression is this one and this is this particular expression is this one and this expression is this one right. This is here in the green box.

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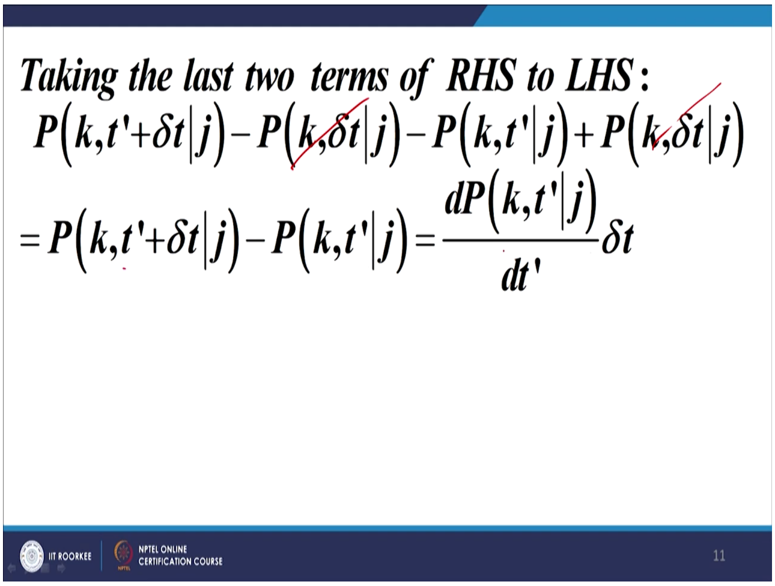

Now, what I do is I shift this last two terms. This is term A. This is term B I shift both these terms to the left hand side. When I shift both these terms to the left hand side look at what happens. This is as it is this and this cancel out right and what are we left with.

We are left with P k t dash plus delta t sub subject to j minus P k t dash subject to j written in the differential form this is nothing but d P k t dash which is subject to j upon d t dash into delta t this minus this this minus this is nothing but in this expression we written in the form of a differential.

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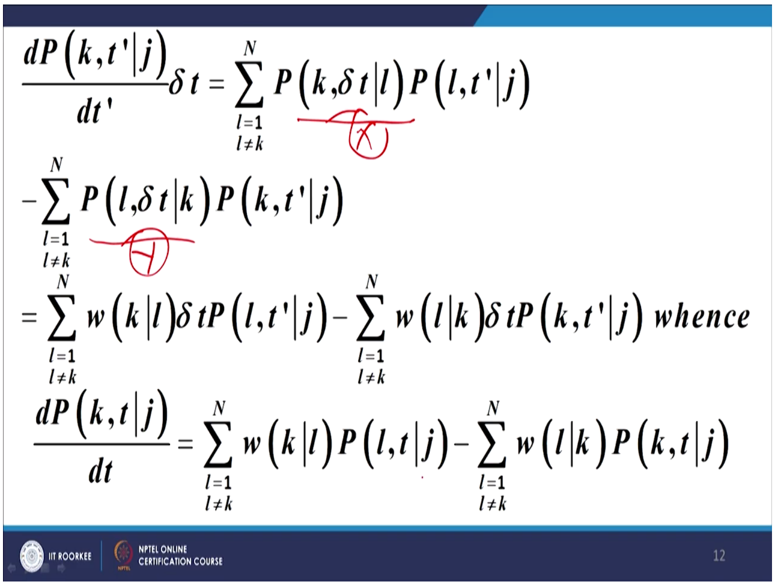

So, the left hand side as now simplified considerably. Left hand side we have d P upon d t dash delta t and look at the right hand side. The right hand side is this expression minus in this expression. We now incorporate the concept of transition rates; because we are talking about a stationary process.

So, when we incorporate the transition rates for this expression, this is let us call it term X and this expression let us call it term Y. We incorporate the transition rates this becomes w of k subject to l delta t and this becomes w of l subject to k delta t and this delta t cancels with the delta t on the left hand side and the we get the final expression d P upon d t is replaced t dash by t all through and we get d P upon d t is equal to this expression.

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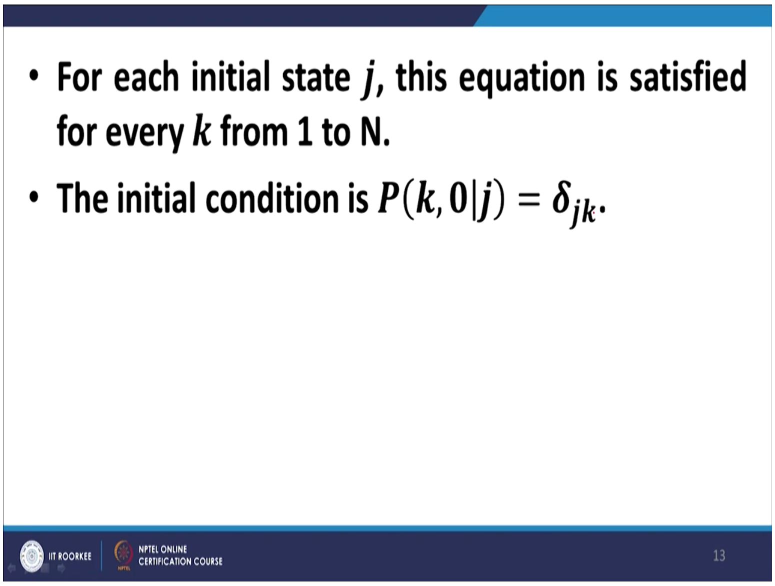

Now, very interestingly you can interpret this term as a rate equation where one side represents the inflow and the other side represents the outflow of probabilities. And the initial condition of course is P k t c comma 0 subject to j is equal to delta k the delta initial condition which is commonly used. And of course the summation it is satisfied for every k from 1 to N for a given initial strategy. Now, given an initial strategy the equation is satisfied for k equal to 1 2 3 for all the states for all the final states 1 to N.

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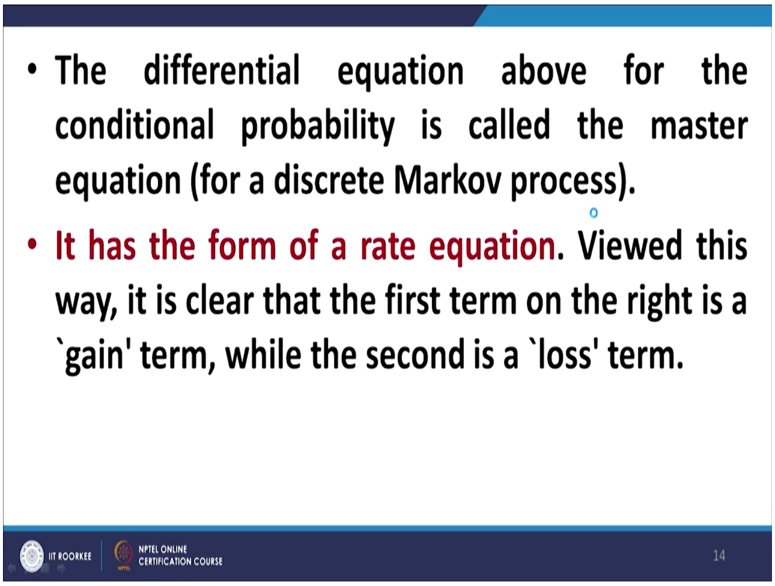

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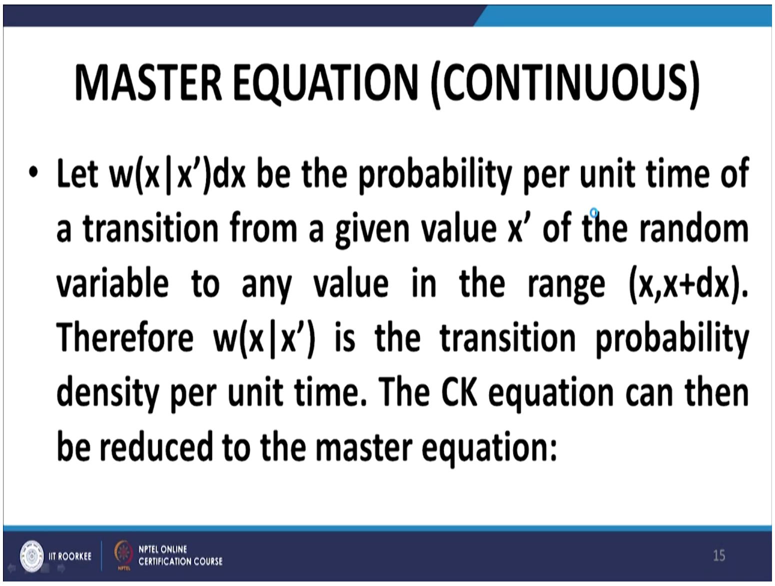

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$$\frac{dP(k,t|j)}{dt} = \sum_{\substack{l=1 \\ l \neq k}}^{N} w(k|l)\, P(l,t'|j) - \sum_{\substack{l=1 \\ l \neq k}}^{N} w(l|k)\, P(k,t'|j)$$

$\Downarrow$ *Continuous variable Markov process*

$$\frac{\partial p(x,t|x_0)}{\partial t} = \int dx' \left\{ p(x',t|x_0)\, w(x|x') - p(x,t|x_0)\, w(x'|x) \right\}$$

with the initial condition $p(x,0|x_0) = \delta(x - x_0)$

This is the master equation as I have told you. Now, in the continuous case the master equation takes this form. As I mentioned we replace probabilities by probability densities and we also replace summations by integrals the rest of it is more or less parallel. Of course the initial condition instead of the Kronecker delta, it shifts to the direct delta condition delta x minus x naught.

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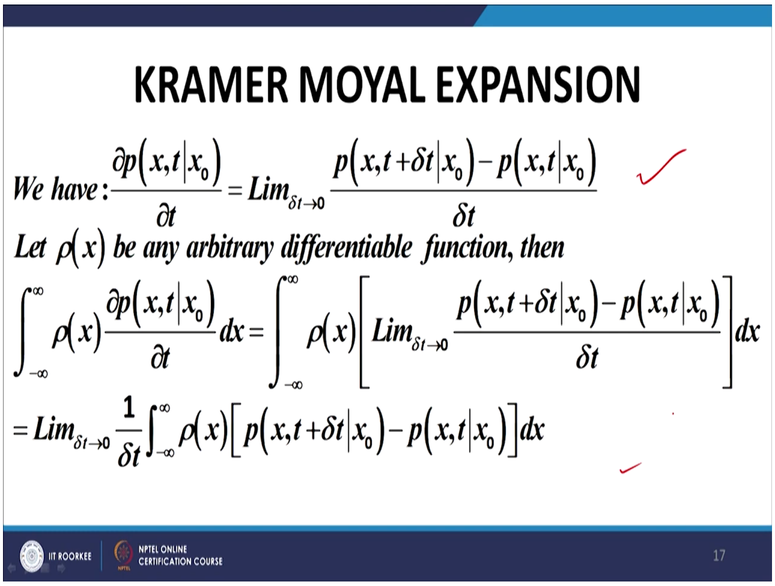

Now, we come to a very interesting extension of the master equation which is called the Kramer Moyal expansion; slightly it elaborate in terms of mathematics, but it is very interesting let us do it through let us go through it slowly. We have this is quite simple the first step it is simply the definition of a differential delta p d p by d t is equal to limit delta t tending to 0 p x t plus delta t subject to x 0 minus p x t subject to x 0 upon delta t. This is simple nothing much about it.

We multiply this expression by an arbitrary differential function rho x and we integrate over the entire spectrum minus infinity to infinity with respect to x. I repeat we take any arbitrary function rho x multiply both sides of this equation by rho x and then integrate it with respect to x from minus infinity to plus infinity this is what we get.

Now, this limit can be brought interchange with the integral sign there is no problem. The integral is with respect to x the limit is with respect to t. So, we can take the limit outside and take the integral inside the inside the limit then we get this expression. So, this is straightforward.

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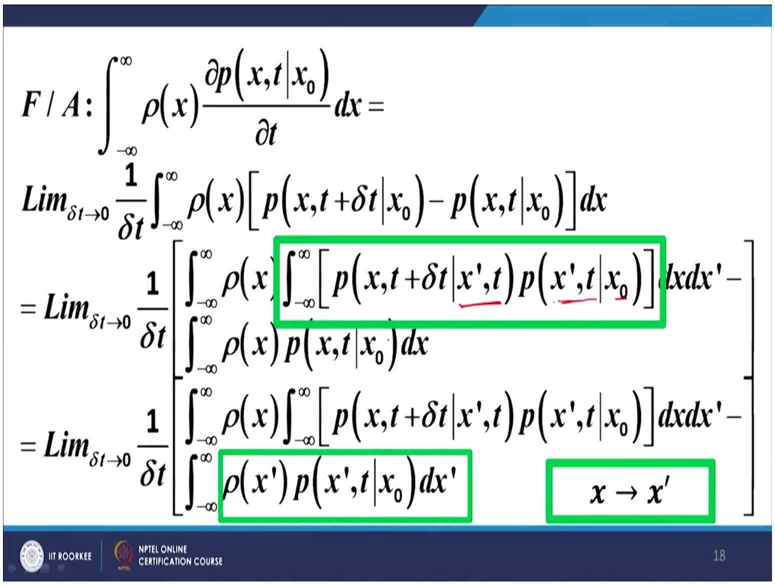

Now, comes the manipulation part. If you look at this this integral this was the left hand side. If you look at this right hand side, on the right hand side what have we done? We have introduced this expression if to draw a parallel is something similar to what we do in quantum mechanic; when we introduce a decomposition of identity between two states to make the calculation similar.

It is something very similar to that. See we have this particular probability p x t plus delta t subject to x 0. In fact, if we have used in a sense in a sense we have used the Chapman

Kolmogorov equation and this because p t plus delta t x dash t. This is an intermediate state and into p x dash t x 0 and integrated over the all the values of the intermediate state which is x dash.

So, to that extent it can be understood as incorporating here in the decomposition allowed by the Chapman Kolmogorov equation. That is what we have done here the second part we have not changed. Now, what we do is we do a similar thing in the second case. What we simply do is we replace the integrity we simply change the integration variable from x to x dash. This is a definite integral. So, we have no problem.

We simply instead of using x we use x dash. We make the substitution x goes to x dash in the second integral not in the first integral. First integral we retain as it is.

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$$F/A: \int_{-\infty}^{\infty} \rho(x) \frac{\partial p(x,t|x_0)}{\partial t} dx$$

$$= Lim_{\delta t \to 0} \frac{1}{\delta t} \left[\int_{-\infty}^{\infty} \rho(x) \int_{-\infty}^{\infty} \left[p(x,t+\delta t|x',t)\, p(x',t|x_0) \right] dx dx' - \int_{-\infty}^{\infty} \rho(x')\, p(x',t|x_0)\, dx' \right]$$

$$= Lim_{\delta t \to 0} \frac{1}{\delta t} \left[\int_{-\infty}^{\infty} \rho(x) \int_{-\infty}^{\infty} \left[p(x,t+\delta t|x',t)\, p(x',t|x_0) \right] dx dx' - \int_{-\infty}^{\infty} \rho(x')\, p(x',t|x_0)\, dx' \int_{-\infty}^{\infty} p(x,t+\delta t|x',t)\, dx \right]$$

$$\text{since} \int_{-\infty}^{\infty} p(x,t+\delta t|x',t)\, dx = 1$$

$$= Lim_{\delta t \to 0} \frac{1}{\delta t} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\rho(x) - \rho(x') \right] \left[p(x,t+\delta t|x',t)\, p(x',t|x_0) \right] dx dx' \right]$$

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So, we are we are here we get this from the previous slide. Now, what do we have. Now, we introduce this expression. Now, please look at this carefully. Look at this first p x t plus t dash subject to x dash t dash dx. Now, we are integrating over x. In other words, what are we saying the system is moving from the state t sorry it is state x dash t to the state x at from time t to time t plus delta t.

In other words, the system started at state x dash at time t and it moved to the state x at time x t plus delta t. If you sum it over all the values of x; that means, all the possible states the system can take on transition and the net result has to be 1. Now, that is precisely what we have used because you are summing over all the possible values of the random variable in the sample space. So, if you sum the probabilities it has to give a value of 1. So, that is precisely what we have done we have introduce this as the unity decomposition of the unity.

Now, if you compare these two expressions, I can take a lot of things common p x plus t x t plus delta t x dash t dash I can take common and I can also take this expression common x dash t subject to x 0. It is also there in both the integrals. So, whatever remains is rho x minus rho x dash into this whole thing is common to both of them and the integration is with respect to x and with respect to x dash.

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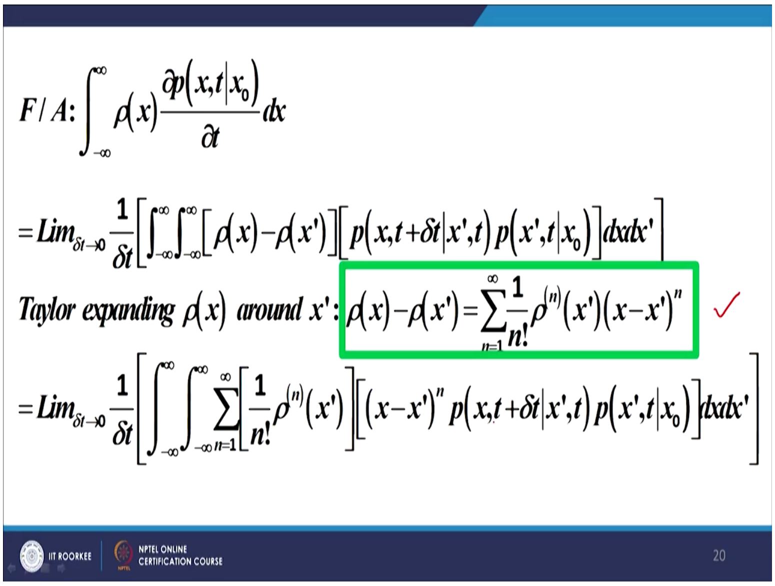

Now, what we do. We expand this rho x around x dash as a Taylor series. We expand this rho x around rho x dash as a Taylor series and this is precisely what we get and we substitute this in our expression. So, rho x minus rho x dash can be written in this form as a Taylor expansion which is given in this box rho x minus rho x dash summation 1 upon n factorial nth derivative of rho x dash x minus x dash to the power n.

We substitute this in the main equation on the right hand side and we get this expression for rho minus rho x minus rho x dash.

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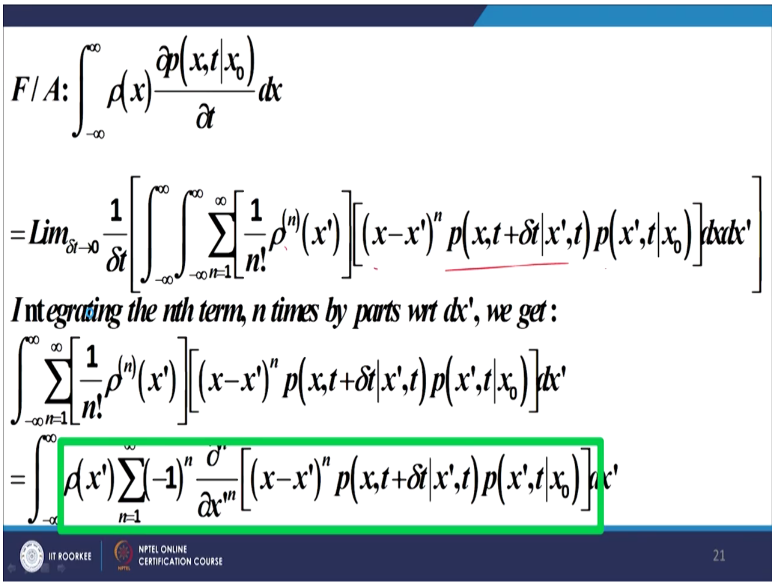

Now, what do we have. Let us say let us say accumulate things. On the left hand side, we had this expression we had rho x and we had this partial derivative and we integrated with respect to x. On the right hand side, outside the outside the integral we had a limit to 1 upon delta t and inside the limit we have this expression for the rho x minus rho x dash and then we had this expression.

Now, if I integrate the nth term this is a summation please note that this is a summation infinite series summation n is from 1 2 n is to infinity. So, this this is a summation of a series of derivatives of order of order n equal to 0 n equal to 1 and n equal 2 3 and so on n equal to 1 to infinity. I am sorry n equal to 1 and n 2 n equal 3 and so on.

Now, what do we do. We integrate the nth term n times. First term one second term twice third time 3 times and nth term n times. We integrate them by parts n times. We take the we differentiate this in this particular expression and we integrate this particular expression.

So, because you are integrating this n times the nth derivative n times you get this expression back and we assume that the boundary conditions are such that the surface term vanishes. And therefore, what we get is what we get is because this is the nth derivative and it is integrated n times.

So, I end up with rho x dash the integration by parts shifts the integration to the other side and we are captured a factor of minus one for each integral for each path for each paths integral. So, that becomes minus 1 to the power n and this differential shift from the first factor to the second factor and this is precisely what is done here.

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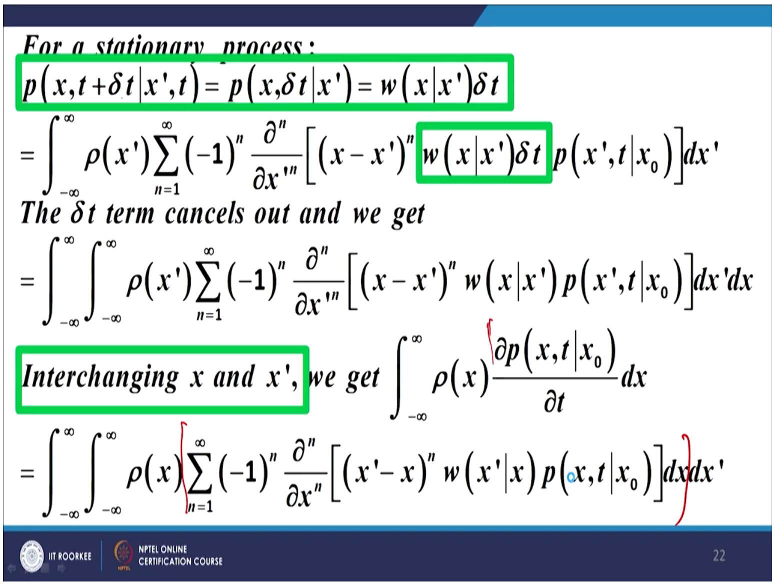

Now, for a stationary process I can introduce what. I can introduce the transition rates right and the I can write this expression as w x p x t plus t dash x dash t. I can shift the origin of time and I can write it as p x delta t subject to x dash at 0 ignoring the 0 not writing it. It is very much there all know and this can be written in terms of the transition rate as w x as x dash delta t.

We substitute this here and we get this expression. The delta t of course cancels out this delta t which I have got by introducing the transition rate cancels out with a delta t which I had when taking the limit. So, that delta t and this delta t goes and we have this expression and now we are almost through the left hand side and the left hand side involves integration of rho.

I will just let us go back. The left hand side is this one integration of rho x and the partial derivative of dp by with respect to d t and the right hand side gives you this whole expression.

Simply interchange x and x dash and we get this is my left hand side this is my right hand side I can that implies if I take this to the left hand side then integral of rho x into this minus this expression take rho x common and you get this integral this minus this whole thing is equal to 0.

That implies dp upon dt is equal to summation the summation term the entire thing. This entire summation this dp this whole thing will be equal to this whole summation of course with the integral.

You see what I have done is let me repeat. This is my left hand side. This is my right hand side. I take this right hand side to the left hand side take rho x common right, then 1 integral is left the after interchanging x and x dash 1 integral is left rho x is outside the integral. This and 1 integral here rho x rho x is outside this summation with the integral is entirely on within the brackets and the whole thing has to be equal to 0.

And that means what? That means, the term within the bracket because rho x is an arbitrary function arbitrary differential function. Therefore, the term that remains within the bracket has to be 0 that is del p by del t or d p by d t minus this summation from here on with the integral with the integral dx has to be 0.

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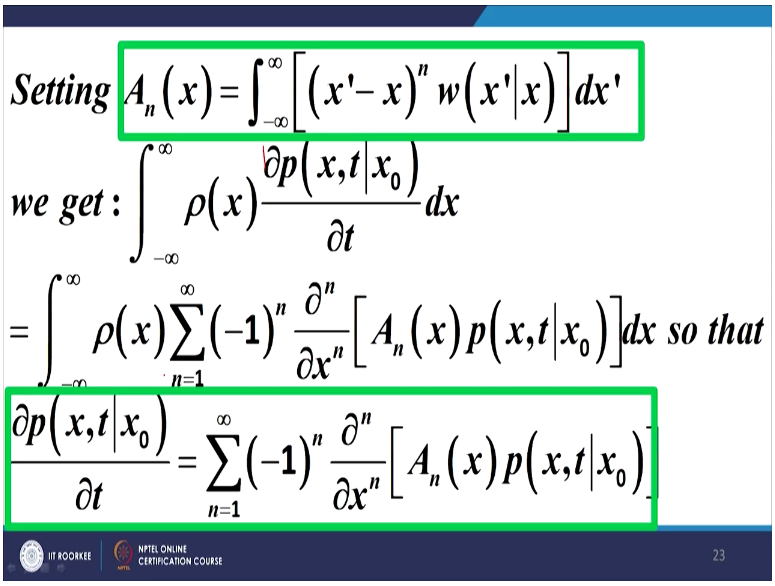

Or we can say that dp by dx is equal to this expression. Now, we simply make certain substitutions to make things simple for us. We write A n x is equal to this expression and dx and we get. In fact, this has been done later. This equation equality has been done later now although it amounts to the same thing. In this particular case, what I have done is first I put the values of A 1 and A 2 and so on and x here.

And then I have taken the things to the left hand side and equated to 0. Either way the result would be the same and what I end up with is here we have rho x into this. I take this to the left hand side and once I take this to the left hand side I take rho x common and what I get this expression is equal to this whole expression. Of course, if you substitute the value of a n x it comes to what we discussed earlier is the same thing.

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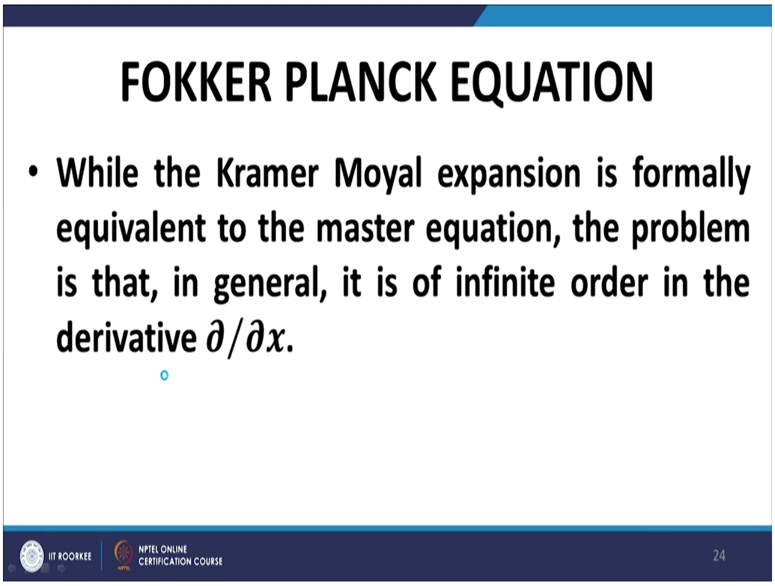

So, this is what is called the Kramer Moyal's expansion. Seems quite involved, but the there is a very interesting aspect to it which we talk in the next slide. Now, formally this is equivalent to the master equation number 1 and number 2 it is an; it has an infinite series of derivatives. So, it is of infinite order in terms d by dx.

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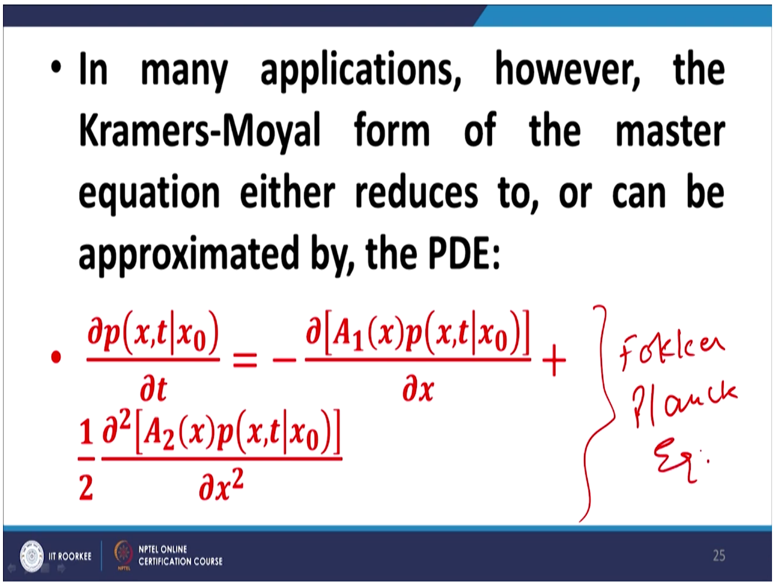

But, the important thing is there is a very interesting result here and that interesting result is that either this equation ends at the second at n equal to 2 or it continues indefinitely.

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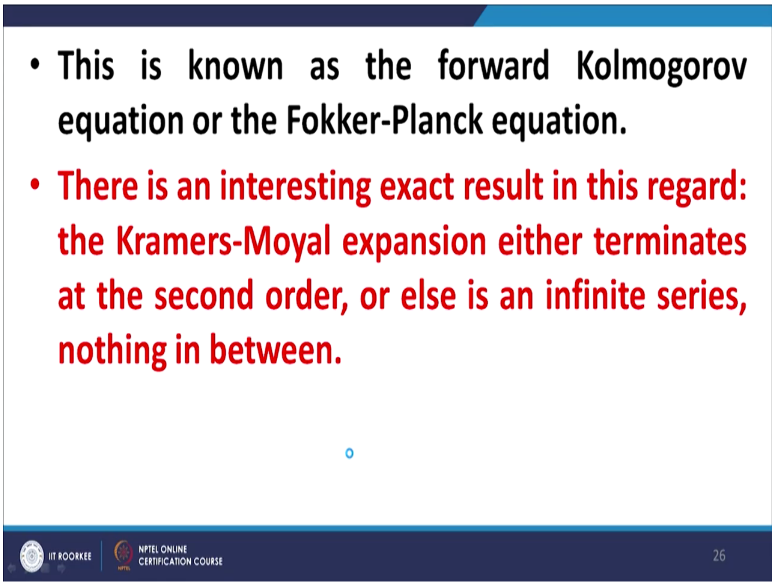

So, therefore, what happens? Either you take the even if you take the first 2 terms in the series this term of course in on the left hand side and this term and this term on the right hand side this serves the pretty good approximation. In most of the cases or in many of the cases the series would terminate at this point and if the series do not terminate then the theorem says that they series would continue indefinitely.

But, even if the series can continue indefinitely the use of only these two terms make up pretty good approximation. So, this is this expression is called the Fokker Planck equation. Which equation? This particular equation. The restriction of the Kramer Moyal form to this particular form is called the Fokker Planck equation.

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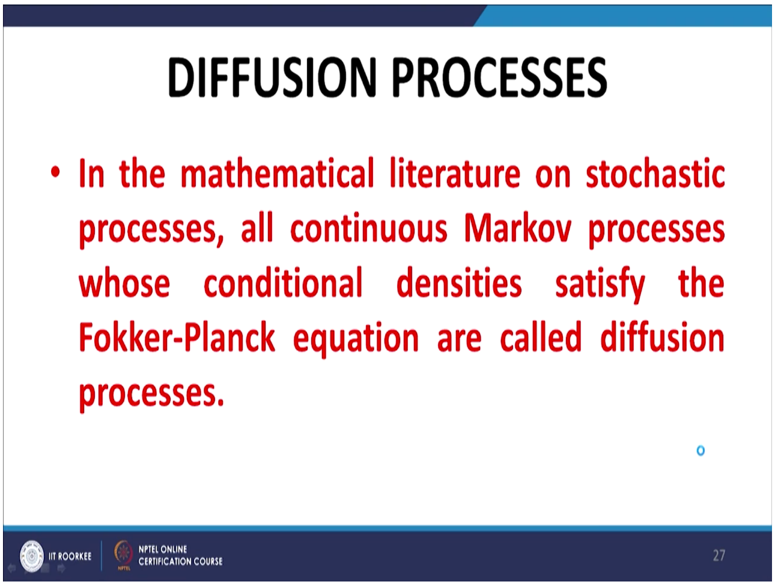

Now, diffusion processes. All continuous Markov processes whose probability densities satisfy the Fokker Planck equation they are called diffusion processes. Drift in the first coefficient relates to the diffusion.

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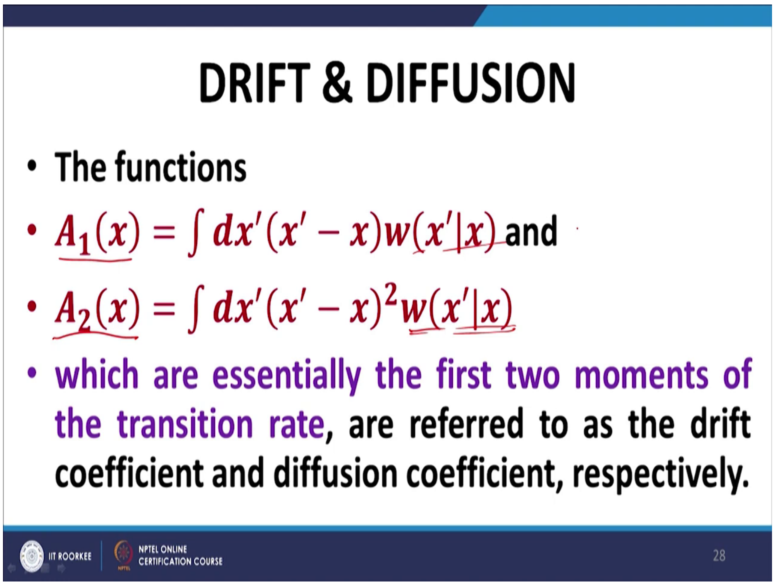

The second coefficient A 2 x relates to this I am sorry the first coefficient A 1 x relates to the drift and the second coefficient A 2 x relates to the diffusion. They are in fact they are the moments of the transition rates.

If you look at it carefully this is the first moment, this is the second moment and these the first moment is called the drift and the second moment constitutes the diffusion and processes that satisfy the Fokker Planck equation, they are called diffusion processes. So, I think we will conclude now and in the next lecture I propose to take up the concept of Brownian motion and then we will move to green functions.

Thank you.