Path Integral Methods in Physics & Finance
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Lecture – 60
Misc Financial Applications of Path Integrals

Welcome back. So, before the break I discussed the Path Integral solution for a path independent option Black-Scholes option. An option which is whose payoff is independent of the path followed by the price of the underlying asset.

And also the option is excisable only on maturity. We now relax this assumption and we say that we want to evaluate an option, which is path dependent. In other words payoff on the option depends on the path followed by the price of the underlying assets. So, now let us look at this problem.

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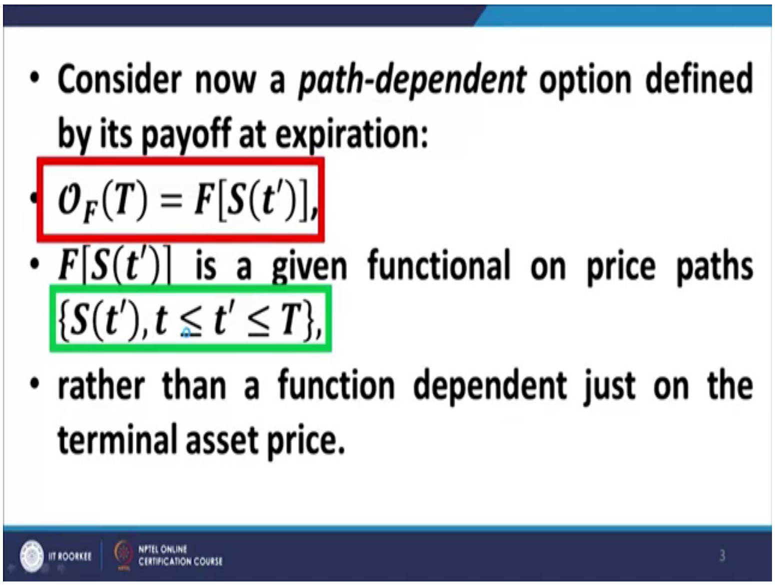

So, in this case the pay off at expiration of the option payoff at maturity of the option will be written in the form of the expression, given in the red box here. Please note the difference compared to what we had in the earlier in earlier case.

In the earlier case we had the payoff be written as F as a function of S of t, where t is the maturity date. Here, we are having F as a function of S of t dash, where t dash is any point between the up to the maturity of the option.

So, F S t dash is now a functional on the price paths given by these expressions, that is the price that is the value of the stock or the price of the stock. From today onwards up to the date of maturity of the option.

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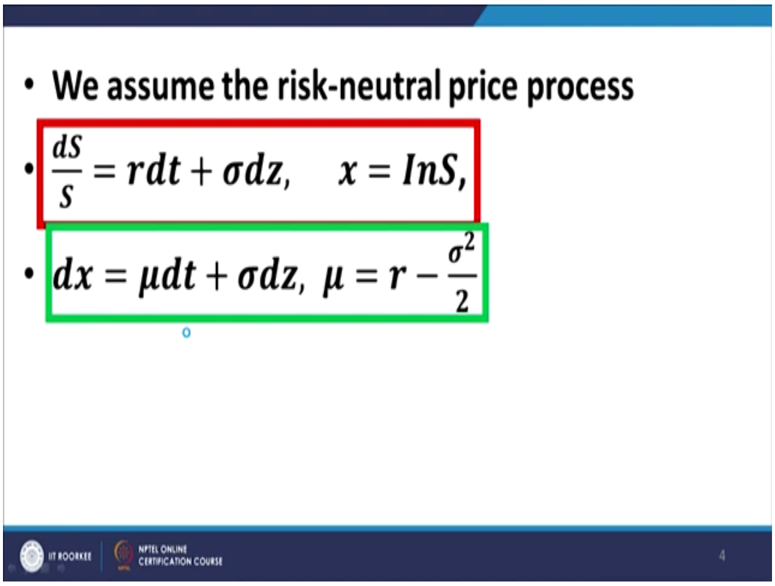

We assume that the stock price follow the risk neutral price process, that is given by this expression. Just to recall dz is the infinitesimal Brownian motion increment standard Brownian motion increment. And it same as d w. And, we have mu is equal to r minus sigma square upon 2, that is the drift term and the diffusion term is given by the component or the by the term sigma.

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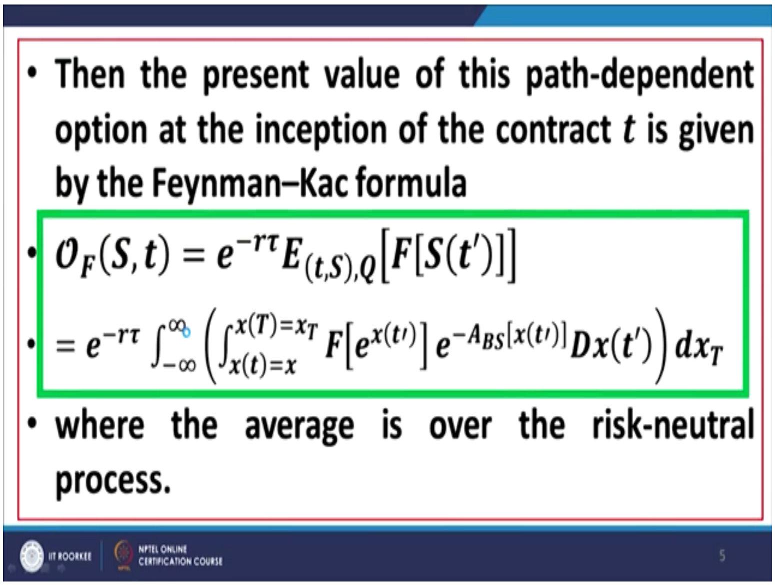

The present value of the path dependent option, at the inception of the contract t that is as of today is given by the Feynman Kac formula we seen this earlier. The only difference is that in the payoff in the earlier case is substituted by the payoff, now which is path dependent.

And the path integral is also change path integral is again the expression within the round bracket. Within the round brackets here, the change is essentially if you look at it carefully, it is essentially in the payoff term F e to the power x t dash. Now, it is a function not of x capital T, but of x t dash.

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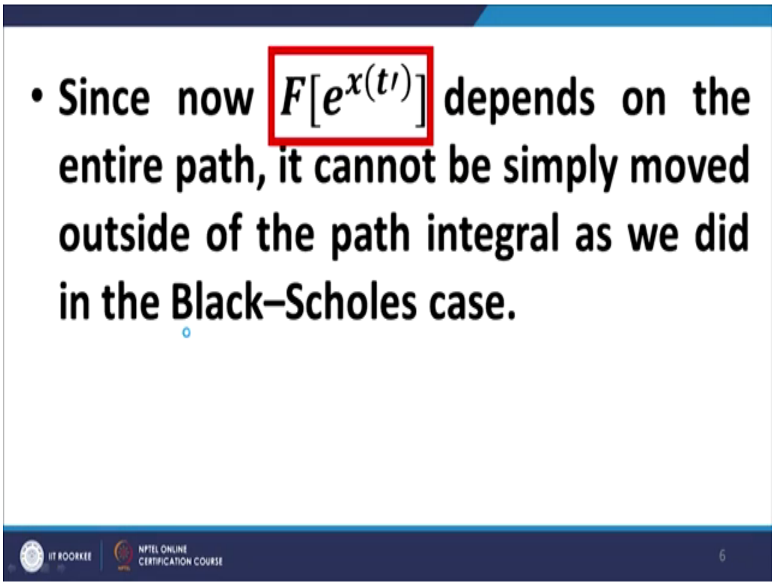

Now, the important now because F e to the power x t dash remember x are logarithms of prices. So, please take note of that F e to the power x t dash depends only on the depends on the entire path, you cannot simply move it outside the path integral. As you did in the Black-Scholes case as you did in the path independent case.

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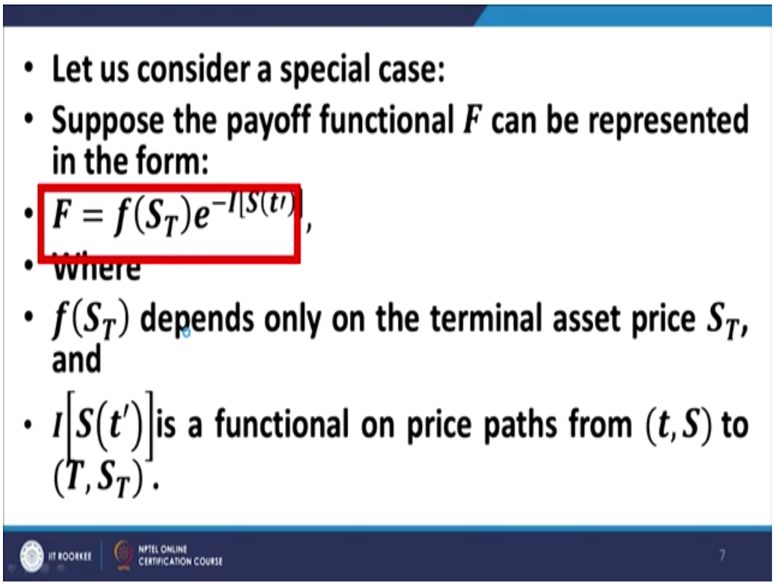

So, now we consider a simplifying case to illustrate the methodology in the sequel. We take a simplification, we take a special case, we assume that the payoff functional F can be represented as a combination of two terms. We write F as small f S of T, where small f is the payoff that relates to the stock price at maturity that is dependent on the stock price of maturity. And another component which relates to the path followed by this stock price.

So, f S capital T depends only on the terminal asset price and I S t dash is a functional on price paths from T comma S that is today to capital T to S capital T. Remember this is this particular term the second term captures the path dependence. And, the first term is a scaling term that relates to the or that is the dependent only on the terminal price of the underlying asset.

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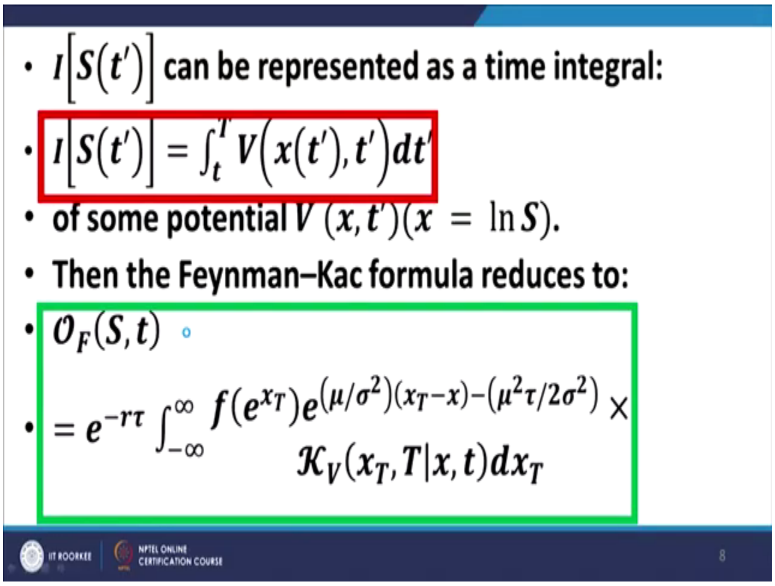

And we can write this I S t or we assume that I S t can be written as a time integral of a potential. We assume that I S t can be written, in terms of a time integral of a potential given by the expression, in the red box here.

We introduce a potential term such that I S t becomes the gradient of that potential with respect to time, or putting it the other way around, I S t is the time integral of the potential with respect to is the time integral of the potential. Is that or you can say that, we introduce a term potential such that this equation is met.

Then the Feynman Kac formula or the path integral formula, can be written in the form which is given here please note the difference compared to the earlier case. We because f because small f e x T depends only on the terminal price, or the terminal logarithm price or logarithm prices mean the same thing.

Therefore, it can be taken outside the path integral. This component of the price can be taken outside the path integral. And the second part that is I S t that is the path dependent component is captured, or it is retained in the path integral.

And that is incorporated therein by this factor of the potential as you shall see just now. And, we have this transition green function or transition amplitude K V x T, T x t. This is the; this is the path integral component K V x T T x, t comma t is the path component path integral component.

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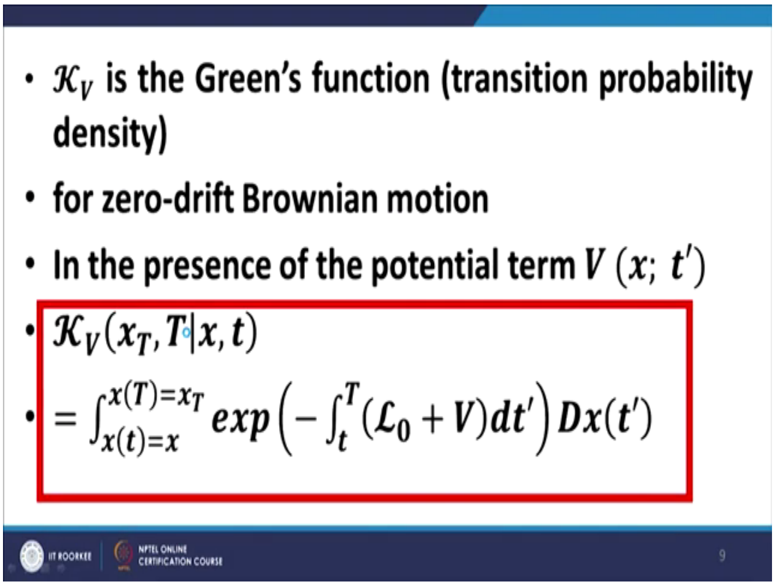

So, K V is the green function that relates to the zero-drift zero drift Brownian motion. But, in the presence of a potential term V x t, where the potential term its is itself related to the payoff of the underlying asset.

Payoff of this derivative I am sorry payoff of the option by this particular expression in the red box. So, K V is given by this expression this path integral, which is here in the red box here right at the bottom of your slide.

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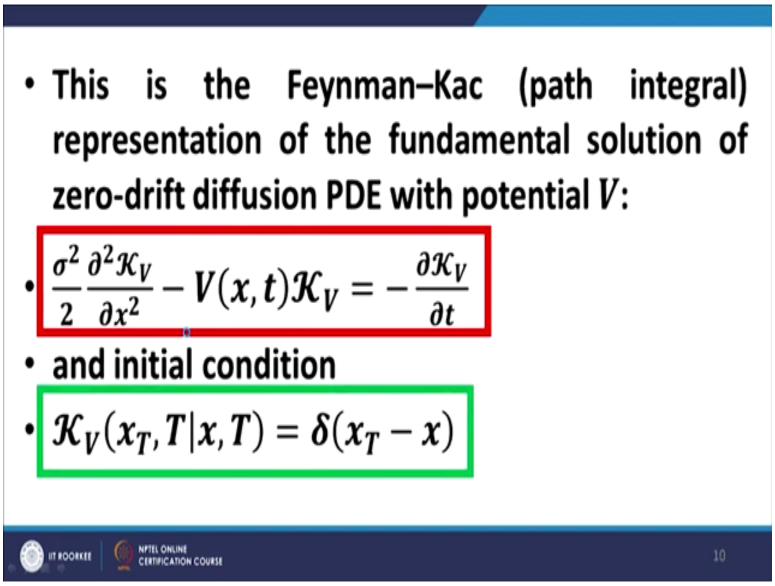

And so, this is the Feynman Kac path representation and it is the fundamental solution, path integral representation of the fundamental solution of the zero drift diffusion equation with potential. Zero drift diffusion equation with potential give V x comma t, and the diffusion equation is shown in the red box here. The initial condition as was the case earlier is shown in the green box here.

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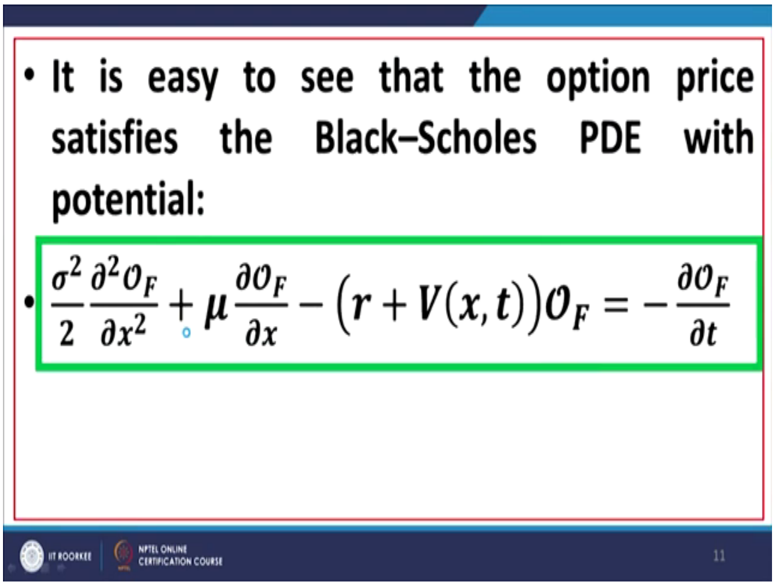

The option price, which we will obtain on solving this particular set of equations. will satisfy the Black-Scholes equation, but with an additional potential term which is shown in the green box here.

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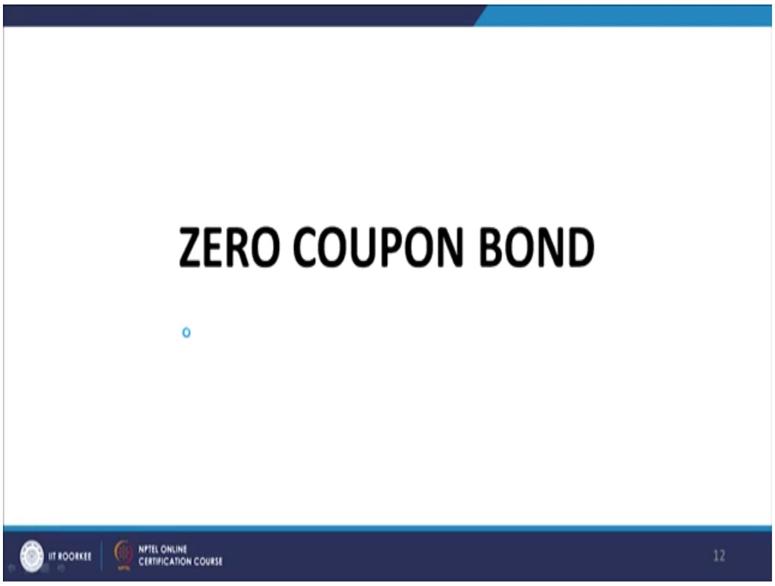

So, this is the way we can this is the skeleton of the way, which in which we can obtain solutions to the path dependent option problem. You see the problem is that in such situation in such cases, it is more often than not the case that closed form solutions are not possible or not obtainable.

And therefore, the only way out is to obtain numerical solutions, through numerical integration or numerical solution of the differential equations or numerical estimation of the path integrals. But, nevertheless the message that is conveyed here is that by using the concept of path integrals.

We are able to have a versatile tool by which we can attack such problems and resolve such problems, in cases where this the payoffs are highly complex highly intricate. Now, what I proposed to do is? I proposed to highlight some more applications of the path integral, formalism in the context of finance and related areas.

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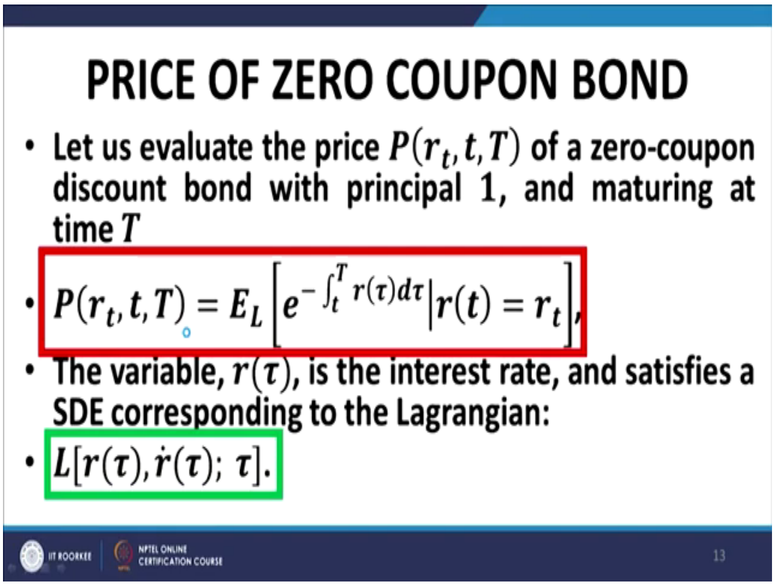

Now, let us talk about the Zero Coupon Bond. A zero coupon bond is a bond that is usually sold at a discount and it is redeemed at par and, it does not pay any coupons it does not pay any interest during its life. In other words it is sold at a price below its par value and on the redemption date it is redeemed at its par value. And the difference between the two constitutes the yield, constitutes the return for the investor.

The zero coupon bond as its name implies does not pay any interest coupons during its life. So, suppose we have to evaluate a zero coupon bond, we have to work out the current price of a zero coupon bond. Let r t be the instantaneous interest rate, t is at time t r t is the instantaneous interest rate at time T.

And capital T is the maturity of the bond at which it will be pay 1 1 unit of money, in which is its phase value which is its redemption value. And capital T is its maturity as we have been denoting earlier.

Then, the current price of the bond is the present value of the is the present value of the future cash flow that is the redemption value because, there are not going to be any coupon. So, it is going to pay only the redemption value. So, the current price will be the present value of its redemption or face value redemption value as the case may be if in case they are not identical then of course, it will be the redemption value.

And, but the important thing is when we work out the present value, the interest should be the relevant interest rates prevailing during each instant of the period of time, between today and the maturity of the bond. That is what is captured by r t.

And therefore, put I am putting it into the form of an equation. What I have just stated? Is that the current price of the bond is the expected present value this is what is present value of the because, the principal value is 1, or the redemption value principal value face value is 1.

So, we in that factor goes and this is expected expectation of the interest rates prevailing over the life of the bond. Now, the have the variable r tau r tau is the instantaneous interest rate prevailing at time t equal to tau. Today, is small t and maturity is capital T keep that at the back of your mind.

And this stochastic differential equation can be obtained from corresponding to the Lagrangian, which is given in the green box here.

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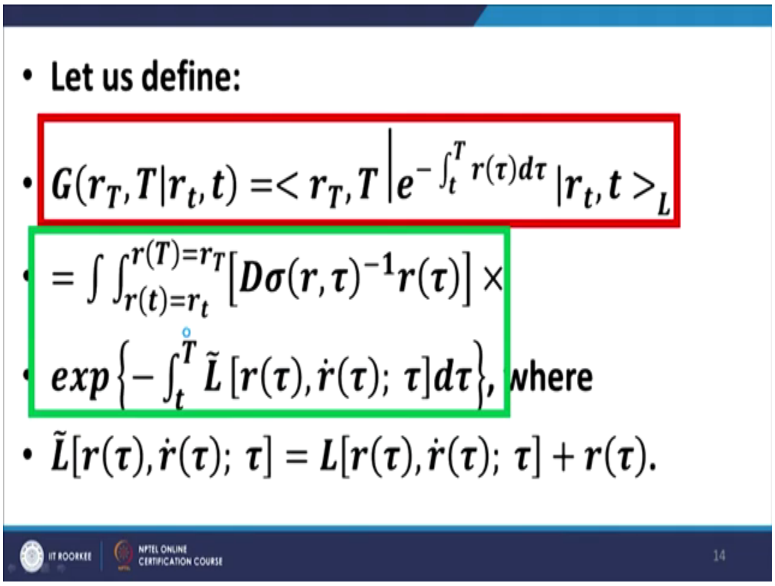

And the we define the expression the matrix elements in fact, as the as we have done in the case of quantum mechanics. As the expression that is here in the red box.

And, in terms of the path integral this matrix terms or the matrix elements take the form, which is given in the in green box here. Where L tilde is the Lagrangian modified Lagrangian, which is given by the expression right at the bottom of your slide.

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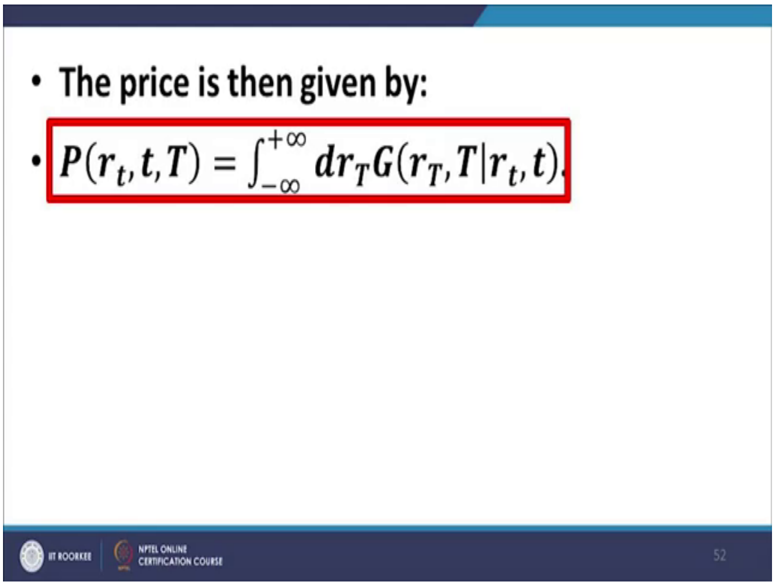

A and using this expression using this expression for G. We can obtain the value of the price by integrating, this expression between minus infinity and plus infinity with respect to the interest rates. With respect to the interest rates at maturity.

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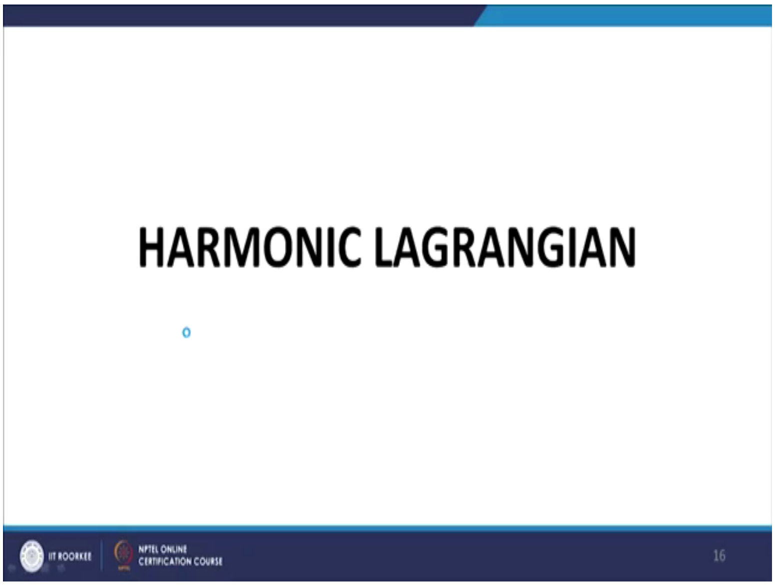

Now, we talk about the harmonic Lagrangian because, it has an application to a very important interest rate model.

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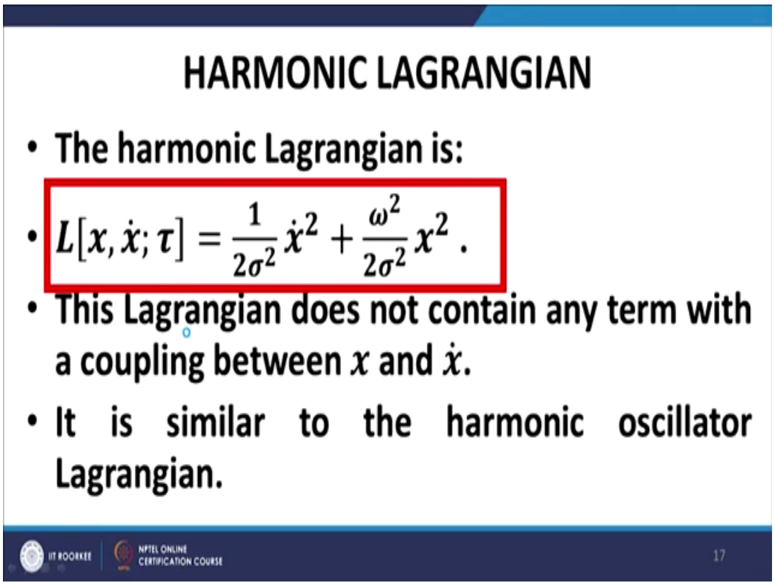

The harmonic Lagrangian is given by this expression here. Here, just recall that this Lagrangian is very similar to the Lagrangian that we worked out for the harmonic oscillator, which we solved for in the context of the quantum mechanics its absolutely similar.

The Lagrangian is absolutely similar except for a sign difference here, there we had a minus sign here we have a plus sign here. And the Lagrangian does not contain any cross terms here.

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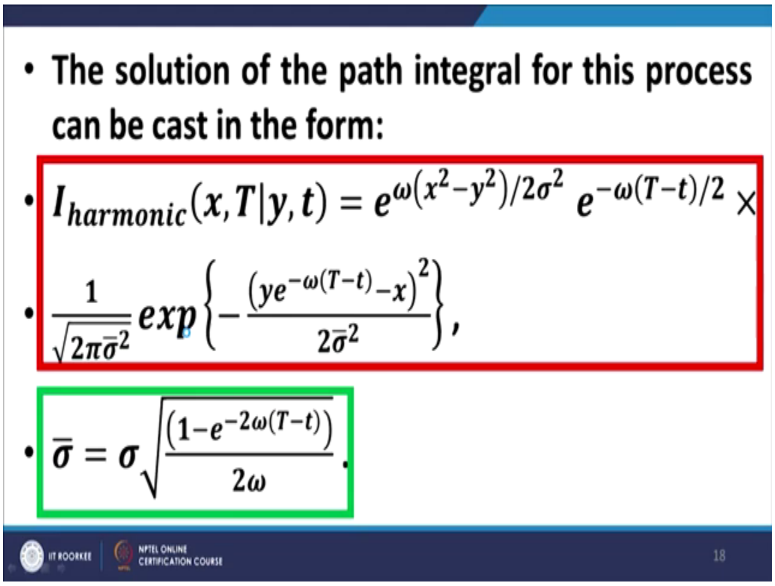

The solution to the path integral can be obtained, in absolutely the same way, as we have obtained had obtained the solution using Fourier transforms. In the case of the quantum mechanical problem of the harmonic oscillator. And the solution that we obtained is given in the red box here, where sigma bar is taking the expression given in the green box here.

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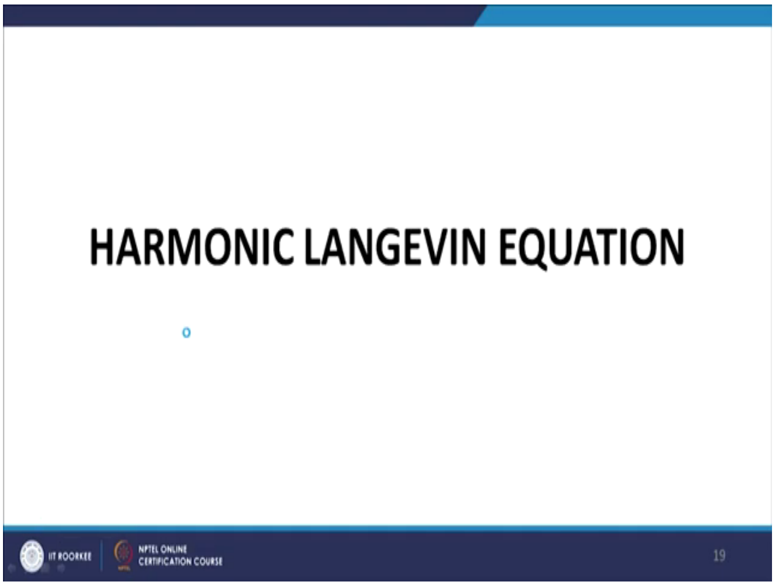

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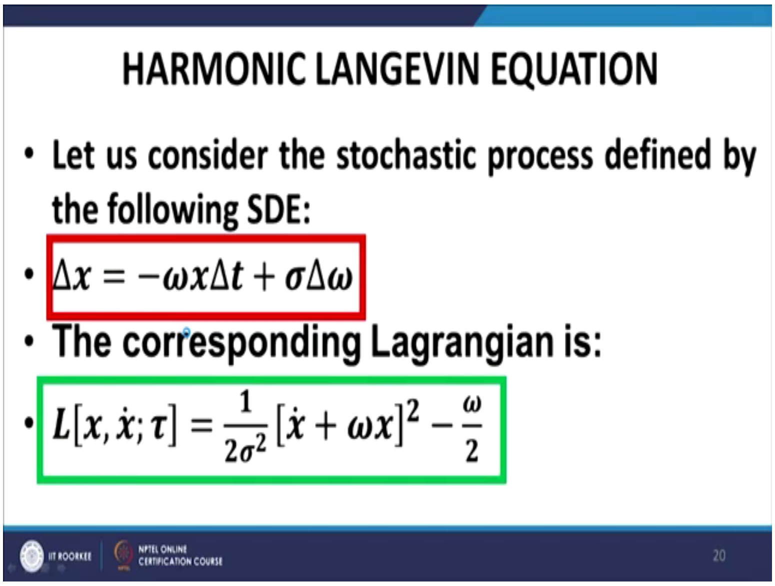

Then, now the Harmonic Langevin Equation this is the harmonic the equation that is given in the red box here is the harmonic Langevin equation. And the corresponding Lagrangian is given by the expression here in the green box.

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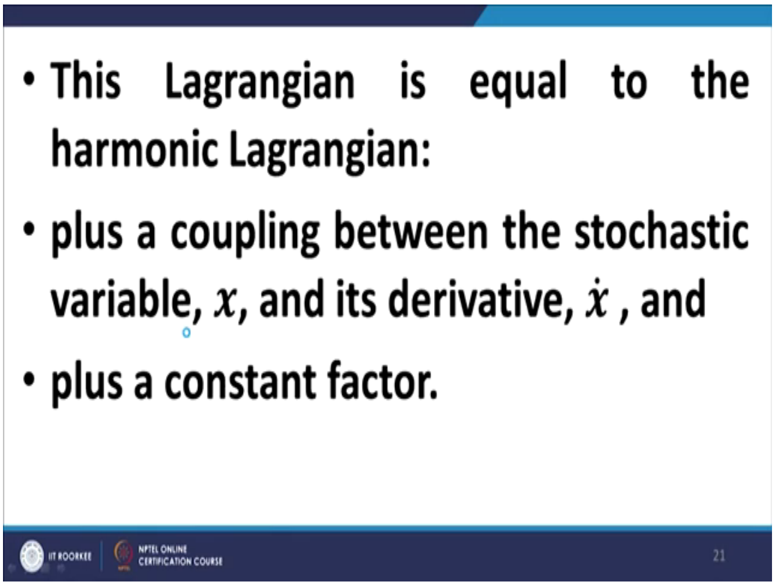

And now, this expression if you look at the Lagrangian here carefully. This Lagrangian is very similar to the Lagrangian the in the earlier case, immediately preceding case. The harmonic Lagrangian except that there, is a coupled coupling term between x and x dot and there is a constant factor as well.

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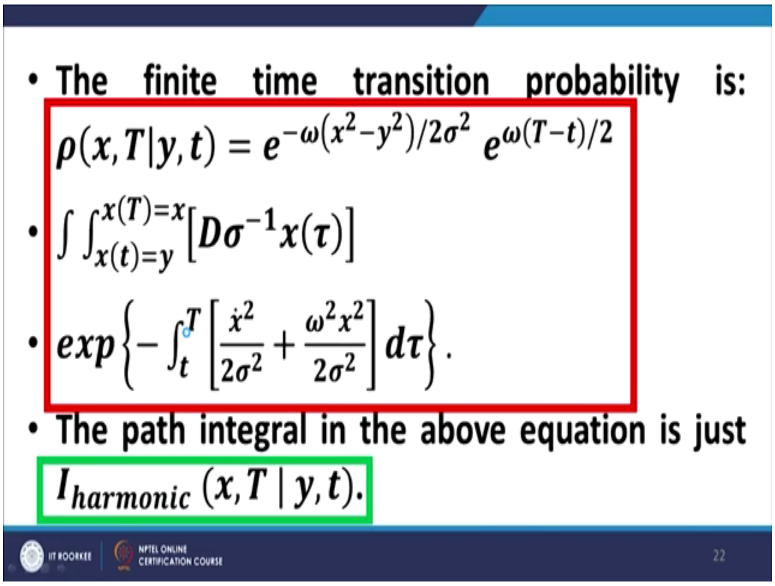

The finite time transition probability, we can obtain again in the same way, and it is given by the expression that is here in the red box. And the path integral the path integral in this case path integral, that we have here in this finite time transition probability rho x capital T y t.

This particular part the second part, that is the path integral part is the same. As we have for the harmonic case for the harmonic oscillator or the harmonic case, that we discussed just a few minutes back.

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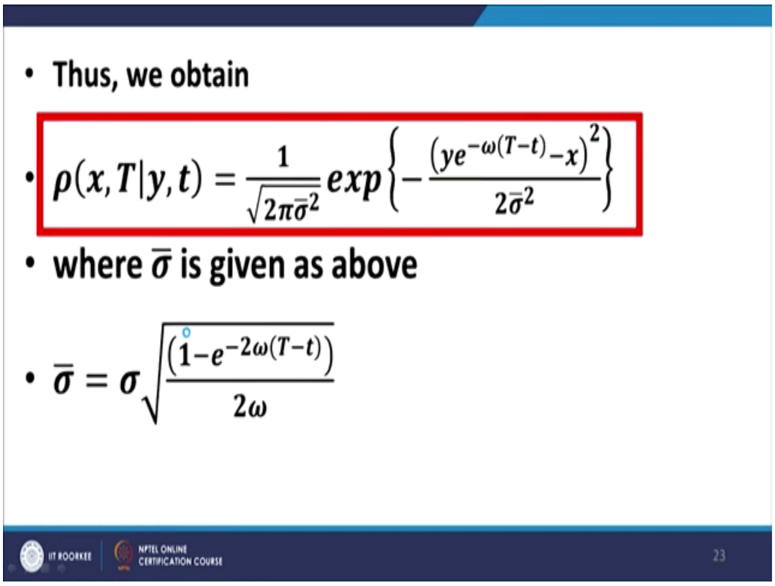

Just before the Langevin equation one. And putting that value here, we get the expression for the transition, for probability as the expression that is here in the red box. Where, sigma bar is given by this expression right at the bottom of the slide.

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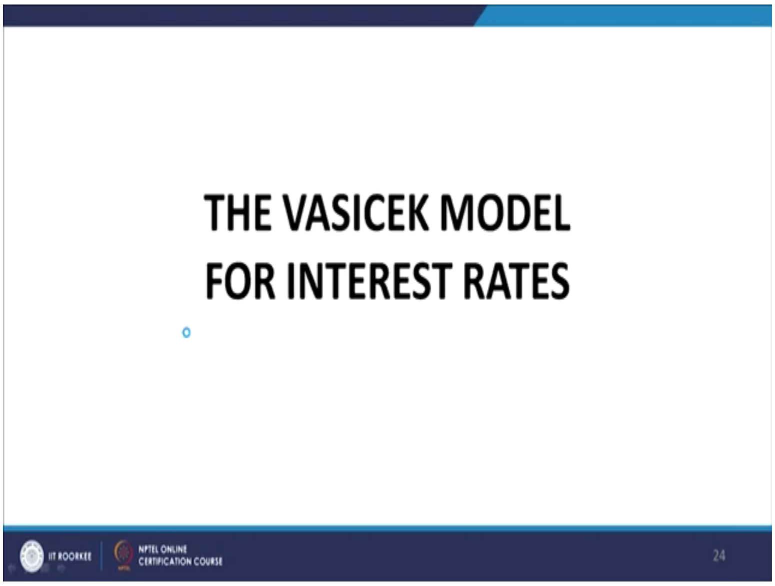

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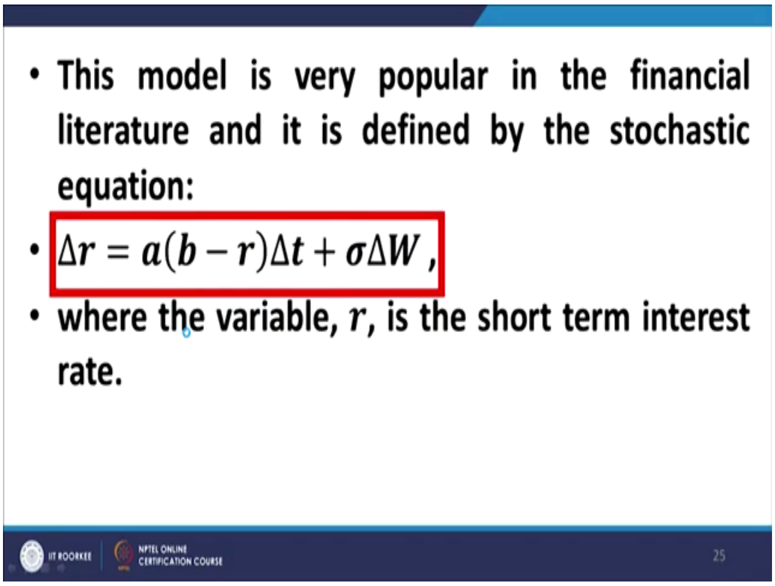

Now, Vasicek model is a very important model for short term interest rates. And the model is described by the equation, that is given here in the red box. Where the randomness is captured by the say these second term and the first term is giving the drift.

The second term gives the randomness as here, delta W is again the infinitesimal Brownian motion, increment delta r is the change in interest rates.

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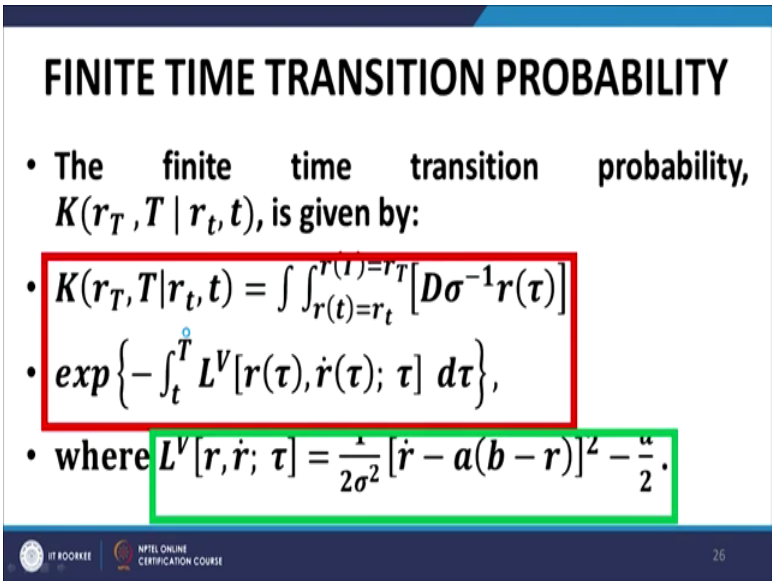

And, we obtain the finite transition probabilities in this model the Vasicek model by the path integral which is given here, in the expression in the red box. Where the Lagrangian takes the form which is given here in the green box.

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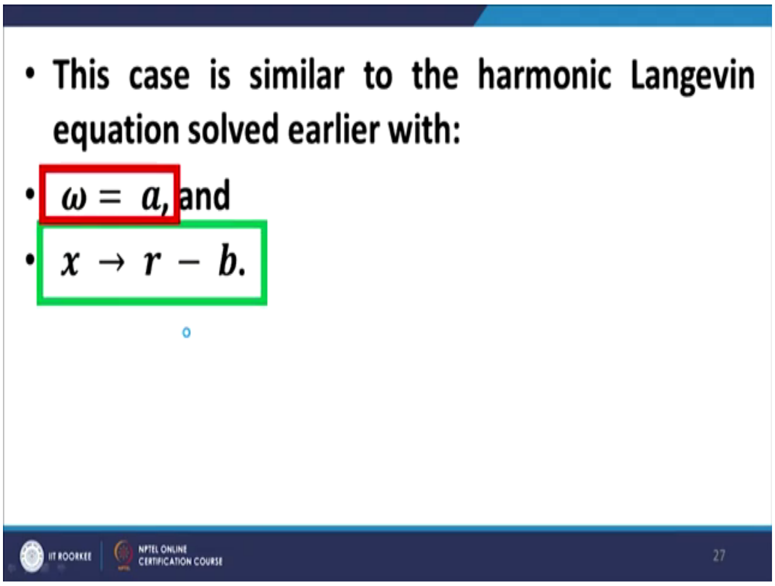

Now, as you can see it if you simplify this case is similar to the harmonic Langevin equation. With the substitutions omega is equal to a and x is equal to r minus b. The path integral can be worked out simply by using the results of the harmonic Langevin equation. And, we get the expression which is here in the red box. Where L tilde is the modified Lagrangian, which is given by the expression in the green box here.

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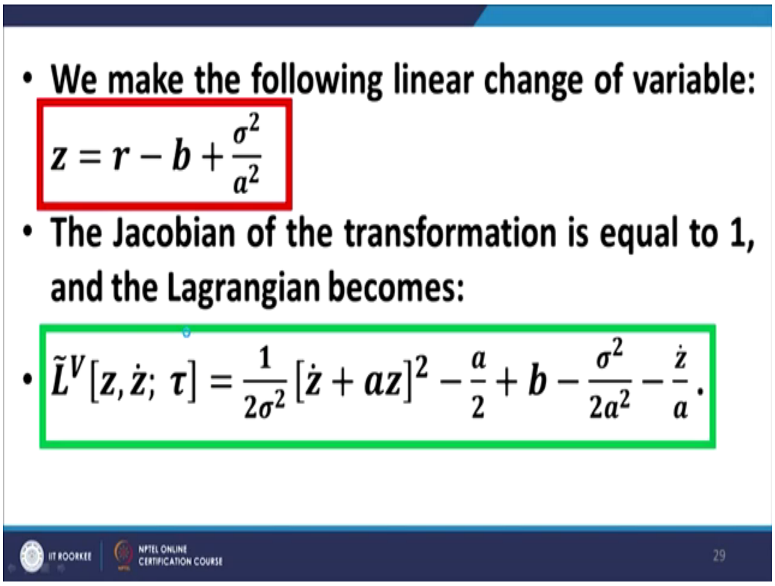

By making this linear change in variable, variable that is given in the red box here. We are able to do some simplification, we can write the modified Lagrangian L. L tilde in the form which is given in the expression that is in the red box here. Now, if you look at it the first three the last three expressions.

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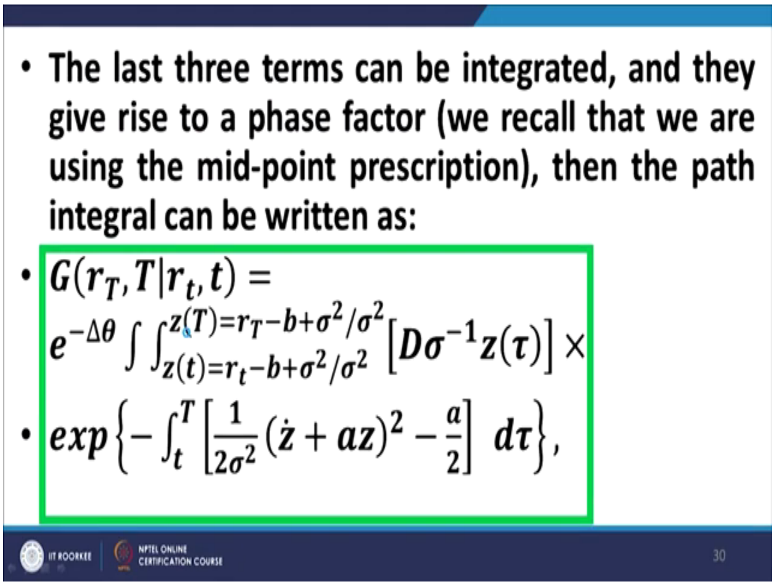

The last three expressions can be integrated straightaway. And, they give rise to a phase factor and the phase factor is captured in the pre factor that is e to the power minus delta theta. And the rest of the terms are retained in the path integral.

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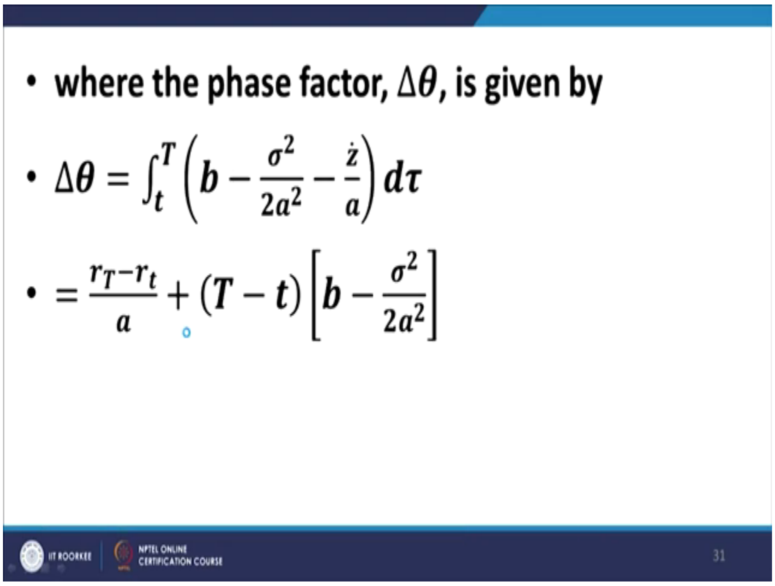

Delta theta is given by this expression, which when simplified can be written in the form of this expression.

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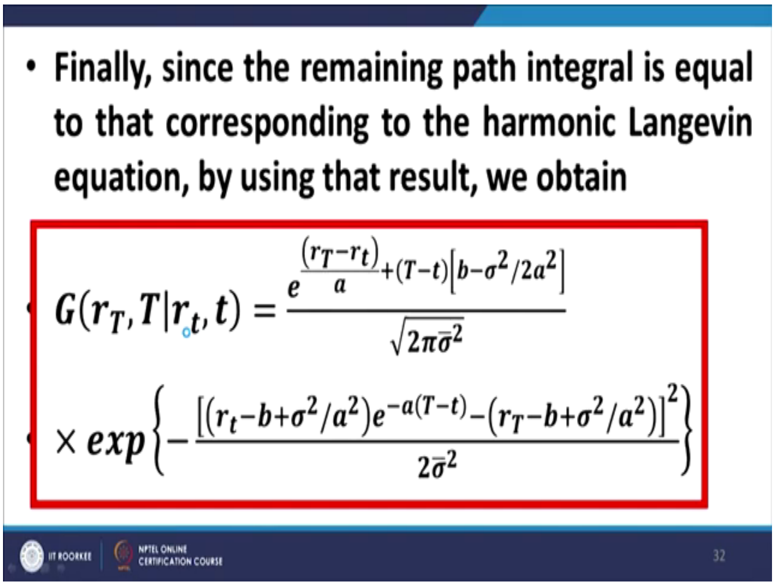

And, as far as the remaining path integral is concerned, we can use the results of the harmonic Langevin equations. And, we can arrive at the result which is here in the red box.

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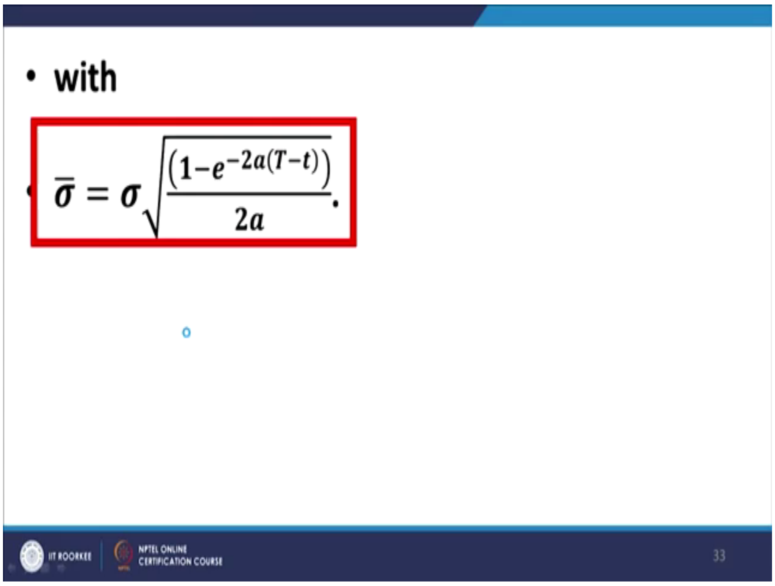

With sigma bar given here in the red box, as the modified standard deviation volatility.

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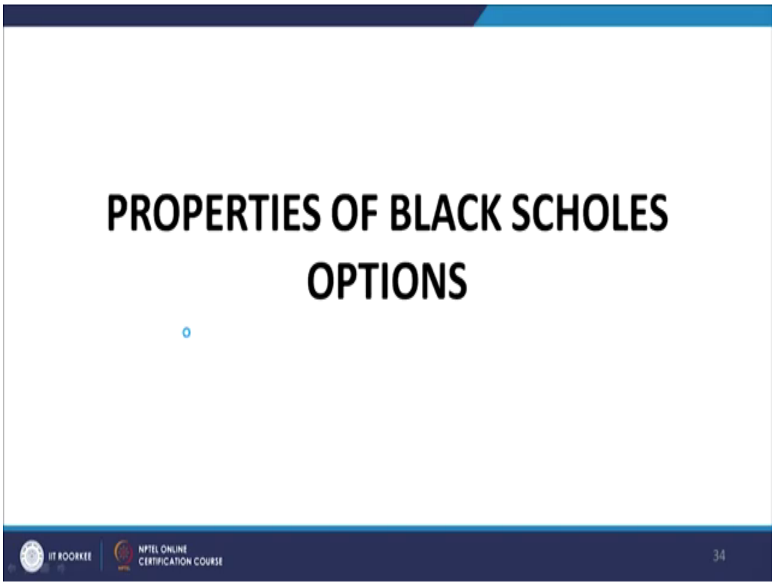

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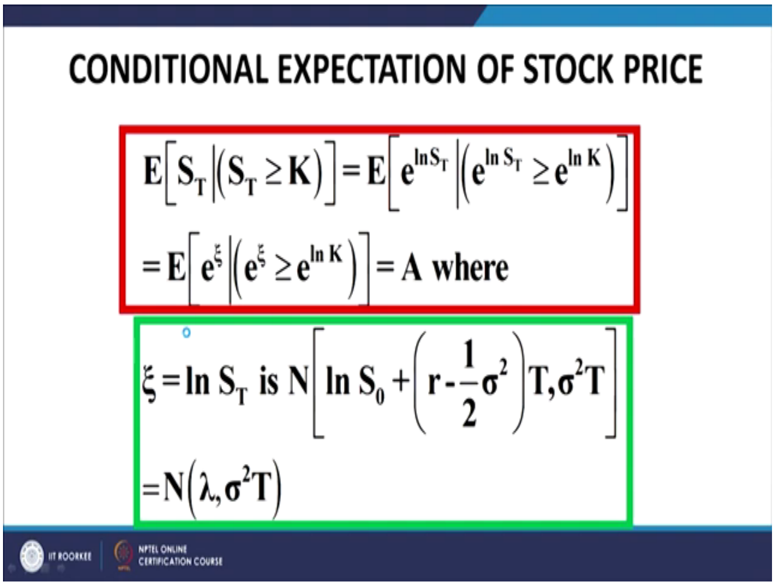

Now, some important properties of Black-Scholes equations quickly, let us run through them. Now, if the conditional expectation of stock price. If you look at this if you the conditional expectation of stock price E of S T remember, we are talking about as on maturity. So, E of S T subject to S T is greater than equal to K log of x is a monotonic function of x.

We can write it in the form E expectation of e log of S T, E S T is greater than equal to e log of K. The same thing they mean the same thing because, log of a variable is a monotonic function of the variable. And therefore, that being the case we can now substitute log S T as xi and we get the expression that is given in the a second line of the red box, which we write as a.

Where xi is equal to log of S T and remember log of S T is normally distributed with a mean of log S naught plus r minus 1 by 2 sigma square T comma sigma square T. Mean, of the mean of this and a variance of this remember we are working in a risk neutral world. We substitute this log S 0 plus r minus 1 by 2 sigma square T, as lambda and sigma square T as we retain as it is.

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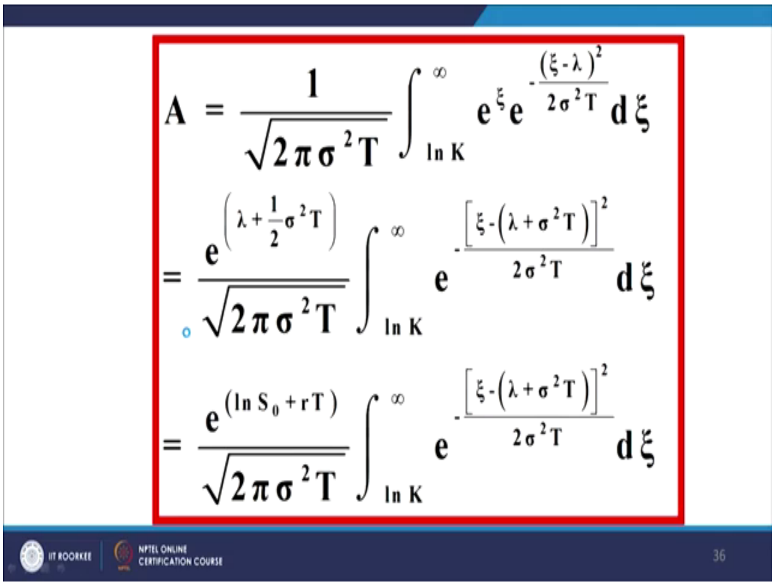

So, a is equal to this expression, now the integration variables why log K to infinity. Because, we want to find out the value of S T subject to S T is greater than K, which in the in terms of the new variable means..

That we want to find out the expectation of e to the power xi subject to xi greater than log K. e to the power xi subject to xi greater than log K. That is precisely what is here and when we do the simplifications, when we do this lot of simplification lot of algebra here.

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$$\text{Set } z = \frac{\xi - \left(\lambda + \sigma^2 T\right)}{\sigma\sqrt{T}} = \frac{\xi - \ln S_0 - \left(r - \frac{1}{2}\sigma^2\right)T - \sigma^2 T}{\sigma\sqrt{T}}$$

$$= \frac{\xi - \ln S_0 - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \text{ so that } dz = \frac{d\xi}{\sigma\sqrt{T}}$$

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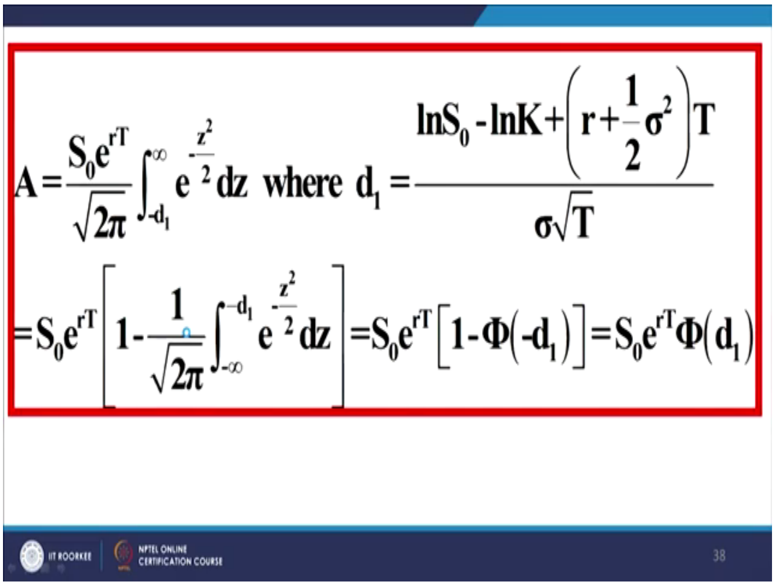

And after doing all these algebra, what we end up with is S 0 e to the power r T phi d 1. So, this is what this is the; this is the conditional expectation of the stock price finishing and greater than the exercise price of the option S 0 e to the power r T phi d 1. Now, what is the need for this let us see?

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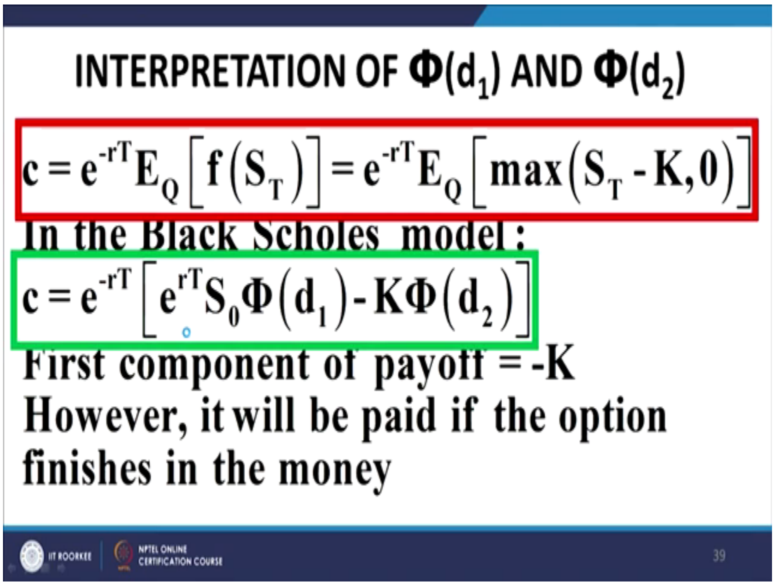

Interpretation of phi d 1 and phi d 2 and the Black-Scholes solution. That this is an important point. Now, let us quickly do it c is equal to this is the formula, that we have standard used it again and again.

So, we can before the Black-Scholes call the payoff is maximum S T minus K comma 0. So, we write it as it is. So, c can be written as e to the power minus r T, e to the and in the Black-Scholes model c is equal to S naught phi d 1 minus K e to the power minus r T phi d 2.

You take e 2 the power minus r T common you get this expression inside this square bracket. And the first component of payoff is let us analyze first component of payoff is minus K, minus K is what the payment of the exercise price by the holder of the call option. That will be payable if the option is exercised. And if the option is not exercised it will not be pay.

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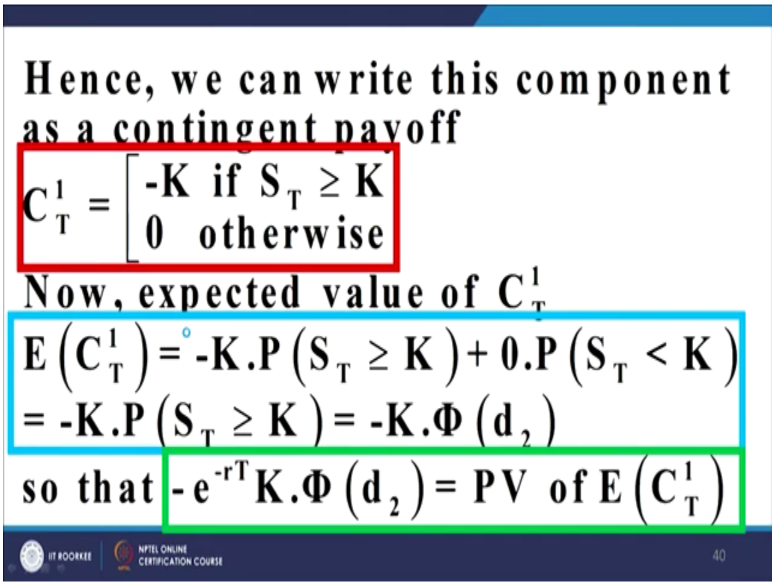

So, this component of the payment let us call it C 1 will be minus K, if S T is greater than K if the option is exercise. Then I will pay this particular exercise price and 0 otherwise. Now, if the probability of S T greater than K is given by phi of d 2. So, expected value of this is equal to minus K P S T greater than K into 0 into this is simple.

So, minus K into this is minus K phi d 2 because, probability S T is greater than K is nothing, but phi of d 2. And therefore, it this is as far as the first component, but because this is going to happen at maturity its present value is e to the power minus r T K phi of d 2. This is as far as the exercise price is concerned.

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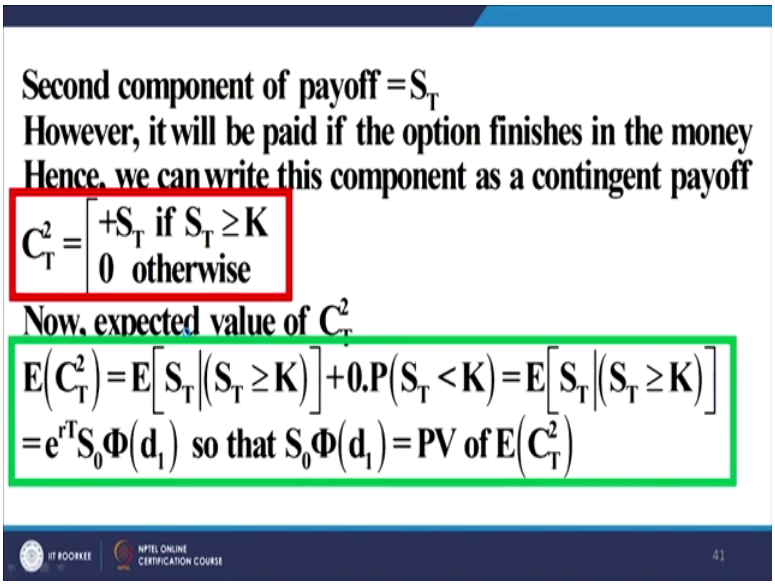

Second component of payoff is what I am going to receive, I am going to receive the stock which I can sell in the market at S T. But therefore, the second component is S T, if S T is greater than K and 0 otherwise therefore, the expected value of the second component is E S T S T greater than equal to K plus 0 this thing.

And this is equal to E S T subject to S T is greater than K, which we have just now evaluated which is e r T S 0 phi d 1. So, S 0 phi d 1 is equal to present value of this second component.

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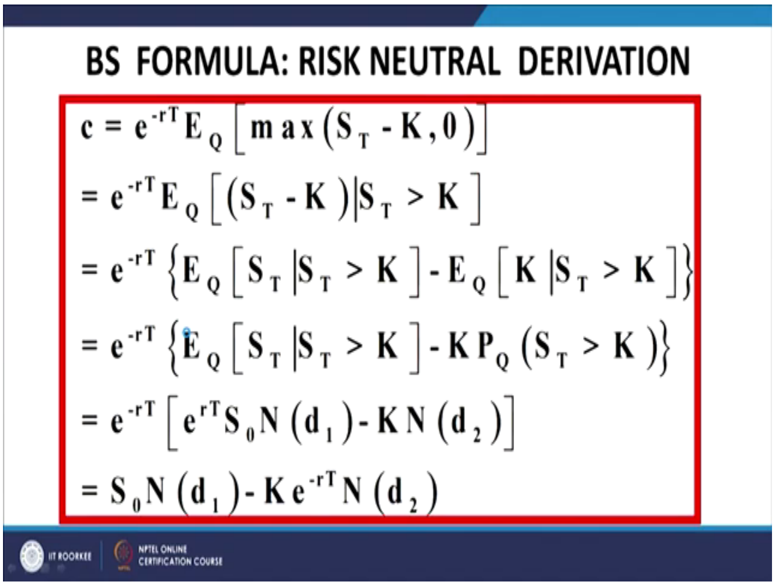

And therefore, when we substitute these values when we substitute these values, what we find is that the Black-Scholes formula is also based on the premise. That the value of the option is the present value of the expected payoff from the option at maturity.

Risk neutral valuation of Black-Scholes, this is we start with this formula. We, simplify this start with this formula, which we have been using e to the minus r T risk neutral probabilities. We, substitute the payoff for the call option maximum S T minus K comma 0, that gives us e to the power minus E Q S T minus K subject to S T greater than K.

Because, the option will be exercised only if S t is greater than K otherwise the payoff will be 0. So, and because the payoff will be 0, it will not contribute to the expectation. And therefore, if S T is less than K there will be no contribution to the expectation value. This can be written as E Q S T subject to S T greater than K minus E Q K subject to S T is greater than K.

That is E Q this we have already obtained in previously, as this expression and this is E Q that is K into S T greater than K. And that is nothing, but n phi d 2 or n d 2. And again the we arrive at the same formula using the risk neutral derivation.

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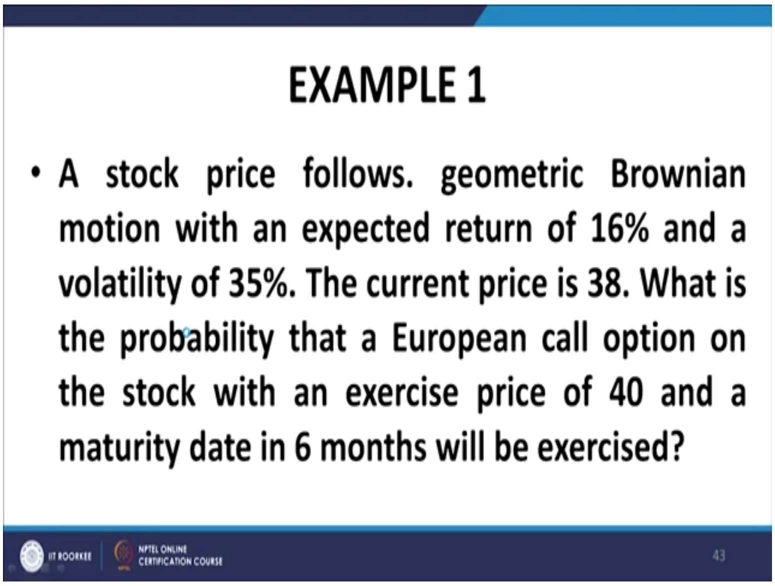

Then, we have got some problems here some examples if a stock price geometric. If a stock follows geometric Brownian motion, with an expected return of 16 percent mu is 16 percent volatility is 35 percent the current price is 38. What is the probability? That a European call option on the stock with an exercise price of 40. And a maturity date of 6 months will be exercised.

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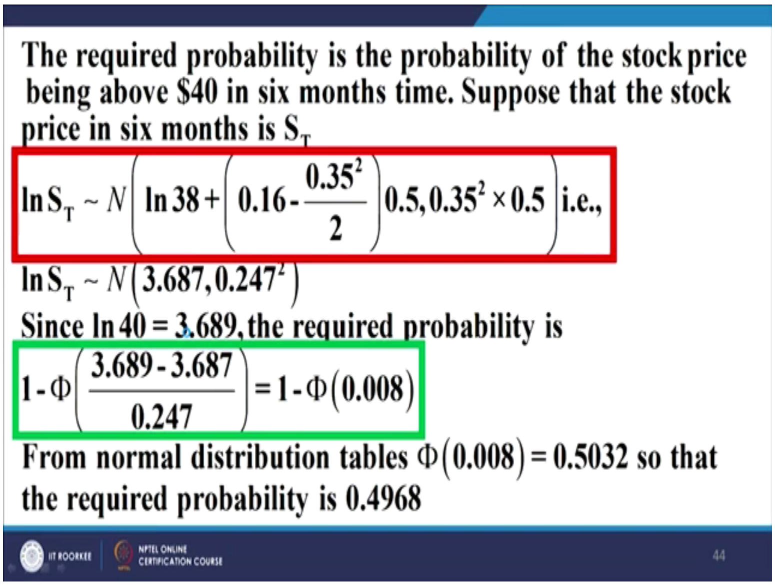

So, that is the question, we see remember the possibility of the possibility of exercise or the probability of exercise is equal to the probability. That the stock price exceeds the exercise price, stock price exceeds the exercise price the probability of that is phi d 2.

So, we simply have to work out phi d 2. So, we can we work out, we can work out d 2 work out phi d 2 and that will the solution. The other alternative the alternative problem is to start from first principles, we want the probability of the stock price being more than 40 in 6 months' time.

Suppose this stock price on maturity or at the end of 6 months is S T, then log of S T is normally distributed with a mean of the expression here. And a variance of the mean of the expression first expression and variance of the second expression. In other words, it is normally distributed with a mean of 3.687 and a variance of 0.247 squared log of S T.

Now, since log 40 is equal to 3.689 our required probability is 1 minus 3.689 minus 3.687 divided by 0.247, that is equal to 1 minus phi of 0.008. And which is equal to 0.4968.

So, with this we come to the conclusion of this course. In this particular course attempted to bring to you a package on path integrals. This path integrals constitute novel approach, to contemporary problems in a variety of areas. In multiple disciplines like statistics quantum mechanics quantum field theory, quantum gravity.

And, then they are also finding application in economics, in finance and condensed matter of physics and so, many areas. The path integral is essentially a tool a variation of traditional integration, where we try to integrate over all possible paths between two points in a particular underlying space.

Please do connect with me on the discussion forum. And I will also post supplementary notes covering each and every lesson, thank you thank you so, much for being with me and please be in touch on the discussion forum.

Thank you.