Path Integral Methods in Physics & Finance
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Lecture - 06
Gaussian Integration, Central Limit Theorem

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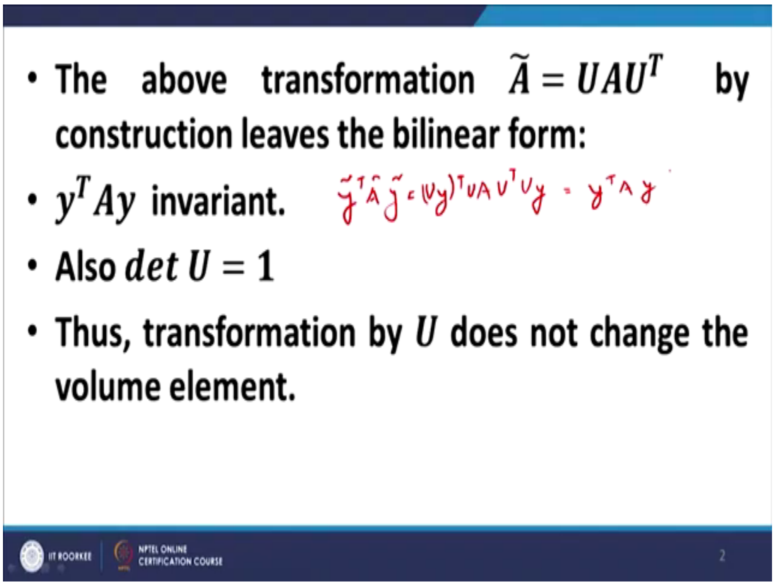

Let us continue from where we left off, but before that one point determinant U of course, can also be minus 1, but if we do not consider that aspect and because we want to retain the positivity of the integration volume. So, determinant U equal to minus 1 we ignore and we retain determinant U equal to 1.

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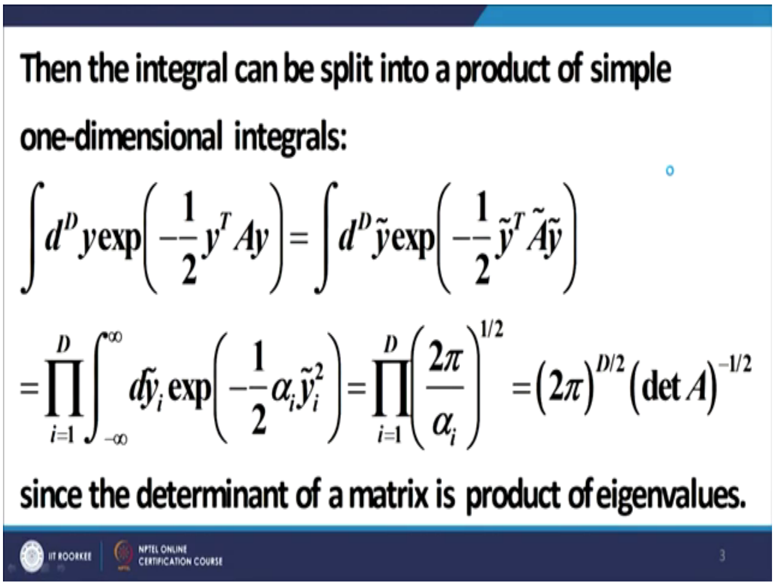

The net result is that when we look at the integration the bilinear form in the exponential is unchanged, by the rotation and at the same time the integration volume is also remaining unchanged integration; element is also remaining unchanged. So, now, the important thing is that A tilde is the matrix which is diagonal. And the diagonal elements represent its eigenvalues.

Therefore, because it is a diagonal matrix, we can split this integral into d into a product of D independent integrals. Each of them containing 1 eigenvalue and that is what it is displayed in this expression. Making use of the previous formula for 1 dimensional case, we apply that formula here and we end up with the expression here where for each eigenvalue alpha i we get a factor of alpha i to the power 1 by 2 in the denominator.

So, we have the product of 2 pi 1 or 2 to the power 1 by 2 corresponding to each integral. So, there will be d factors of 2 pi to the power 1 by 2. So, that is 2 pi to the power D by 2. And in the denominator we will have a product of alpha 1, alpha 2, alpha 3 up to alpha d to the power 1 by 2.

Now, the product of these eigen values in the denominator is nothing, but the determinant of the matrix A. So, we can write this whole expression as 2 pi to the power D by 2 determinant A to the power minus 1 by 2 because, it is in the denominator. Now, the important point, the integral will only exist, if all the eigenvalues of the matrix are positive. And which will only happen if the matrix is positive definite.

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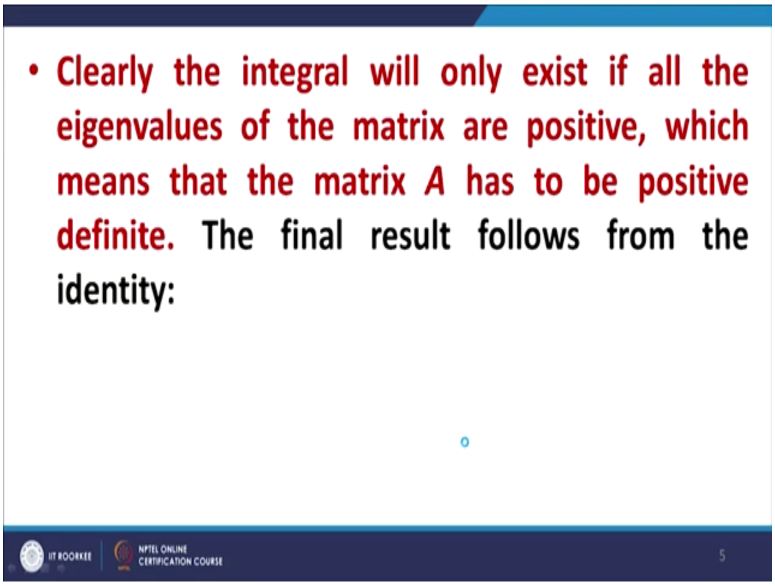

I mentioned earlier that the matrix A has to be real symmetric and positive definite. Now, because it is symmetric we can diagonalize it, had it not been symmetric we could not have been diagonalize it. And had it not been positive definite, we would not have had this eigenvalues of the matrix, in this form we could not have put the integral in this form. So, that is where the relevance of these terms comes into play.

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$$\ln \det A = \ln \prod_{i=1}^{D} \alpha_i = \sum_{i=1}^{D} \ln \alpha_i = Tr \ln \tilde{A}$$

$$= Tr \ln U^T A U = Tr \ln U^T U A = Tr \ln A$$

$$Hence, (\det A)^{-\frac{1}{2}} = \exp\left(-\frac{1}{2} Tr \ln A\right)$$

Now, let us look at this the log of determinant A. Now, the determinant of A is the product of the eigenvalues of A. Therefore, log of determinant A can be written as log of the product of eigenvalues of A and that is equal to this sum because, when you take the log of the product it translates to a sum. So, it becomes the sum of the eigenvalues of various eigenvalues that alpha 1, alpha 2, alpha 3, log alpha 1 plus log alpha 2 plus log alpha 3 and so on.

Now, recall A tilde is the diagonal matrix, all of whose elements all the diagonal elements are the various eigenvalues. So, instead of alpha I can instead of summing over alpha I can use the trace the expression of trace also. But the important thing is to note that because, it is a diagonal matrix the product of the eigenvalues equal to the matrix.

And if we have a function of the matrix A tilde for example, we want to work up say log of A tilde that will be equal to log of the various eigenvalues. And therefore, when I use sum the log of eigenvalues, in other words log alpha will also be diagonal matrix, with the all the elements of the diagonal matrix being log alpha 1, log alpha 2, log alpha 3 and so on.

And therefore, when I want to sum over all the log alpha i's I can as well use the trace for the log of the matrix A tilde. In other words what I get is log of determinant A is equal to trace log of A tilde. Substitute A tilde equal to its expression in terms of the rotation matrices and apply the what cyclic relationship we end up with this expression being equal to trace log of A.

In other words whatever we established, we are established that log of determinant A is equal to trace log of A. And therefore, determinant A to the power minus 1 by 2 is nothing, but exponential minus 1 by 2 trace log of A we simply substitute this value in the expression that we had and we arrive at the result. And this integral i is equal to the 2 pi to the power D by 2 exponential minus 1 by 2 trace log of A.

So, there is a lot of manipulation here, but it is a very very important result and it displays many important relationships in matrix algebra, which we have to use again and again. Please note this particular point that, if we have a diagonal matrix for example, A is the diagonal matrix, then log of A will also be a diagonal matrix and its elements will precisely be the log of those diagonal elements which constitute the matrix A.

And if we are for example, if, are look looking at the matrix A squared, then again will have a diagonal matrix whose elements are squared alpha 1 square alpha square and so on so that is the property that we have made use here, right.

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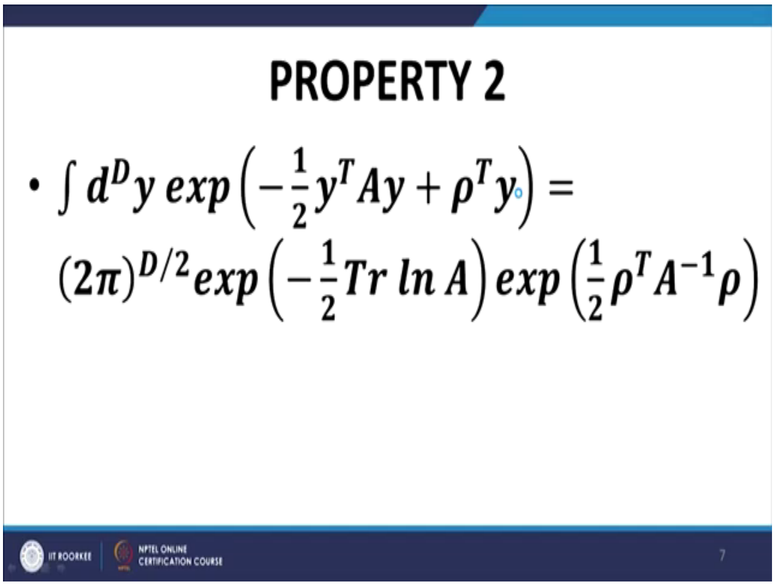

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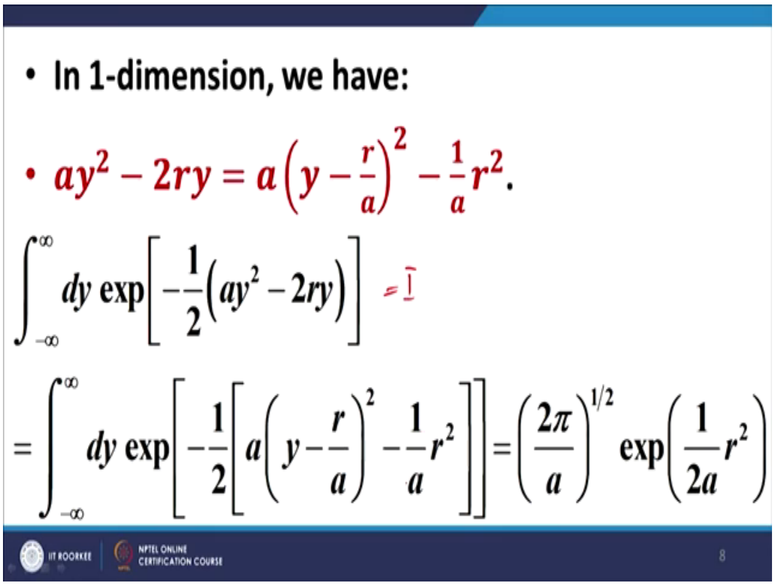

We now look at property 2. Now, property 2 is again let us first look at the 1 dimensional case and then we will generalize. Generalization is in this case it is very elementary we need to evaluate this integral let us call this integral I.

It is simply in fact, we have done this earlier in the context of determining or calculating the moment generating function of the normal distribution dy is equal to exponential minus this. We, convert this to a square a perfect square involving the terms y and 2 r yand an extra piece which is independent of y this taken out.

So, we write this ay square into 2 r a y square minus 2 ry as a into y minus r by a whole square minus 1 by a r square. Now, look at this carefully this factor 1 by a r square is independent of y. And therefore, it can be taken outside the integral and what are we left with? We are left with the expression integral dy exponential minus 1 by 2 ay minus r by a whole square. Recall this is of the same pattern, same form as the pdf of a normal distribution. And what is the coefficient of y in this case? The coefficient of y in this case is a sorry y square is a.

The coefficient of y square is a and when you simplify this expression, just as we did in the 1 dimensional case. This particular portion when integrated gives you 2 pi upon a to the power 1 by 2 and this part exponential minus 1 by a minus minus becomes plus and we have 1 upon 2 a into r square. This goes outside the integration because, it is independent of i and we have this expression.

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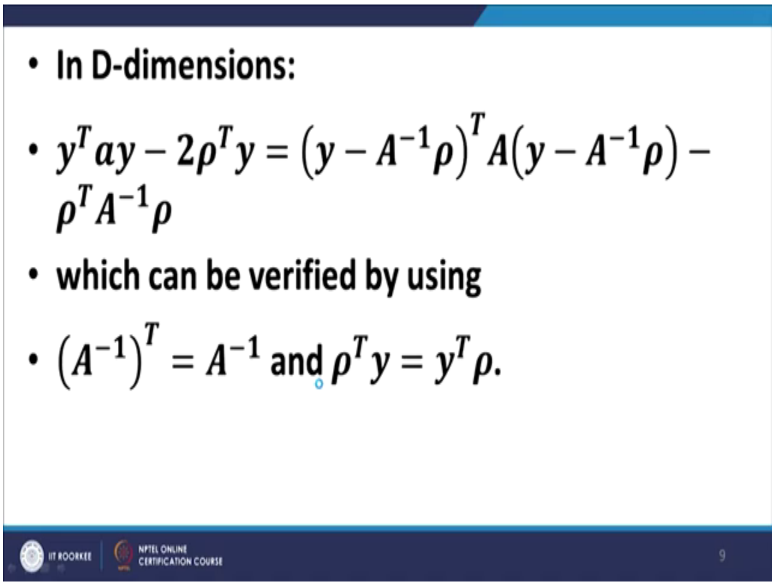

Now, we generalize it to the matrix case. In the matrix case in the D dimensional matrix case, you see what did we have here we had here ay square minus 2 ry, you just keep track of this ay square minus 2 ry on the left hand side. Now, let us look at this ay square is written as y transpose a y, y transpose ay minus 2 we will introduce a new factor rho transpose corresponding to r over there in to y.

And this can be written as; this can be written as y minus A inverse rho transpose A y minus A inverse rho minus this expression. This is absolutely same as we did in the case of the 1 dimensional case, when we completed 1 perfect square. And we took outer term which was independent of integration and we took it outside the integral that is precisely what is done here.

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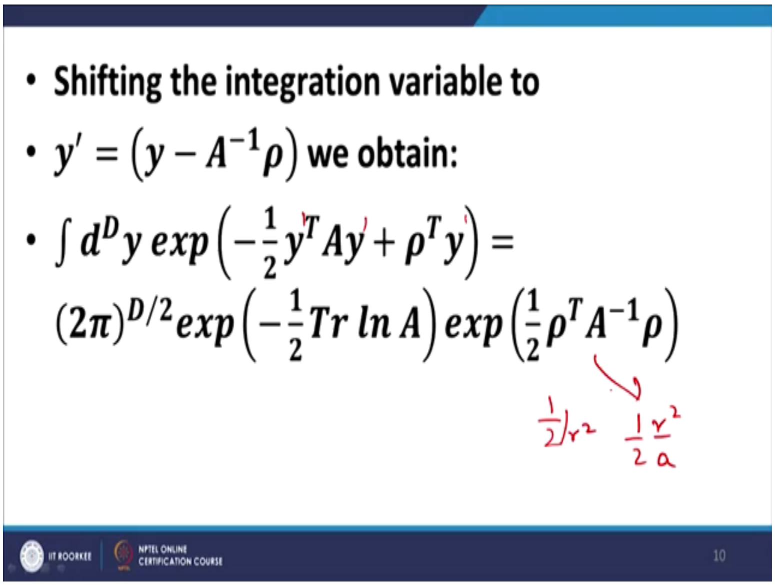

And, now what we do is simple substitution we substitute y dash equal to y minus A inverse rho; obviously, dy is equal to dy dash right. So, that does not make any difference and in terms of y dash we get this expression minus 1 by 2 y dash transpose A y dash plus rho transpose y dash. This is what we get and when we substitute, when we make use of the earlier expression, we get this result this is quite straightforward that is not much ado about it.

If you look at this correlates to what we had done earlier, what was there? In the 1 dimensional case, it was 1 by 2 r square I am sorry, it was 1 by 2 r square upon a and that is precisely what it is matching here, we can have a look at that to confirm. Let us have a look at this 1 by 2 ar square precisely. What I have written here? 1 by 2 ar. So, this matches to this expression this expression was taken outside the integral, then this we had done earlier in the context of the previous problem previous example for a D dimensional case.

So, this part is done this part is the extra piece, that we have which we took outside the integral and we get this result. This integration gives you this part and this part goes out as this part.

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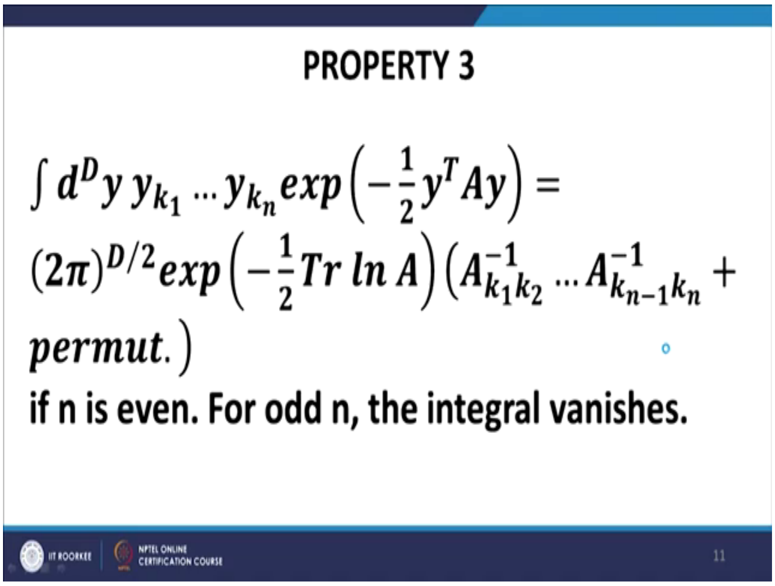

We, have another property another very interesting property is this particular property is slightly more involved, but it is very important and we need to look at it.

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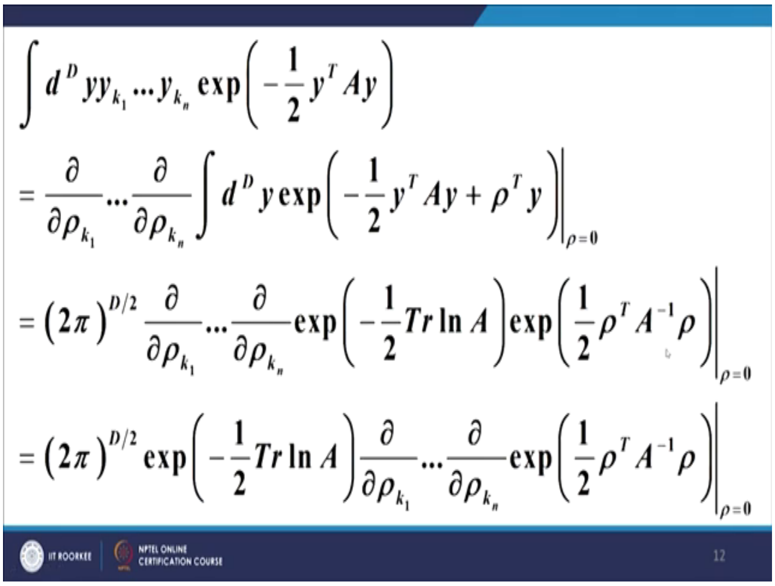

You see, what we do here is see, if you look at this these are factors which are related to this y over here various expressions which are extracted out of this column vectors y and y transpose, which are extracted and put as part of the integral.

So, here what we do is let us start with this matrix. Now, if you look at this if I take it instead of this matrix, I take a new matrix I take this matrix expression. For the moment just to forget about this forget this quantities right. What I do is? Instead of this matrix in this particular exponential, I write this exponential. And then I, what I do is I take the derivative of this whole thing, this whole thing including the integral with respect to del of del rho k 1.

Now, what will I do? When this del rho d of D rho k 1 operates on this it gives you this integral again. And then it integrates this exponent and when sorry, when this differentiates this exponent. And when it differentiates this exponent it pulls out that particular quantity, which is related to this which is and what is the coefficient of rho k 1. If you look at this in this particular expression the coefficient of rho k 1 will be nothing, but y k 1 from this from this particular quantity.

So, instead of writing by k 1 here what I have done is I have introduced a additional factor its little later, it will be called the source term, but for the moment you can in use it an additional quantity and additional factor that I have introduced here in the integral. And then when I differentiate this because, there is no rho here there is no rho in this quantity, there is only a rho over here.

And when I use this rho in this quantity I pull out the corresponding y the rest of the rho's, because they are not in this rho will become 0. And the net result would be that I would precisely get this expression, when I differentiate with respect to rho k 1, rho k 2, rho k 3, rho k n.

When I do all this differentiation every case, I will get one factor. When I differentiate with respect to rho k 1, I will pull in y 1 y k 1 when I differentiate with respect to rho k 2, I will pull on y in k 2 and so on. So, if I differentiate with respect to all these quantities then, I will get this factor in other words these two quantities are equivalent.

And therefore, now we have done this differentiation already, we have done this differentiation already, but at the end of the day we need to throw out this rho. So, after doing this differentiation remember after doing this differentiation the result that I get in that result, I will put rho equal to 0. And then when I put rho equal to 0. This expression will go away and when this expression goes away, I recover this, I recover this I have pulled back factors of this due to the differentiation and I get precisely this quantity.

In other words what is it first of all add the source term step 2 differentiate with respect to rho k 1, rho k 2, rho and so on. Step 3 after differentiating put rho equal to 0 and you get that same quantity is this one. Now, we know the integral of this expression the integral of this expression, we just now done and that is precisely this.

This is the integral of this expression. Therefore, this the integral is nothing, but this integral or this expression differentiated with respect to rho k 1, rho k 2, rho k n and then rho put equal to 0, and then rho put equal to 0. Now, again this factor, this factor is independent of rho and let us take it to the other side.

Let us take it outside the differentiation operators you put 2 pi D and D by 2 a constant. You take this which is independent rho outside and this differentials a differentiation operators, which are differentiation with respect to rhos various rhos are operating on in this quantity. Now, let us look at what happens?

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$$Now,\ \frac{\partial}{\partial \rho_k}\left(\frac{1}{2}\rho^T A^{-1}\rho\right)$$

$$=\frac{1}{2}\sum_{j=1}^{D}\left(A_{kj}^{-1}\rho_j+\rho_j A_{jk}^{-1}\right)=\left(A^{-1}\rho\right)_k$$

Let us take the first case let us differentiate with respect to 1 rho rho k. Now, when you are differentiate with respect to 1 rho you use, this representation of the matrix this is precisely what you get. Summation j equal to 1 to D 1 by 2 and then you get these 2 expressions, which is which can be written compactly as this particular expression.

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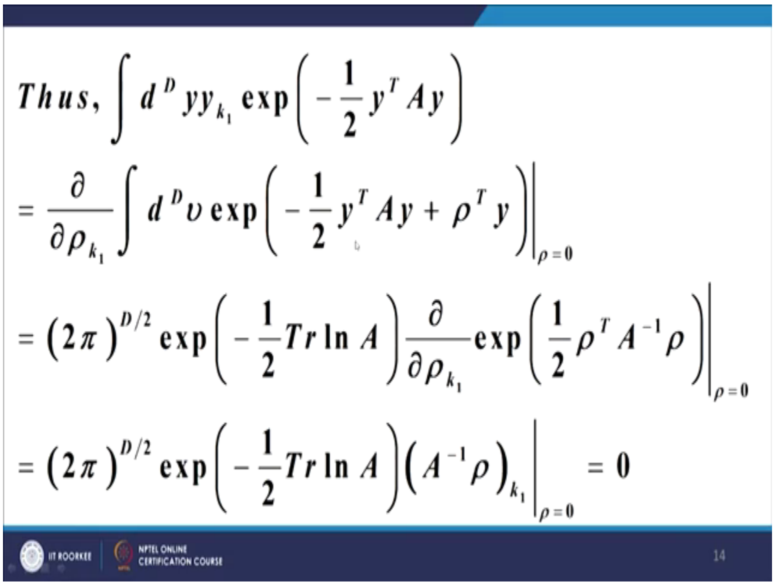

But, remember then I have to put therefore, what do I get? I get this expression is equal to this expression. And this expression turns out to be differentiation of this and when I do this differentiation I get this expression.

Now, when I put rho equal to 0 this term vanishes, even this term vanishes the whole thing vanishes and i end up with 0. So, therefore, if I have 1 factor here by a k 1, the whole integral gives me 0 to reiterate. If I have 1 factor here the whole expression will turn out to be 0. And indeed if you have an odd number of y k is here, the net result will always be 0.

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$$\frac{\partial}{\partial \rho_{k_2}} \frac{\partial}{\partial \rho_{k_1}} \exp\left(\frac{1}{2}\rho^T A^{-1} \rho\right)$$

$$= \frac{\partial}{\partial \rho_{k_2}} \left[\left(A^{-1}\rho\right)_{k_1} \exp\left(\frac{1}{2}\rho^T A^{-1} \rho\right) \right]$$

$$= \left[A^{-1}_{k_1 k_2} + \left(A^{-1}\rho\right)_{k_1} \left(A^{-1}\rho\right)_{k_2} \right] \exp\left(\frac{1}{2}\rho^T A^{-1} \rho\right)$$

Let us look at an even number, let us look at two differences rho k 1, rho k 2. Rho k 1 result we have already have with us. So, we differentiate this again with respect to rho k 2. Now, when you differentiate this with respect to rho k 2 what do I get? I get an expression similar to the previous case with respect to k 1 and I get the expression with respect to k 2.

When this is differentiated with respect to k 1 I get this, when this is differentiated with respect to k 2 I get this. But, when this is differentiated with respect to k 2, I get these coefficients. Now, these coefficients are independent of rho please note this point. These coefficients are independent of rho.

Therefore, when you put rho equal to 0 what happens? These coefficients survive this rho equal to 0 gives me 1, because it is in the exponential exponent. So, the exponent being 0 this gives me 1 these rho becomes 0. And this particular term A inverse k 1 k 2 survives.

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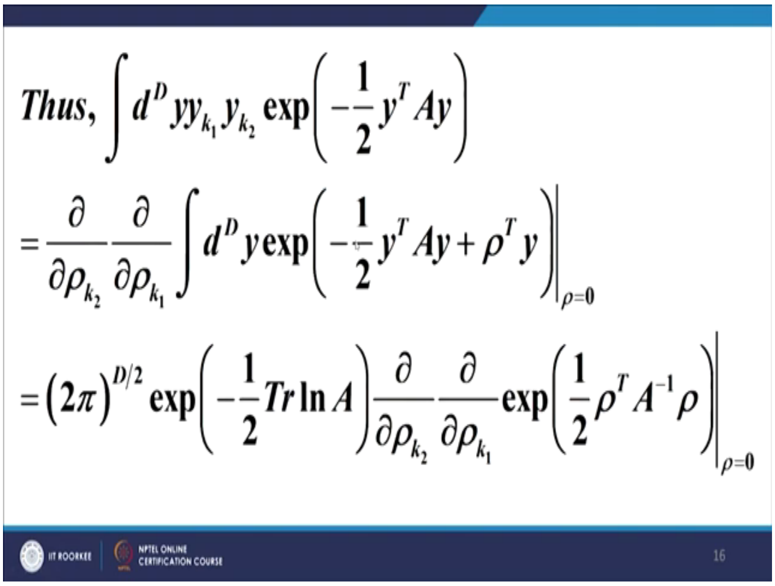

So, the net result is let us see, when you have two differentiations, 2 terms in the integral y k 1 and y 2. Then, what we have in the end of all this manipulation.

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$$= (2\pi)^{D/2} \exp\left(-\frac{1}{2} Tr \ln A\right)$$

$$\left\{\left[A^{-1}_{k_1 k_2} + \left(A^{-1}\rho\right)_{k_1} \left(A^{-1}\rho\right)_{k_2}\right] \exp\left(\frac{1}{2}\rho^T A^{-1}\rho\right)\right\}\Bigg|_{\rho=0}$$

$$= (2\pi)^{D/2} \exp\left(-\frac{1}{2} Tr \ln A\right) A^{-1}_{k_1 k_2}$$

$$\int d^D y y_{k_1} y_{k_2} \exp\left(-\frac{1}{2} y^T A y\right) = (2\pi)^{D/2} \exp\left(-\frac{1}{2} Tr \ln A\right) A^{-1}_{k_1 k_2}$$

We have at the end of all this manipulation is A inverse k 1, k 2 plus this thing plus this. And when you put rho equal to 0 this becomes one the exponential, becomes a 1 this becomes a 0 this becomes 0 and this is retained, because this is independent of rho.

So, what we get is 2 pi to the power D exponential minus 1 by 2 trace law of this thing and A inverse k 1 k 2. So, this is for 2 quantity 2 wise y k 1 y k 2 appearing in the integral yk 1 y k 2 appearing in the integral.

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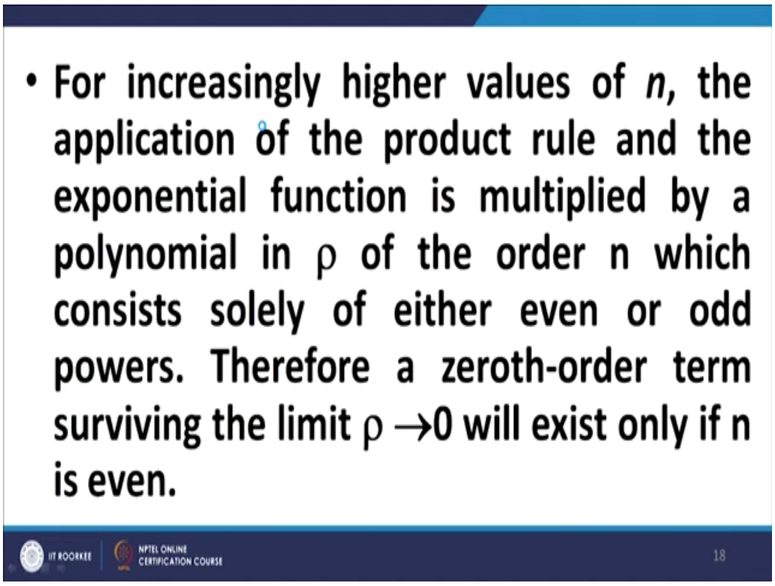

Now, as the number of values yk 1, yk 2, k 1, k 2, k 3 the number of differentiation increases the exponential factor is multiplied by polynomials in rho of the order n. Now, what does it consist of? It consists of either even or odd powers. Now, one zero-th power term remains one zero-th power term remains, which will survive in the case when the limit rho equal to 0 rho tends to 0 is taken.

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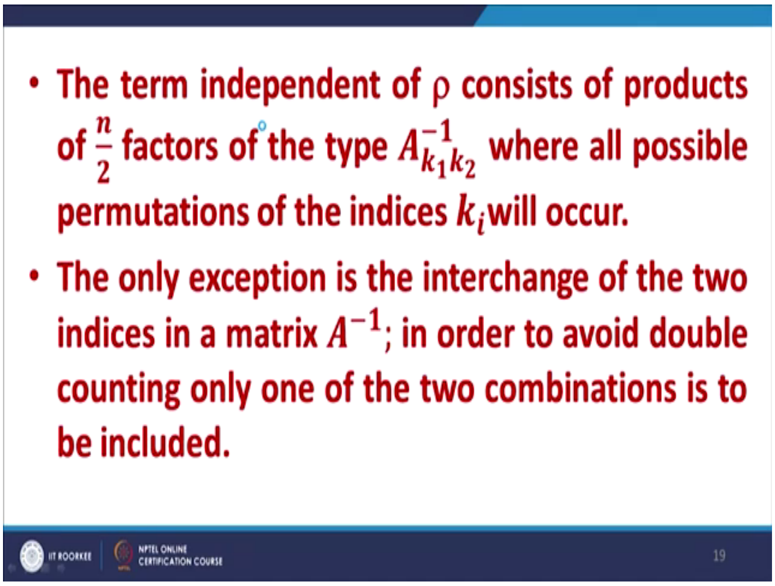

And; however, that zero-th that term which survives it consists of n by 2 factors of this to this type a k 1, k 2, k 1, k 3, k 1, k 4 k k 1, k n, k 2, k 3, k 2, k 4 and so, on k 2 k n and so on. But remember this is symmetric k 1 k 2 is same as k 2 k 1. So, the net result is you get n by 2 terms which survive, when there are n terms yk 1, yk 2, yk n in the integral sign.

So, that this integral is very important for odd numbers it gives you a 0 result. For even numbers it gives you a number of terms n by 2 terms in fact, where n is the number of k 1, k 2, k n you can have a look at this it is given here it is. Yes A inverse k 1 k 2 A inverse k n minus 1 k n plus the permutations of these quantities, there will be n by 2 terms in total.

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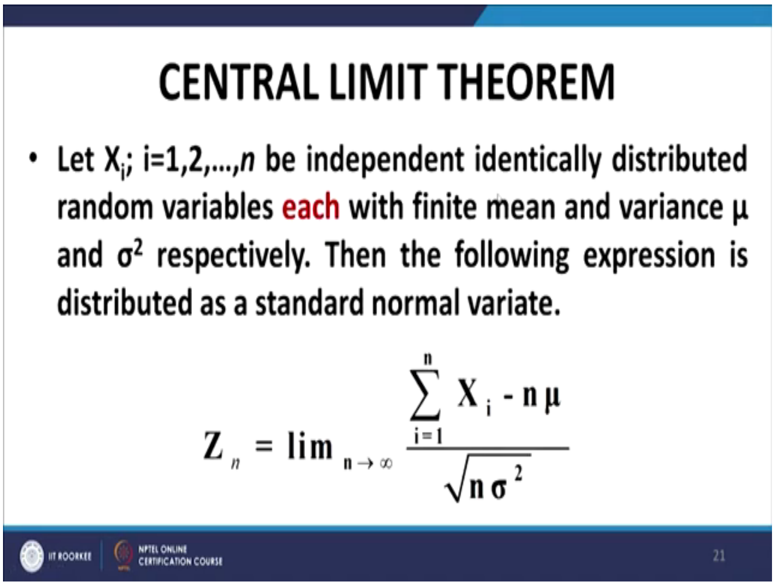

Now, we come so that is about Gaussian integrals these three formula were very important, they find a lot of applications. Now, we come to another beautiful theorem in statistics, which again has a lot of applications, which is called the central limit theorem. What it says is? Suppose x 1, x 2, x 3 x n they are identical independent identically distributed random variables independent x 1, x 2, x 3 all are independent.

And they are identically distributed random variables they have a finite mean and a finance variance. The mean of each of them is mu and the variance of each of them is sigma square. Then, the following expression Z n limit n tending to infinity sigma X i that is the sum of all the random variables minus n mu upon under root n sigma square tends to the standard normal distribution, standard normal distribution. This is a very important theorem let us look at it in more detail.

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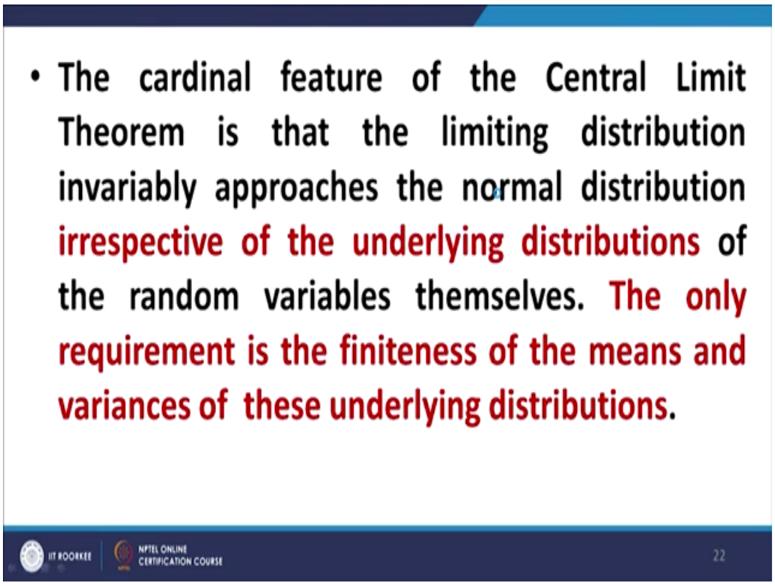

What is the important part a part in it? The important part in it is the variables x 1 x 2 x 3 etcetera are independent identically distributed, but how are they distributed that is irrelevant. In other words the nature of the underlying distribution of x 1 x 2 x 3 is not relevant. Howsoever, they may be distributed they may be uniformly distributed, they may be normal distributed, they may be binomially distributed, Poisson distributed.

Whatever in any distribution they are provided all these variables are distributed similarly and they have the same mean and variance. And they are independent then this condition automatically holds. That is the beauty of this theorem the beauty of this theorem is that it does not bother it, does not worry about the underlying distribution of the variable. It does not say that it was via Gaussian distribution, it does not say that it must be a uniform distribution it holds for all distributions right.

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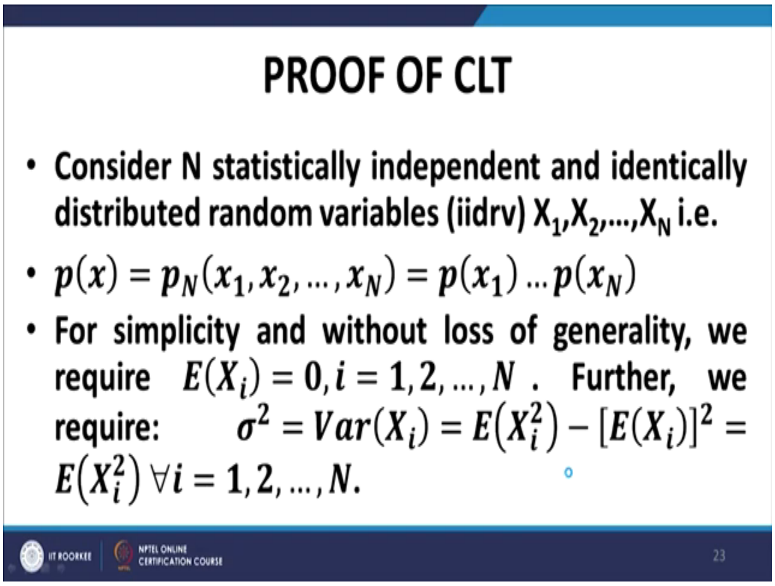

Let us look at the proof quickly. Let us say assume that there are n statistically independent and identically distributed random variables x 1, x 2, x N. Let us assume that p x is the probability density function of x and that is the it is a function of x 1, x 2, x N naturally. And because they are independent because, they are independent p x, x 1, x 2, x 3 can be expressed as px 1, px 2, px 3, px N because they are independent.

Now, for simplicity let us assume that E of X i that is the mean of all these random variables is 0 is simply as a shifting of the origin. And we also assume that sigma square is equal to E X i square minus E. Because, this quantity is 0 this quantity is 0 therefore, we can write E of X i square is equal to sigma square because the mean is 0. So, the second movement about the mean and the second movement of the about the origin do coincide. And we have rewrite it as sigma square.

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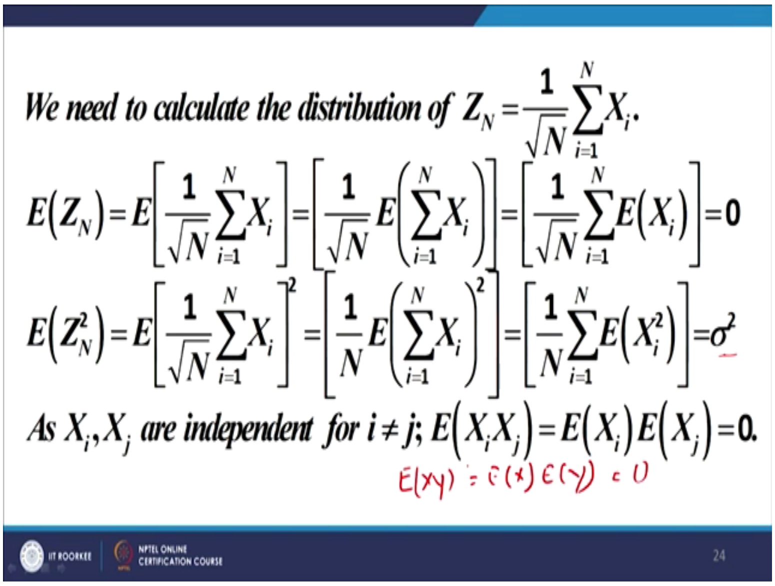

Now, we need to calculate the distribution of Z N, where Z N is equal to 1 upon root N sigma X i. Remember the mean is 0 the variance is sigma square. Let us work out the parameters of Z N mean of Z N is equal to mean of this expression. When you simplify this expression expectation of this you can take this expectation, in inside this root N. Because, this is deterministic again, because the expectation of a sum is the sum of the expectations you can take it here. And each of them as a expectation of 0.

So, the whole thing as the expect expectation of 0 therefore, E of Z N has an expectation of 0. Similarly E of Z N square, when you work it out and this part is straight forward this part is straight forward. Now, because they are independent what do we have? We have E XY is equal to E of X E of Y. And making use of this, we can write this as E summation XY is equal to summation E X square plus Y X 1 square plus X 2 square plus X 3 square plus X 1 E of X 1 X 2 which can be written as E X 1, E X 2.

And using this we find that this summation of this whole expression is equal to sigma square. But, E of X is equal to note please note E of X is 0, E of Y is also 0. So, in this case E X, where the cross terms all the cross terms vanish right. And we have E of X i square is equal to sigma square.

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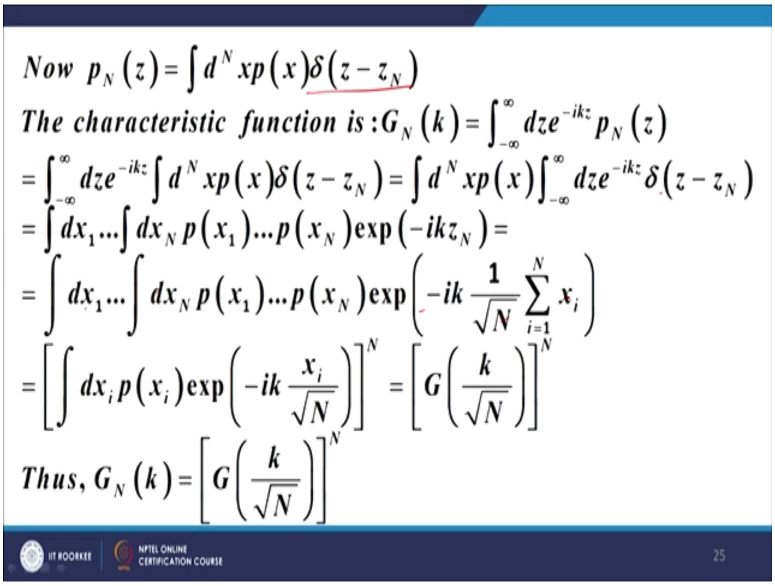

Now, let us look at the probability density function of the of z. We define p N z as just as we did in the example to start this lecture. We, want to work out p N z is what it is a probability of the variable z it taking a particular value small z. Now, because the variable z has a number of variables is the sum of a number of variables which is represented by the sum z N which is represented by the sum z N.

Therefore, you must show we can take all these numbers all these variables, the probability of all these variables which combined to which are present to give you, the values z provided that condition of summability holds. In other words provided that z N is equal to z. Where z N is given by summation z N is given by what? z N is given by the summation. Which summation? z N is given by this summation z n is given by this summation.

So, provided that the summation holds this constraint holds, you can have any values of x 1, you can have any values of x 2, x 3, x 4, x 5 whatever, but this condition must be met. That Z must be equal to Z N where Z N is defined by this. And how do we impose this constraint? Again we go back to under x delta function we impose this by setting this delta function over here.

So, p N x p N z is equal to integral d N p x. This is p x is the collective distribution of all the variables x p x 1, p x 2, p x 3 as you shall see just now. Now, the characteristic function is given by the Fourier transform of p N z therefore, it is given by dze to the power minus ikz p N z. Let us substitute this value of p N Z in the correct expression for the characteristic function.

We get dze to the power minus ikz and this whole expression comes as it is. When its do this integral simply over the delta function z gets replaced by z N. So, this is what I have? Now, I substitute z N in terms of that particular expression that I mentioned earlier this expression. So, when I do the substitution I get this expression.

Now, what I do is this x i e to the power minus i k 1 upon root N x 1, x 1 I take with this integral minus i k 1 upon root N x 2, I take with the second integral and so on. So, each integral has a factor of e to the power minus ik 1 upon root N with a its appropriate x, x i with dxi with x i. So, they are all integrals which are identical.

And therefore, I can write this expression this expression as integral d x i, p x i. This is dx i this is px i and then this x i expression e to the power minus i k x i upon root N to the power N and if you look at this carefully, if you look at this carefully and this is nothing, but the characteristics function of k upon root N. In other words whatever we established, we have established that G N k that is the characteristic function that we wanted or we started with is equal to G k upon root N to the power N. This is simply algebraic manipulation.

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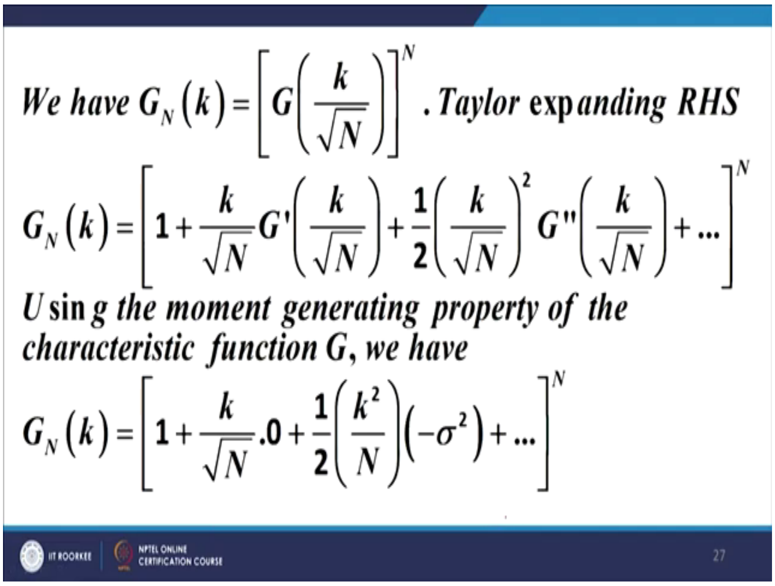

And therefore, this is what we have? G N k is equal to G k upon root N to the power N. Now, you can expand the right hand side as a Taylor series. If you expand the right hand side as a Taylor series and use the moment generating property of the characteristic function, the first moment remember we have taken the mean as 0.

So, the first moment this expression becomes 0. And we have taken the second moment as sigma square; sigma square at the second movement. And therefore, therefore, when you put these values here, you get 1 G dash k upon root N gives you 0 and this expression gives you minus sigma square minus because, we are dealing with characteristic function we are not dealing with moment generating function. So, this to the power N gives me equal to G N k.

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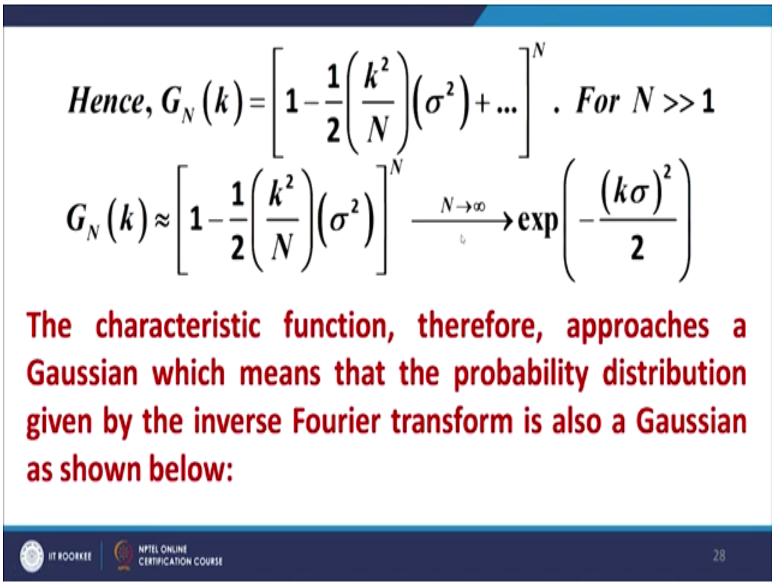

So, a bit of simplification what do we have G N k is equal to 1 minus 1 by 2 k square upon N into sigma square to the power N. Take the limit as N tends to infinity you get this expression, remember this is the Fourier transform, this is not the PDF itself the Fourier transform of the PDF.

But, because the Fourier transform of a Gaussian is a Gaussian therefore, it follows that because this is a Gaussian and this is you can look it carefully it is a simply a Gaussian unnormalized Gaussian expression PDF are normalized, but because it is Gaussian it follows that the p x, that it will correspond to is also Gaussian that can be seen as I will show you in the next slide right.

(Refer Slide Time: 33:51)

$$p_N(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ikz) G_N(k) dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(ikz - \frac{k^2\sigma^2}{2}\right) dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(k\sigma - i\frac{z}{\sigma}\right)^2 - \frac{z^2}{2\sigma^2}\right] dk =$$

$$= e^{-\frac{z^2}{2\sigma^2}} \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(k\sigma - i\frac{z}{\sigma}\right)^2\right] dk = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma^2}}$$

So, p N of z we now we inverse invert the Fourier transform, then we have p N of z is equal to exponential minus ikz G N k dk. Simply substitute the value of G N k which we had here this is my expression of G N k, this is the characteristic function. This is the Fourier transform of p x.

So, I substitute that Fourier transform of p x a to x to invert the Fourier transform. This is the inversion of the Fourier transform in this the inversion of the characteristic function, which is this and when you simplify this as we have been doing throughout this lecture you get this particular expression.

So, it follows it follows now that the PDF of z that we have obtained is nothing, but the normal distribution PDF 1 upon under root 2 pi minus z square upon 2 sigma square which is nothing, but N 0 comma sigma square, alright.

Thank you.