Path Integral Methods in Physics & Finance

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Lecture – 59
Path Integral Solution of Black-Scholes PDE

Welcome back. So, in the last lecture I introduced the Black-Scholes equation, I derived the Black-Scholes a partial differential equation by constructing a hedge portfolio in the continuous time version using a stochastic differential equation for the stock price.

And, then we also proceeded to solve the in Black-Scholes equations and we arrived expressed expressions for the value of a European call option. I now move to the next segment and that is the Solution of the Black-Scholes equation using the Path Integral Approach.

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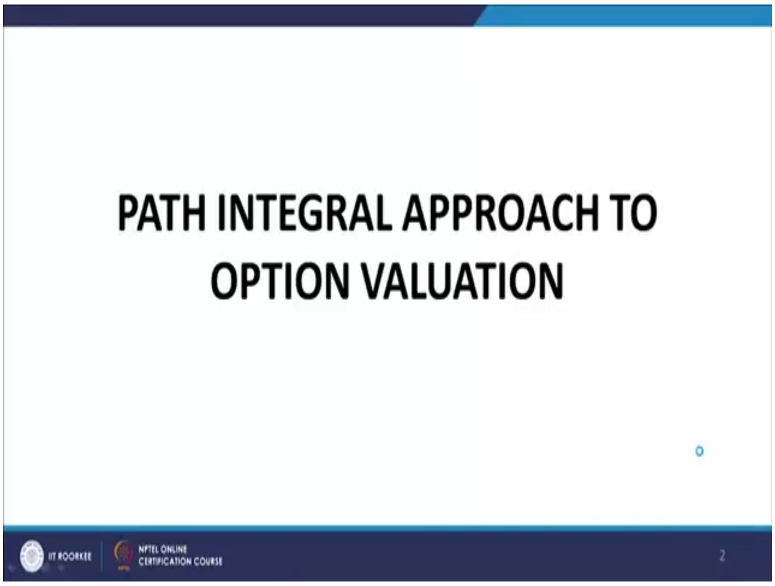

Path integral approach for the option valuation, the initially what I will do is I will take up the solution of the path of the option valuation problem using the path integral approach for a option which is path independent. In other words, the payoff of that option is independent of the path followed by the stock price. It depends purely and solely on the stock price or the value of the stock price at maturity.

The payoff of the option is determined entirely by the value of the stock or the price of the stock on the date of maturity of the option. It is not related to it is not dependent on how the stock reached that particular price. It is not dependent on the path followed by the stock in reaching that particular price. So, let us first attempt that problem, resolve that problem and then we will move on to path dependent options.

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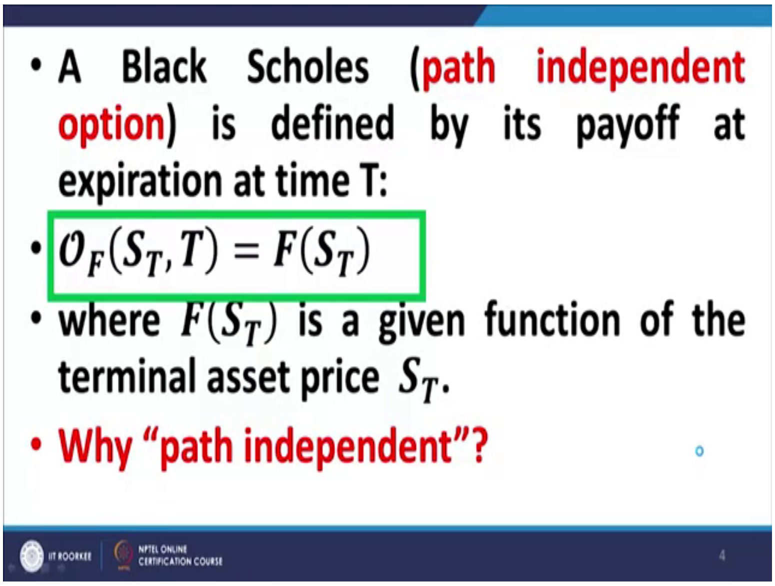

So, in a sense if you recapitulate the problem that we worked out yesterday or in the last lecture rather was the Black-Scholes problem, that is essentially the path independent option.

Because European options are path independent, they are exercisable at a particular date and the payoff on the option is dependent on the price that prevails on that particular date. So, here Black-Scholes option as it is sometimes called or the Black-Scholes formula relates to options that are path independent whose payoff is dependent entirely on the value of the underlying asset or the price of the underlying asset on the date of maturity of the option.

So, let us say resolve or let us solve the Black-Scholes option pricing problem using the path integral approach. So, in as we did earlier we can define the payoff at expiration of a Black-Scholes option or a path in independent option in terms of the expression that is given here in the green box. You may not carefully that the payoff is defined only in terms of the price or the value of these underlying asset at maturity of the option which is represented by capital T.

F S T is a function of the terminal asset price S capital T, S capital T the terminal asset price, the price of the asset on the date of maturity of the option. So, that is important, I reiterate that we are talking about options where the payoff is dependent on the price at maturity and the option is also exercisable on maturity.

In other words we are talking about European options number 1 and we are talking about options whose payoff depends on only on the maturity value of the stock and is independent of the path followed by the stock price in reaching that maturity value.

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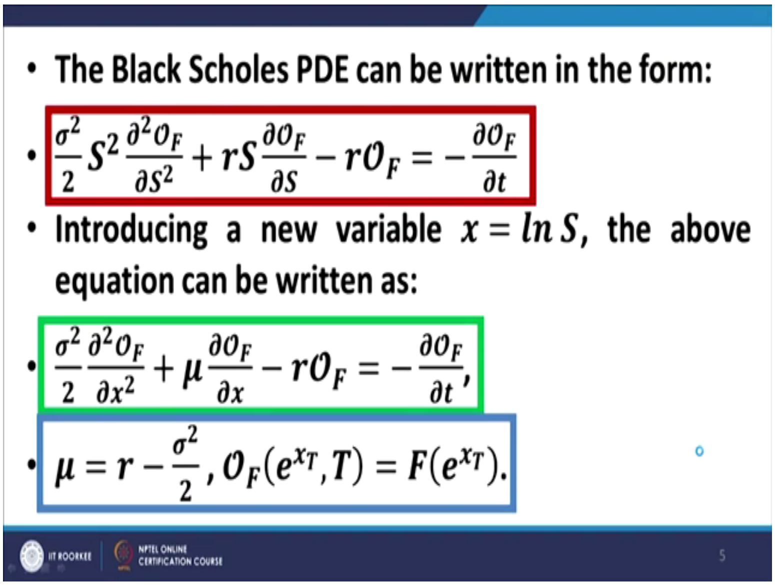

So, the Black-Scholes partial differential equation can be written in the form, we have derived that equation in the last lecture. And, it can be written in the form for the particular product or the for the particular derivative that we are talking about. It can be written in the form given in the red box here. And, in terms of the logarithm of the price log of S, we can simplify this Black-Scholes equation to the form that is accessible in the that is expressed in the green box here.

Importantly you may note that the important difference the fundamental difference between the equation that is there in the red box. And, the equation that is there in the green box is that the coefficients are now independent of the independent variable in the equation that is here in the green box.

They are independent of the x variable, the coefficients as you can see coefficient of the second derivative, second order derivative, first order derivative. And, the term O F also they are all independent of the variable x.

So, that is the simplification; that is achieved by transforming the independent variable from S to log of S. And of course, mu is equal to r minus sigma square upon 2 and the expression for O F also changes in conformity with the substitution of log S as e to the power x.

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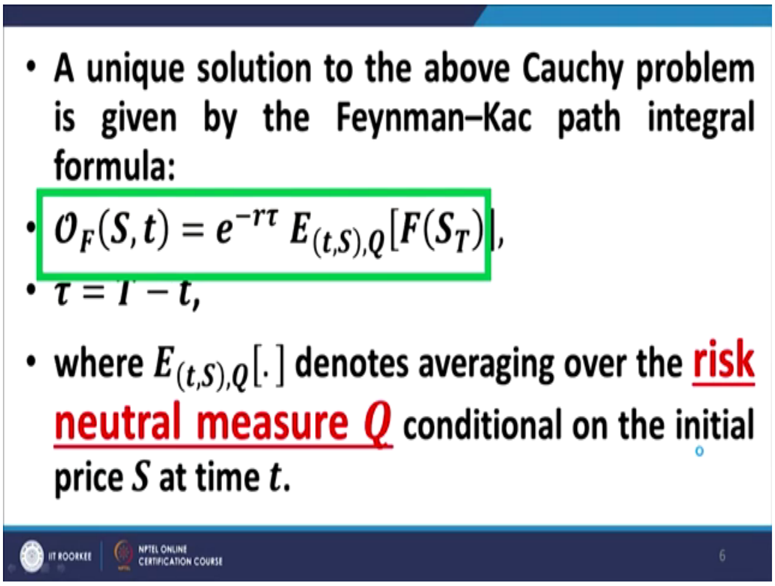

Now, this is the Cauchy problem and the Feynman path in path and Feynman-Kac path integral formula provides the solution to this. And, the solution to this as a in fact, we noted when we worked out the solution for the option problem using the binomial model. This is precisely the same expression, if you notice, if you recall this is precisely the expression that we arrived at using risk neutral probabilities, when we did the option pricing in the binomial model.

So, that is the formula that we have here for the option price and here the important thing to reiterate here is, that Q is the risk neutral measure of probability or the risk neutral probabilities as we may call them or probabilities in the risk neutral world.

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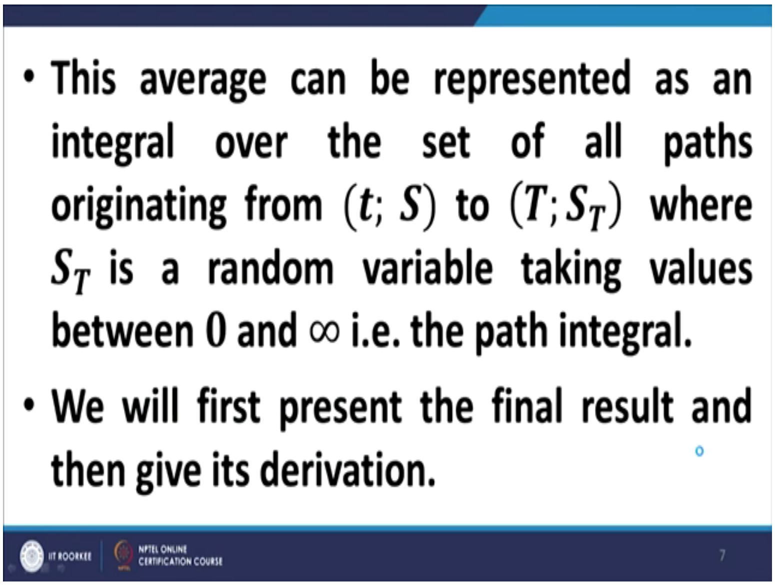

And, this particular average that is the expectation value that we are talking about in the risk neutral world or with respect to the risk neutral probabilities, they can be represented as an integral over the set of all paths starting from the initial point at t which is today when the stock price is S to capital T which is the maturity of the option when the stock price is S capital T, where S capital T is essentially a random variable.

And, where it can take values any value in the set between 0 to any real value between the set 0 to infinity.

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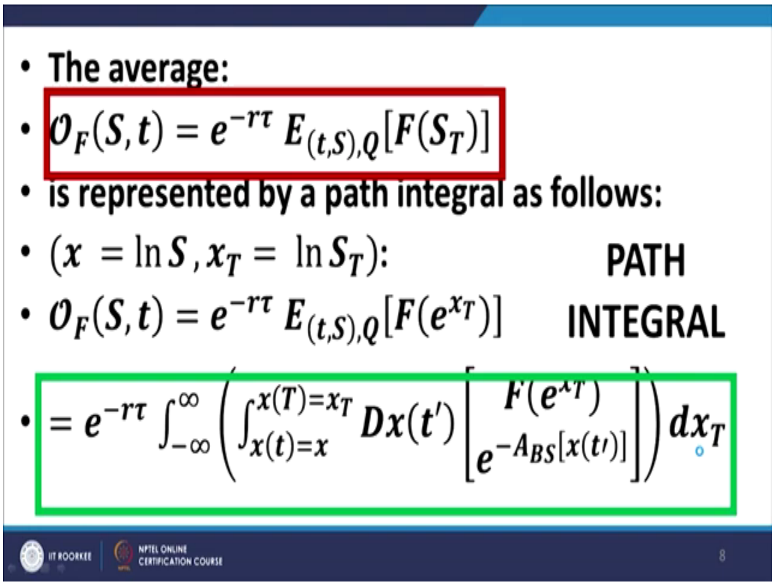

So, continuing from here we have got this expression of as the value of the contingent claim or the value of the option. And, we represent it by a path integral in the form that is given in the green box here. Please note the substitutions x is equal to log of S and x T that is the log of the stock price at maturity log of S T is x T and the path integral component, the pre-factor represents the present value factor.

The bringing back to t equal to the c, the option is exercisable at a future date and therefore, the payoff is going to materialize at a future date. And therefore, we need to when we work out the value as of today, we need to work out the bring back the expectation value to t equal to the present time. And, this pre-factor e to the power minus r tau does this job, it is the present value factor that brings the expectation value of the option payoff to the date of valuation.

The expression within the round bracket constitutes the path integral, the path integral which of the payoff over all paths and between the initial or the current value of the stock and the projected and the value of this stock as on date of the maturity of the option. And, then we are integrating this over with respect to Dx t to cover all possible values of the stock between minus between 0 to infinity.

Please note the stock value can take anything between 0 to infinity the therefore, the log of S T can take values between minus infinity and plus infinity. And therefore, because the variable here is log of S T, that is x t. And therefore, x t can take values between minus infinity to plus infinity.

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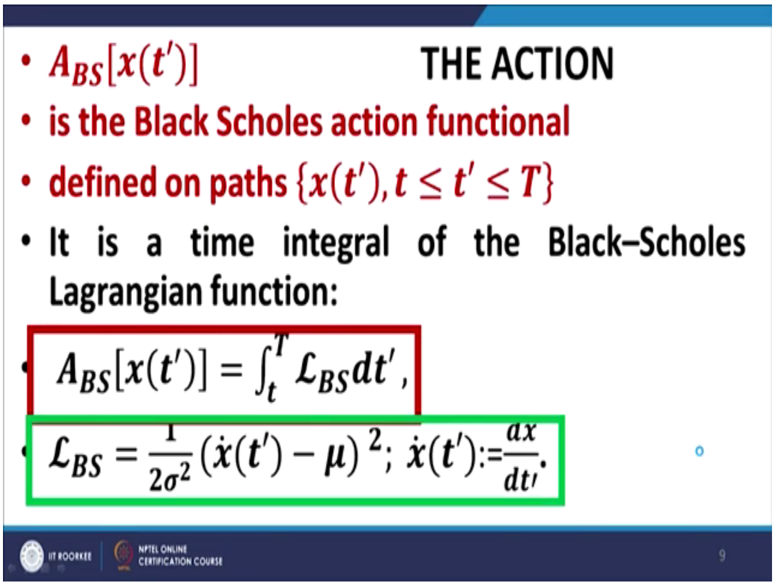

The important term here is if you note carefully is the Black-Scholes action A BS which is this is the action functional. It is defined on paths as was the case in the case of quantum mechanics. And, it is a timing integral of the Black-Scholes Lagrangian function where the Black-Scholes Lagrangian function is given by the expression here in the green box here.

So, the action is the time integral of the Black-Scholes Lagrangian function as shown in the red box here. And, the Lagrangian function or the Black-Scholes Lagrangian function is the given by the expression in the green box here.

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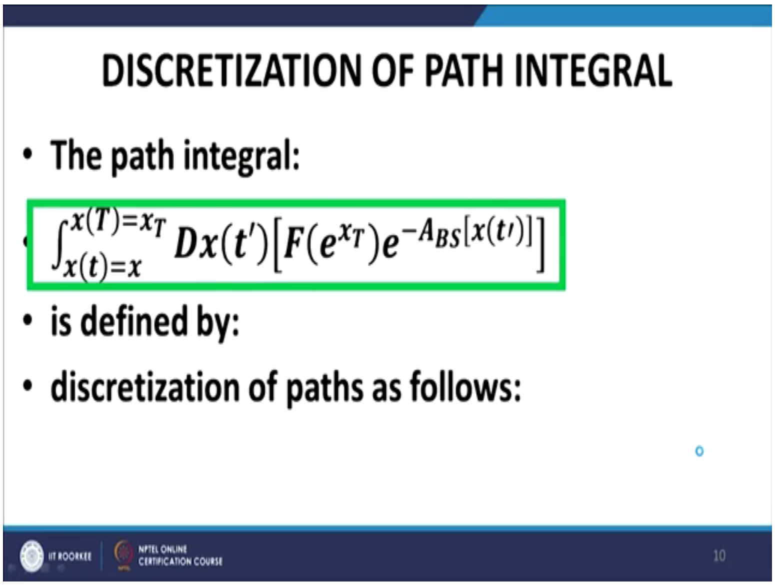

Now, how to evaluate this path integral? We have know we have got the expression for the path integral, how to evaluate this path integral? We use the standard mechanism of time slicing for this purpose.

We have done it many many times in this course in the context of statistics, in the context of quantum mechanics, again in the context of quantum field theory and we repeat the exercise here again. And, we discretize it in the timeline, the we discretize the paths by discretizing the timeline.

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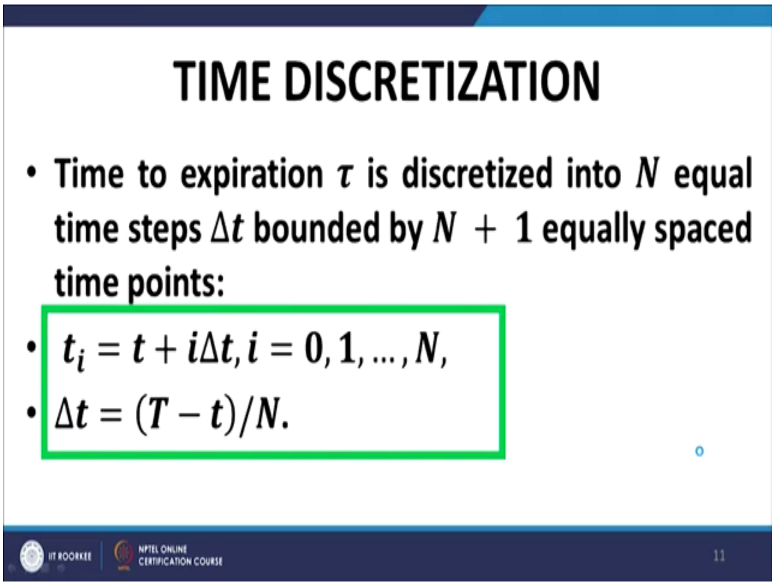

How do we discretize the timeline? As usual we split up the time between the remaining time to maturity that is time between t equal to small t to t equal to capital T, t equal to small t is today is now, the current instant and t equal to capital T is the maturity of the option.

So, this whole time interval between t equal to small t and t equal to capital T which is represented by tau is split up into N equal partitions or time steps, each of length delta t ah; they are all disjoint of course as. And therefore, they are bounded by N plus 1 equally spaced time points and the ith time point is given by this expression here. And of course, delta t is given by capital T minus small t divided by N.

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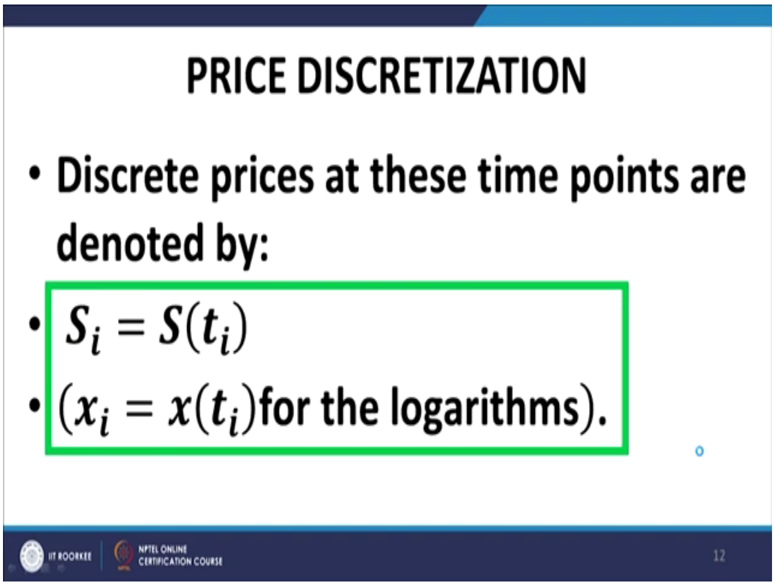

The prices at each of these nodes or each of this. In fact, see we are now having a lattice and the prices at each of these time points or lattice nodes on the time line are expressed in terms of S of t i and which is abbreviated as S of i. And, the logarithm of the prices are expressed in terms of log of S i which is abbreviated as or which is symbolized by x of i.

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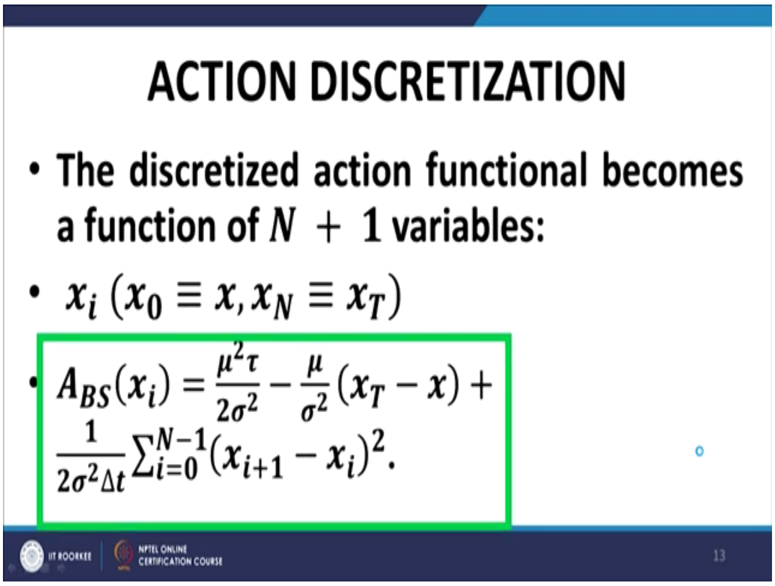

And, now we talk about the discretization of the action variable or action functional. The action functional is now a function of N plus 1 variables starting from x 0 which is x to x N which is x capital T. This is the current upon x is the current logarithm of the current stock price and x capital T is the logarithm of the stock price at maturity date of the option.

So, and of course, there are N minus 1 other intermediate points at each of at intermediate nodes on the timeline at each of which the price is S i on the logarithm of the price is log of S i which is denoted by x i. So, the action functional is a function of all these N plus 1 variables.

Now, let us see let us get the expression for the discretized action. We have got the action for the Black-Scholes for formula or the Black-Scholes from the Black-Scholes Lagrangian.

The action for the Black-Scholes from the Black-Scholes Lagrangian takes the form which is given in the green box here all discretization which is shown in the next slide.

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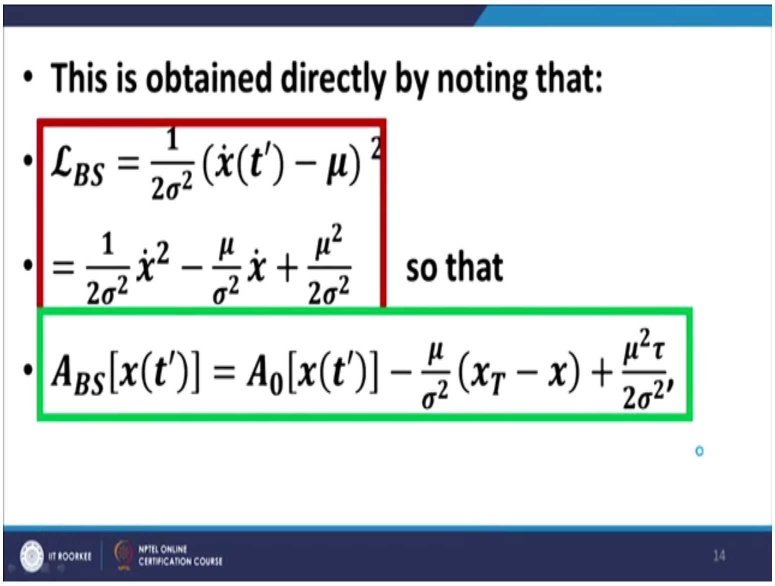

The Black-Scholes Lagrangian as I mentioned earlier is given by the expression in the red box here. When you simplify this, you get the second expression for the Black-Scholes Lagrangian.

And therefore, the corresponding action which is the time integral of the Lagrangian is obtained as the expression that is given in the green box here. The last two terms are self-explanatory, they are quite elementary. The first term we come back to in the next slide.

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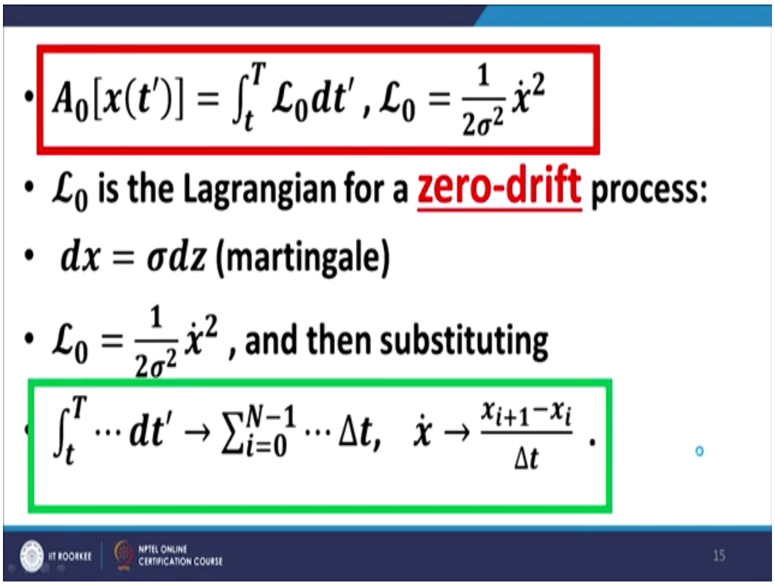

The first term A 0 A 0 of x dash x of t dash is the integral of is the time integral of L 0 or L 0 of where L 0 is the expression that is given in the red box here; 1 upon 2 sigma square x dot square. Now, what is this L 0? L 0 is the Lagrangian corresponding to the zero-drift Brownian motion or zero-drift process which is given by the expression here. d x is equal to sigma dz or d where dz is the infinitesimal Brownian motion increment which we have been denoting so far by dw.

So, dx is equal to sigma dz, this is a driftless Brownian motion and the Lagrangian corresponding to this driftless Brownian motion is denoted by L 0. And, which takes the; which takes the expression 1 upon 2 sigma square x dot square which when we discretize this expression; we get the expression as you can see in the green box.

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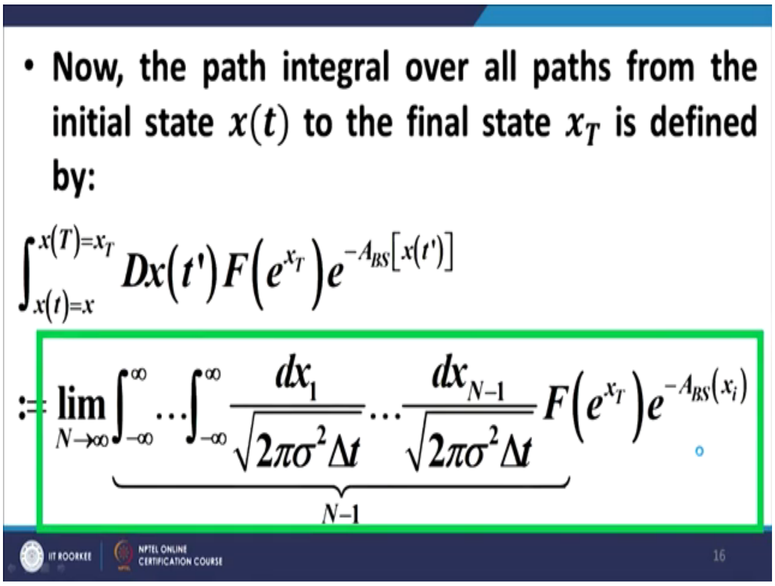

The process of discretization is explained in the green box here and the expression that we end up with is precisely what we have in the previous slide, which is by making this substitution integral in these integral substitutions, the expression that we have here for A 0 is; I think we will have it in the next slide. I will come back to it in a minute. So, this is this substitution there we substitute the integral by the summation with the dt being substituted by delta t.

x dot being substituted by x i plus 1 minus x i upon delta t. So, this is standard process and that gives us on; yes here it is I am sorry, here it is. This is the and this last term is corresponds to the term that I have been talking about. This is the expression that we get for the action corresponding to the driftless Brownian motion or driftless Lagrangian.

So, this let us call it the driftless, driftless action. And, this expression the last term we have obtained this expression by making the substitutions that are given here in the green box. And in using the expression for the Lagrangian, the driftless Lagrangian which is given here in the red box.

So, let me repeat, using the Lagrangian that is here in the red box and making the substitutions that are given in the green box; we arrive at the expression which is here in the green box here; the last term in the green box here. And, we call this A 0 which is the action corresponding to the driftless Lagrangian and the driftless Brownian motion right. So, now we move to the path integral. When we do the path integral, we first simplify the path integral term.

The path integral term is given by the expression in the green box here, again and the viewers would be similar; would be familiar with this expression. We have used it many many times, this is simply extension of the path integral over the various paths the which are denoted or which are obtained on the discretization or the time slicing and the corresponding slicing of the prices.

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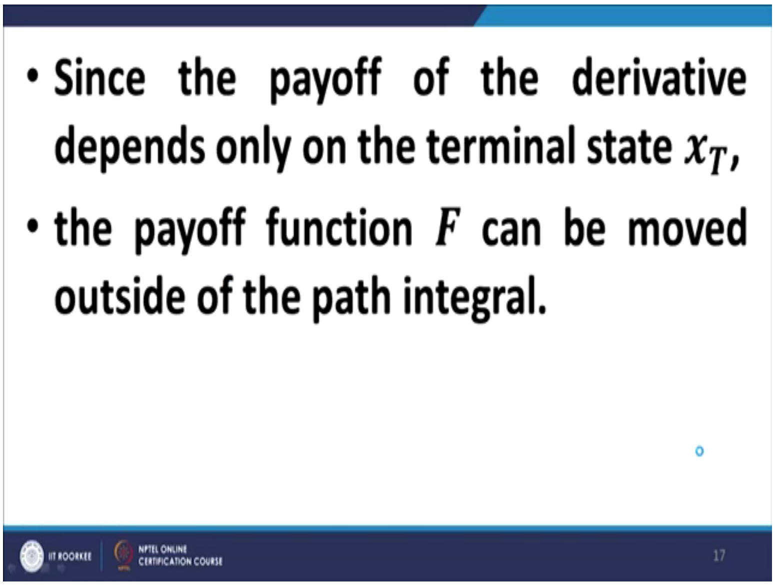

Now, we have got the some simplifications to do. Firstly, we are I reiterate we are talking about or we are working on derivatives or working out the price of an option which is path independent.

In other words, the payoff of the derivative is independent of the path followed by the stock. And therefore, it depends only on the terminal price. And therefore, it can be taken outside the path integral and in other words what I am trying to say that that F a F e x T can be taken outside the path integral.

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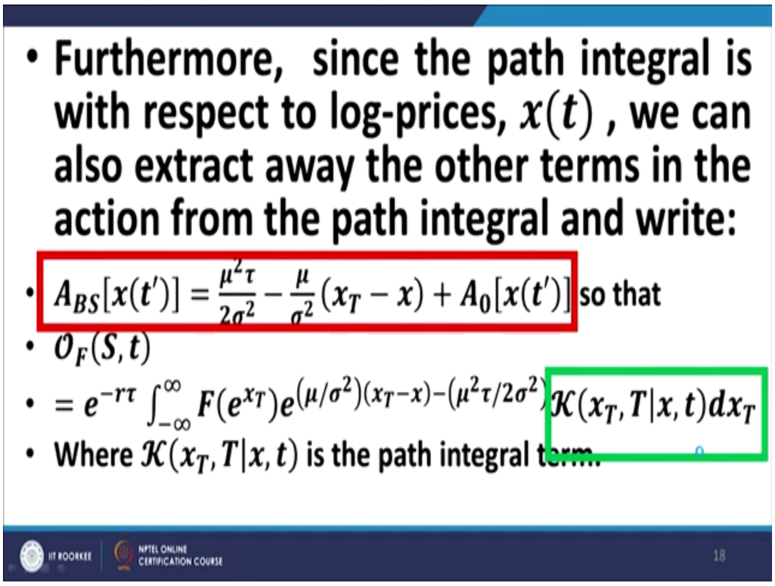

And the another simplification can be done, the log prices x t we are the path integral that we are interested in is with respect to the log prices x of t. And therefore, we can take out all the other terms which do not involve x of t and from the action outside the path integral. In other words, we can from the path integral or we can split the path integral into two parts. We can have a pre-factor consisting number 1 of the payoff of the option which is the independent or which depends only on the terminal price.

And number 2 other terms which do not relate to or which are not dependent on the prices or the log prices of the underlying asset. So, that being the case we are able to simplify the action. If you look at this action, the first term and the second term both terms are independent of the prices x t. And therefore, they can be taken into a pre-factor and similarly we can take F of a; F e to the power x T also into the pre-factor.

And, we can write the path integral as a separate quantity as K x T T subject to x comma t and d x T and where this term and the term in the green box here constitutes the path integral component. So, this is the path integral component and all the rest of it is not related to it the x t, the paths it is not related to the path. And therefore, they have nothing to do with the path integral and therefore, they can be taken as a pre-factor.

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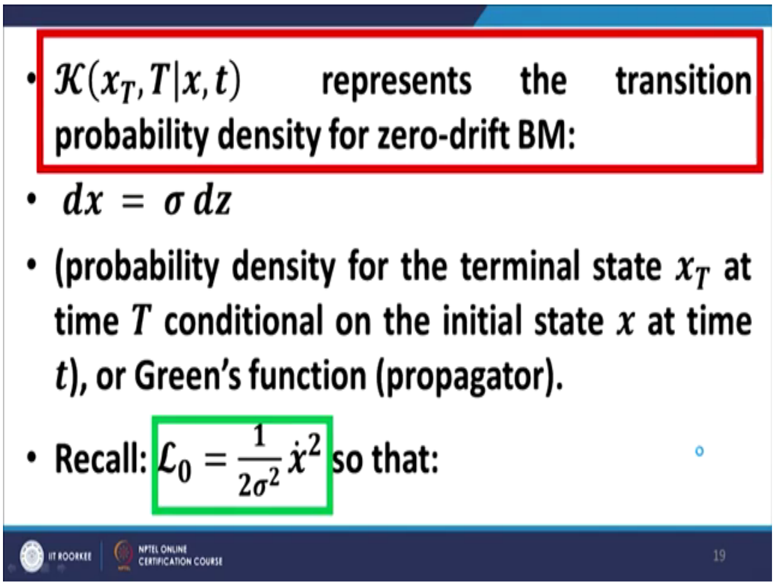

Now, what is this K x T? The K x T represents a transition probability density. It is the transition probability density of what? It is the transition probability density of the price to move of the log price to move from x at time small t to x capital T at time capital T; with respect to; with respect to when the stock price follows or when follows the stochastic differential equation, driftless stochastic differential equation dx is equal to sigma dz.

So, the stock follows this K represents the transition probability with respect to the zero-drift Brownian motion, with respect to the driftless Lagrangian as we have just mentioned. So, and recall that the zero-drift Lagrangian or the driftless Lagrangian is given by 1 upon 2 sigma square x dot square.

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$$\mathcal{K}\left(x_T,T\left|x,t\right.\right)=\int_{x(t)=x}^{x(T)=x_T} e^{-A_0\left[x(t')\right]}Dx\left(t'\right)$$

$$:=\lim_{N\to\infty}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{dx_1}{\sqrt{2\pi\sigma^2\Delta t}}\cdots\frac{dx_{N-1}}{\sqrt{2\pi\sigma^2\Delta t}}}_{N-1}$$

$$\times\exp\left(-\frac{1}{2\sigma^2\Delta t}\sum_{i=0}^{N-1}\left(x_{i+1}-x_i\right)^2\right)$$

So, we now evaluate this factor and we leave the other factors for the moment, we leave them alone, we evaluate this factor. We for this purpose we have we write the action as A 0, where A 0 is the action corresponding to the driftless Lagrangian, the driftless Brownian motion.

And it can be written in the form as I mentioned earlier A 0 takes the form expression; takes the expression given in the round brackets here. So, we have simply substituted the value of A 0 here and our next exercise is simply to simplify this particular integral.

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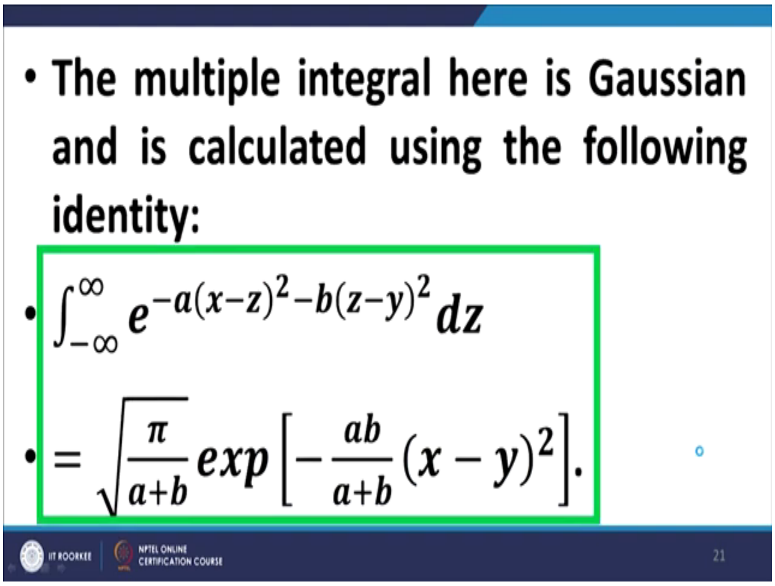

We this integral is Gaussian and we evaluate this integral using the formula, that we have in the green box here. You would recall that we had used this formula earlier, when we worked out the quantum mechanical path integral for a free particle which follows absolutely the same lines. And, the formula indeed used therein and here is happens to be the precisely the same. So, we quickly run through it, this integral of course, is Gaussian.

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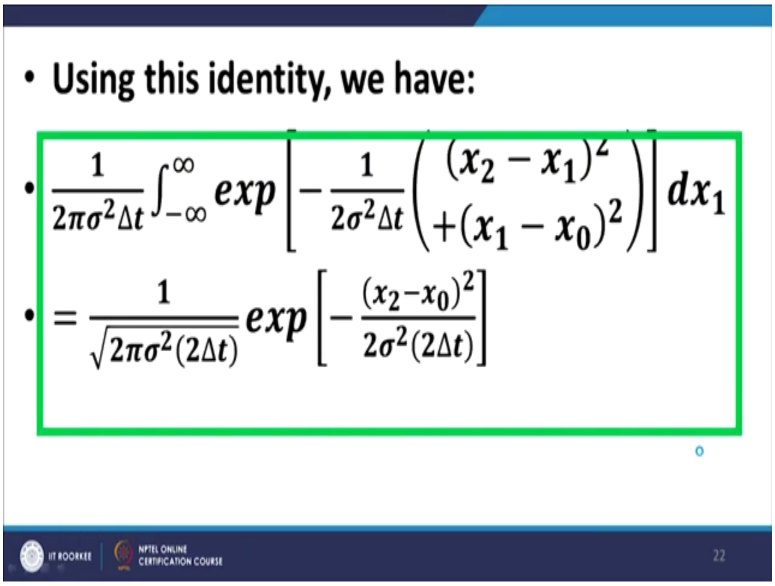

So, using this formula, using this identity we can write this expression. If we pick up any two term, if we pick up the two terms x 2 minus x 1 square plus x 1 minus x 0 square on the left hand side. If I take x as x 2 and z z as x 1 and y as x 0 and a and b both as 1 upon 2 sigma square delta t and simplify the expressions, what I get is the result that is there in the green box.

Now, it can be seen from this, if you look at this carefully; the by using this formula what has happened is this delta t has changed to 2 delta t here in the exponential as well as in the pre-factor. And, the second thing is the x 2 minus x 1 and x 1 minus x 0 have given as x 2 minus x 0. In other words, when we have integrated this with respect to x 1, the x 1 has gone out and we are left with x 2 minus x 0.

So, what we I repeat there have been two changes; delta t has become delta 2 delta t in the pre-factor and in the exponential denominator of the exponential. And, number 2 the x 1 term has disappeared and we are left with x 2 minus x 0 squared of course.

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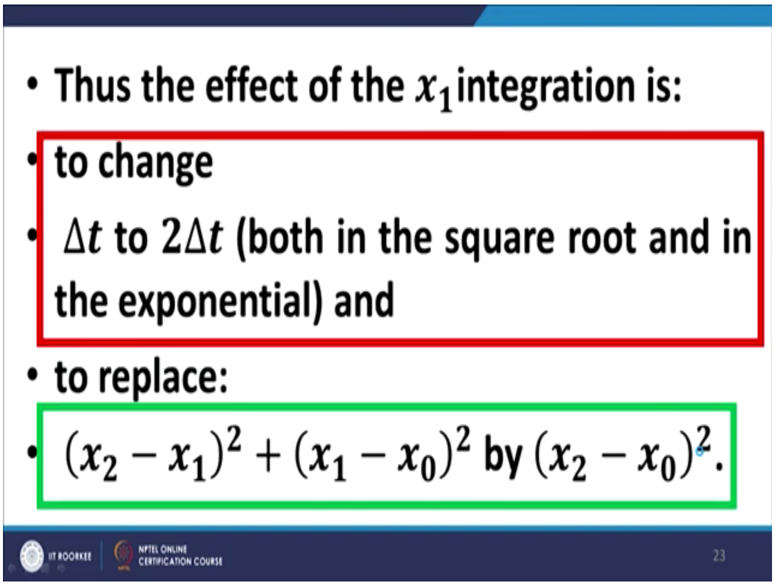

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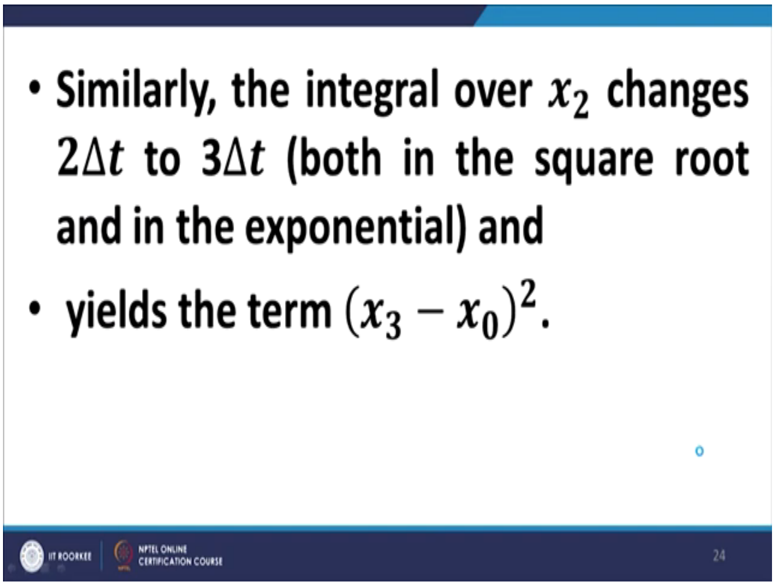

So, now we again proceed the same way, we now in pick up x 3 and we evaluate the integral with respect to x 2. We pick up the term involving x 3 and we evaluate the integral over x 2; the result would be that wherever 2 delta t appears, it would be replaced by 3 delta t. And, at the same time instead of the x 2 term would disappear and we would now have x 3 minus x 0 square.

So, proceeding in this same way iteratively we can do all the N minus 1 integrals that are there in this particular path integral. If you look at this, there are N minus 1 path N minus 1 integrals there.

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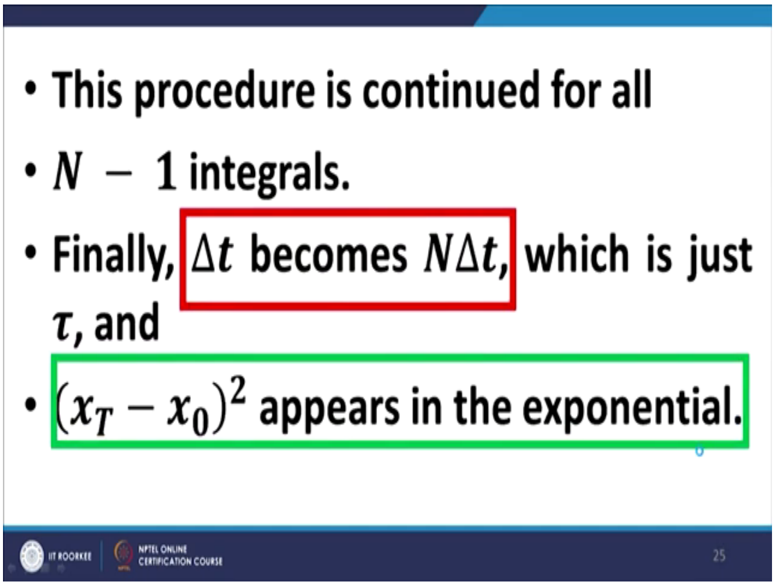

So, when we are when we complete all the N minus 1 integrals what we end up with will be; number 1 delta t will become N delta t and that is nothing, but tau. Because what is delta t? Delta t is equal to t minus tau upon N. So, N delta t is nothing, but tau and the squared expression that we will end up with is x capital T minus x 0 whole square which will appear in the exponential.

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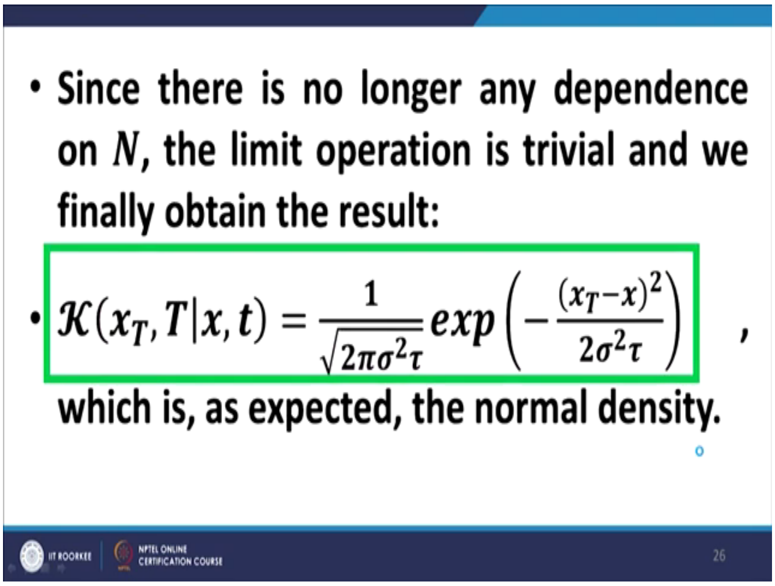

So, there is no dependence on N left after all the N minus 1 integrals are done. So, the limit N tending to infinity can be taken straight away, it is trivial and the expression that we end up with is the expression that is given in the green box. So, this is the expression for the kernel that we have here, that we for in the context of driftless Lagrangian. This corresponds to the driftless Lagrangian, please note this.

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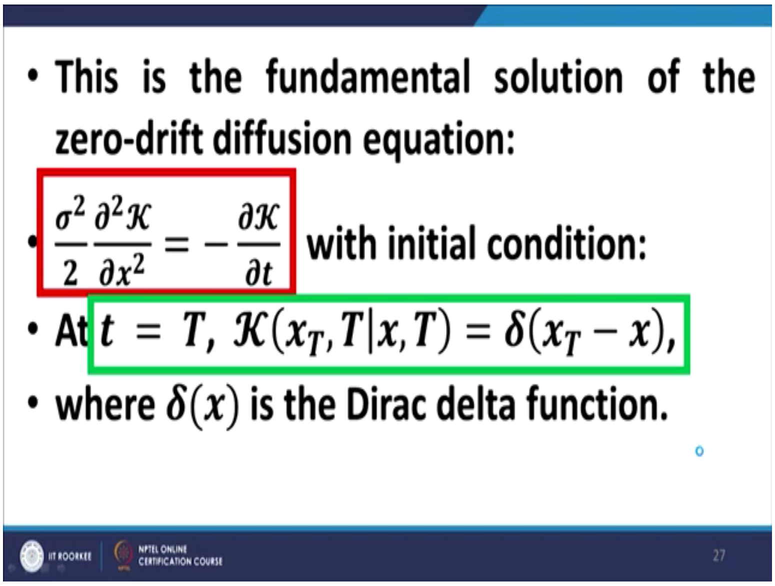

So, it can be easily shown, it can be easily shown that the expression for K that we have obtain K script that we have obtained satisfies the diffusion equation with zero-drift as you can see which is there in the red box here with the condition that at t equal to capital T K x T, T will be equal to the direct delta function condition.

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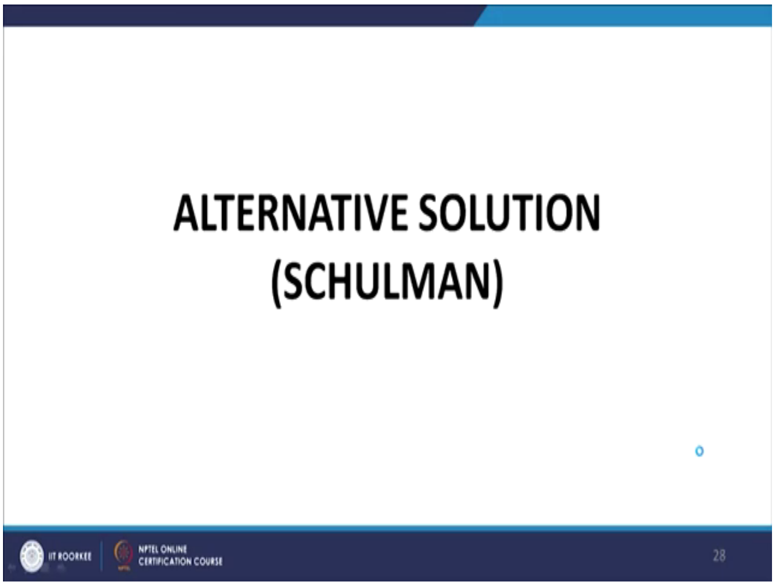

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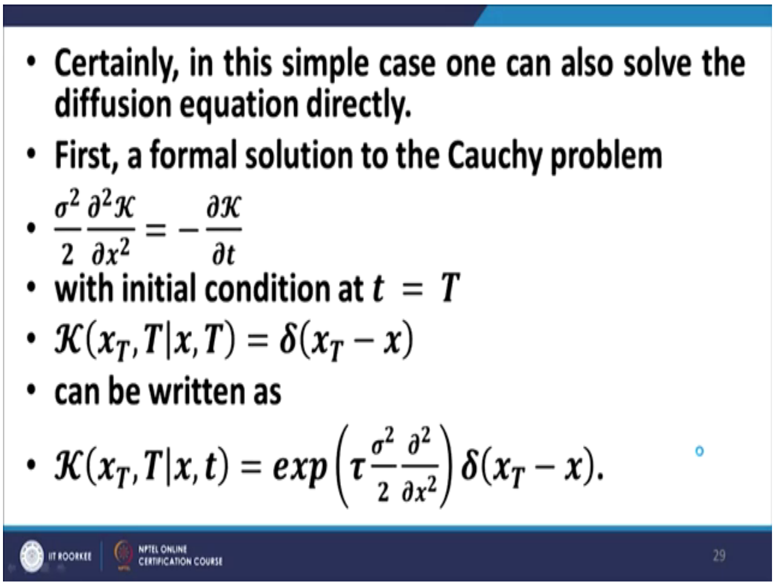

This can be verified by solving the diffusion equation without drift in another manner. And, then comparing the two results, we use the initial condition that we had postulated earlier. And, that enables us to write the value of K as exponential of this operator into the into the direct delta function as shown in the last equation here.

So, this expression immediately is the immediate solution of to the driftless diffusion equation given here in the middle of the slide.

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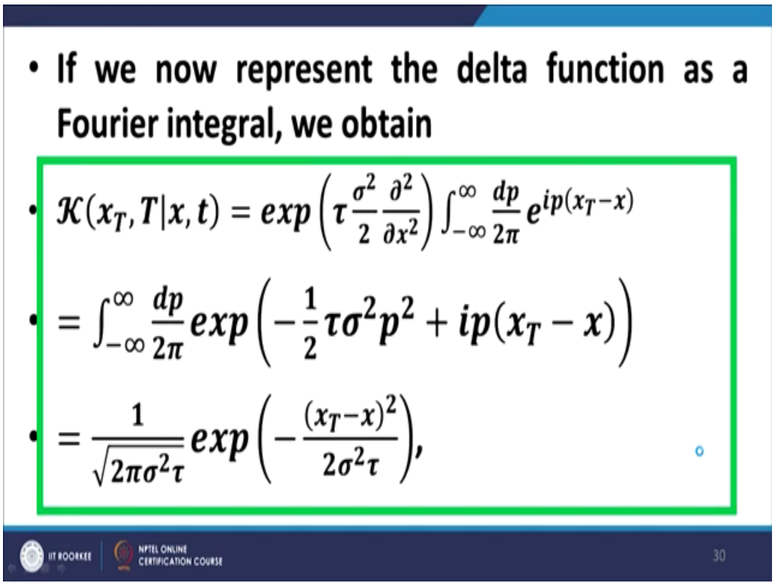

This can be easily verified by applying this operator and writing down delta in terms of its Fourier form or Fourier integral. And, then applying this operator on the delta function written in terms of the Fourier integral. And, then evaluating the Gaussian integral, we get the same result we get the same expression for K as we obtained in the through the path integral formalism.

We get precisely the same expression for the transition amplitudes as we obtained in the case of; in the case of the path integral; precisely the same expression. And, that shows the or that indicates the velocity of the path integral approach.

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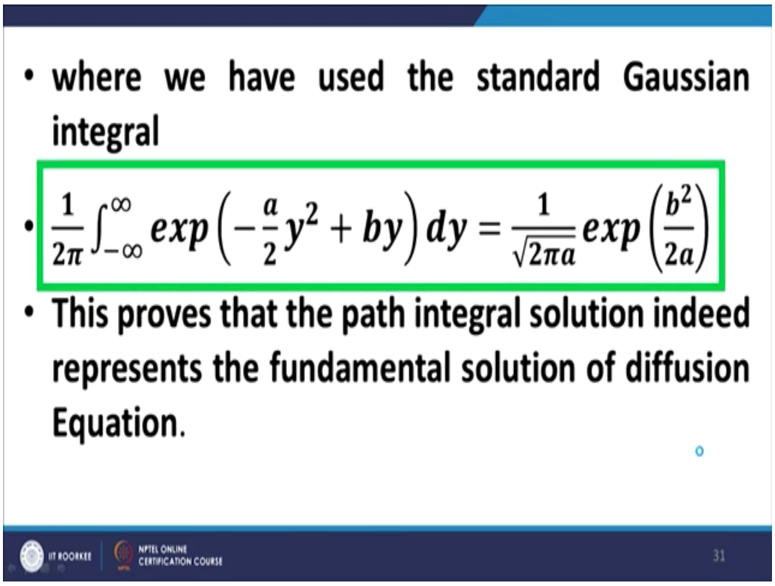

So, this is the evaluation of the integral, Gaussian integral that is done in this particular in this particular slide. We have used this particular formula which is there in the green box.

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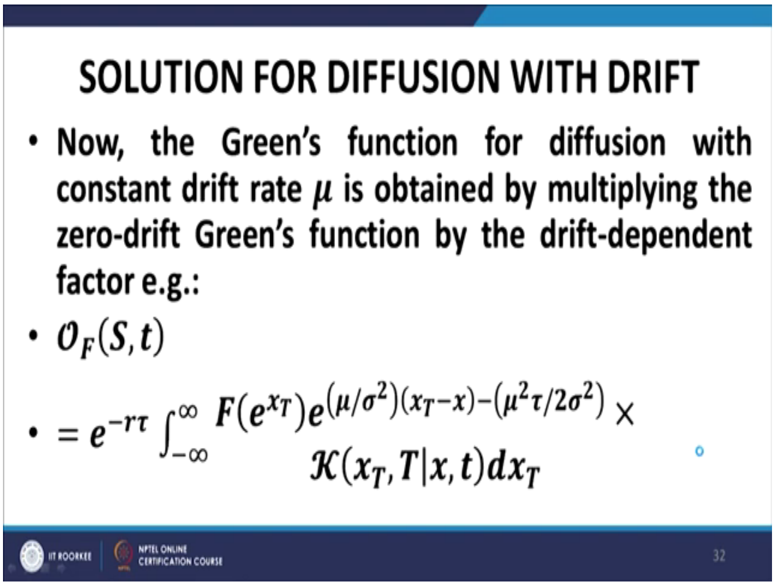

So, now we have to go back because what we have done so far is we have obtained the Green function or the transition amplitude or the transition probability in respect of the driftless Brownian motion. But, remember our stock price follows the a Brownian motion or a stochastic differential equation with drift.

So, we need to incorporate that what the drift term also into our solution. Now, the Green function for the diffusion with constant drift rate mu is obtained by multiplying the zero-drift Green function that we obtained right now, that is the transition amplitude.

They are synonymous to a green transition amplitude or the or the Green function by the drift dependent term and this is easily seen. We have shown that O F is equal to the expression that is there on the right hand side, this is brought forward from earlier in this expression.

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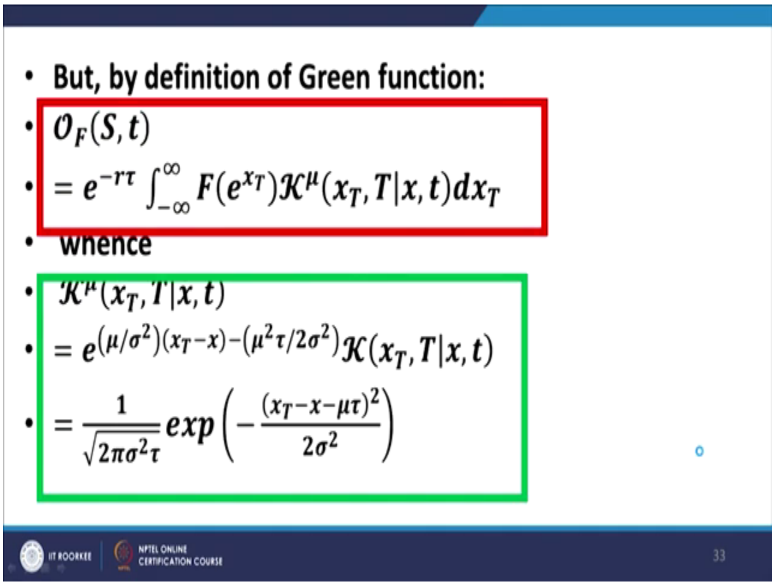

By the definition by the definition of Green functions we can write O F in the form which is given in the red box here. So, by comparing these two expressions, by comparing this expression which we have here in the red box and by comparing the expression which we have here which we are brought forward from earlier. We get an expression for K mu, K mu is the transition amplitude corresponding to or the Green function corresponding to the Brownian motion with the drift coefficient of mu.

So, that being the case that is obtained as the expression which is here in the green box here. And, when we substitute the expression for K we or this substitute the expression for the driftless Green function K; we obtain the expression that is right at the bottom of your slide in the green box.

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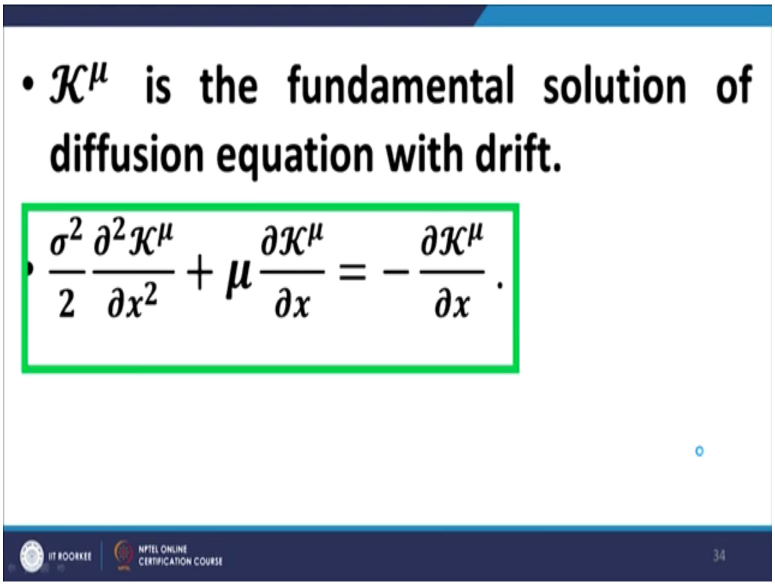

So, it can easily be shown that K mu is the fundamental solution of the diffusion equation with drift which is given here in the green box here. It is a straightforward exercise parallel to what we had done earlier in the case of in the diffusion equation without drift case of without drift. It is absolutely synonymous or absolutely similar, we can do it in the same way.

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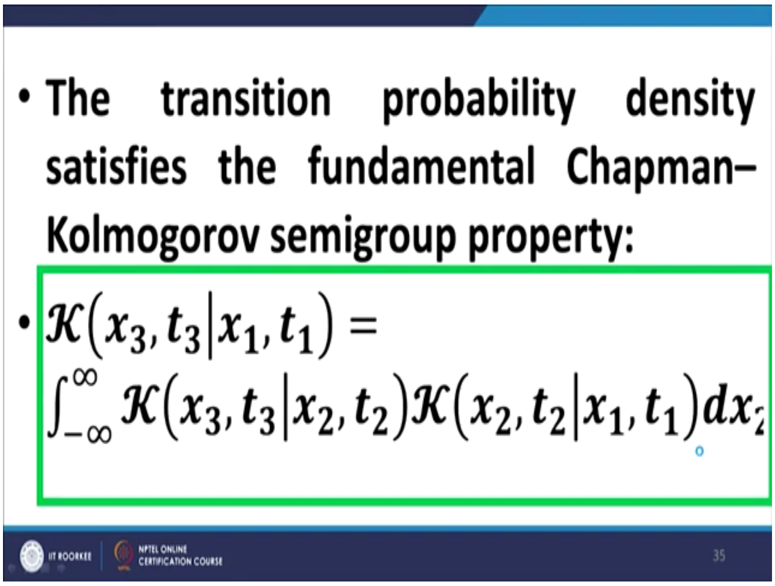

Now, another important thing the transition probability density or the transition amplitude also satisfies the Chapman-Kolmogorov equation which also we had discussed at an earlier point in time and the semigroup property or the translation semigroup or the property of Chapman-Kolmogorov equation, that is the probability of a particular state moving from state 1 to state 3 at time t equal t 1 to time t 3.

If you intervene at a particular point in time t 2 and you have you can integrate over time t 2, integrate over all possible paths at time t 2 that connect the state 1 to the states at time t 2. And, then from states t 2 to the states at time t 2 to state 3 at time t 3 integrated over all possible states at time t 2 will give you the transition probability of or transition amplitude of moving from state 1 to state 3.

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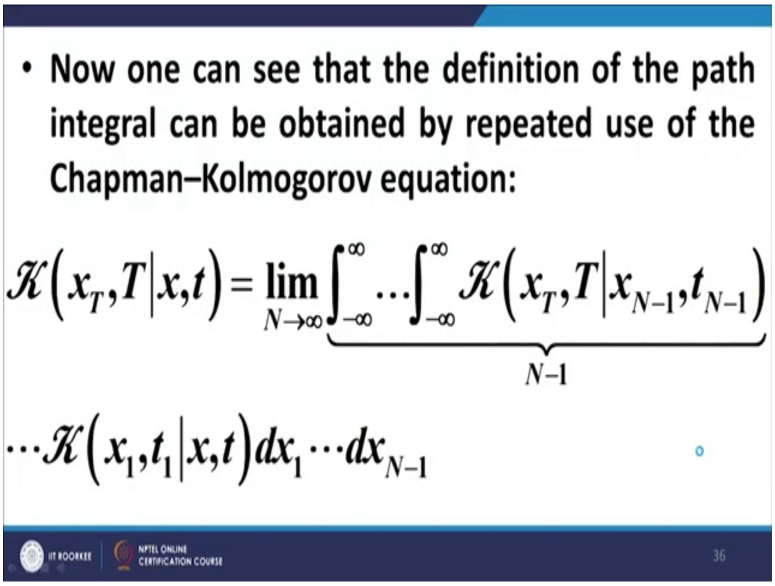

So, this is also this also shows that we can obtain the path integral by repeated application of the Chapman-Kolmogorov equation; at you see Chapman-Kolmogorov equation relates to one time slicing or one slice being introduced between t 1 and t 3, only one time slicing. Here we are having an infinite number of time slicing in being introduced. So, it amounts to repeated applications of the Chapman-Kolmogorov equations.

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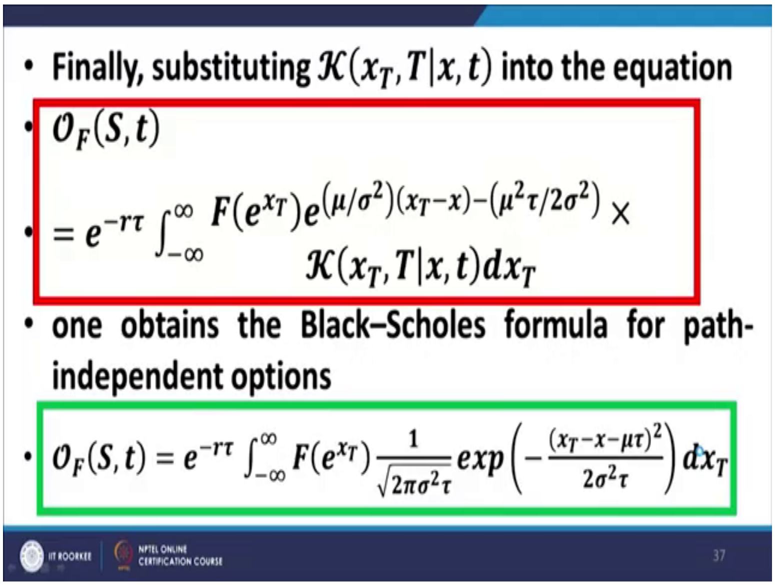

And, from this expression which we have obtained earlier, when we substitute the expression for when we substitute all the terms; we obtain the expression for path independent options in the form that is given here in the green box.

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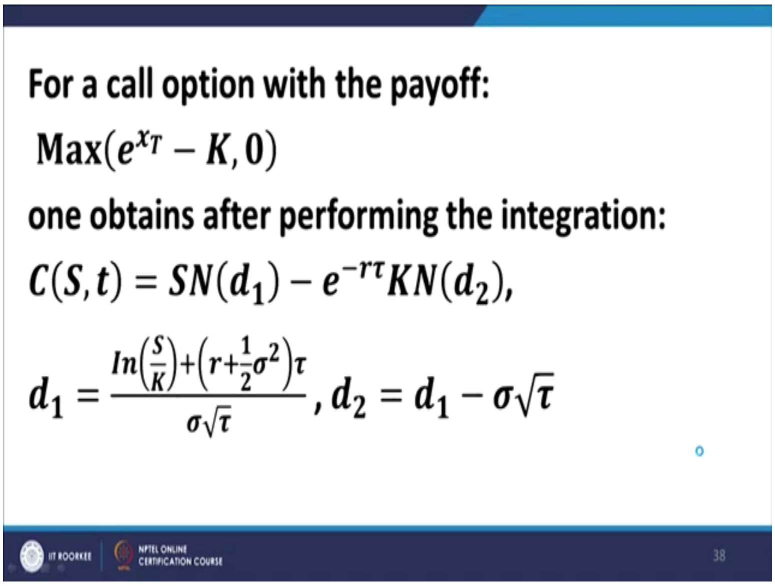

And, for a call option which in this expression simplifies to the expression that is given here on this slide.

Thank you, we shall continue after the break.