Path Integral Methods in Physics & Finance
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Lecture - 58
Black Scholes Model

Welcome back. So, before the break I was talking about the binomial model. And, we also discuss some examples of the binomial model for the valuation of European options and how it needs to be modified needs to be amended, in the context of valuation of European options.

So, now having done that the binomial model works in a discrete framework. Now, let us move on to the continuous version or the continuum version of this option pricing. And we now; I now introduce to you the black Scholes model.

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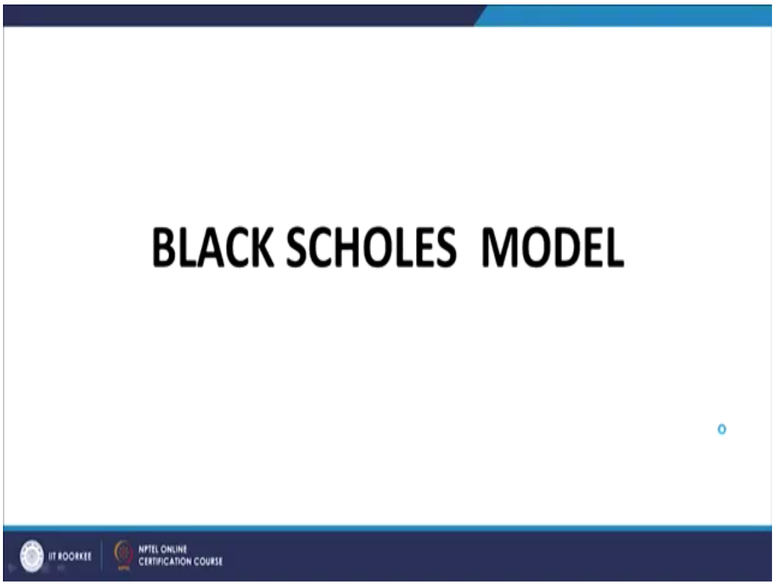

The black Scholes model is the cornerstone of contemporary valuation, theory of contingent claims. In fact, a huge chunk of finance on finance theory, rests on three fundamental propositions or three fundamental works done by Harold Markowitz enter in the context of mean variants portfolio optimization, William Sharpe who introduced the capital as a pricing model and then the black Scholes model for option pricing for contingent claims valuation. So, let us now get into the black Scholes model.

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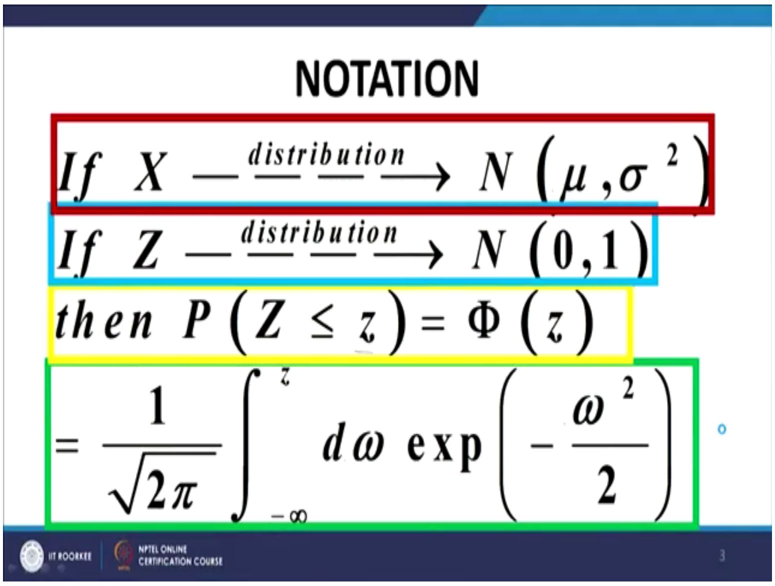

Before I get into the black Scholes model a bit of assumptions notation, I am sorry. Now, if X is normally distributed right because, there is some ambiguity among the literature on how we represent normal distribution? So, I have put it here for the sake of completeness. If X is normally distributed with a mean of mu and a variance of sigma square, we write it as N bracket mu comma sigma squared.

If Z is a standard normal variate we write it as N comma 0 comma 1. Because, the standard normal variable is has a mean of 0 and a variance of 1. Now, if Z is a standard normal variate in other words if it is distributed as a standard normal variate. Then, we define the cumulative distribution function capital phi of z as P capital Z is less than equal to Z.

In other words the random variable Z that is capital Z the random variable Z is the is capital Z and small z is a particular value that the random variable z can take. So, capital phi of small z represents the probability of capital Z that is the random variable Z taking values up to and including the small the value small z. And that is given by this expression which is given at the bottom of your slide in the green box.

So, this is the notation that I will follow here, phi z is the cumulative normal distribution please note this. This is cumulative normal distribution is the distribution, which represents the total probability of the random variable taking a particular value. Taking a values up to a particular value; taking values up to a particular value from the downside. So, that please note this notation, let us now go to the assumptions.

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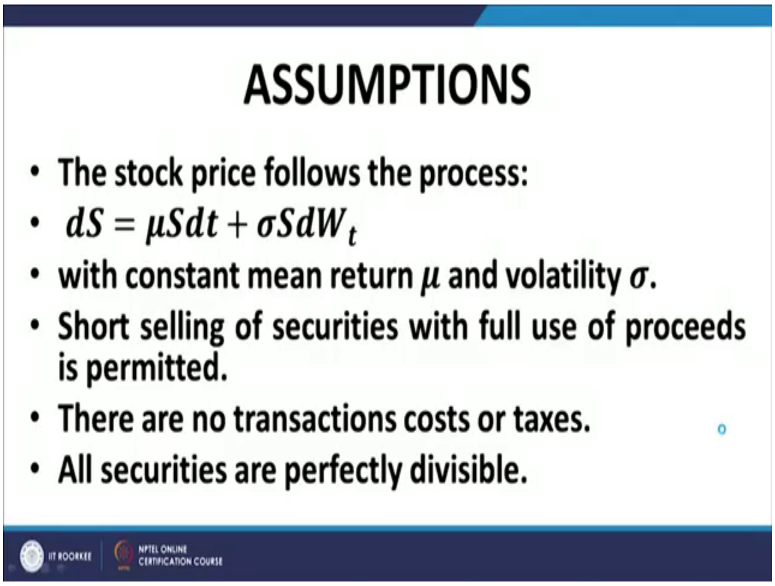

We assume that the stock price follows the stochastic differential equation d S is equal to mu S dt plus sigma S dt W t d W t is remember d W t is the infinitesimal Brownian motion increment and mu and sigma are constant parameters; mu is the mean return and sigma is the volatility. Short selling of securities will with full use of the proceeds allowed.

Short selling I explained earlier it is the process of borrowing the securities and selling it in market at t equal to 0 in anticipation that at a future point in time the price would go down. You, could buy the securities in the market and replenish it to the owner from whom, you had borrowed that securities and sold them in the first place at t equal to 0. So, that is called short selling.

Selling securities which are not owned by you by borrowing them; there are no transaction costs or taxes; in other words, there is no market frictions then all securities are perfectly divisible securities form a continuum in that sense. They can be traded in any values of any real number ah any positive real number of course.

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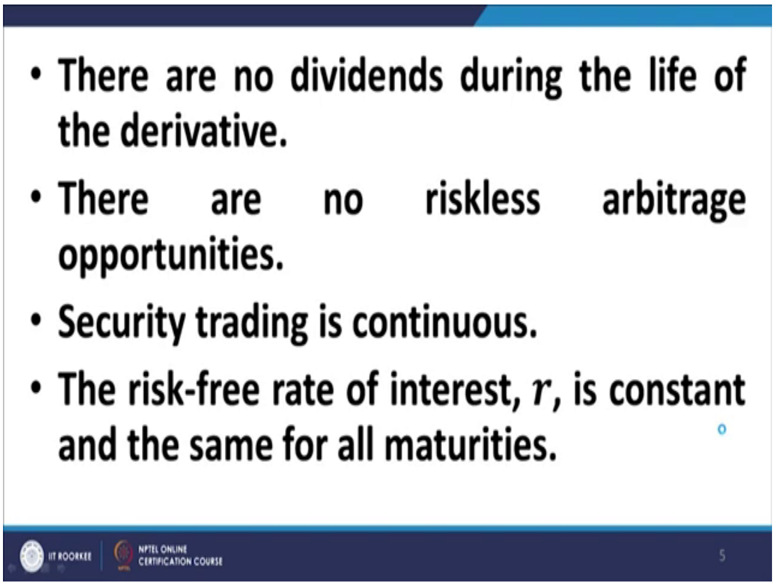

There are no dividends during the life of the derivative, this is an important assumption of course the black Scholes model can be modified to accommodate this assumption. But, for the moment the fundamental model assumes that, there are no dividends during the life of the derivative.

There is no riskless arbitrage opportunities in other words the market is efficient. So, if there is any arbitrage opening it would be syphoned away immediately, security trading is continuous and the risk-free rate r, is constant and is the same for all maturities. In other words, the yield curve is flat and r is continuous compounded please note this.

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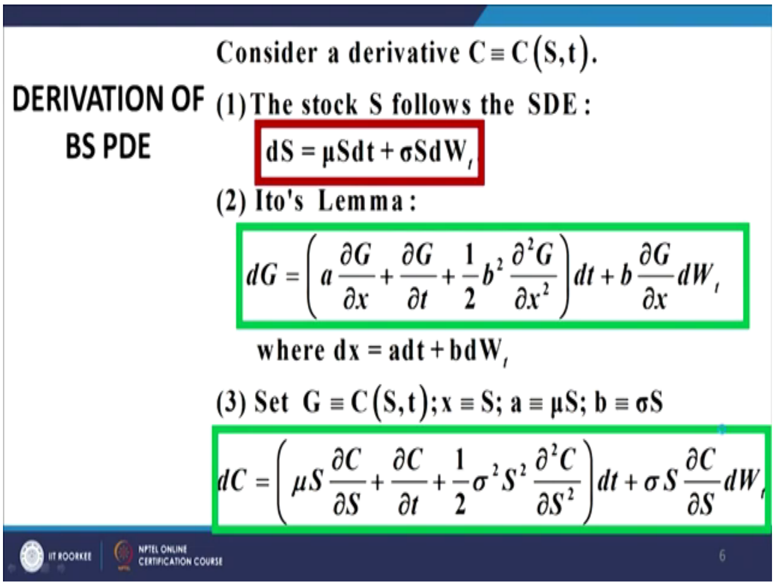

So, we now start we now do the derivation of the black Scholes partial differential equation. What is the data that we have let us start with that the stock S follows the stochastic differential equation, I mentioned just now. It follows the stochastic differential equation given in the red box d S is equal to mu S dt plus sigma S d W t remember d W t is the infinitesimal Brownian motion increment

We also have proved the Ito’s Lemma; Ito’s Lemma for the total derivative of a function of a continuous at least twice differentiable function, of a stochastic variable. And maybe explicit dependence of time is also there. But, the important thing is that there is a x is a stochastic variable, following the stochastic differential equation dx is equal to a dt plus b d W t; and if that is so and if G is a function of x and t. Then d G is given by the expression given in the green box here.

Now, setting G equal to C x t C S t, where C is a derivative. And the derivative price or the derivative value is a function of the stock price or price of the underlying asset. And may have explicit dependence of time, I have discussed that issue and x is equal to S in this case a is equal to mu of S.

And mu into S I am sorry and b is equal to sigma into S. And these values you can obtain by comparing the equation in the red box, by the expression that is given right below the green box. d S is equal to mu S dt plus sigma S d W t and dx is equal to a dt plus b d W t; comparison of these two clearly gives us, a is equal to mu S and b is equal to sigma S. We said G is equal to C S t.

ah Then, we get from Ito's equation that is from the equation that is in the upper green box we get the equation d C is equal to the expression, that is given in the lower green box here. So, this is the expression for the total derivative of the price of the derivative.

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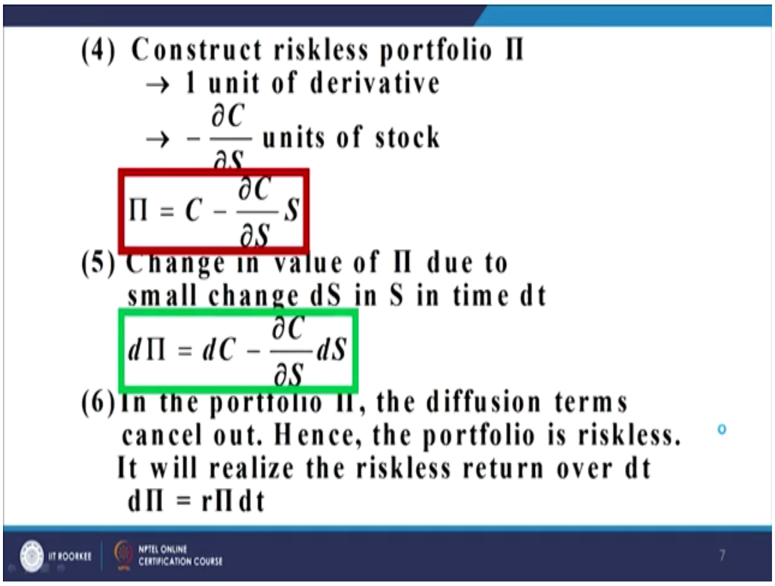

Now, now recall that when I started talking about the pricing of options, I emphasized very strongly the fact that the pricing of options bases itself or the strategies for the pricing of financial derivatives bases itself on the presumption that we could create a riskless asset, by taking a position in the derivative and a corresponding position in the underlying asset. That is precisely that is precisely what happened?

When I talked about the binomial model, when we constructed the binomial hedge and that is precisely what is going to happen just now. What we do is? We construct a portfolio capital pi which consists of 1 unit of the derivative C that is that is our derivative here. We have symbolically represented the derivative by C. So, the portfolio that we have constructed consists of 1 unit of the derivative at minus d C by d S units of the stock, minus d C by d S units of the stock.

So, the value of the portfolio becomes C minus d C by d S into S, where S is the instantaneous stock price and C is the instantaneous price of the derivative and d C by d S is the number of units of the stock that has incorporated in the portfolio pi. So, pi is equal to C minus d C by d S into S.

Now, the change in value of the of the portfolio pi due to a small change d S in S in time dt would be given by d pi is equal to d C minus d C by d S into d S d C minus d C by d S into d S. Now, here we are making a very strong assumption, we are making the assumption that during this during this time period dt that we are considering.

Or due to this change in price the portfolio composition does not change that is d C by d S does not change, d C by d S remains constant during the small time period for which, we are considering the change in the value of the portfolio; the composition between the portfolio that is 1 unit of the derivative and d C upon d S units of the stock.

This composition remains unaltered as a consequence or due to a change, in the price of the underlying asset. We are making that strong assumption and so, that being the case we get d pi is equal to d C minus d C by d S into S.

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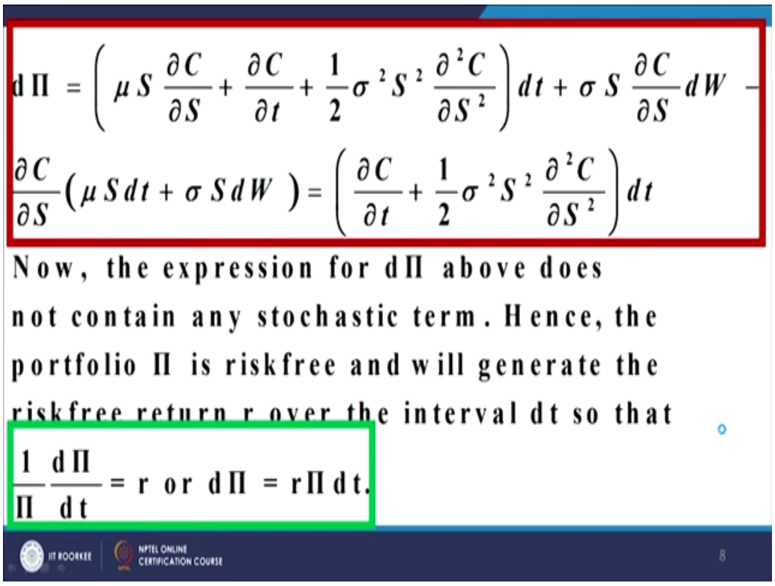

So, next step d is now putting the values you see what we have here is? d pi is equal to d C minus d C by d S into d S. We have got expressions for d C here with us we have got expressions for d S with us. Let us recap d C is given by the expression in the bottom red bottom green box here, and d S is given by the expression that is given in the upper red box here.

I repeat d C is given by the expression that is given in the bottom green box here. And d S is given by the expression that is in the upper red box here. Now, putting these values there what we get dpi is equal to the whole thing simplifies out and what I get is d C d pi is equal to this expression on the right hand side in the red box, with a with a port factor of dt this is very interesting.

This is very interesting for the simple reason that the this d pi now, does not contain any term with d W here the terms in d W are cancelled out. Are inter se cancelled out between the d C term and the term involving d S. That is precisely what I mentioned in the beginning that, the randomness in the derivative. And the randomness in the underlying asset will annul each other.

We, construct the portfolio in such a way we construct the portfolio with such composition that the randomness in the derivative and the randomness in the underlying asset annul each other. Why this is possible? Because, you can see clearly that both are being modelled by the same Brownian motion. The d W that is appearing in d C with the same d W that is appearing in d S.

So, the Brownian motion is same and therefore, we are able to by adjusting the coefficients of d W. We are able to construct a portfolio such that the outcome or the emerging portfolio Ah the change in value of the emerging portfolio is independent of d W. Now, what does it mean? Because, the randomness is only there is modelled only by the d W term. There is no other random term the only term that that incorporates randomness in our analysis is the Brownian motion term.

In other words, we are modelling the randomness purely by Brownian motion. So, because that Brownian motion term is missing here in d pi; it means what? It means that this portfolio is not a risky portfolio because, it has no randomness. The randomness that was initially there has been annulled has been removed by inter se adjustment through the portfolio. And therefore, now we have a portfolio which has no randomness.

In other words which is risk free no randomness means no risk, no risk means risk free and therefore, my portfolio pi is risk free and because, my portfolio pi is risk free it will generate the risk-free rate of return. And that is precisely what we have done here in the green box d pi is equal to d 1 upon pi d pi upon d t, this is the return this is equal to the risk free rate or d pi is equal to r pi d t.

So, that being the case we have d pi is equal to r pi d t. Now, we have one expression for d pi which is given in the red box d pi is equal to the expression that is at the right hand side in of the red box. The other expression for d pi, we have from the expression in the green box.

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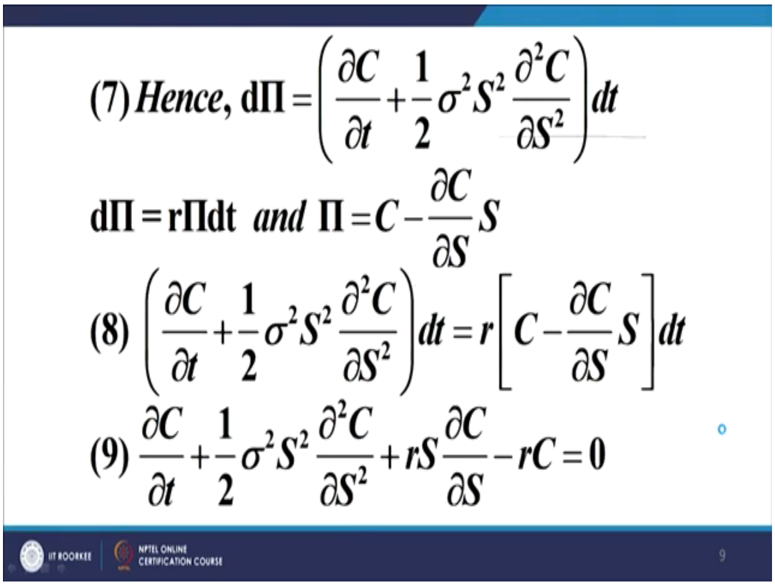

And we also know that pi is equal to C minus d C by d S into S. So, we have d pi from here d pi from here and we have pi equal to this. So, equating the two values of d pi and using pi is equal to C minus d S by d C by d S into S, what we end up is equation number 9 which is the black Scholes partial differential equation. So, equation number nine is called the partial black Scholes partial differential equation. And it is it has revolutionized the valuation of contingent claims in finance theory.

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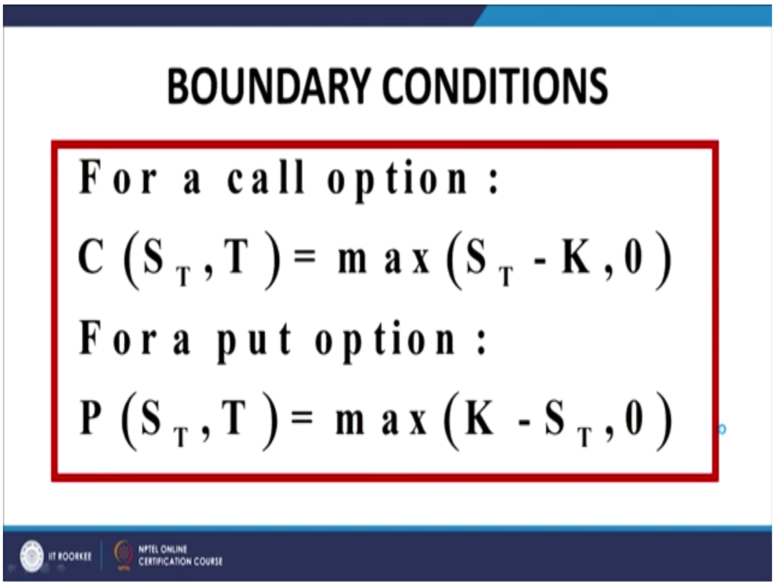

The boundary conditions in the case of a call option European call are quite straightforward C S T, must be equal to maximum of S T minus K comma 0. And for a put option, it has to be the other way around P S T comma K is equal to max m K minus S T comma 0.

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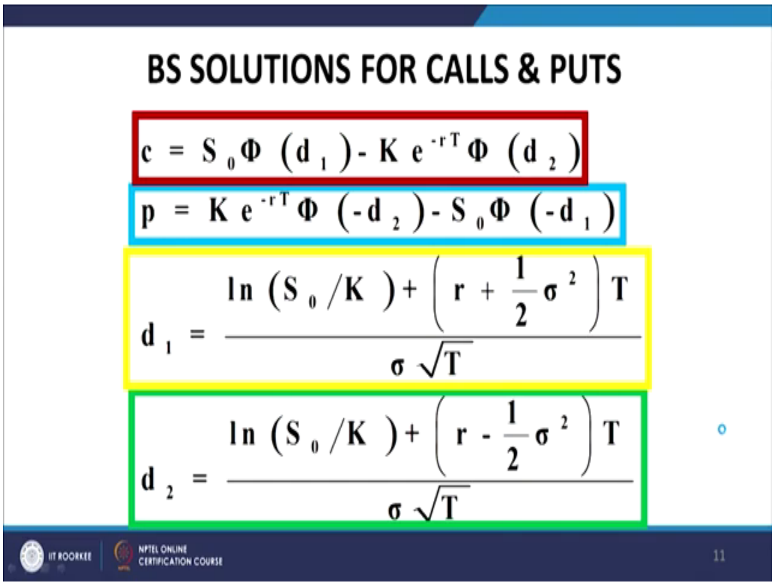

These are the boundary conditions and the black Scholes solution for calls and puts take the form C is equal to S naught. And this is the value at T equal to 0 that is today C is equal to S naught phi d 1 minus K e to the power minus r T phi d 2. And correspondingly they have a value of the put, where d 1 is given by this expression in the yellow box. And d 2 is given by this expression in the green box. Now these are solutions of the black Scholes equations.

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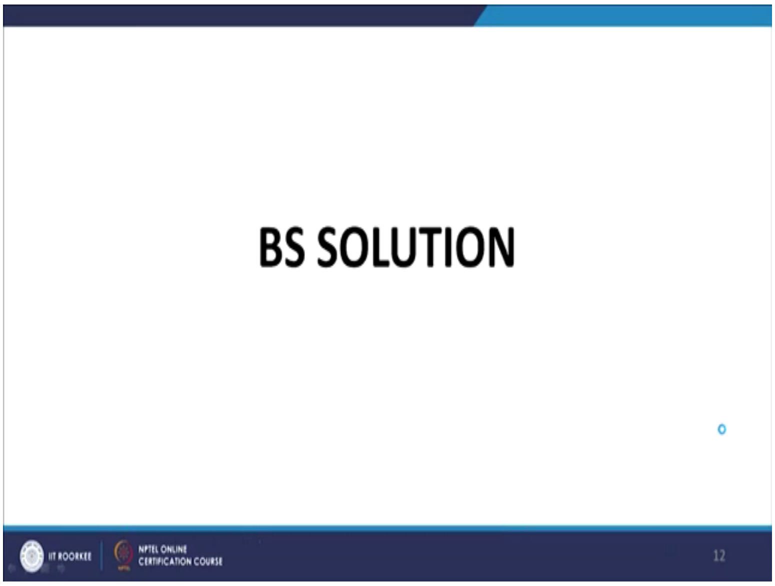

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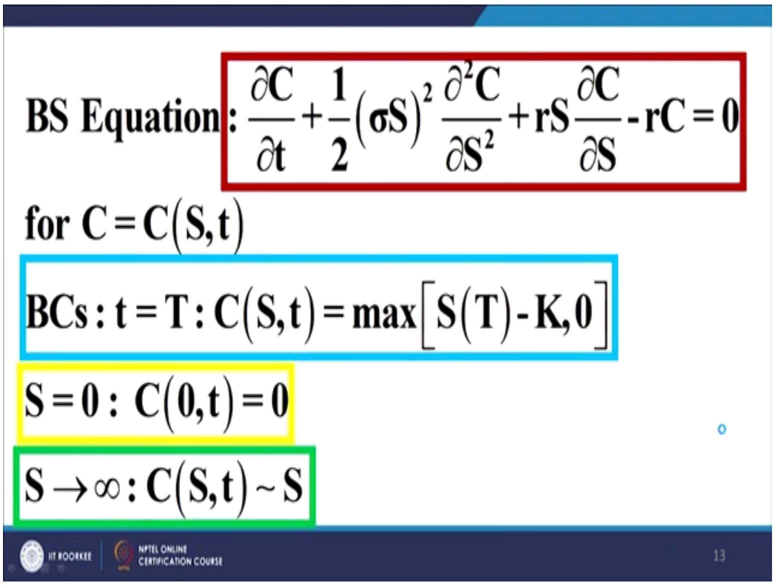

Now, let us try to solve the black Scholes equation, the black Scholes equation that is that we have just worked out that we just arrived at is given here, in the red box. The boundary conditions at t equal to capital T C S t is equal to maxima S T minus K 0, that is what we discussed.

We have yes we are solving this black Scholes equation for a European call please take note of this, we are solving this black Scholes equation for the case of the European call. So, that is why we are writing C as C we are writing this as the boundary condition maximum S t minus K comma 0, at maturity of the option.

If stock price is 0, then the call value will be 0 because the call value will be entirely out of the money. And therefore, its value will be 0 and if stock price is infinity the call will mimic the stock. And therefore, the call price will also correspond to the stock price.

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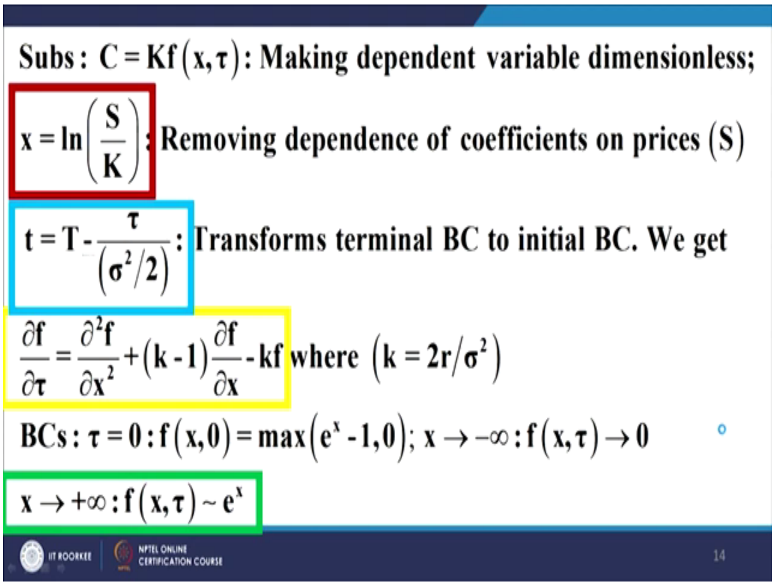

We make certain substitutions to first step. The first step is to make certain substitutions. What are those substitution let us start with we make the dependent variable dimensionless, we make a dimensionless by expressing it in the units of the exercise price. In other words, we write C upon K is equal to f x tau, where x tau will be defined later, but we instead of using C the independent as its dependent variable henceforth. We use f as the dependent variable henceforth where f is equal to C upon K.

We write x is equal to log S upon K, this enables us to remove the remove the dependence of the of the coefficients of the derivatives on the independent variable. As you can see here that the coefficients of the derivative depend on S here. So, by making the substitution x equal to log of S upon K, we are able to remove this dependence of the coefficients of the derivatives on the independent variables.

Then, we make the substitution given in the blue box, which converts the final value problem to an initial value problem. And on making these substitutions, we get the expression, we get the corresponding expression for the black Scholes equation as the equation given in the yellow box here. And the boundary conditions take the form that are given below in the below this expression, in the yellow box by making the substitutions that are made above.

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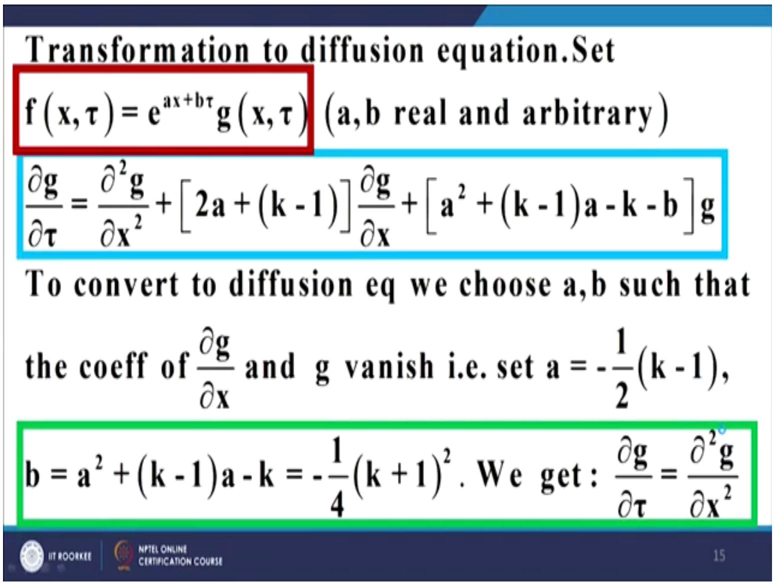

The second step is to transform, this equation for f that is this equation in the yellow box here that we have. The we transform this equation to a diffusion equation; first order in T and second order in x first order in tau and second order in x. How do we do that?

We make this substitution that is given in the red box here, where a and b are free variables which we can set as per our requirements. When, we when we substitute this expression for f in the equation that is given in the yellow box, we get the expression that is given in the blue box here clearly.

In order that the coefficients of dg upon dx. And the and the coefficient of g both vanish we require that a is equal to 1 by 2 k minus 1 and b is equal to 1 by minus 1 by 4 k plus 1 whole square.

The derivation is given here and we need to write a equal to minus 1 by 2 I am sorry a is equal to minus 1 by 2 k minus 1 and b is equal to minus 1 by 4 k plus 1 square, in this equation in the red box. And, if we do that we are able to convert, this equation into a diffusion equation for g.

Because, the drift term and the term of containing only g, both terms vanish and what we are left with is a first order term in tau and a second order term in x square, which is given at the right hand corner of the green box in the slide.

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Solve: $\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2}$

BCs: $\tau = 0 : g(x,0) = \max\left(e^{(k+1)x/2} - e^{(k-1)x/2}, 0\right) \Rightarrow$

$g(x,0)e^{-\alpha x^2} \xrightarrow{|x|\to\infty} 0 \; (\alpha > 0)$

$\tau > 0 : g(x,\tau) \xrightarrow{|x|\to\infty} e^{(k+1)x/2-(k+1)^2\tau/4} \Rightarrow$

$g(x,\tau)e^{-\alpha x^2} \xrightarrow{|x|\to\infty} 0 \; (\alpha > 0)$ where α is a real positive constant.

So, the boundary conditions take the form that are given in the slide. So, now, we need to solve the diffusion equation that is given here subject to these boundary conditions.

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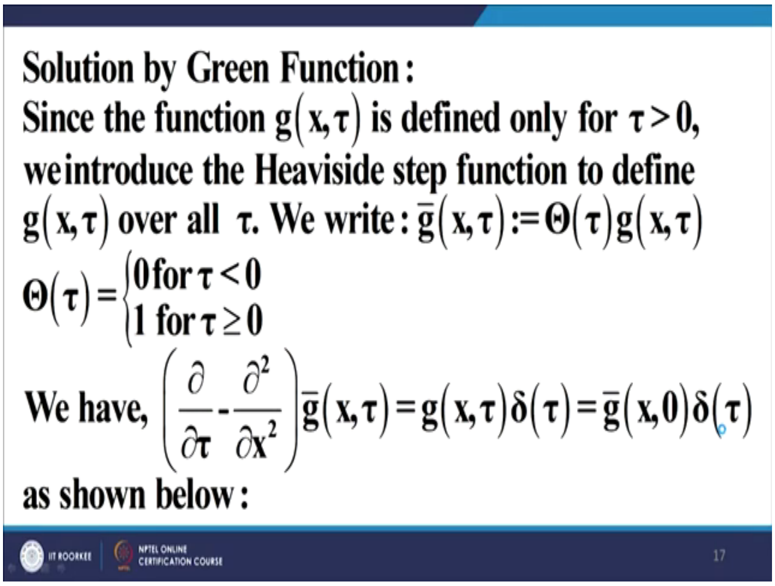

We solve we solve this diffusion equation by the green function. Now, but before we do that we need to note that the function g is defined only for tau greater than 0; so, in order that this function is defined over the entire real line, we introduced the Heaviside step function.

And defined g bar in turn by incorporating, there in with g the Heaviside step function as g bar is equal to theta tau g x comma tau where theta is the Heaviside function, which takes the value 0 for tau less than 0 and takes the value 1 for tau greater than equal to 0.

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We have, $$\left(\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial x^2}\right)\bar{g}(x,\tau)=\left(\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial x^2}\right)\Theta(\tau)g(x,\tau)$$

$$=\delta(\tau)g(x,\tau)+\Theta(\tau)\frac{\partial g(x,\tau)}{\partial\tau}-\Theta(\tau)\frac{\partial^2 g(x,\tau)}{\partial x^2}$$

$$=\delta(\tau)g(x,\tau)+\Theta(\tau)\left[\frac{\partial g(x,\tau)}{\partial\tau}-\frac{\partial^2 g(x,\tau)}{\partial x^2}\right]$$

$$=g(x,\tau)\delta(\tau)=\bar{g}(x,0)\delta(\tau)$$

since $\text{RHS}=\Theta(0)g(x,0)\delta(\tau)=g(x,0)\delta(\tau)=g(x,\tau)\delta(\tau)$

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So, we then have the expression that is given at the bottom of your slide. And this particular expression is derived on the next slide as you shall see here, it is the expression is derived here on the slide. And please note the important thing here is that the derivative of the Heaviside function is the delta function Dirac delta function.

So, when you differentiate it with respect to tau, you get the delta function this is the first term in the second equation. The rest of it is quite straightforward and that gives us the expression delta tau g x tau, which can be simplified and written in the form of g bar x 0 comma delta tau keeping in view the property of the Dirac delta function.

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From above, $\left(\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial x^2}\right)\bar{g}(x,\tau)=\bar{g}(x,0)\delta(\tau)$ with solution

$\bar{g}(x,\tau)=\int_{-\infty}^{+\infty}dy\,\bar{g}(y,0)\,p(x,\tau|y,0)$ where the kernel

$p(x,\tau|y,0)$ satisfies $\left(\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial x^2}\right)p(x,\tau|y,0)=\delta(x-y)\delta(\tau)$

This has the solution: $p(x,\tau|y,0)=\frac{1}{\sqrt{4\pi\tau}}\exp\left[-\frac{(x-y)^2}{4\tau}\right]$

So, that being the case now the solution of this expression, solution of this equation is readily obtained from the green function. And we have the solution as the expression given here. In the second line in this slide g bar x tau is equal to integral minus infinity to infinity d y g bar y 0 p x tau subject to y 0. Now, what is p x tau, this expression p x tau subject to y 0 is nothing but the transition with the kernel.

And it is that represents a transition probabilities transition probabilities of the happening of x at of the variable y, taking the value x at time tau given the value y at t equal to 0. And, it satisfies p x tau the transition probabilities satisfies the condition satisfies the delta function condition given here in the slide in the second last line of the slide.

This equation is the diffusion equation with the delta function on the right hand side. In other words the p, p represents the green function for the diffusion equation. And therefore, p can be written in the form of this expression given at the bottom of the slide.

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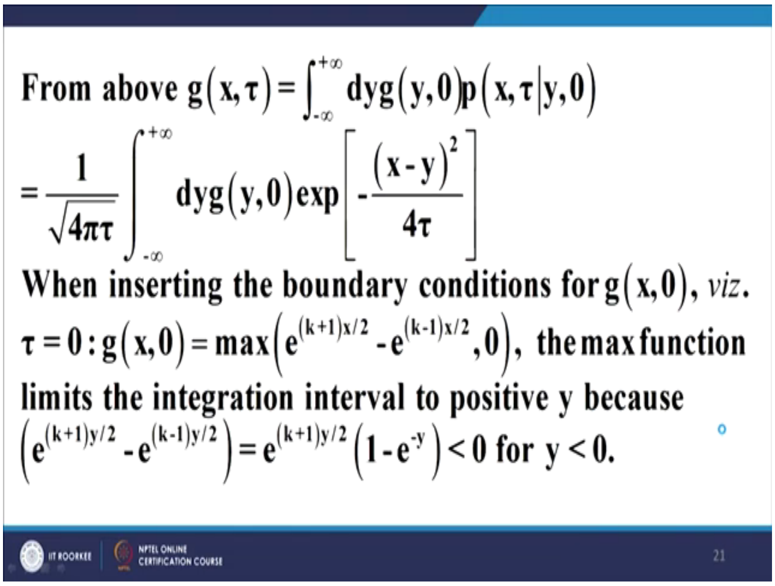

Now, one word about the boundary conditions; if you look at the boundary conditions the boundary conditions are given here, at tau equal to 0 g x 0 is equal to maximum of this expression within the round brackets. And this maximum function limits the integration to positive y only. Why does it do so?

You can see right at the bottom of the slide that, if y is less than 0, then the this whole expression, this whole expression becomes less than 0. And therefore, the because it is because it is we need to take a maxima of this expression, this expression in the round bracket which is now less than 0 for y less than 0 and 0.

So, for y less than 0 the 0 will predominate we will, this maxima function will capture the 0 from the 2. And there therefore, for y less than 0 the contribution to the integral will be 0.

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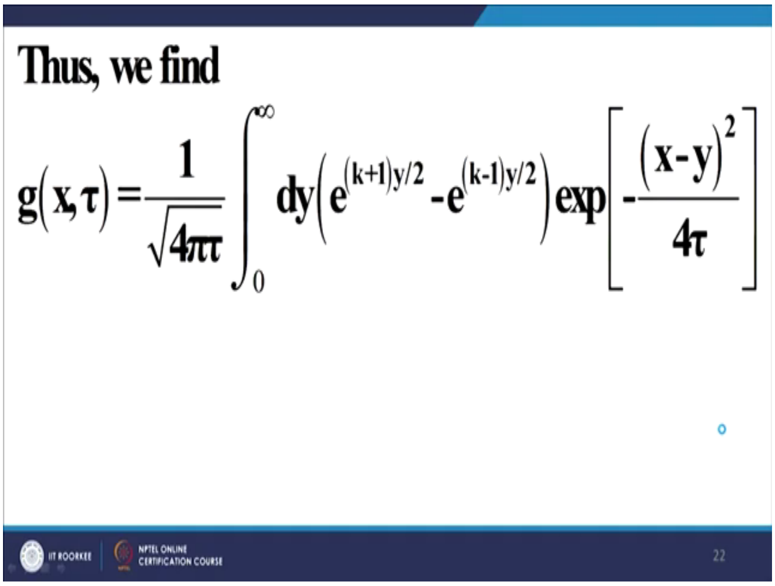

$$g(x,\tau)=\frac{1}{\sqrt{4\pi\tau}}\int_0^{\infty} dy\left(e^{(k+1)y/2}-e^{(k-1)y/2}\right)\exp\left[-\frac{(x-y)^2}{4\tau}\right]$$

So, this is what needs to be taken care of and therefore, we can write the integral in the form that is given in the expression here on the slide.

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From above $g(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_0^{\infty} dy \left(e^{(k+1)y/2} - e^{(k-1)y/2} \right) \times \exp\left[-\frac{(x-y)^2}{4\tau} \right]$

Setting $z = \frac{y-x}{\sqrt{2\tau}}$, we get

$$g(x,\tau) = \int_{-x/\sqrt{2\tau}}^{\infty} \frac{dz}{\sqrt{2\pi}} \left(e^{(k+1)\left(\sqrt{2\tau}z+x\right)/2} - e^{(k-1)\left(\sqrt{2\tau}z+x\right)/2} \right) e^{-z^2/2}$$

And from this we start doing a lot of algebra, we set z is equal to y minus x upon under root 2 tau to introduce the standard normal variants. Because, at the end of the day we need to represent the equation, in term represent the solution in terms of the standard normal variate and therefore, we write z is equal to 1 y minus x upon under root 2 tau, which introduces the standard normal variate here.

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$$= e^{(k+1)x/2+(k+1)^2\tau/4}\int_{-x/\sqrt{2\tau}}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(z-\sqrt{2\tau}(k+1)/2\right)^2} -$$

$$e^{(k-1)x/2+(k-1)^2\tau/4}\int_{-x/\sqrt{2\tau}}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(z-\sqrt{2\tau}(k-1)/2\right)^2}$$

The integration variable is also the standard normal variate, and after doing a lot of algebra here.

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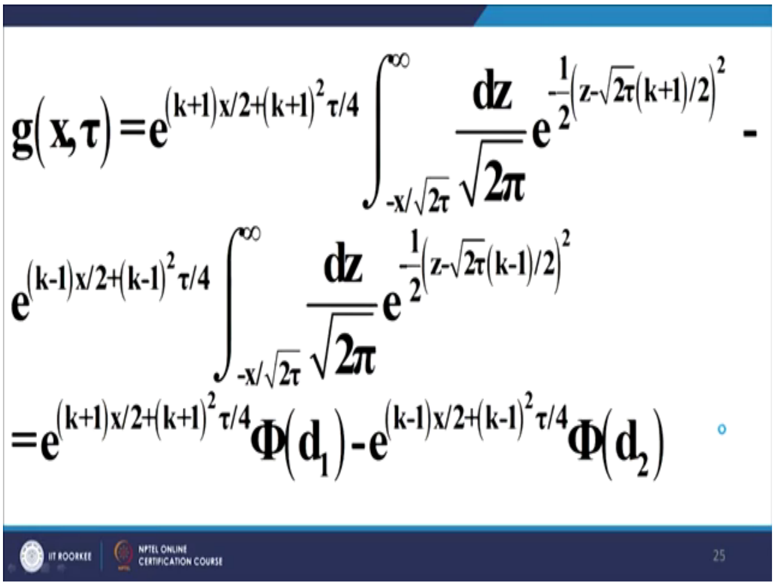

As you can look at it there is a lot of algebra going on, but its purely algebra purely manipulation of algebra. And what we end up with? At the end of the day what we find is the expression that is given at the bottom equation of your slide

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$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

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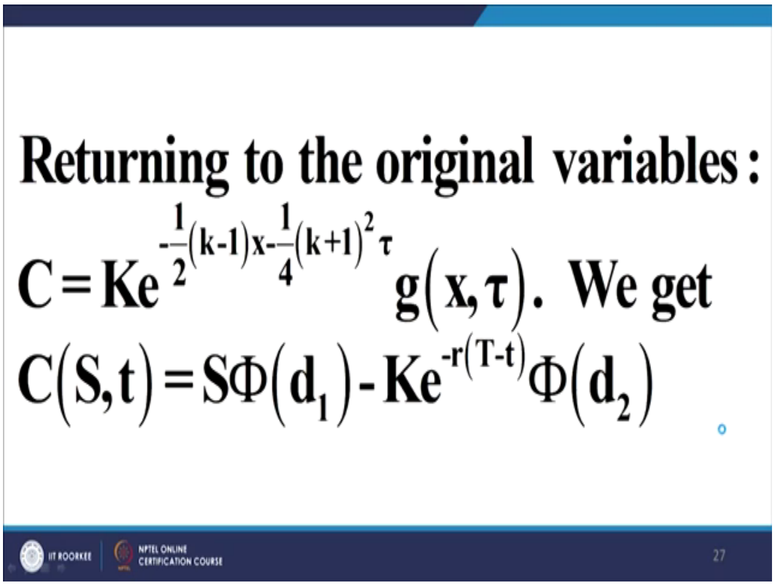

Which can be simplified there is one more step, which can be simplified further and written in the form using d 1 and d 2 in this form. We can write them as the expression, which is here on the slide as C S t is equal to S phi d 1 minus K into e to the power minus r T minus d phi d 2.

So, this is the expression that this is the ultimate solution of the black Scholes equation. That we started with that we obtained from the concept of a constructing a hedge portfolio and having done that having the solution is of course, very involved, but at the end of the day this is what we get from solving the equation and doing a lot of algebra.

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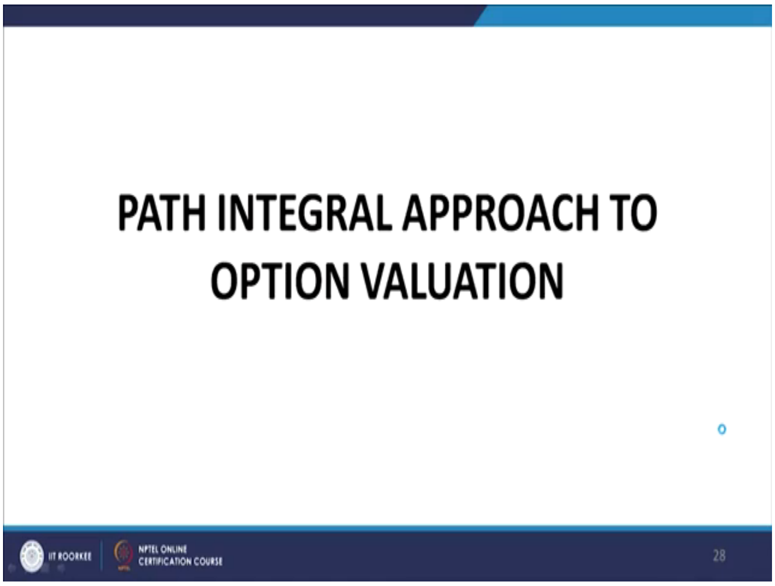

So, with this we complete the black Scholes model. In a sense we complete the traditional approach to the black Scholes model. Now, we get into the path integral approach to option valuation. We have done the black Scholes, we have done the black Scholes solution black Scholes partial differential equation, we saw we derived and we solved the black Scholes partial differential equations using the green function approach.

We now do it solve the black Scholes equation using the path integral approach. So, that is our next step. Now, we start with the path independent case which is a for which the European call is a typical example, because the payoff from the European call is dependent only on the on the stock price at maturity. It does not depend on how the stock price goes reaches the maturity point.

So, in that sense the option value is independent of the path, that the stock price takes or the underlying takes in in reaching the maturity value. This is the simplest concept and we shall start with that in the next lecture.

Thank you.