Path Integral Methods in Physics & Finance
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Lecture - 57
Pricing of Options: Binomial Model (2)

Welcome back. So, in the last lecture I introduced the concept of risk neutral probabilities which forms the cornerstone of valuation in contemporary finance theory.

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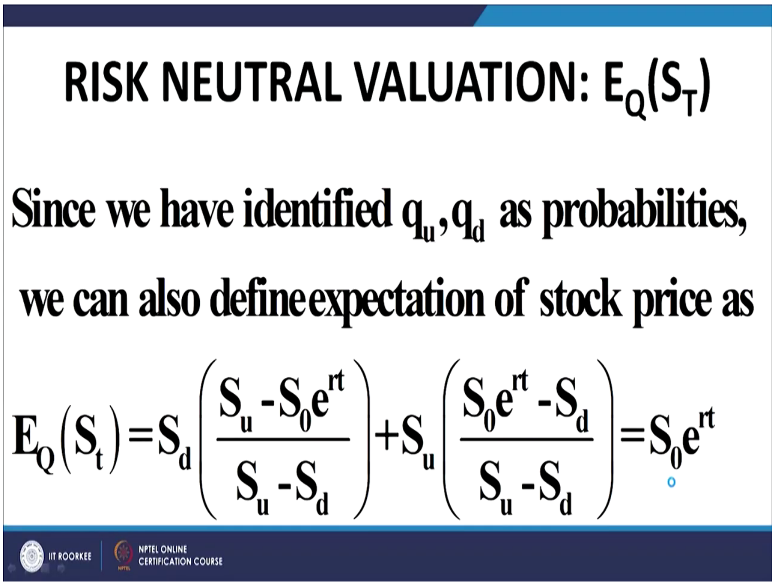

Let us briefly recap. When we work out the expected value of the stock price at a future point in time at a future date on the basis of risk neutral probabilities. We find that this stock is providing the risk free rate of return as you can see in this particular slide in the equation that is given at the bottom here. What does it mean, that that can is a very important message.

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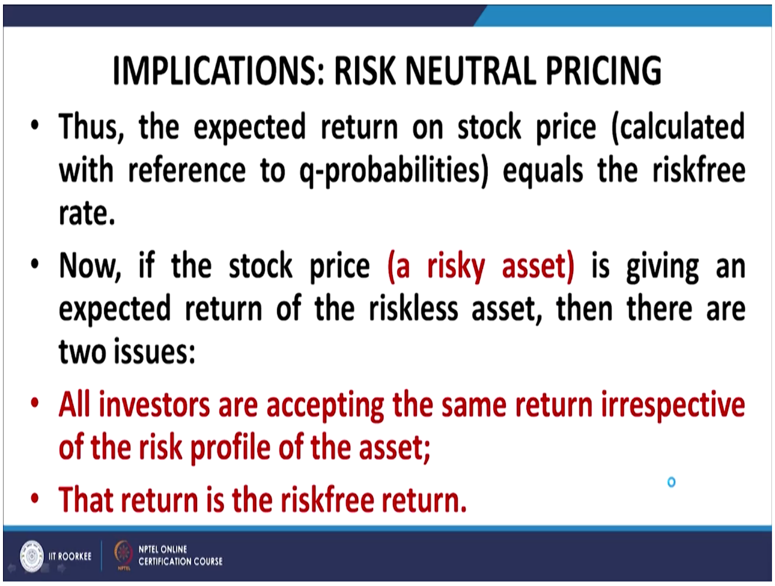

The message is that because the stock epitomizes a risky asset what we are saying is that the risk neutral probabilities are such probabilities that when expectation value of a risky asset is computed with respect to those probabilities we still end up with a risk free return. And it follows as a corollary that the investment philosophy of the populace of the an investors that populate the risk neutral world shall we say is such that they are concerned purely with the return component of the security and they are not bothered about. They are indifferent to they are not concerned with the risk component.

In other words, if 2 assets provide you one of the assets of out of 2 provides a higher return irrespective of what level of risk it contains the investor would go for the asset which provides a higher return. The fall out of this of this this theme is twofold. Firstly, that all investors will accept the same return. And secondly that same return because of arbitrage considerations as I enumerated in the last lecture because of arbitrage considerations that it return has to be the risk free rate of return it cannot be a higher return.

Because if again if there are 2 assets prevailing in the market one of them providing a higher return than the and the risk free return and notwithstanding the fact that it has a much more a significant level of risk people would go for investment in that particular risky asset. And thereby creating buying pressure on that particular asset and selling pressure on the risk free asset and at equilibriums the 2 would neutralize or 2 would converge. So, that being the case these are the 2 important followers of risk neutral pricing. We shall be encountering this risk neutral pricing again and again in the sequel. So, let us move on right.

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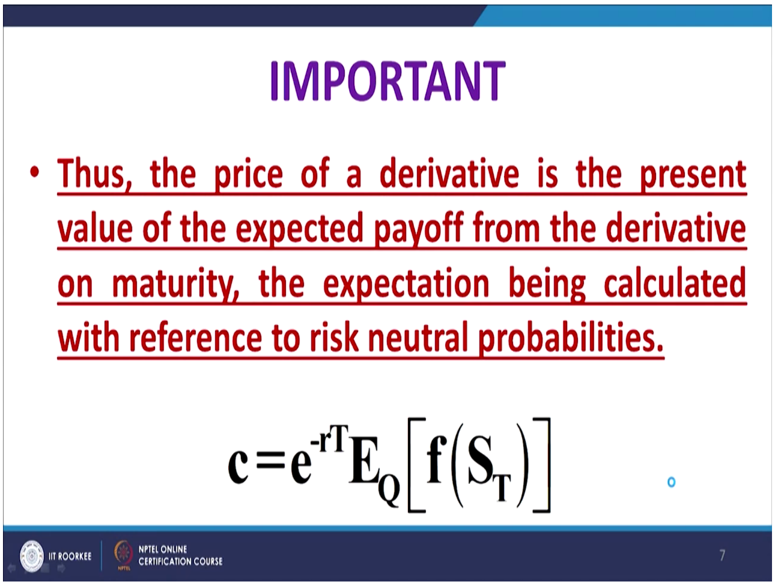

So, we have discussed this. So, so where do we stand now let us recap where we stand now. Where we stand is captured by the equation given at the bottom of this slide. What does it say? It says that the value of a derivative the value of a of a contingent claim at a point in time let us say t equal to 0 or whatever it is equal to the discounted value of the discounted expected value of its future payoff of its payoff at maturity, but that expected value being calculated with respect to risk neutral probabilities that is very important.

The expected value must be calculated with respect to risk neutral probability. So, ft represents the payoff the S T is the stock price and maturity f S T is the payoff of the derivative E Q E is the expectation operator.

So, we are working out the expectation of the payoff and we are working out the expectation of the payoff with respect to Q probabilities the risk neutral probabilities which is a denoted or which is indicated by the suffix Q with the expectation and then of course, we are discounting to today's date.

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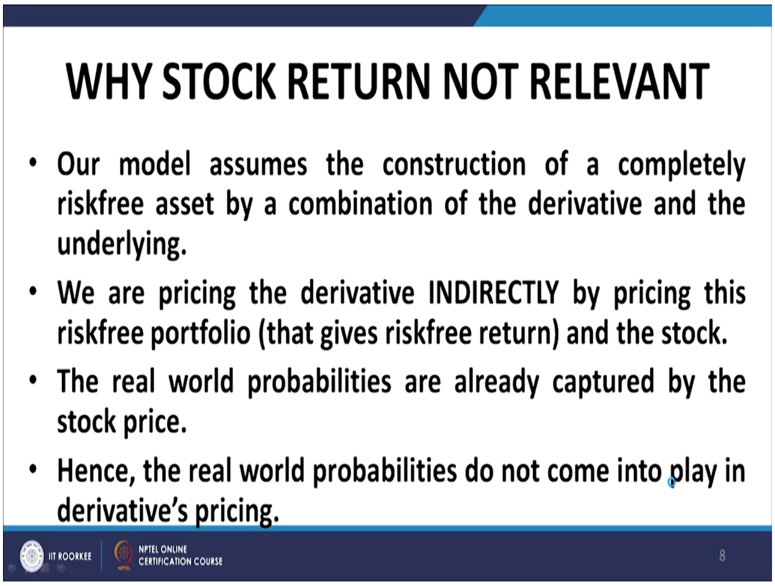

Now, a very interesting feature arises here a very interesting point arises here is that why the stock return is not relevant in the analysis. We are talking when we work out the expected return on the on the asset we find that it is a risk free rate, when we are discounting we are using the risk free rate why this variables of stock return go on.

The answer to this is that let us go back to how we on what basis we worked out the price of the derivative or how did we start going about working out the price of the derivative or the valuation of the derivative. What we did was we constructed a risk-free asset. We how did we construct a risk-free asset.

We worked on the premise that the there can be a combination we can create a combination of the risky asset and the derivative in such a form; in such a form that the 2 together constitute a risk-free asset. And then of course, we valued the risk-free asset and we valued the stock or the or the risky asset and thereby we arrived at the value of the derivative. Indirectly, we valued the derivative on the basis of valuing the risky asset and valuing the risk-free asset. We did not value the derivative directly.

Now, when we value the stock the stock return or the probabilities of up and down moments of the stock, the behavior of the stock prices of whatever you may say are encoded in the stock price. And therefore, they are captured at that point in time when we incorporate in the valuation the value of this stock price. And therefore, we do not need to reconsider the issue of the probabilities of up swings or down swings and therefore, the stock return when we work out the value of the derivative.

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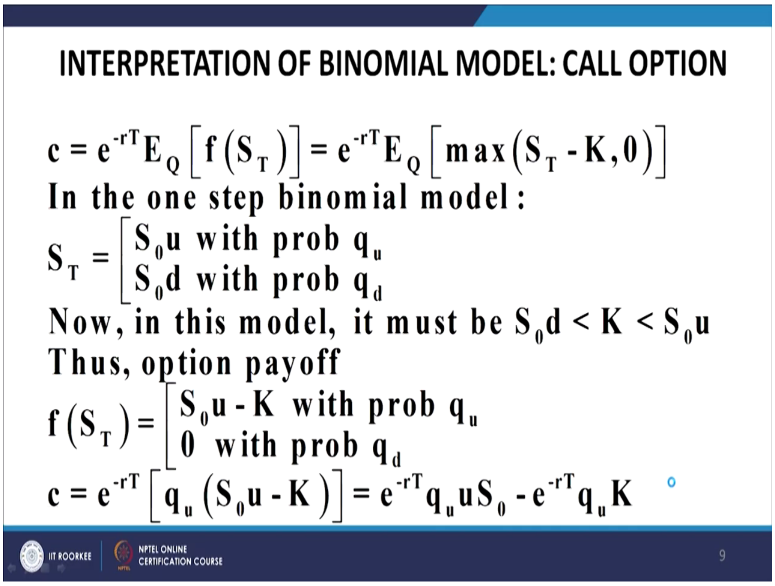

Now, an interpretation of the binomial model. This is important again. So, let us quickly run through it. The value of the derivative for the moment let us call it a call option as the call option being the same the most common derivative used in financial literature for explaining the concept of derivatives. So, let us stick to call options.

So, call option is given by the formula that I explained just now. The present value of the expected future payoff, the expectation calculated with respect to the risk neutral probabilities and that works out to when we put in the value of f of S T that is the payoff of the call.

Remember we are talking about European calls. So, that works out a maximum of S T minus K comma 0, this is the value of the payoff of the call option at maturity. Remember again to reiterate we are talking about European calls.

So, in the one step binomial model what do we have the stock price can take the value as S 0 u that is the it can go to the upper node with probability q u or it can go to the lower node with probability q d. It can take only these 2 values. It cannot take any other value at maturity we are talking about a single step binomial model. So, at T equal to capital T that is at the end of the first step it can the stock price can jump either to S 0 u or it can jump down to S 0 d. The probability of an up swing is q u if we are talking about risk neutrality or probabilities the probability of an upswing is q u and the probability of a down swing is q d.

Now, in this model it must be that S 0 d is less than K. Remember K is the exercise price is less than S 0 u. Why is that? Well, if K is less than S 0 d; obviously, it has to be less than S 0 u because S 0 u is greater than S 0 d. So, that being the case if K is less than S 0 d then in the call will invariably we exercise it will essentially we exercised and if K is greater than S 0 u then again the call will never be exercised.

So, in both these extreme cases the call losses the character of being an option and therefore, they become redundant. So, for for real life cases we must necessarily have S 0 d is less than K is less than S 0 u.

The option payoff therefore, becomes if the stock price goes up if the stock price goes up then the call would be exercised and we will buy at K and will sell in the market at S 0 u would that would be the prevailing market price because the stock price is going to the upper node.

So, the payoff would be S 0 u minus k the probability is q u and if the stock price goes down the call would not be exercised and therefore, the payoff would be 0. Putting these values in, we arrive at the expression that is given in the bottom right hand corner of this slide. Now, let us see what these two components represent that is and this is what she has been worked out to be equal to. Let us see what is the interpretation of each of these two components.

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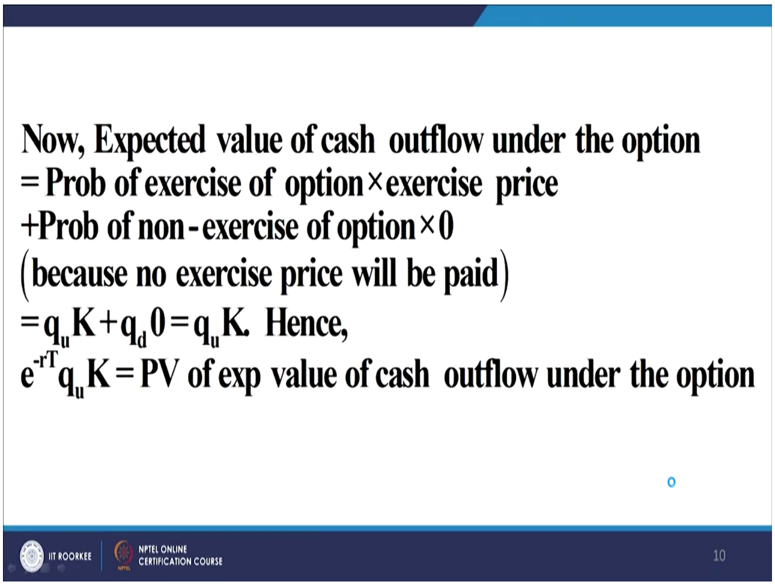

So, the expected value of cash outflow at maturity of the option what would it be? It would be the probability of exercise of the option into the exercise first, but plus probability of non exercise of the option into 0. Why?

Because if the option is not exercised there would be no outflow on account of exercise price the option would lapse without exercising and therefore, there would be no outflow on account of exercise price.

So, probability of exercise of the option as we mentioned is q u because the it will be exercised only if the stock price goes up it goes to the upper node. And therefore, the expected value of the exercise price becomes q u into K plus q d into 0 that is q u into K, but remember this exercise price is going to be paid at maturity of the option. So, the present value of this exercise price is e to the power minus rT q u K.

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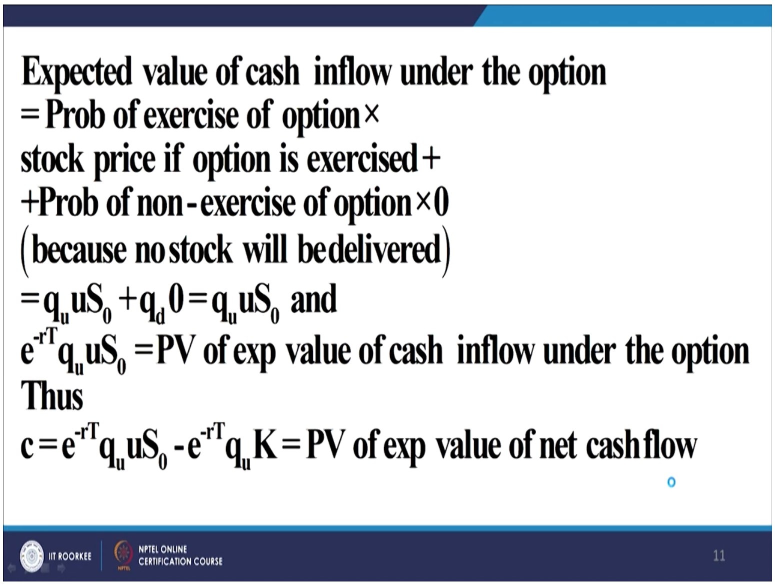

Now, let us look at the other component. What is you are you are going to pay the exercise price provided you are assuming that you are long in the option. What will happen? In return, what you are going to get in return? You are going to get a 1 unit of the stock in return which you are going to sell in the market presumably.

So, if you sell that stock in the market the cash inflow that you are going to get under the call option is S T. So, in that case if the option is exercised the pay in you may say, the cash inflow for you is S T and if the option is not exercised the pay in is 0; because the option is going to lapse and you are not going to get the stock.

So, in this case the expected value of the cash inflow on account of receiving the stock would be equal to the stock price into the probability of exercising of the option. What is the stock price that is u S 0 and the probability of exercising the option is q u if that is if the if the stock goes to the upper node and if the stock goes to the lower node the option would not be exercised and in that case you will not get the stock.

So, the pay in would be 0 in that case. So, combining the 2 the expected value of the pay cash inflow here is q u into u S 0 and the present value of this expected value is e to the power minus r T q u uS 0.

When you substitute these values here what we find is that we get the exact expression that was there in the earlier slide and that is c is equal to this expression which is there at the bottom of this slide which we derived on the basis of taking the call as a whole wholesome entity. And then we split up the components of the cash flow at maturity of the call option and we obtained the present values of those expected values of those components and we find that the value of the call is nothing but the present value of the expected net cash flow or the net payoff from the option at maturity.

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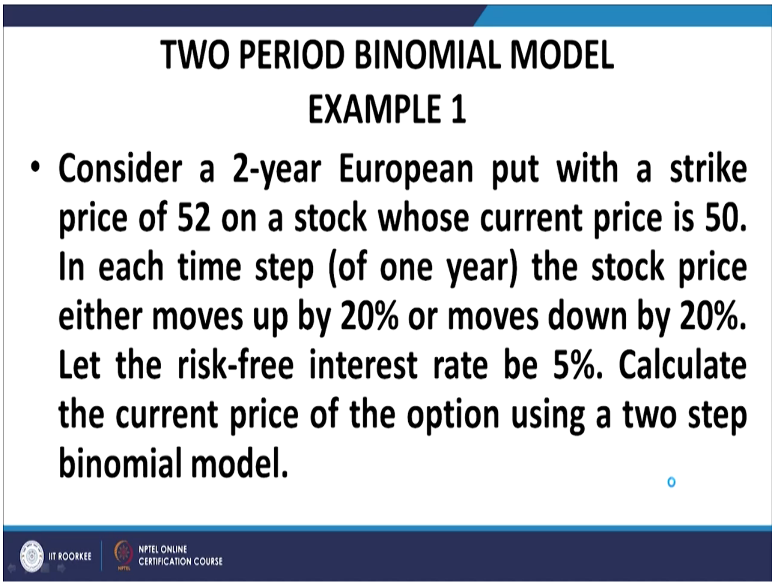

Now, and this is we worked out so far on the premise of the stock falling a one step binomial model. On the premise of the stock falling of the one time step the stock makes a move either upwards or downwards and either to the upper node or to the lower node. However, and this model can be extrapolated. It can be shown by mathematical induction just as we proved the binomial theorem. On similar analogy we can prove that this model holds for 2 period 3 period and n periods as well and therefore, it this generalization to n periods is quite straightforward of this model.

However what I will do now is I will illustrate a 2 period binomial model that will not only serve as an example of an extension of the one period binomial model it will also show that the model holds for more than one period situations.

So, let us do this problem before we proceed further. Consider a 2 year European put with a strike price of 52. The strike it is a put option recall and it is a European option. European option, put option, the life of the option is 2 years. These are cardinal features. The strike price is 52, K is equal to 52, the current stock price is 50 S naught is 50 and each time step is assumed to be of 1 year. So, there will be 2 time steps each of 1 year and the stock price can move at the end of each step that is at the end of 1 year and then again at the end of 2 years up by 20 percent or down by 20 percent.

This in this particular example we are assuming that the stock price the amplitude of the up jump and the down jump is the same which is 20 percent. So, it goes either up by 20 percent or it goes down by 20 percent. The continuous compounded risk free rate is given as 5 percent and per annum of course, rate in interest rates are always given per annum basis. So, the risk free rate is 5 percent per annum continuous compounded. We need to calculate the current price of the option using the two step binomial model. So, let us see how we go about it.

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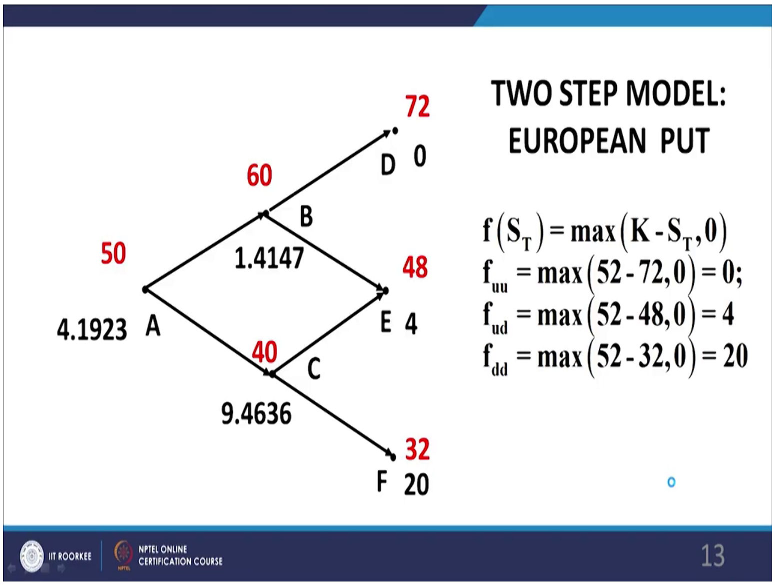

This is the diagram that is there on this slide. This is the two step binomial model we will start at T equal to 0. The stock price is 50. The stock price is given in red letters. So, the stock price at T equal to 0 at node A which is the starting node which is today which is now A is 50.

At the end of 1 year the stock price can go up by 20 percent either to 60 or it can go down by 20 percent that is down to 40 which are given by the nodes B and C respectively. Again at the end of the second year the stock price can make another jump from where it was at the end of the first year if it was at 60 it can go up to 72 another jump of upward jump of 20 percent or it could go down by 20 percent from 60 to 48. And if it was at 40 it could go up to 48 upward jump of 20 percent or it could go down to 32 that is a downward jump of 20 percent.

Now, what are the option free offset maturity. If the stock remember it is a European put option and the exercise is 52. So, if the stock price at maturity is 72; obviously, the option is out of the money and you would not exercise it and therefore, the payoff would be 0. If the stock price at maturity is 48, the exercise price is 52. You would exercise it and you would get a payoff of 4; because you would be able to buy the asset at 48 and sell the asset in the by the asset from the market at 48 and sell the asset under the option at 52.

Thereby pocketing a payoff of 4 that is if the stock price at maturity ends at 48. And if the stock price at maturity ends at 32 again you would definitely exercise the option in this case you would buy the asset at 32 from the market and you would sell the asset under the put option at 52 and the payoff that you would get is 20. So, these are the payoffs 0, 4 and 20 corresponding to the stock values taking 72, 48 and 32 respectively. Now, we work out the risk neutral probabilities. Let us see what we get.

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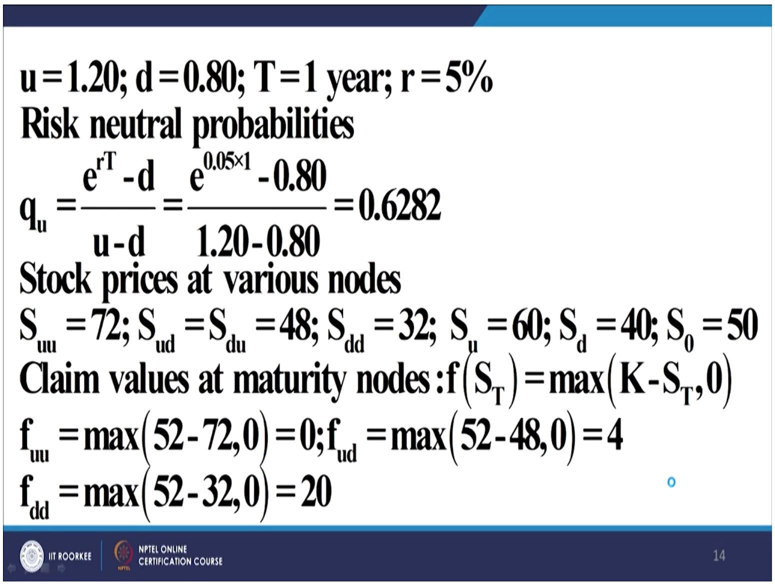

The risk neutral probabilities are given by a S 0 e to the power r T and the upswing probability is given by S 0 e to the power r T minus S 0 d upon S 0 u minus S 0 d d S 0 factors cancel out throughout and we are left with e to the power r T minus T upon u minus d. When I substitute the respective values I get the q u is equal to 0.6282 and that gives a corresponding value of 0.3718 for q d. So, just to recap the probability of an upswing is 0.6282. The probability of a downswing is 0.3718. Let us now work out the value of the option at each of the each of the nodes.

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$$f_u = e^{-rT}\left[qf_{uu} + (1-q)f_{ud}\right]$$
$$= 0.9512(0.6282\times 0 + 0.3718\times 4) = 1.4147$$
$$f_d = e^{-rT}\left[qf_{ud} + (1-q)f_{dd}\right]$$
$$= 0.9512(0.6282\times 4 + 0.3718\times 20) = 9.4636$$
$$f_0 = e^{-rT}\left[qf_u + (1-q)f_d\right]$$
$$= 0.9512(0.6282\times 1.4147 + 0.3718\times 9.4636)$$
$$= 4.1923$$

Now, as far as the nodes one maturity are concerned that is nodes D, E and F the value of the option is obviously equal to its payoff which is expected at 0, 4 and 20. When we work out the value of the option at the node B we have to work out the expected value we have to work out the expected value of the payoffs at D and E with respect to risk neutral probabilities. So, it would be 0 into 0.6282 plus 4 into 0.3718. This would be the expected value of the payoff of this leg and then we have to discount it for 1 year because the length of the length of this leg time step is 1 year.

So, what do we get here is 0.9512 into 0.6282 into 0 plus 0.3718 into 4 that is 1.4147. This is what is the value of the option at the node B. Similarly, for the node C we use the option payoffs of 4 at node E and payoff of 20 at node F.

We use the probability is 6 0.62824 node E and 0.3718 for node F. Of course, we have to discount them and discount the expected value that we get for 1 year and the result that we get is 9.4636 and again. So, we now got the value of the option at the nodes at the nodes B and node C. Using these values and value at node B equal to 1.4147 value of the option at node C equal to 9.4636.

We can now work out the value at node A using the same probabilities for 1.4147 we use the probability 0.6282 and for node C we that is 9.4636 we use the probability 0.3718. And that and then discounting it for 1 year again and we get the value 4.1923 which as the value of the option as of today. So, this is this is how the evaluation of the option 2 period binomial model can be used for valuing the option. I reiterate this is the example of the two period binomial model with the kind of computing power that we have exercise to nowadays.

We can extend this two period binomial model to as large number of periods as we want and thereby we can enhance the spectrum of values of the underlying asset at maturity of the option to whatever level whatever spectrum that we desire or we expect for the modeling of the stock price.

Now, this is this was the example this was the illustration of the evaluation of a European option, an option that could be exercised only at maturity. Remember when we talked about European and American options the European option was the option that could be exercise only at maturity and not earlier. However, the American option could be exercised at any of the earlier dates at any point in time up to the date of maturity of the option.

Suppose we were to do the same problem and instead of the option being the being of the European variety suppose it was an American option then how would the analysis change. That is an interesting point. The point here is that because the American option can be exercised earlier. The American European option can be exercised only at a fixed rate. American options can be exercised earlier and because the American option can be exercised earlier we need to examine the possibility of that earlier exercise being optimal for the investor.

In other words, compared to the holding of the option in other words retaining the option without exercising if at any point in time during the life of the option the investor feels or it can be justified that the exercise of the option would be more optimal to the investor then he should exercise the option. This possibility needs to be incorporated into the analysis. Let us see how this possibility is incorporated into this analysis. What we do now is we do the same problem on the premise that the option is a an American option. So, we have the same figures here I will not repeat the payoffs. The payoffs are same 0, 4 and 20 and working back the risk neutral probabilities that we have worked out are 0.6282 and 0.3718.

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$$f_u = 1.4147;\ f_u^{\text{early exercise}} = \max(K - S_u, 0) = \max(52 - 60, 0) = 0$$

$$f_d = 9.4636;\ f_d^{\text{early exercise}} = \max(K - S_d, 0) = \max(52 - 40, 0) = 12$$

$$f_0 = 0.9512\left(0.6282 \times 1.4147 + 0.3718\ \overbrace{9.4636}^{12.0000}\right) = \overbrace{4.1923}^{5.0894}$$

$$f_0^{\text{early exercise}} = \max(K - S_0, 0) = \max(52 - 50, 0) = 2$$

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Stock prices at various nodes

$S_{uu} = 72;\ S_{ud} = S_{du} = 48;\ S_{dd} = 32;\ S_u = 60;\ S_d = 40; S_0 = 50$

Claim values at various end nodes:

$$f(S_T) = \max(K - S_T, 0)$$

$$f_{uu} = \max(52 - 72, 0) = 0;$$

$$f_{ud} = \max(52 - 48, 0) = 4$$

$$f_{dd} = \max(52 - 32, 0) = 20$$

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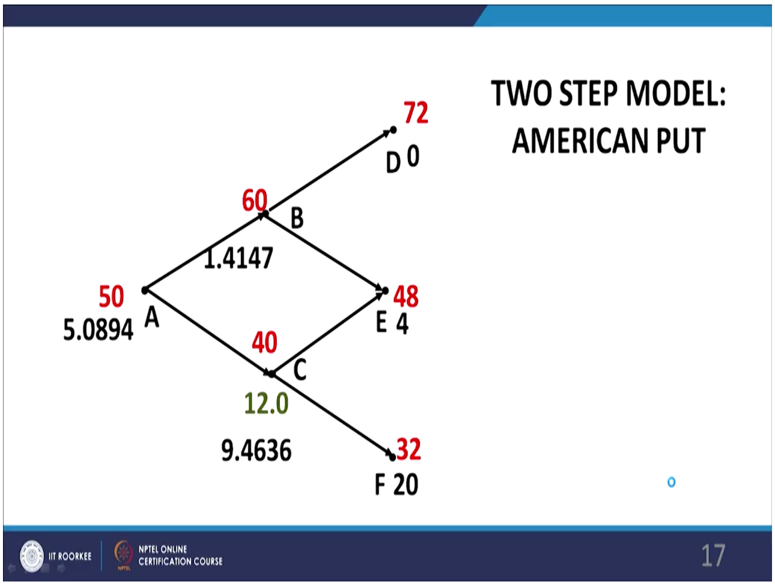

So, we worked out all of that and that gives us the values at the at of course, at the nodes D, E and F the values remain unchanged. They are 0, 4 and 20 which are the respective payoffs as some maturity.

So, at these nodes the payoffs would be equal to the value of the option would be equal to the payoffs quite naturally. At the node B, the worked out value as per the binomial tree worker comes to 1.4147 we have seen that and at the node C the worked out value comes to 9.4636 we have done that. Now, the important point is when we work out the value at node B or at node C at these two points the investor has the discretion has the choice to exercise this option to exercise this American option.

So, we need to compare this value which is the worth of the option in the event that it does not exercise the option. He holds the option vis a vis the payoff that he could receive, the payoff that he could realize if he were to exercise the option at these nodes. Now, if you look at this carefully the projected stock price at the node B is 60 and the exercise price of the option is 52.

So, the obviously the projected stock price at node B the projected stock price is higher than the exercise files. And as a result of it the payoff from the option in the event that it the investor does decide to exercise the option would be 0. He would not gain anything by exercising the option; because the market price is higher you would rather sell in the market than selling the asset under the option contract.

So, that it makes no sense to exercise the option. And therefore, he would be better off retaining this option and thereby carrying it forward at the calculated or the book price of 1.4147, but let us see what happens at the node C. Now, at the node corrosion, this stock price the book price of the option book value of the option worked out on the basis of the binomial tree comes to 9.4636 just keep it at the back of your mind. Now, the stock price is 40, the exercise price of the option is 52. In other words and remember this is a put option.

So, he could very well make a profit or make a payoff and a payoff of 12 rupees or 12 units of money by exercising the option at this point; because the market price of the underlying asset is 40. By say asset in the market and he sells the asset under the option contract by early exercising the option which is entitled to do because it is an American option.

So, he earned the profit of 12 rupees and thereby he makes a higher profit, then what he would what is the book value of the option. What is the value of the option if the option is sell is retained and carried forward without exercise. In that event, the worth of the option turns out to be 9.4636. However, if you were to exercise the option we were you would get a payoff of 12 rupees.

So, that means, at this point the worth of the option because anybody in the market could do this strategy. Therefore, the worth of the option worry to sell the option without; worry to sell the option without exercising he could still get a price of 12 rupees buys by selling the American option without exercising it.

Because any because if he sells the option and somebody to the buyer the buyer could well agree exercise the option and make a profit of 12 rupees. So, ignoring transaction costs and all other frictions it implies that the worth of the option at this point at the point C would be 12 rupees in the market and it would not be 4, 9.4636.

And therefore, the value that needs to be carried backwards when we compute the value of the option at the point A is not 9.4636, it has to be replaced by the payoff the higher payoff that is 12 at 12 rupees. So, let us see how this goes about. We work out the. So, at the end of the day what are we done? We worked out the payoff of the option at every point where early exercise is possible and we find that at the at the node C if we do the early exercise we get a benefit which is higher than the book value of there. And therefore, the market valuation of the option also at the node C would be correspondingly higher than what the binomial tree is giving us.

And therefore, when we work out the value of the option at the node A we need to replace the payoff or the worth of the option at node C by the at the projected market value of the option which is 12 and correspondingly. And correspondingly the value of the option at T equal to 0 that is node A now changes from 4.1923 to 5.0894.

The correct value of the American option is 5.0894. But, wait there is the possibility of early exercise is also available to the investor at the point A. So, we still need to examine whether he should exercise the option at the point A or he should be satisfied with the with the book value of the option or the binomial value of the option which is 5.0894.

Now, the stock price at the point A is 50. The exercise price of the option is 52 and remember it is a put option. So, what happens he gets the payoff of 2 rupees or 2. Worry to exercise the option worry to sell the option in the market at the point A he would get a payoff of 2 rupees.

Worry to retain the option, the option would be worth 5.0894. So, in this case at the point A it is not worth for the worthy for the investor to early exercise the option. He should retain the option keep it with him and in the event that the stock price really reaches the point C, he should exercise the option and that is how this this two step binomial model gets modified in the context of an American option.

We need to examine the possibility of early exercise at each and every node where there does exist and the investor has the chance to dispose of the option in the market. If there is any chance of doing that we have to put that bring that into the analysis and compare that with the binomial value of the option and carry forward carry backwards rather the value whichever is the higher of the 2. We will continue.

Thank you.