Path Integral Methods in Physics & Finance
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Lecture - 55
Properties of Options

Welcome back. So, in the last lecture, I introduced the concept of financial derivatives as financial instruments whose price process is a dependent variable and depends on the price process of another underlying asset.

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Then, I introduced various financial derivatives common financial derivatives. We talked about forwards, we talked about futures options and swaps. Thereafter, my focus turned to options because at the end of the day the path integral formalism is more relevant in the context of pricing of options.

I explained the features of an option contract. The option contract involves two parties party A and B. The speciality of the option contract is that it gives a right to one particular party either of the two parties, the party that is long in the option say party A, to buy or sell the underlying asset at a particular predetermined price. And therefore, whereas, party a has a right a discretion, a power which he may exercise or he may not exercise. On the other hand party B, which is the who is normally termed as the writer of the option has the obligation to meet his leg of the contract. In other words, in the event that a decides to exercise the option party B is mandated is a prescribed that he must commit or he must honor his leg of the contract.

So, that is what an option contract is and that is where the option contract differs from a forward or a futures contract. A forward or a futures contract entails obligations in both parties. Both parties need to honor their legs of the contract there is no discretion on either party to get out of the contract or let the contract lapse.

Thereafter, I introduced certain terminologies a certain concepts in relation to options. We talked about European and American options European options being the options that are exercisable at a particular date and American options that are exercisable up to a particular date, in that particular date is termed as the maturity date of the option.

We have the call options which gives the right to buy the asset under the option contract to the holder of the option and we have the put option which has which gives the right to the holder of the option to sell the underlying asset at a predetermined price to the party who is the writer of the option. And the underlying asset is the asset that forms the substratum of the contract and that is to exchange hands under the option contract.

The strike price or the exercise price is the price at which the asset can be sold or bought as the context may dictate in the case of put option or call option. The strike date as I mentioned is the maturity date on with the call option on with the European option becomes exercisable and up to which the American option becomes exercisable.

The holder of the option is the is the party that hold that has the right and therefore, he being at a superior pedestal he pays the price for getting that right and that price is called the market price of the option or the premium of the option. And the holder of the option is also sometimes termed as being long in the option and the writer of the option is sometimes termed as being short in the option.

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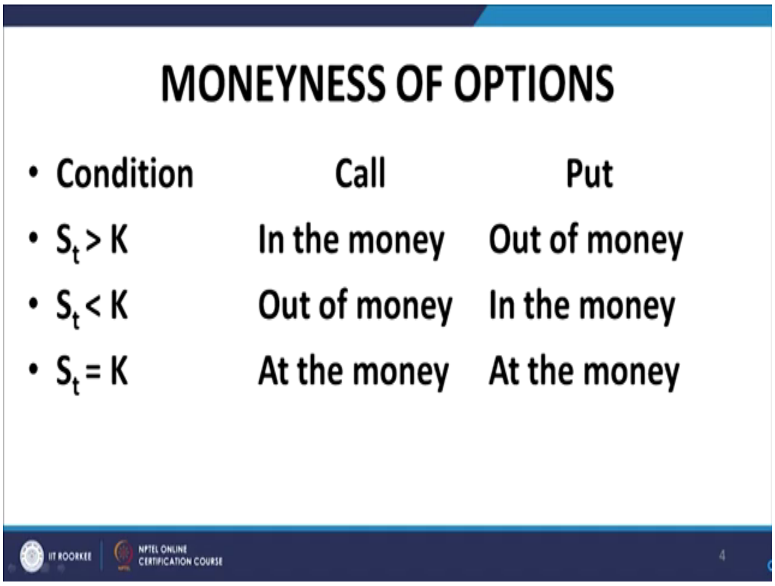

Then, I talked about moneyness of options. Moneyness of options is related to the payoff of the options if S t is greater than K, then obviously in the case of a call option it would be better or it would be advantageous for the parties long in the option to exercise the option because he can buy the asset at the price K, where K is the strike price you will usually denote the strike price by K.

So, K being the strike price you can buy the asset under the option contract at the strike price K and then sell it in the market as the prevailing market price which is S t. And therefore, and a positive profit or a positive return from the option contract and that is why in this scenario the call option is called in the money.

The put option in this context is out of the money because it will not yield a similar payoff it will give a 0 payoff in that case and conversely if S t is less than K the put option becomes in the money and the call option becomes out of the money. And if S t is close to K or equal to K both options are said to be at the money.

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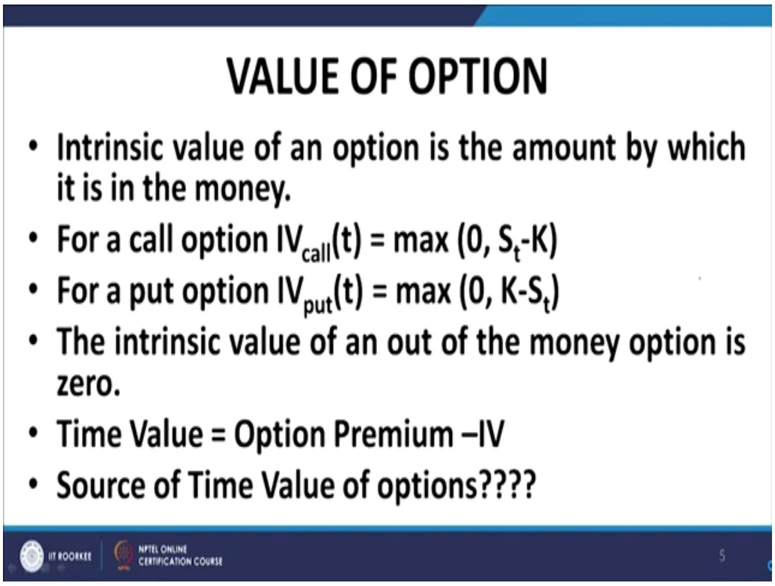

The value of the option I explained last time, the concept of intrinsic value, the intrinsic value is defined as the maximum of 0 comma S at a particular point in time t the intrinsic value of the option is given by maximum 0 comma S t minus K. Obviously, the intrinsic value of the option cannot be negative. And for a put option it takes the expression 0 comma K minus S t.

It is usually the case, not necessarily so, but it is usually the case that the market price of the option exceeds the intrinsic value and the differential between them is termed as the time value of the option. And the time value is generally generated from the possibility of the option yielding a higher payoff on the date of maturity should the should the stock price or the price of the underlying move in a favorable direction.

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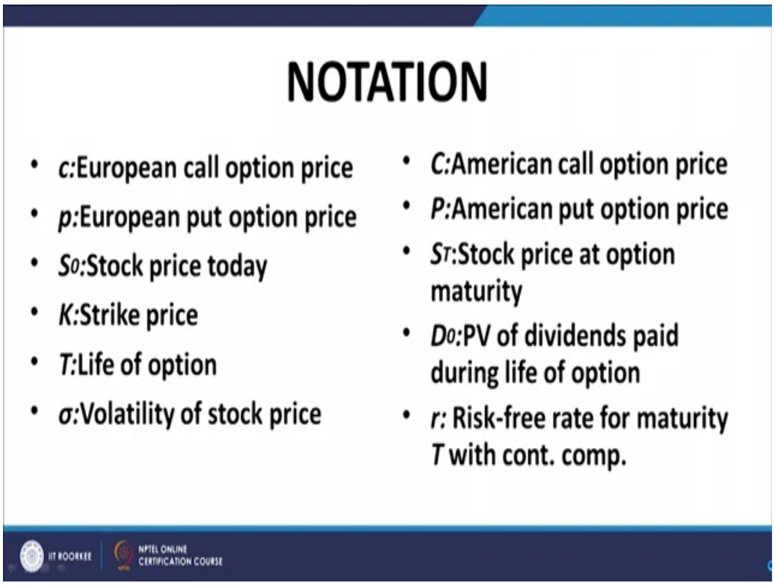

This slide contains the notation. The important things are that small c denotes the European call option price. The small p denotes the European put option price and the corresponding capital letters denote the American call option price and the American put option price, respectively. The stock price, instantaneous stock price is denoted by S naught. The strike price by K, the life of the option by capital T and sigma is the volatility of the stock price which we shall be encountering later on.

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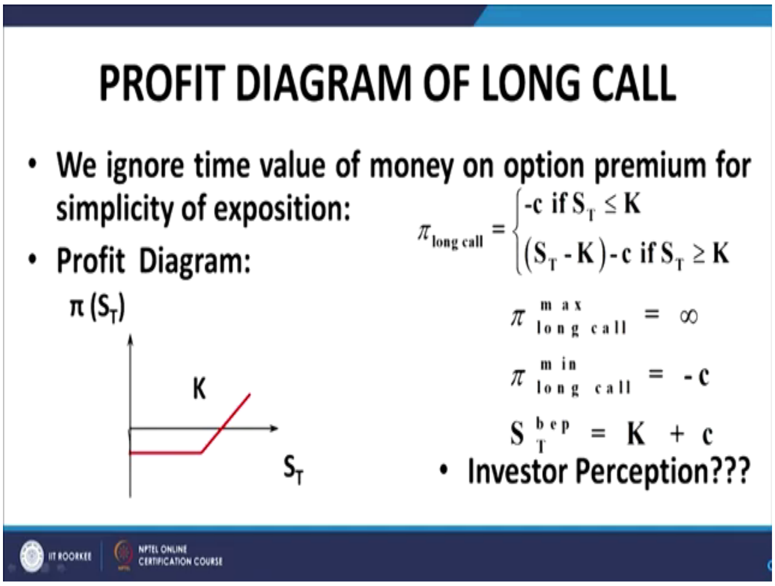

Now, by analyzing options or analyzing the behavior of options and preparing an investment strategy investors usually use a concept called a profit diagram. A profit diagram or a payoff diagram is the depiction, is the graphical depiction of the payoff from the option under various you see the and the stock price which forms the underlying asset is a random variable. So, it can take any values from 0 to plus infinity at least theoretically so.

Now, that being the case what we try to do in this case is we try to work out the payoffs or the profits from various options or combinations of options at various levels of stock prices that that graphical depiction is called the profit diagram or the payoff diagram. And the mathematical function and that represents that payoff diagram or the profit diagram is referred to as the payoff function or the profit function.

As an illustration we have got here the profit diagram of a long call. Now, clearly if the as an maturity if the stock price is lower, is lower than the exercise price of the call option, the investor would not exercise the call option because you would rather buy; if the I am sorry.

If the stock price is lower than the exercise price of the call option then the investor would rather buy in the market rather than exercise the call option and therefore, the payoff from the call option would be 0. And as a consequence there too what will happen is that the premium that he had paid for acquiring the call option becomes a loss for him that is precisely what is depicted in this diagram here.

So, long as the stock price; stock price is the is represented on the x axis and the profit is represented on the y axis pi S t is the profit and S t is the stock price that maturity of the option. We are talking about European options here.

So long as the S t is less than K you can see that the investor will not exercise the option because it is advantages to buy the asset buy the stock from the market rather than buy the stock by exercising the option; because if he exercises the option he has to pay a higher price.

So, that being the case he does not exercise the option and because he does not exercise the option he loses the amount of premium that he has paid there on. And that is represented by the loss that we have here, in the red line before the kink. And then, as soon as the stock price starts exceeds the exercise price the investor, the option becomes in the money. And therefore, the investor exercises the option and his profit, his profit increases linearly with the increase in the stock price which is represented by the 45 degree line right from the point S t equal to K.

The profit function if you look at it represents clearly what I have explained just now. It is pi of long call is equal to minus c, where c is the premium that he has paid at the point of acquiring the option. Please note we are ignoring time value of money, but it has to be noted that c would be would have the cash outflow for acquiring the option would have occurred at a much earlier date at the point in time then he would have acquired the option.

However, we are doing the analysis at a date on which the option becomes mature or on which the option becomes exercisable. So that being the case we should actually to be more precise to be more accurate we should account for the time value of money, but we normally do not do so. And in doing this simplistic analysis and we assume that the time value of money is insignificant

So, the profit or rather the loss in the case when S t is less than K is confined to the premium that he has paid and when a S t is greater than K. Of course, it earns the profit of S t minus K the differential because he can buy the asset under the option contract at K and sell it in the market at S t provided S t exceeds K and make a profit of S t minus K. Of course, out of this S t minus K differential the cost of acquiring the option would have to be would have to be deducted in order to arrive at the premium at the profit of the strategy.

The break even points and other parameters are given here. The important thing to discuss is the investor perception. As you can see here an investor with a who is investing or who is taking up the strategy ah is clearly bullish about, bullish means that he is he is expecting that on the date of maturity of the option their stock price would be significantly higher than the exercise price because only then he would make more and more profit. In fact, the profit increases linearly with the increase in stock price beyond the exercise price and therefore, his perception is that the stock price would be very high and as such the strategy is essentially a bullish strategy.

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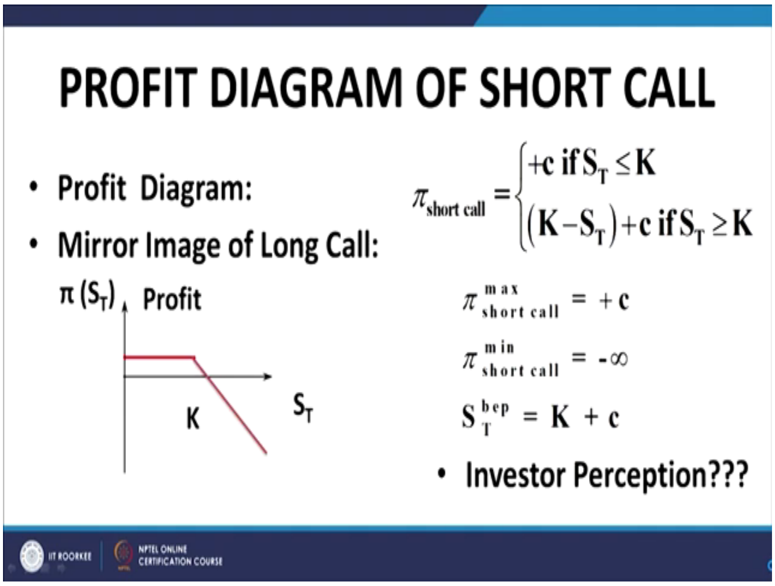

Now, looking at it, we looked at it from the perspective of a party there who has bought the option, who holds the option, who acquired the option. Let us look at it from the perspective of the party who is sold the option, who is written the option, who is short in the option. From the perspective of the party who is shot in the option what happens. If the party was long in the option as I mentioned the party who is short in the option has no discretion. The discretion entirely lies with the party who is long in the option. So, if the party who is long in the option does not exercise the option.

And when will that happen? It will happen when S t is less than K. So, if S t is less than K the party who is long in the option will not exercise the option and as a result of which the party who is short in the option will not have to beat his obligation, he will go scot free and he will pocket the premium that he has received from the party who is long in the option for buying the option from the party who is short in the option. And that constitutes his profit, that constitutes his compensation for taking the obligation for taking the risk. When? When, the party who is long in the option does exercise the option.

As you can see he is making the profit up to the point he is making the profit up to the point of where S t is equal to K. The S t equals K, what happens? The party who is long in the option exercises the option and because the party who is long in the option exercises the option the party who is short in the option as an obligation to meet the his leg of the contract, he has to deliver the underlying asset to the party who is long in the option. And therefore, because he will buy the asset from the market, he will, it will cost him a price of S t and he will receive a price of K. So, in a sense a loss of K minus S t occurs to him once the option is exercised. I am sorry; once the option is exercised by the party who is long in the option.

So, in a sense you can say that that the long position and the short position are mirror images of each other about the x axis. So, the payoff the profit function also depicts the same thing, it is plus c. If it is S t less than equal to K he gets the premium and goes caught free. And if the option is excised he has to incur a loss of K minus S t out of which he is compensated by the amount of premium that he has received on writing the option.

And as far as the investor perception is concerned it will be contrary or it will be inverse of the investor perception or the party who is taking a long position. He believes that the option price would lie between 0 and K, and therefore, he could pocket the premium without incurring the possibility of the option being exercised by the long party.

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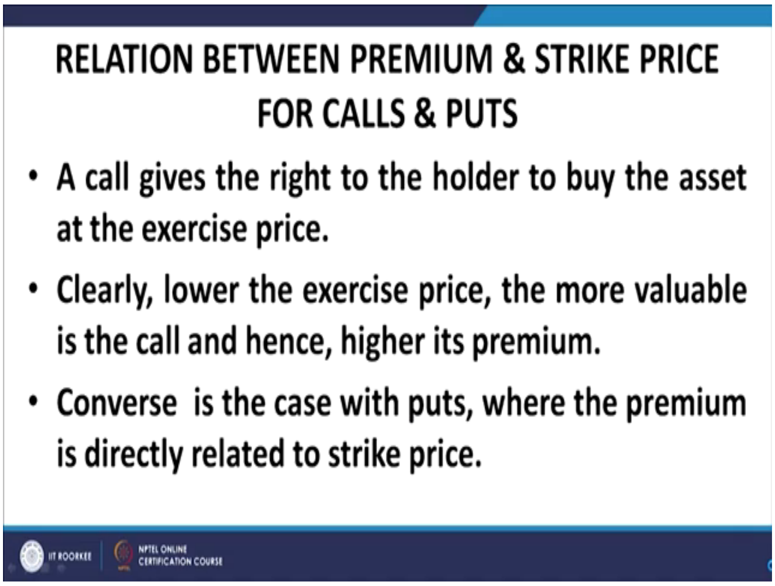

Now, this this is a very interesting issue. The relationship between the premium and the strike price for calls and puts. As you know, what are calls? Calls gives you the right to buy the underlying asset at a predetermined price. Obviously, lower that predetermined price, lower the exercise price, the predetermined price is called the exercise price. So, lower the exercise price the cheaper is the price that which you will acquire the asset and therefore, the more valuable is the option.

In other words, in the case of a call option the lower the exercise price the more valuable is the option. And therefore, the higher would be the premium that would be payable on the option. So, the premium and the exercise price in the context of call options move inversely to each other.

The the opposite is the case of a put option. Why? Because the put option entails a right to sell the asset. So, higher is the exercise price, higher is the price at which you can sell the asset, and therefore higher is the value of the option and therefore, higher is the market premium of the option.

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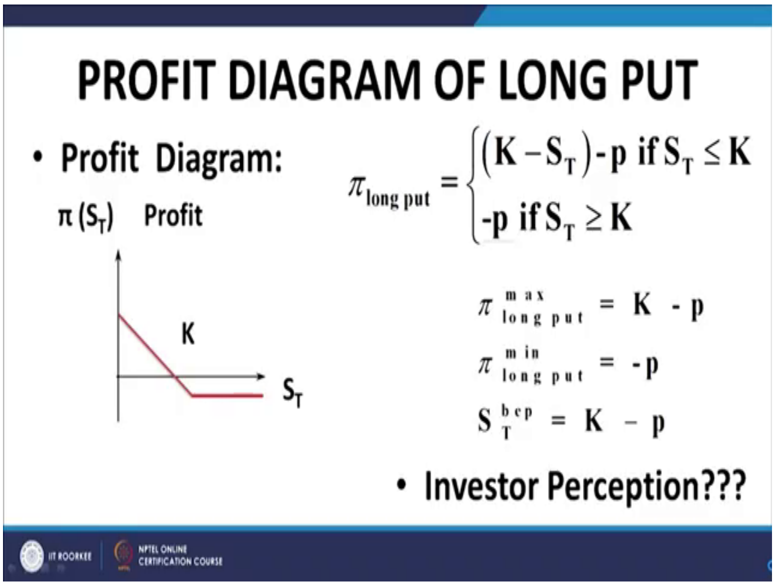

This is the profit diagram of a long put as I mentioned. The put option gives you a right to sell the asset at a predetermined price. Obviously, if the market price is very low and say let the market price is 0, then you can buy the market or buy the asset from the market at 0 price and sell it under the put option at the exercise price K and thereby make a profit of K, get a payoff of K. Obviously, the cost again the premium of acquiring the option will have to be deducted. So, this is the analysis of this diagram. It is absolutely in an analogy with the analysis of the payoff diagram or the profit diagram in the context of the long call.

Just to go back, the difference between the profit diagram and the payoff diagram, the payoff represents a total cash flow whereas, the profit represents the net cash flow that is the cash flow after deducting the amount of premium that you paid for establishing or setting up of a particular strategy. So, the payoff is the total cash flow. For example, in this in this case K minus S T is the payoff and K minus S T minus p is the profit. So, that is the difference between the two. We can use either the profit diagram or the payoff diagram as is convenient, they really convey the same message in a sense.

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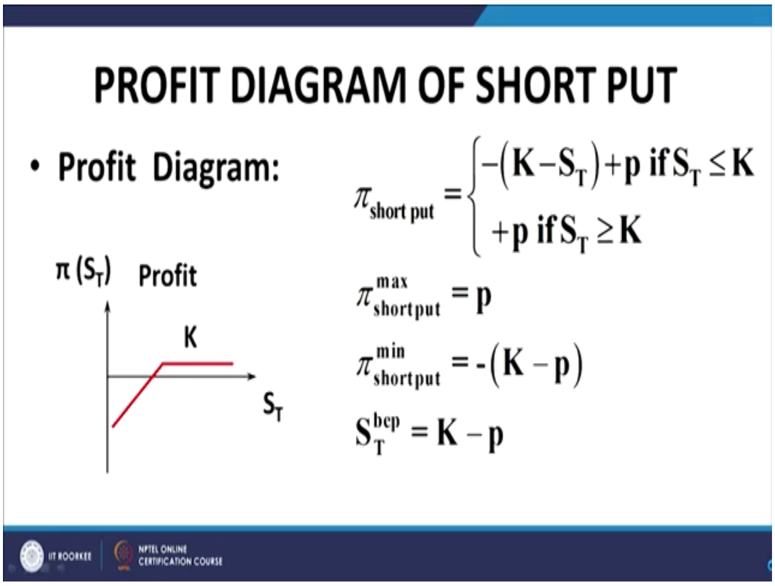

So, this is about the profit diagram of the long put. This is the profit diagram of a short put. Obviously, again they are they are mirror images of each other about the x axis and the payoff function is also given here, it becomes the inverse of the payoff function of the long put.

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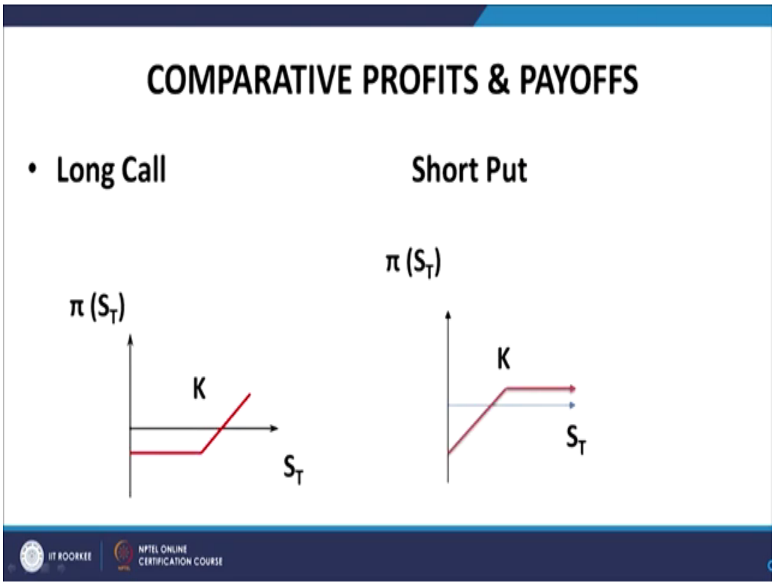

This is the comparative statement of the long call and the short put. As you can see they have similar structures. But there is there is another intrinsic difference that in the case of a long call initially you have a loss, and once the price increases beyond the exercise price you start making profit.

Here again the once the price increases beyond the exercise price you start making profits, but here the profit is confined to the amount of premium in the case of the short put, in the case of a long call of course, the profit theoretically at least is unbounded because the larger the stock price higher, the stock price in the market on the date of maturity of the option higher is the payoff from the option.

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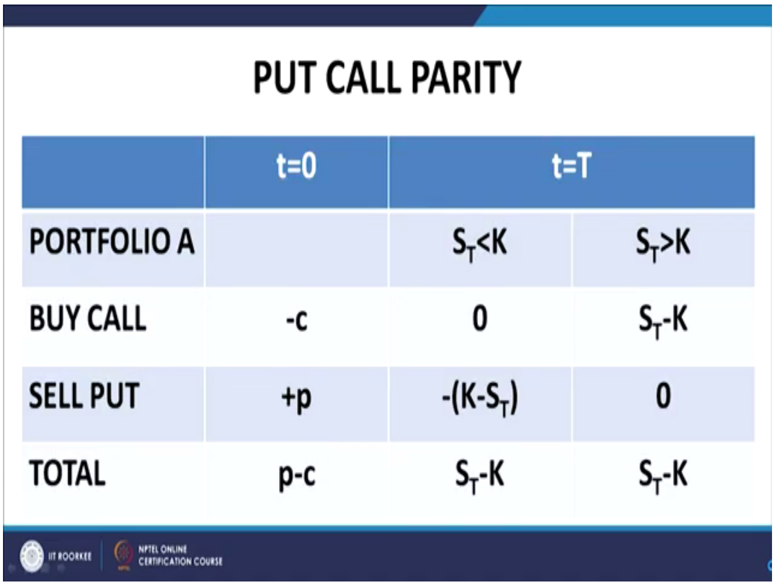
PUT CALL PARITY

	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
BUY CALL	-c	0	S_T-K
SELL PUT	+p	-(K-S_T)	0
TOTAL	p-c	S_T-K	S_T-K

Now, we come to a very important relationship between the price or the premier of the put option and the call option. It is called put call parity. What is the parity relationship between the put option and the call option? Let us see how we analyze it. And by virtue of this put call parity, I will also try to explain to you the concept of arbitrage which forms one of the fundamental concepts in finance for the pricing of a assets. So, let us start with put call parity.

Let us assume that I have got two portfolios, an investor has invested in two portfolios portfolio A and portfolio B. Let us look at the constitution of portfolio A and portfolio B. The portfolio A consists of a long call, a call that you have bought and a short put that you have written. I repeat a call that you bought by paying a price, so that cost is reflected as negative because it is a cash outflow.

And you have bought it at t equal to 0. So, it is the cash outflow at t equal to 0 and you are representing it by minus c. And you are selling a put and now because you are selling a put, you are writing a put therefore, you will get a get the premium on that and the premium is represented by plus p; because it is a cash inflow. So, the net cash flow at t equal to 0 is p minus c is quite elementary.

Now, what we do is you see the stock price on maturity it can take at least theoretically, it can take any value between 0 and infinity. We partition this interval between 0 and infinity to into two disjoint intervals between 0 and up from 0 to K and then greater than K. In other words, we are simply partitioning the stock price spectrum into two parts, the part between 0 and K and then between from K to infinity.

Now, let us look at the payoffs if the stock price on maturity. Remember, it is a random variable. So, let us look at how the payoffs behave, how the portfolio Behaves, how the constituents of the portfolio Behave when the stock price ends up in one leg of the partition one segment of the partition and when it ends up in the other segment of the partition because it can end up in either segment of the partition, it is a random variable.

Let us assume that in this stock price at maturity at the exercise date of the options remember both of them are European options first of all and they have the same excise date. The call and the put both have the same exercise date and both of them are European options. So, they cannot be early exercised right.

So, if S T is less than K then the call option will not be exercised because the investor will rather prefer, if he wants, if he desires, he would rather buy the asset in the market then exercise the option you pay a higher price for acquiring the asset. So, if S T is less than K, the payoff from the long call strategy or the long call component of the portfolio will be 0. And if S T is greater than K, then obviously, you would exercise the option pay a price K and sell the underlying get the underlying asset under the option contract and sell it in the market at S T and thereby make a payoff of S T minus K, which is what is represented here.

So, in other words as far as the as long call is concerned the payoff is 0, if S T is less than K; if S T ends up in the first segment and if S T ends up in the second segment of the partition then it is S T minus K. And what about the short put? As far as the short put is concerned the payoff if S T is less than K, because if S T is less than K the party who is long in the put option in this particular, remember this is a short put. So, you have written this put option. So, you do not have any discretion, you have the obligation.

The party who is long in the option as the discretion and because the party who is long in the option finds that the market prices is lower and the and the exercise price is higher, he would exercise the option and thereby he would make the party who is long in the option would make a profit of K minus S T. And that would be because it is a 0 sum game. So, that would be the amount of loss that you are going to make. So, the loss that you are going to make is minus K minus S T which is shown here in this in this diagram.

And obviously, if S T is greater than K, then the party who is long in the option will not exercise the option because he would rather sell in the market at a higher price, then exercise the option and therefore, the payoff from the option would be 0 the option would lapse unexercised.

So, the total payoff as you can see here on the portfolio A in both scenarios in both segments of the of the partition or S T less than K and S T greater than K turns out to be S T minus K. And the and the cost of establishing the portfolio in a sense or rather the cash outflow in term at the T equal to 0 for establishing the portfolio is p minus c. Let us move forward. So, this is as far as portfolio A is concerned.

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	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
TOTAL	p-c	S_T-K	S_T-K
PORTFOLIO B			
BUY STOCK	$-S_0$	S_T	S_T
BORROW	$Ke^{(-rT)}$	-K	-K
TOTAL	$-S_0+Ke^{(-rT)}$	S_T-K	S_T-K
$c+Ke^{(-rT)} = S_0+p$			

Now, we constitute or we look at the alternative portfolio or the second portfolio that I have talked about. The second portfolio, as we take a long position in the underlying asset by the stock that constitute the underlying asset of the two options, remember the options I have the same underlying. So, I buy this stock which constitutes the underlying asset of the two options that that constitute portfolio A.

Obviously, that will cost me a price of S naught. So, minus S naught because it is a cash outflow, so it occurs at T equal to 0. I pay the price the current market price then prevailing the then prevailing market price of this stock which is S naught, so I pay a price of minus S naught. And at T equal to capital T that particular stock or the unit of stock would be worth S T and that is now this S T would be independent of the partitions that we have created whether S T is less than K or S T is greater than K in either case the stock would be worth then prevailing market price which is S capital T.

And in addition to this partly for acquiring this stock I borrow a certain amount of money. The amount of money that I borrow is the present value of K that is Ke to the power minus rT. This is remember we are using continuous compounding. So, we use we borrow an amount Ke to the power minus rT.

Now, because I have borrowed this is an inflow and because it is an inflow it carries a positive sign. So, this is the amount that I borrow and the amount that it that will have to be repaid to satisfy the obligation created by this borrowing will be equal to K, the future value of this at time capital T which turns out to be K.

Remember, continuous compounding and because it will be repaid, the repayment will be a cash outflow. And therefore, the minus sign is there and the amount will be K and again it this particular outflow or this particular cash repayment is independent of the state of evolution of nature. In other words, it is independent of evolution of this stock price whatever the stock price may be this amount has to be repaired. So, in either case if S T is less than K or S T is greater than K the amount to be repaid is minus K and minus K.

And what we find now is that the total cash flows at T equal to capital T for each segment of the partition turn out to be equal for portfolios A and portfolio B. You can clearly see on this slide if S T is less than K portfolio A and portfolio B both give a payoff of S T minus K, and when S T is greater than K again the same thing happens. In other words, for each leg of the partition each segment of the partition the payoffs between portfolios A and B remain this same; what does it convey about the cost.

Now, it is here that the concept of arbitrage comes into play. What arbitrage means is that there is no free (Refer Time: 29:01) in the financial domain or financial markets. It essentially means that in a more refined manner what it means is that a particular asset or bearing a particular set of a risk return characteristics cannot command different prices in different markets. Or putting it the putting it in another form two different assets having different having identical risk written characteristics two different assets having identical risk return characteristics will command the same price in this in the market.

So, either way you can look at it. But the important thing is the assumptions that underlie this arbitrage concept are very important the fundament the most important requirement for the existence of arbitrage is that there should be no frictions, no market frictions like transaction cost, buy cell spreads, lending borrowing spreads, and so on.

So, in the absence of any such ah frictions and the existence of a efficient market that is without any information distortions, spontaneous and smooth flow of information if the all these conditions are fulfilled then an asset having a particular set of risk written characteristics cannot command cannot command different prices or putting it the other way around two identical assets must command identical prices.

So, here again we have a similar situation we have got two portfolios, we have got portfolio A we have got portfolio B. Both of them have identical payoffs, in identical domains and therefore, they must command the same price. And or in other words the cash outflows corresponding to the creation of portfolio A and portfolio B must necessarily match which gives us the relation that is written in the bottom line of this slide c plus Ke, the present value of K that is Ke to the power minus rT is equal to S naught plus p. This is the put call parity relationship.

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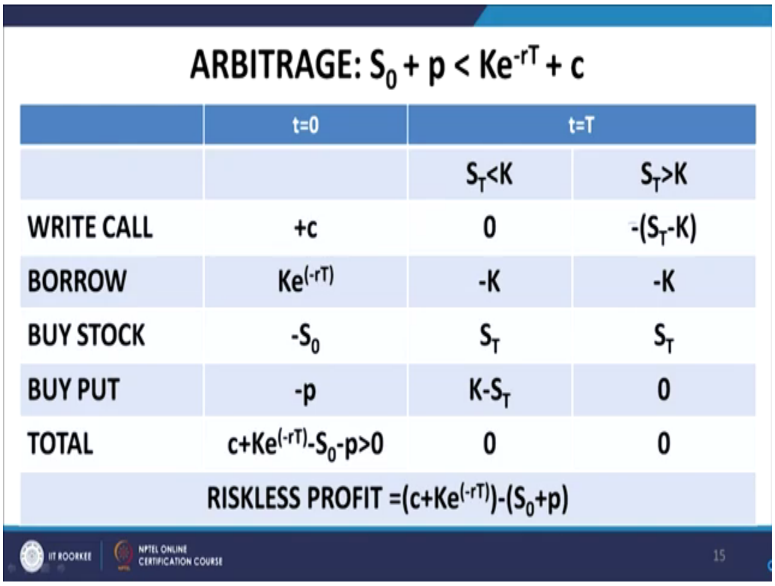

ARBITRAGE: $S_0 + p < Ke^{-rT} + c$

	t=0	t=T	
		$S_T<K$	$S_T>K$
WRITE CALL	+c	0	$-(S_T-K)$
BORROW	$Ke^{(-rT)}$	-K	-K
BUY STOCK	$-S_0$	S_T	S_T
BUY PUT	-p	$K-S_T$	0
TOTAL	$c+Ke^{(-rT)}-S_0-p>0$	0	0
RISKLESS PROFIT $=(c+Ke^{(-rT)})-(S_0+p)$			

Now, this is the, what we discussed just now was the put call parity relationship. Now, what happens is suppose in a situation we put call parity is violated, if for an instant in a particular efficient market put call parity is violated, put call parity does not hold then how can an market player, how can a particular market player benefit from such a situation. This particular procedure would give you more insight into what is meant by arbitrage.

For the moment let us assume that S naught plus p although the converse can also be established in a similar manner, let us assume that S naught plus p is less than Ke to the power minus rT plus c. What we do is we create a portfolio, we write a call option.

We write a call option that entails a cash inflow because you are writing the call options, so you will get the premium. So, you get a cash inflow a plus c at t equal to 0 and again we partition this stock price as earlier into S T less than K and S T greater than K and because its writing a call option the payoff in the first case obviously is 0 because the call option will not be exercised it will lapse. And in the second case, it will be minus S T minus K because it is writing it is writing the option it is a short option, it is the short strategy, so the minus sign is accounted for because of the short strategy.

You borrow an amount Ke to the power minus rT, when you borrow this amount this amount has to be repaid and this repayment is independent of the market scenario or the or the stock price evolution. So, in either leg of the partition, either segment of the partition the repayment will be minus K and minus K. You buy the underlying stock, when you buy the underlying stock the stock again will be worth S T, irrespective of what the market is in market segment we are talking about. It will be the price of the stock will be what the price is at that point in time.

And as far as the put you taking a long position in the put, because you are taking a long position in the put you are paying a price for that and therefore, the cost is minus p it is a cash outflow. And because it is a long put it generates a ah cash inflow or a payoff of K minus S T if S T is less than K and of course, if S T is greater than K it goes unexercised.

The net result is if you look at this if S T is less than K, the aggregate payoff of the portfolio is 0. If S T is greater than K the aggregate portfolio aggregate payoff of the entire portfolio is again 0. So, in other words the portfolio is generating no payoff under any circumstances on maturity.

Let us look at the price um. If you look work out the cash flows you find that the cash flows are positively there, because of the condition that is given here S naught plus p is less than Ke to the power minus rT plus c. This implies that the there is a net cash inflow at T equal to 0. In other words, what you have done is you have generated a net cash inflow at T equal to 0 without a corresponding obligation at T at the maturity of the portfolio and that is that is what is a risk less profit, that is what is called an arbitrage profit.

So, if that put call parity is violated in the manner that we have seen here it results in the generation of a riskless profit. And therefore, what will happen is more and more players will venture into the particular strategy and as a result of which creating demand supply disequilibrium and gradually what will happen is that arbitrage profit will be sucked out of the of the market and which the put call parity will be restored.

Thank you.