Path Integral Methods in Physics & Finance
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Lecture – 53
Ito Equation, Stock Price Modelling

Welcome back. So, today we go on to the last segment of this course which deals with the financial applications of path integrals.

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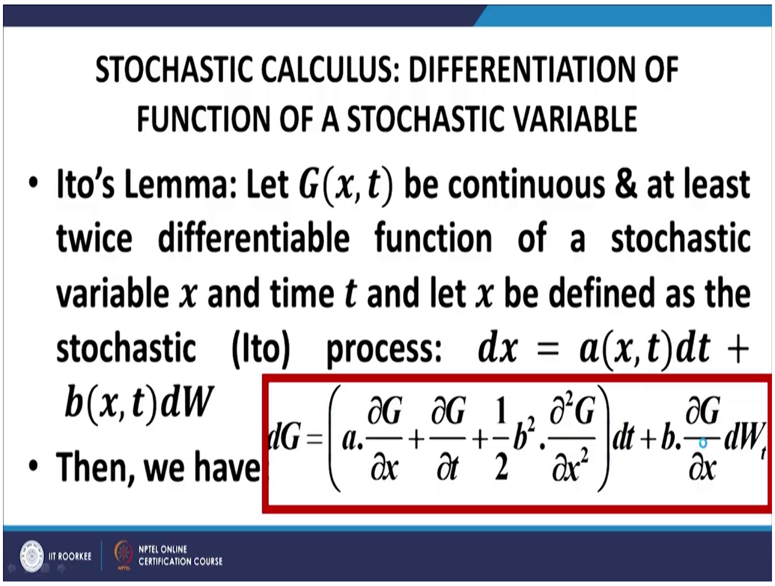

But before we get into that, we need some prerequisites and one of them is going to be Ito's lemma which enables us to do calculus of functions of stochastic variables.

So, let us get into that let us get into the Ito's lemma. Ito's lemma says that if we have a function G x, t where x is a stochastic variable and this function is continuous and at least twice differentiable. The function the variable x is stochastic and t is; of course, time. And we can express x in terms of the expression that is given in the slide dx is equal to the stochastic differential equation dx is equal to a x, t dt plus b x, t d W.

Remember d W is a infinitesimal Brownian motion increment t of course, as I mentioned is the time and then, the total derivative of G can be expressed by the expression given in the red box at the bottom of the slide.

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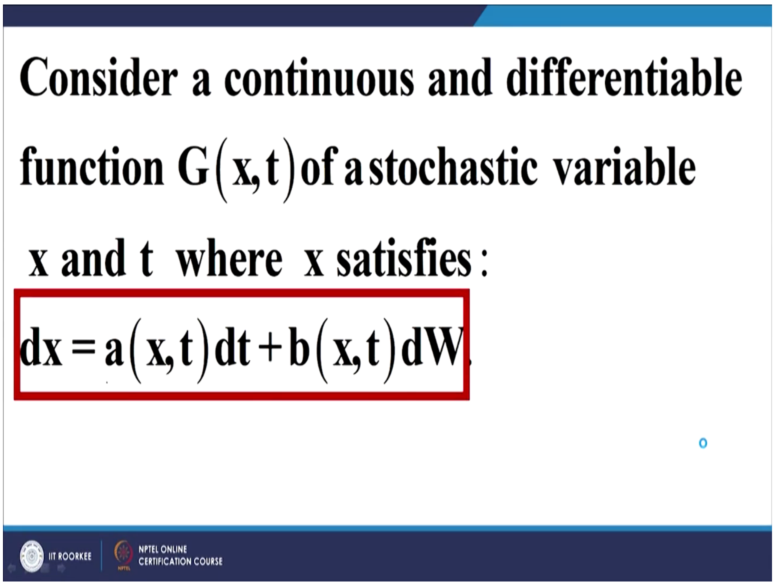

The proof is not too involved. It is quite simple in fact. We consider a continuous and differentiable function, twice differentiable at least function G x, t of x. And t where x is given by the expression that we I mentioned just now and that is given here in the red box.

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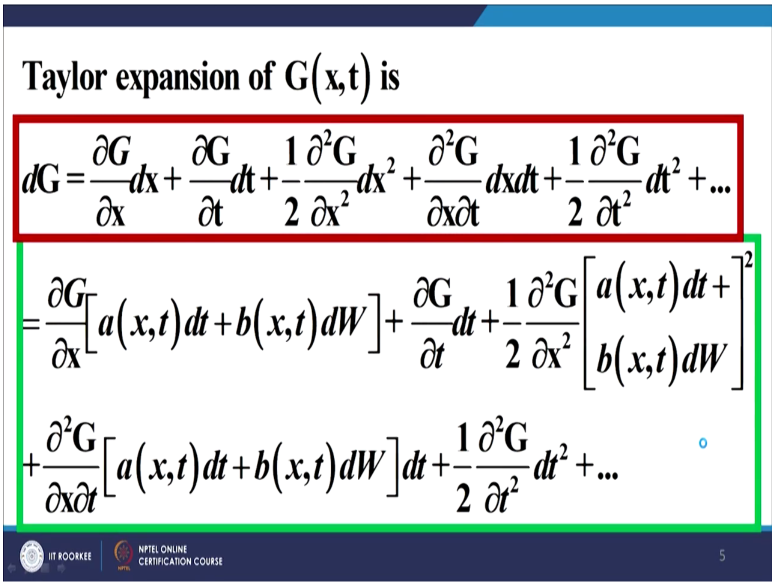

And we expand G as a Taylor series. When we expand G as a Taylor series, we get the expression and that is given in the red box here knowing that G is a function of x and t.

The most of the terms are simple and we retain terms only of the first order in dt, but the important term which we need to analyze is the term that involves the second derivative of G with respect to x that is the term involving the square of t x dx whole squared that I am used to have a look and let us get into that.

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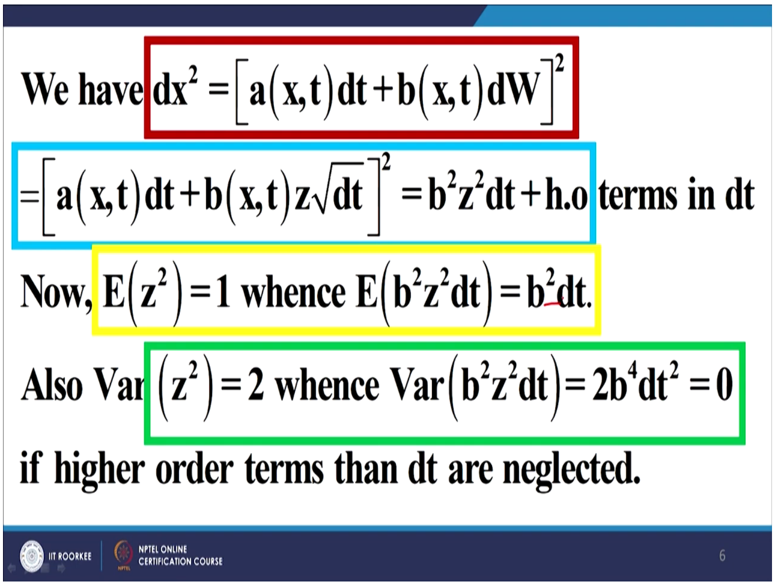

So, dx square is given by the expression in the red box here dx square is equal to a x, t dt plus b x, t d W t whole squared.

Now, when we do this squaring and of course, d W is nothing but the Brownian motion increment. And therefore, it can be written as z under root dt where z is the standard normal variate. This is we have discussed it when we talked about Brownian motion some time back.

So, when we open up the square, the expression that we get is b square z square dt for the squaring of the first term for the squaring of the second term; I am sorry and other terms are of order dt square or are higher orders in dt.

In other words, let me repeat when I open the square of this particular expression given in the blue box here. The one term that is linear in dt is the expression that I get is b square z square dt that is by squaring the second term and the other terms are of higher orders in dt, then the linear term.

Now, let us look at this term. Of course, the higher order terms in dt we shall be ignoring. So, we need not carry them forward in the analysis. We need to study and review the term b square z square dt.

Now, the expectation value of z square where z is the standard normal variate is given by 1 and therefore, the expectation value of b square z square dt is nothing but b square dt. The expectation value of z square being 1 as shown in the yellow box here.

Now, talking about the variance of z square, the variance of z square is 2 it can be shown by working out the using the normal distribution and the variance of z square turns out to be 2. And therefore, the variance of b square z square dt turns out to be 2 b's to the power 4 dt square.

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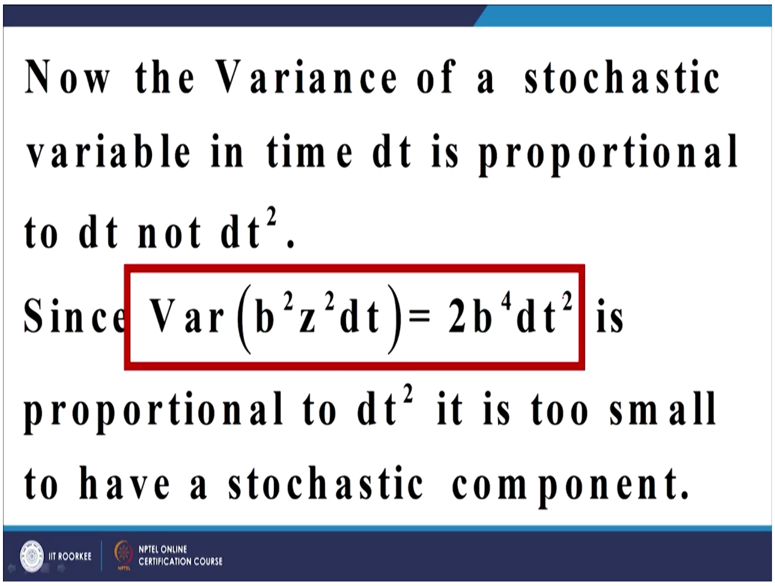

Now, the important point is the variance of a stochastic variable in time dt in an infinitesimal time dt has to be proportional to dt and not dt square. But what we find here is that the variance of; variance of this term E, variance of this term b square z square dt turns out to be of the order of dt square and that means what? Because we are talking about infinitesimal times that means the variance is very small, variance is much smaller than what is expected.

If the process or if this expression b square dt and the b square dt if this expression or b square z square dt rather if this expression were to be a stochastic variable. Had it been a stochastic variable, the variance would should have been of the order of dt in time dt. However, the variance is found to be of the order of dt square which is very very small and therefore to first approximation, we can take the variance to be 0 and as such we can assume that b square z square dt is deterministic and carries the value b square dt.

So, that is the logic behind this and that means, that means, at the end of the day what we have is that this squared component. This entire expression which is nothing but dx square contributes the value, contributes an expression, a deterministic expression, a non random expression equal to b square dt and because its expectation is b square dt. And its variance is close to 0 or insignificant by the parameters or by the yardstick that we are evaluating a stochastic process.

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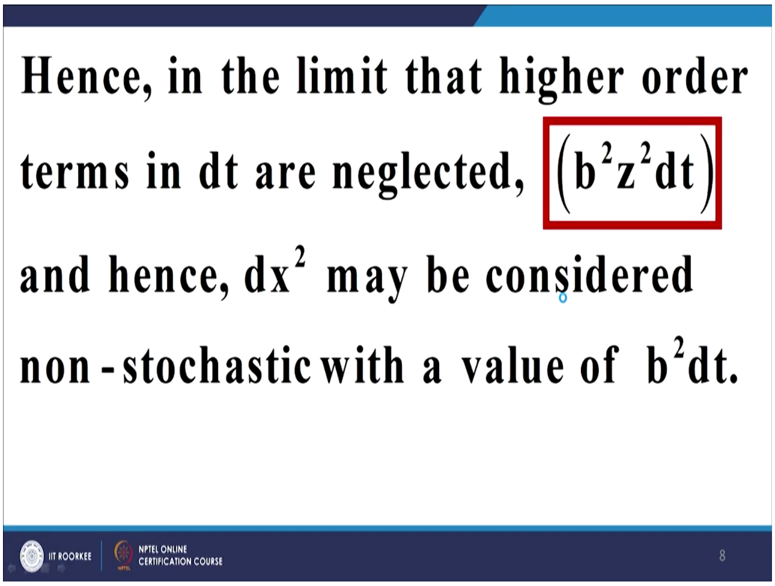

So, substituting this expression here for dx square and retaining terms in to first order in dt what we have is, the expression given in the red box here and which is called Ito’s lemma.

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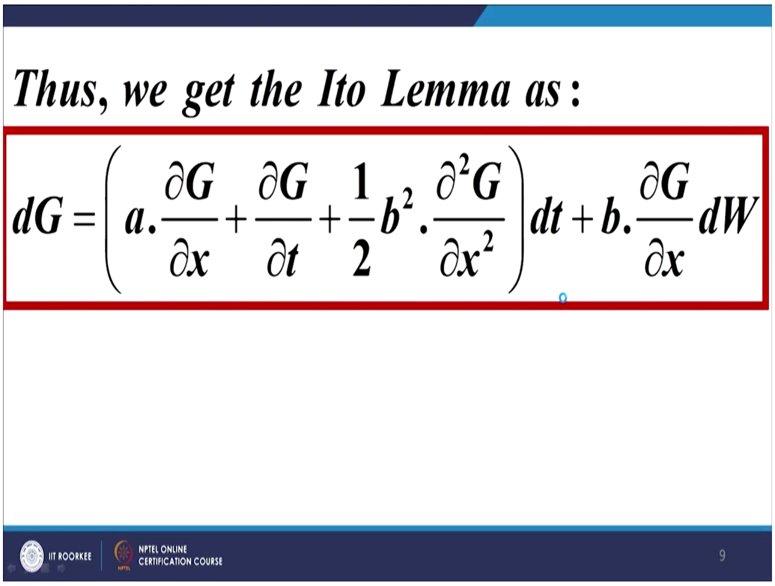

So, we have here an expression for the total derivative of a function of a stochastic variable with an explicit time dependence. And this the term, the coefficient of dt is called the drift coefficient and the coefficient of d W is called the diffusion coefficient.

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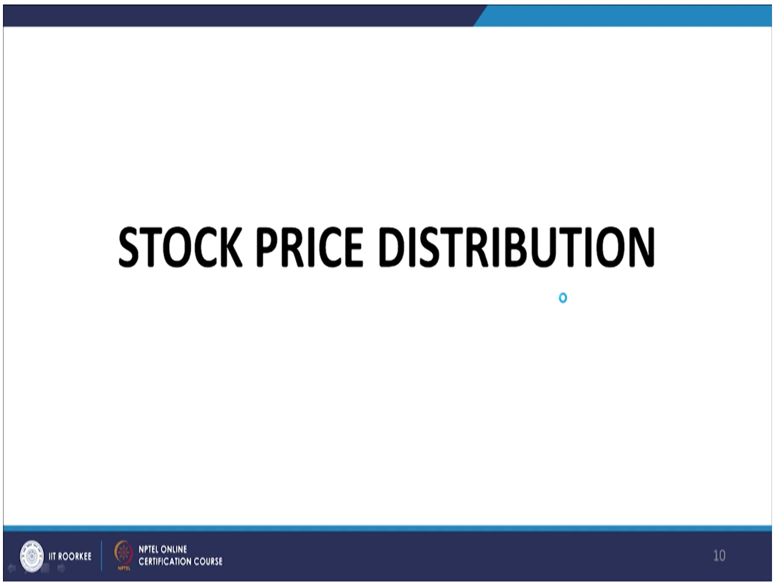

Now, we come to the stock price distribution. Now stock price distribution we shall be talking about is basically a model which is which is known by the Black-Scholes model in fact. But it epitomizes the price processes of most of the financial assets with variations and this is the fundamental process, the simplest process that is and that meets two reasonable rigor the empirical data. And it is if it forms the basis or the underlying of the Black-Scholes model of option valuation or contingent claim valuation that we shall be talking about in due course.

So, let us now get into the stock price behavior or the stock price distribution or the modeling of the stock price howsoever you may like to represent it.

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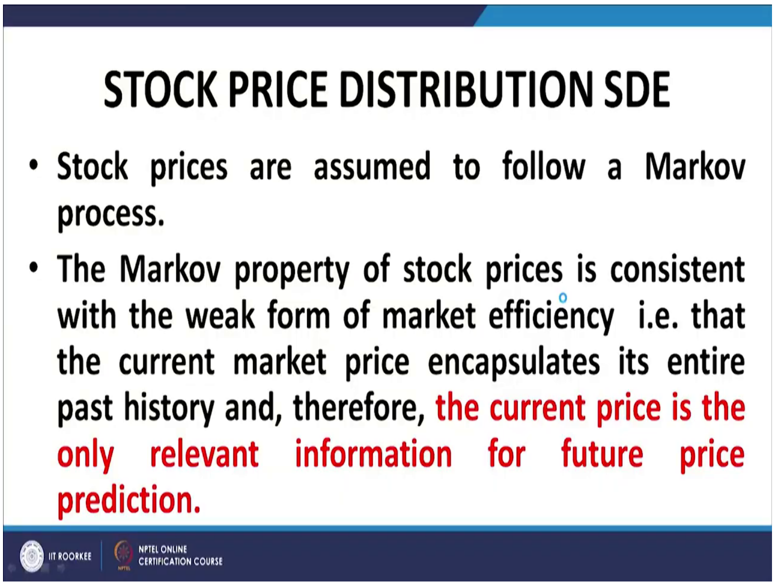

First of all, the fundamental thing is that the stock prices are assumed to follow a Markov process. This particular property is in tandem with or is in conjunction with, is compatible with, consistent with the weak form of market efficiency.

What does the weak form of market efficiency say? The weak form of market efficiency says that the current market price, the current market pattern applies encapsulate the entire past history. And therefore, the current market price is the relevant information that is useful, valuable or should be considered for predicting future prices that is what is the definition of a Markov process indeed and that is what enables us to classify stock price as a Markov process.

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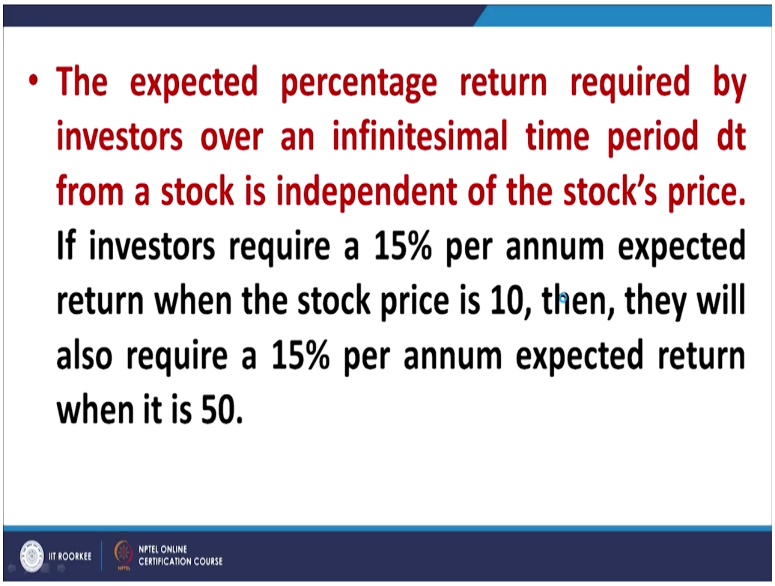

Now, we have two axioms on the basis of which we do the modeling of stock prices which we believe that a rational investor should obey on and which are supported to a large extent by empirical analysis. The first one is that the expected percentage return required by investors over an infinitesimal time period, small time period dt from a stock is independent of the stock's price.

Let me repeat, the expected percentage return required by investors over an infinitesimal time period dt, small time period from a stock is independent of the stock price. In other words, the if an investor requires a 10 percent or a 15 percent or a particular return from a stock when its price is x, it he requires the same return when the price is y. The return over infinitesimal time period is independent of the prevailing market price of the stock.

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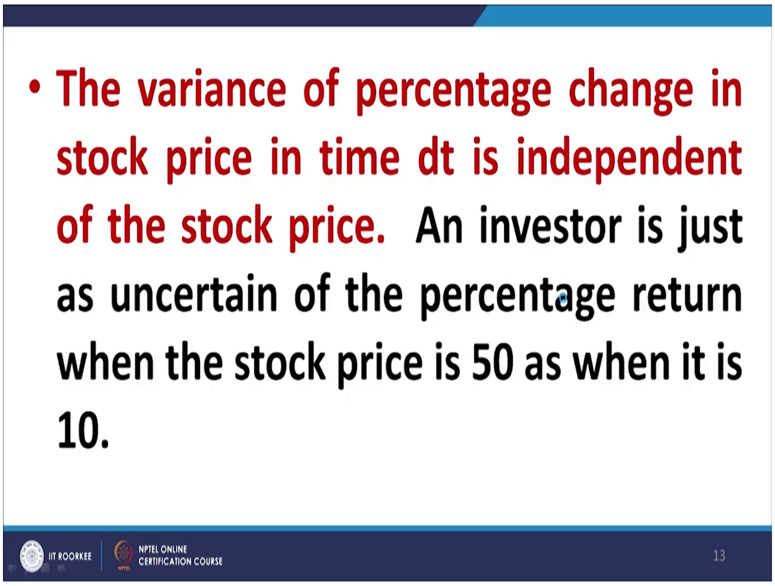

The second argument or the second axiom that is accepted as part of rational behavior is that the variance of the percentage change in stock price, the variance of the percentage change in stock price in time dt is independent of the stock price. Again what we are simply trying to say that an investor is just has an certain of percentage return when the stock price is say x as when he when the stock price is y.

So, that uncertainty in the; in the stock price or the stock return or the stock price or the stock return is independent of the prevailing stock price. These two axioms combined together from the fundamental model of stock price which is carried forward into the Black-Scholes option pricing model.

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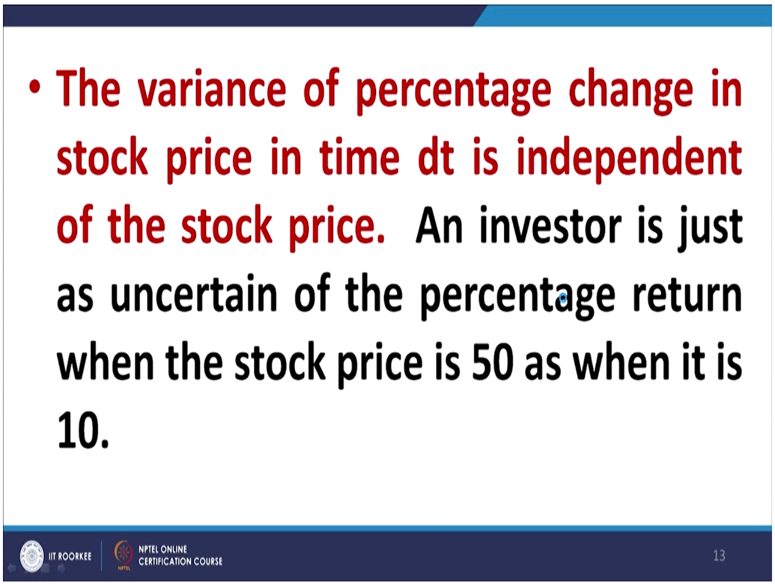

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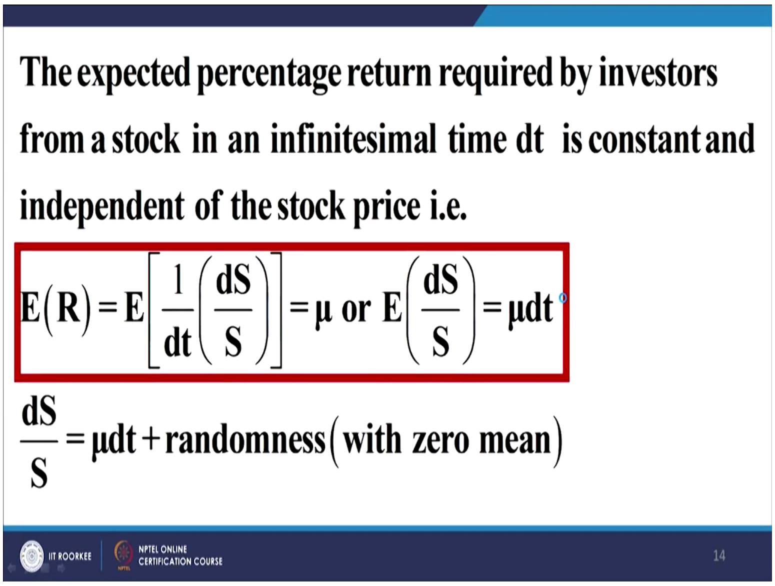

So, how does it translate into mathematics? How does how do these two axioms develop into differential equation or a stochastic differential equation? Let us see the first axiom is what? The first axiom is that the expected percentage change ah percentage return the expected percentage return required by investors from stock in infinitesimal time dt is in constant and independent of the stock price.

The expected return is given by over time dt is given by the expression in the red box here E in the square brackets here 1 upon dt into the percentage change d S upon S, d S upon S is the percentage change divided by the time period you get the percentage change per unit time which is the expected return.

And this return he says, it says the axiom says is independent of the price. And therefore, we call it mu and that enables us to write E or the expected change percentage change in price that enables us to write the expected change in price over time dt as mu dt. Because dt is non stochastic and that enables us to write d S upon S as mu dt plus some random term some stochastic term which has a 0 mean.

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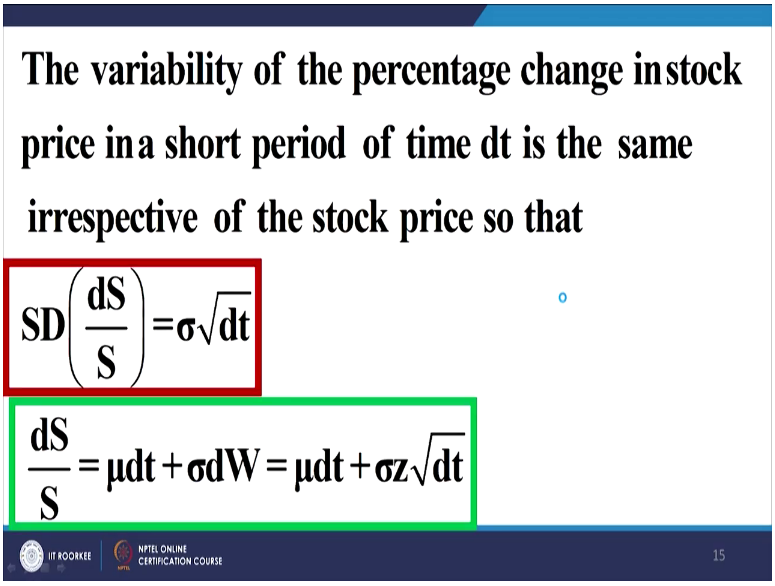

Then, we come to the second axiom. The second axiom is that the variability of the percentage change in stock price in a short period of time dt is the same irrespective of the stock price this gives us the condition that is given in the red box here.

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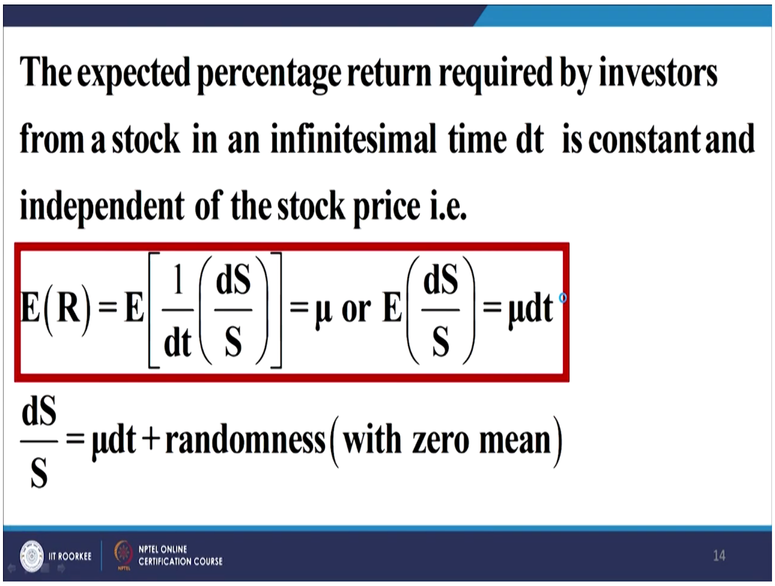

The standard deviation of the percentage change in stock price d S upon x can be written as sigma under root dt. So, combining the two, combining the expression that is given here in the red box and the expression that is given in this slide in the red box we can write this percentage change in the stock price over time dt where dt is infinitesimal in the form that is given in the green box right at the bottom of your slide.

Remember just to recall d W is the infinitesimal Brownian motion increment and therefore, it can be written as z under root dt z is the unit normal variate.

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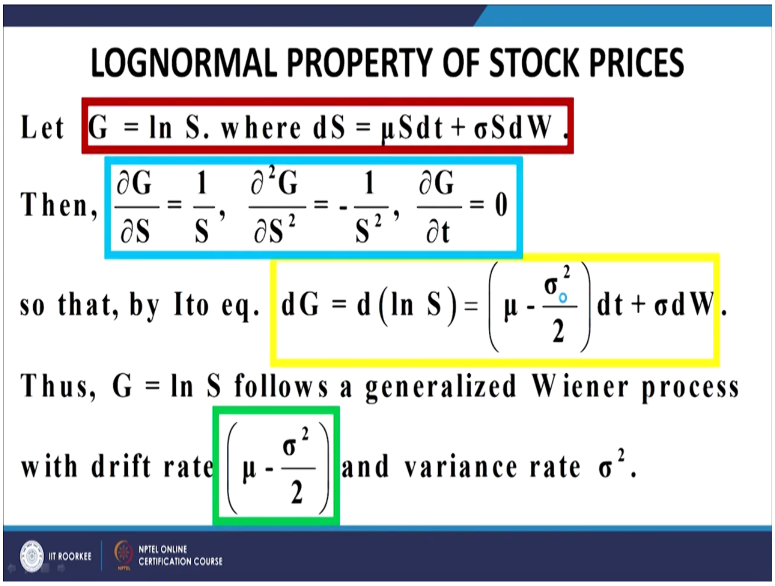

Now, this is the stochastic, the expression that we have obtained here in the green box is the is the stochastic differential equation and governing the or describing the stock price, defining the stock price. What does it translate in terms of the probability distribution of stock prices? How does it or what is the probability distribution of stock prices corresponding to this stochastic distribution stochastic differential equation?

Now, there are two ways of going into this. One way is simply to apply this Ito equation and the other way to go into this is through the Fokker Planck equation. We shall consider both of them.

Let us start with the Ito equation. This and this approach is relatively simplistic. We write G as log of S and where S is given by the expression that is given here in the red box in the green box. I am sorry and that can be written in the form d S is equal to mu S dt plus sigma S z under root d W or you can also write it as d S is equal to mu S dt sigma S z under root dt. Or mu S dt plus sigma S d W either way you can write it in on depending on how you choose to represent the stochastic term infinitesimal stochastic increment.

So, writing G as log S we get the first derivative of G with respect to S as 1 upon S and we get the second derivative of G with respect to S as minus 1 upon x, S squared. And of course, the derivative of G with respect to time is 0; because there is no explicit time dependence in G with when we define G as log of S.

And substituting this in the; in the Ito equation, what we get is the expression given in the yellow box here d G that is d log of S is equal to mu minus sigma square upon 2 dt plus sigma d W.

Now, we can write this as this this clearly shows that the; that the log of the stock price the increment or the change in the log of the stock price has a mean of mu minus 1 by 2 sigma square and dt we can write d log of S as log of S T minus log of S 0 and that enables us to write an expression for the probability distribution of log of S S T.

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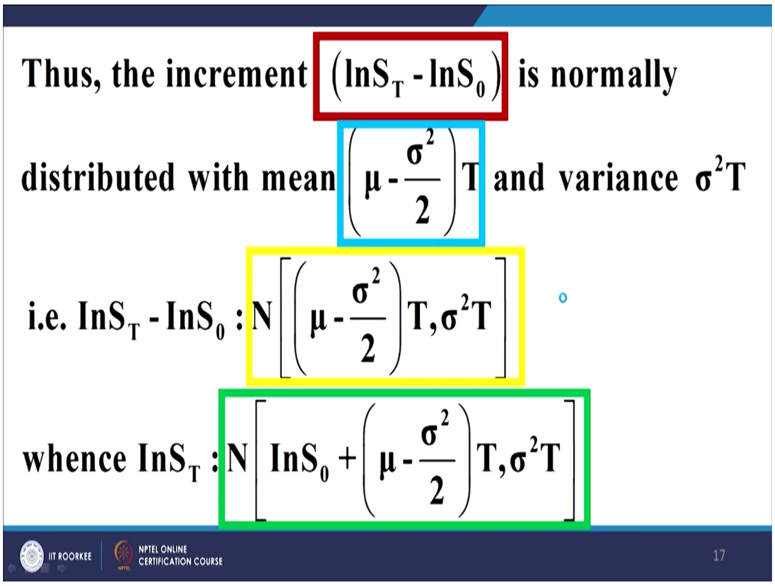

The log of S T is then following normal distribution with a mean of log of S 0 plus mu minus sigma square upon T sigma square upon 2 I am sorry into I and a variance of sigma square T.

That is obtained by integrating this differential equation between the limits S 0 and S T and there we thereafter we get this expression. Log S T minus S 0 is equal to mu minus sigma square upon 2 T plus sigma W T and that implies that log of S T is normally distributed with a mean of log S 0 plus mu minus sigma square upon T and a variance of sigma square T which is the expression that is shown in the green box right at the bottom of the slide.

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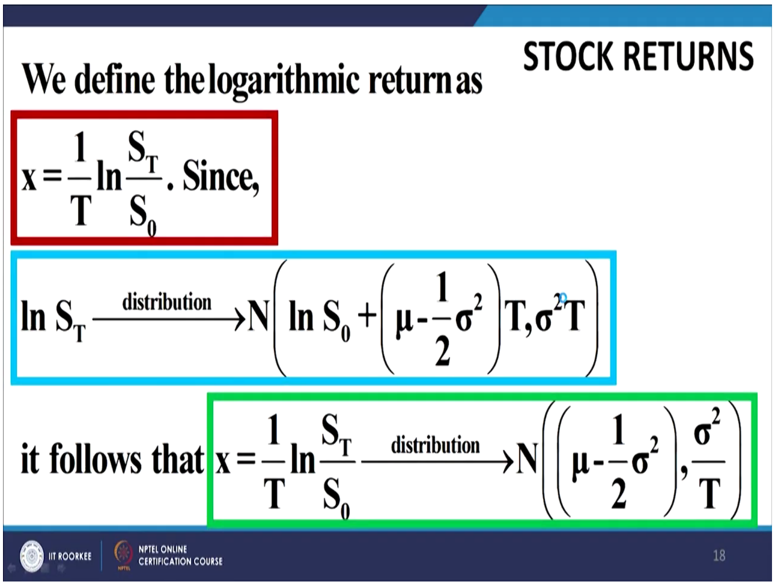

Now, what about now this is the expression for the price? This is the probability distribution of the logarithm of the price log of S T we have obtained the probability distribution for the log of S T.

Now, we obtain the probability distribution for the logarithmic return. We write x equal to 1 upon T log S T upon S 0 this is the definition of logarithmic return the expression in the red box here. This is the expression for the logarithmic return and this knowing the that log S T follows the distribution, that is given in the blue box here normal distribution with a mean of log S 0 plus mu minus 1 by 2 sigma square T variance of sigma square T.

We can immediately obtain the distribution followed by x where x is the logarithmic return x follows the normal distribution with a mean of mu minus 1 by 2 sigma squared. And a variance of sigma square upon T that is it is straightaway obtained by setting the log S 0 to the left-hand side and then dividing throughout by T.

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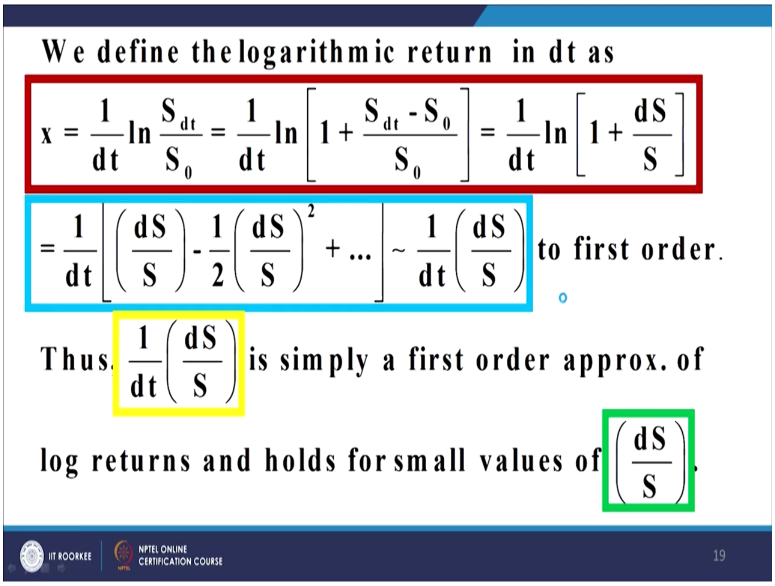

Now, just an issue about logarithmic return. In this slide, I will try to show that the return that we have talked about in this presentation so far in an earlier slide is the basically a first order approximation of logarithmic return. This can be easily seen when you follow the sequence of computations given in the; in this slide.

We have x is equal to 1 upon dt log S dt upon S 0 that is equal to 1 upon dt log 1 plus S dt minus S 0 upon S 0 S dt minus S 0 is nothing but d S so, we put d S here and then, we expand the expression in 1 plus x log 1 plus x form and retain the first order term. And that is nothing but the expression for the return that is in sometimes termed as the arithmetic return, the change in price per unit time that is what we normally use when we talk about return on a financial asset.

So, this is the relationship. In this particular slide, what I have tried to show the viewers is the relationship between the logarithmic return. And the arithmetic return that is usually is presented in the literature on financial instruments.

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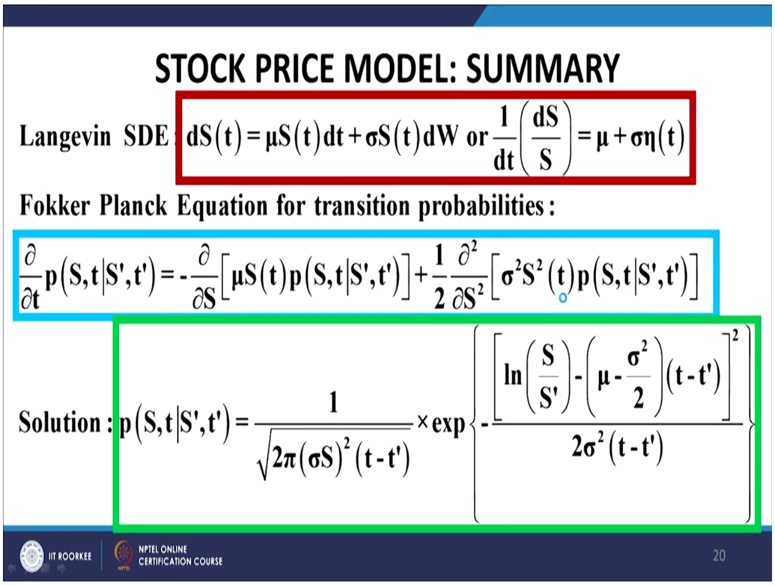

So, this is where we stand at the moment. The our stock price model is given in the red box here. Now the Fokker Planck now the second approach that I mentioned involves the Fokker Planck equation. If we write down the Fokker Planck equation corresponding to the stochastic differential equation that is given in the red box here.

We get the expression that is given in the blue box here. The solution to this Fokker Planck equation that is given in the blue box here is written out in the green box here and we shall follow it, but we shall touch upon it in a sense in the following slides ah, but little bit quickly because of paucity of time.

So, let us get into this. Let me repeat the red box gives us the stochastic differential equation or the Langevin form equation for the stock price. The blue box represents the Fokker Planck equation that corresponds to that stochastic differential equation of the red box and the green box is the solution to the Fokker Planck equation that we shall show in the sequel.

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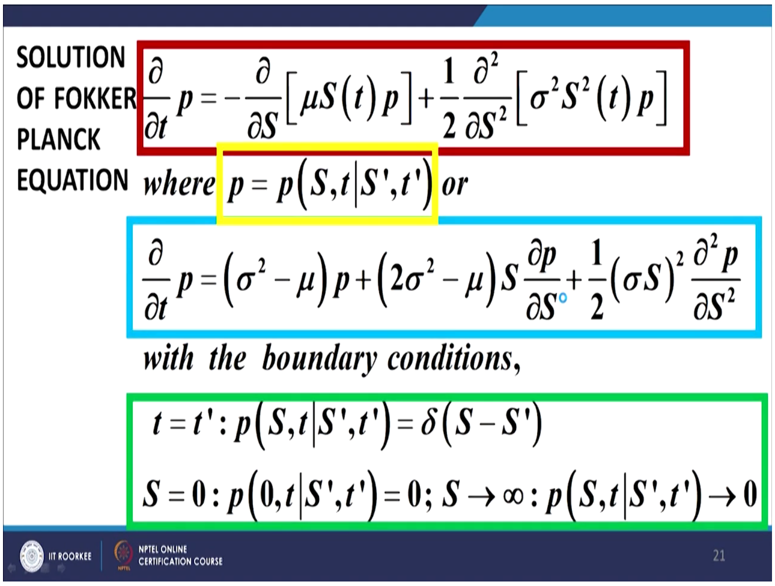

Now, this is the abbreviated form of the Fokker Planck equation that I had shown in the previous slide where the conditional probability p S, t subject to S dash, t dash is simply abbreviated as p and the rest is same. So, that is there in the red box here on simplification on carrying out the differentiations of the various terms here the above red box equation simplifies to the equation that is given in the blue box here.

And the boundary conditions at t equal to t dash p S, t S dash, t dash is equal to delta S minus S dash is quite straightforward boundary condition. If t is equal to t dash, then S must be equal to S dash and if S is equal to 0, then p 0, t S dash, t dash must be equal to 0. That means, the stock price if it is non-zero to start with it will never reach 0 and S tending to infinity p S, t S dash, t dash tends to 0 that means, the stock price cannot be unbounded in a finite interval.

So, these are the boundary conditions, the conditions that are specified in the green box here constitute the boundary conditions. The expression in the blue box here represents the equation that needs to be solved subject to the boundary condition let us see how to do it.

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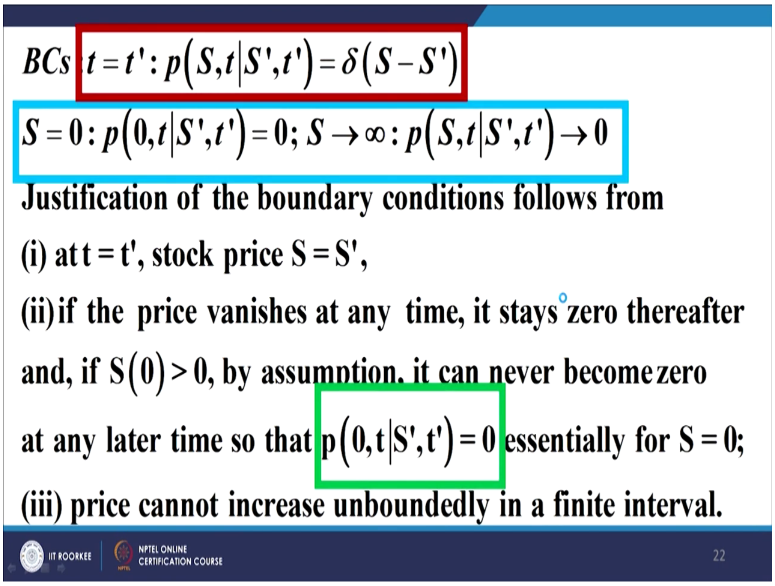

The first step is to do a change in variables we write this is the justification of the boundary conditions which I have already mentioned. So, the viewers can go through it.

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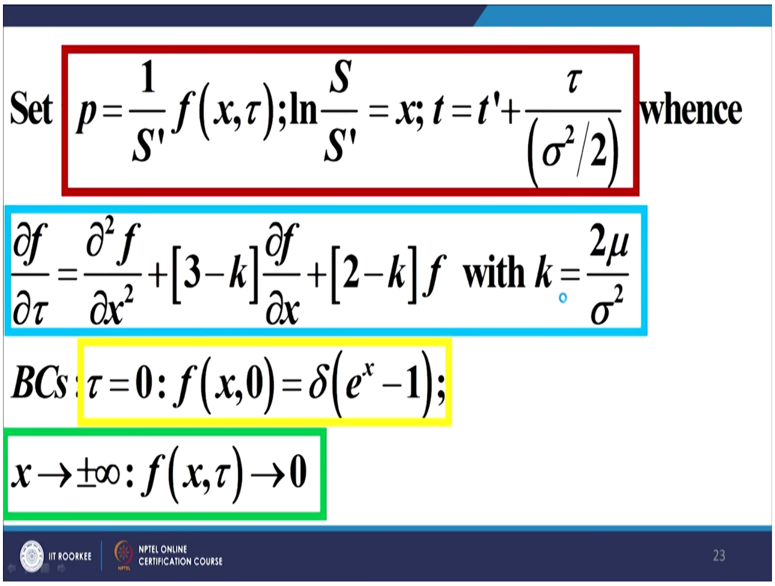

Now, we make the substitutions or changes in variables which are given in the red box here. The impact of this change in variables as you can see here is that the coefficients of the various differential derivatives are now independent of the S or the or the independent variables x in this case it is x. So, the coefficients of all the derivatives are now independent of x we have df upon d tau is equal to d square f upon dx square plus 3 minus k df upon dx plus 2 minus k into f. So, every coefficient is independent of x.

The boundary condition is also get modified due to the transformation of variables and are given in the yellow box and the green box at the bottom of the slide.

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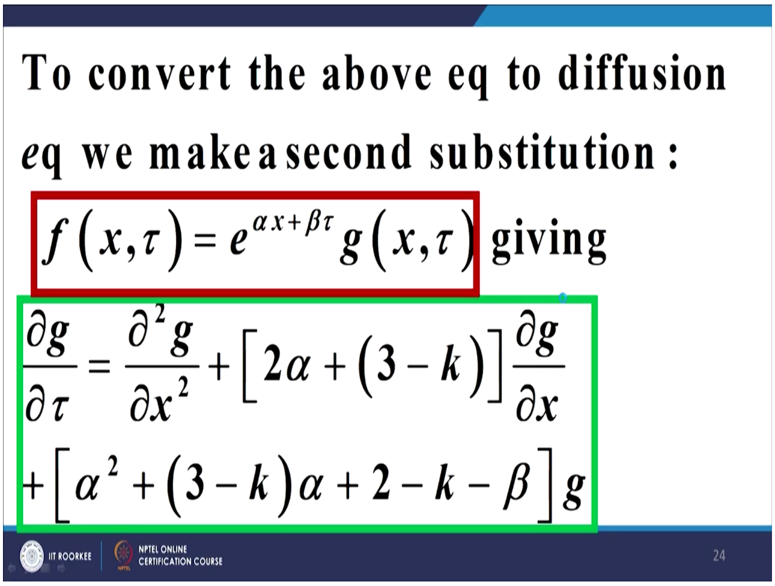

Now, we convert this equation that we have a here in the blue box here, we convert this equation into diffusion equation. In order to convert this into the diffusion equation, we need that the where the coefficients of the first order derivatives and the coefficient of x should vanish. We do that by introducing a new variable g x, tau which is and free variables alpha and beta expressed by the equation given in the red box here where alpha and beta are free variable to be determined by us in such a way that we get the; that we get the diffusion equation.

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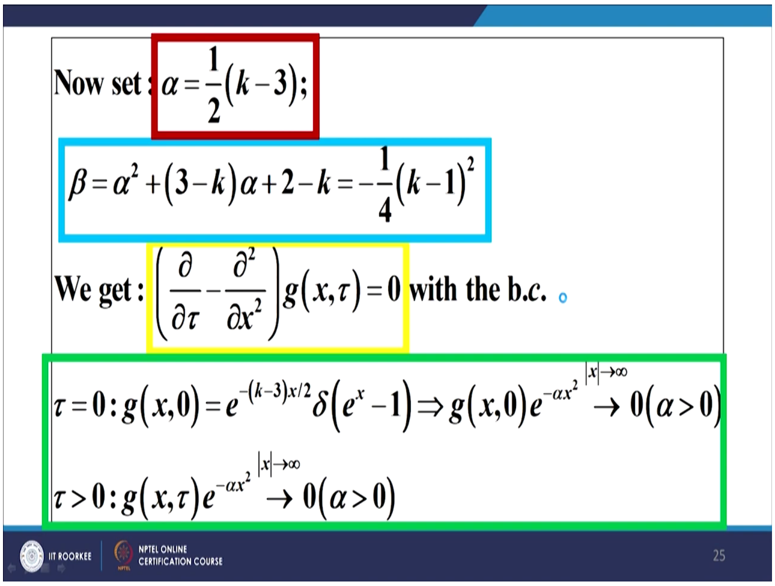

And when we make the substitutions, we get the required values of alpha and beta that that enable us to convert this equation to a diffusion equation given by the expression. Here in the red box and the blue box alpha becomes equal to 1 by 2 into k by 3 and beta becomes equal to alpha square plus 3 minus k alpha plus 2 minus k that is equal to minus 1 by 4 k minus 1 squared. And when we substitute these values of alpha and beta, we end up with the diffusion equation that is given in the yellow box here and the boundary conditions also get modified accordingly in terms of the; in terms of the variable g.

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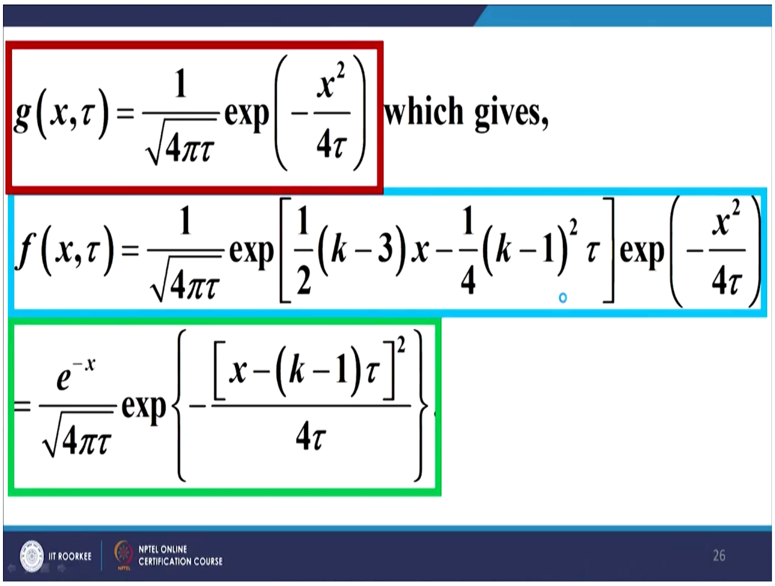

Now, the solution of the differential diffusion equation is I am sorry the solution of the diffusion equation is quite straightforward. It can be done by Fourier approach or by using Laplace transforms by using Fourier transforms. It is a straightforward exercise and the result that we get and in fact, we have done this earlier in one of the earlier lectures I recall and the result that we get is the expression that is given in the red box here.

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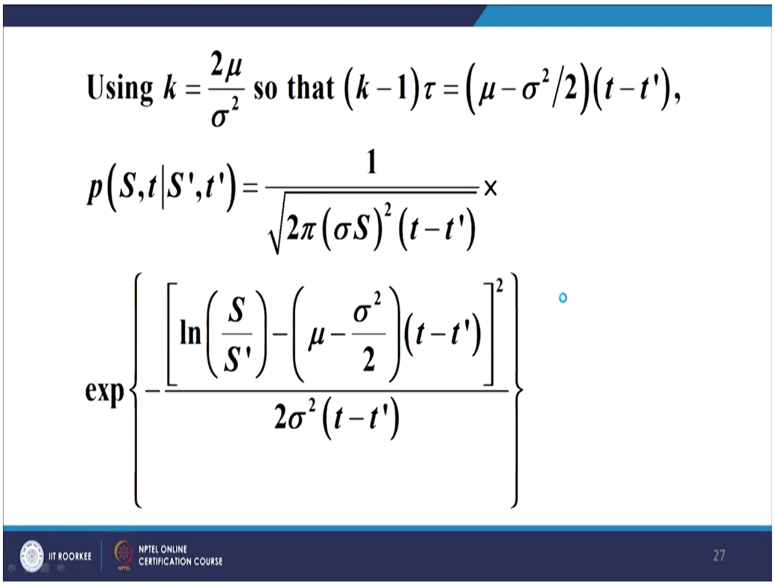

Reverting that to a original variables in step wise, what we end up with is the expression that is given here in this slide. We get the expression for the conditional probability p S, t subject to S dash, t dash as the expression given in the bottom equation on this slide and we which we clearly recognize, which we immediately recognize as the pdf of a log normal distribution; pdf of a log normal distribution. So, this endorses the fact or indicates the fact that the stock prices follow a log normal distribution.

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Setting $\xi = \ln S$

$$p\left(\xi, t \mid \xi', t'\right) = \frac{1}{\sqrt{2\pi\sigma^2\left(t - t'\right)}} \times$$

$$\exp\left\{ -\frac{\left[\left(\xi - \xi'\right) - \left(\mu - \frac{\sigma^2}{2}\right)\left(t - t'\right)\right]^2}{2\sigma^2\left(t - t'\right)} \right\}$$

And this in fact, this is this makes it more explicit if I substitute xi equal to log x and the expression that I get is; the expression that I get here that is in a sense the logarithm of the price. The logarithm of the price follows a normal distribution as you can see this is the pdf of the normal distribution.

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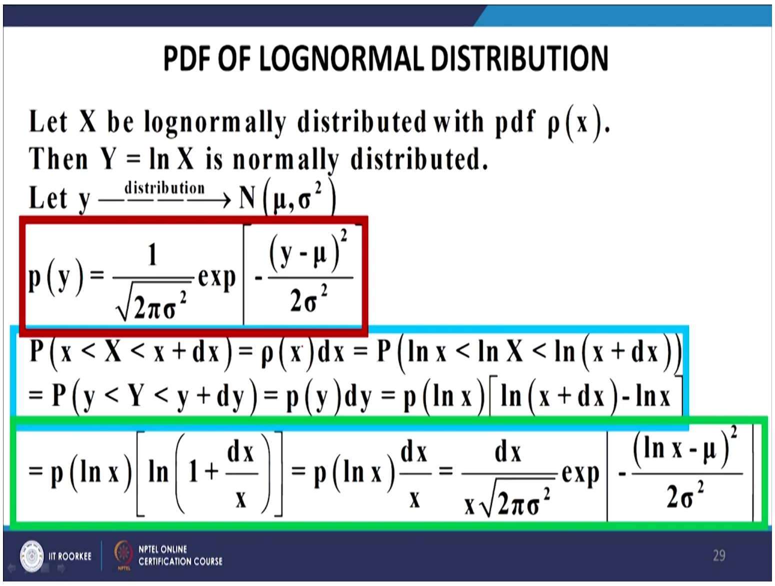

Now, just since, we have talked about the log normal distribution, just a word about the log normal distribution. How do we define log normal distribution? Well, if x is log normal if x is a random variable and if x follows the log normal distribution or x is log normally distributed with a pdf p x, then y equal to log x is normally distributed that is the definition of log normal distribution. If x is log normally distributed, then log of x is normally distributed that is the definition of log normal distribution.

So, let us find the pdf of the log normal distribution. We have y is as I mentioned, y equal to log x is normally distributed and we assume that y is normally distributed with a mean of mu and a variance of sigma square. Remember mu and sigma square are the parameters of log x not of x. X is log normally distributed, log x equal to y is normally distributed and it is normally distributed with a mean of mu and a variance of sigma square are of course, the pdf of y would be given by this expression that is in the red box here that is well known.

Now, the probability that my random variable x lies between an infinitesimal interval X and x plus dx is given by p x dx where p x is the probability density function of x. So, it is the probability that my random variable x lies between X and x plus dx is given by p x dx and that is also equal to p log of x is less than x is less than log X is less than log x plus dx because log X is a monotonic function of x.

So, if x is less if x lies or the random variable x lies between X and x plus dx, then log of the random variable log x capital X also lies between log small x and log small x plus dx. At log small x is nothing but small y, log capital Y is the random variable y and we write log x plus dx as y plus dy and this is nothing but p y and dy or rho y dy whatever you may like to take.

So, so, this expression carrying on the equalities this is equal to or substituting Y equal to log X, we have p log of x into dy, dy can be written as log x plus dx minus log x; because dy can be written as y plus dy minus y and that can be written as x plus log x plus dx minus log x.

And simplifying this and taking we get the expression p of log x into log 1 plus dx upon x and which is of the form 1 plus x and which can be written up to first order as p log x into dx upon x and which has the probability distribution given as the last expression on the bottom of the slide.

So, this is the probability density function of log of x because the this dx and this dx will cancel out this dx and dx will cancel out and what we are left with is p x. So, p x is equal to 1 upon x under root 2 pi sigma square exponential minus log x minus mu whole square upon 2 sigma square. This is the probability density function of the log normal distribution.

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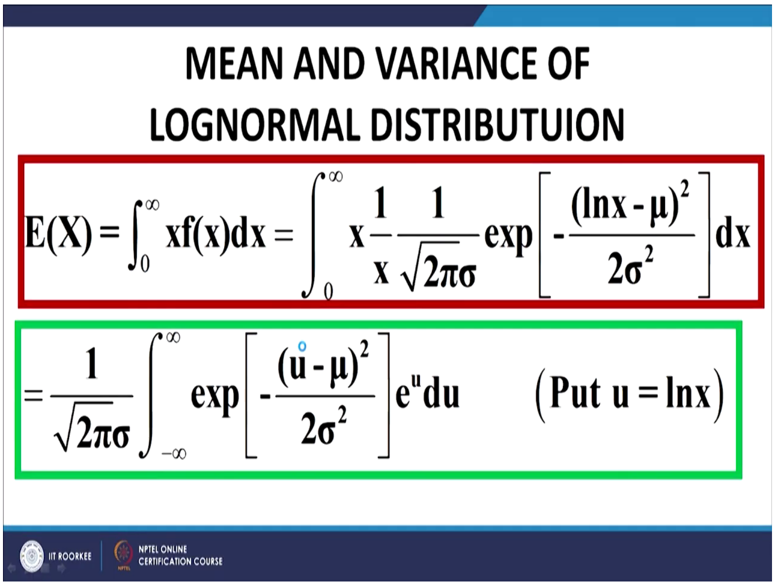

The mean and variance of the log normal distributions are easy to calculate. They are straightforward exercises in calculus simple integration. So, I will not go through them in detail. I put them in the presentation for the convenience of the viewers.

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$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left\{-\left[\frac{(u-\mu)^2 - 2u\sigma^2}{2\sigma^2}\right]\right\}du = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left\{-\left[\frac{(u-\mu-\sigma^2)^2-2\mu\sigma^2-\sigma^4}{2\sigma^2}\right]\right\}du$$

$$=\exp\left(\frac{2\mu\sigma^2+\sigma^4}{2\sigma^2}\right)\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left\{-\left[\frac{(u-\mu-\sigma^2)^2}{2\sigma^2}\right]\right\}du = \exp\left(\mu+\frac{\sigma^2}{2}\right).$$

The last step follows because $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left\{-\left[\frac{(u-\mu-\sigma^2)^2}{2\sigma^2}\right]\right\}du = 1$

as $\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\left[\frac{(u-\mu-\sigma^2)^2}{2\sigma^2}\right]\right\}du$ is the probability density of a normal variable with mean $\mu+\sigma^2$ and standard deviation σ.

But the results that we get is that the mean is equal to exponential mu plus sigma square upon 2, but please recall mu and sigma square are the parameters of log of x and not of x.

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$$E(X^2) = \int_0^{\infty} x^2 f(x)dx = \int_0^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\ln x - \mu)^2}{2\sigma^2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^u \exp\left[-\frac{(u-\mu)^2}{2\sigma^2}\right] e^u du = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\left[\frac{(u-\mu)^2 - 4u\sigma^2}{2\sigma^2}\right]\right\} du$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\left[\frac{(u-\mu-2\sigma^2)^2}{2\sigma^2} - \frac{4\mu\sigma^2 + 4\sigma^4}{2\sigma^2}\right]\right\} du$$

$$= \exp\left(\frac{4\mu\sigma^2 + 4\sigma^4}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{(u-\mu-2\sigma^2)^2}{2\sigma^2}\right] du = \exp(2\mu + 2\sigma^2)$$

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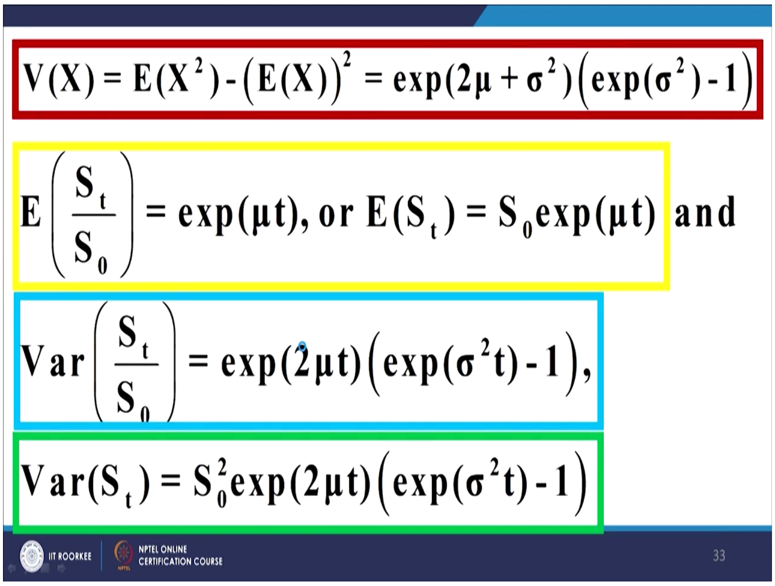

So, this is the mean and the variance, the expression for the variance is given here in this slide, in the red box here and these the expressions in the yellow box and the blue box and the green box are the corresponding expressions when these are applied to the Black-Scholes model or the stock price model that we are investigating for the moment.

Thank you. We will continue after the break.