Path Integral Methods in Physics & Finance
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Lecture – 52
Gauge Fields [2]

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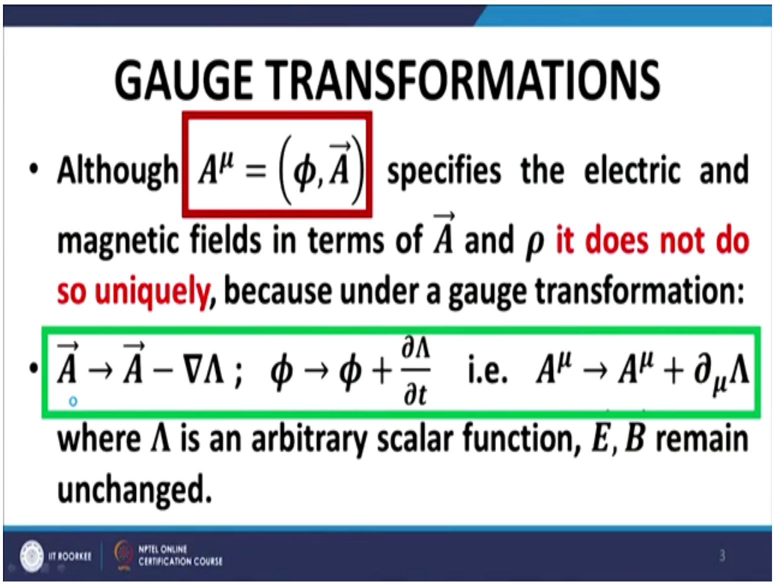

So, let us continue. So, what I was talking about just before the break was the concept of Gauge Transformations. You see we replace the we introduce the four vector A A mu as a (Refer Time: 00:38) representation of Maxwell's equations. And, then we introduce the concept of electromagnetic field tensor.

Now, the point is that the A vector or the A mu vector that we introduced is not uniquely defined. In other words, if I transform A mu by the expression that is given in the green box here my electromagnetic field tensor remains unchanged and therefore, my Maxwell equations still continue to be satisfied. So, that gives us a certain amount of leeway or a freedom or ambiguity in terms of the representation of the 4-vector A or the that we can use for the purpose of representing the electric and magnetic fields, right.

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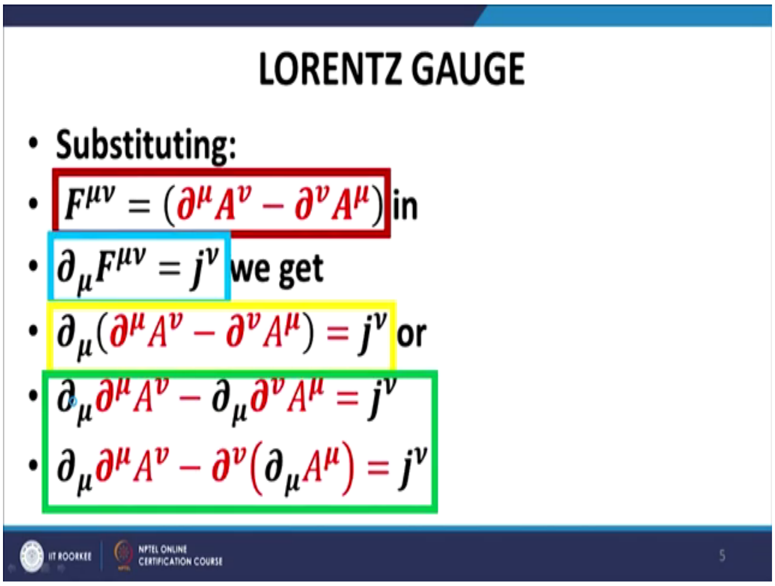

So, now we however, we to arrive at a unique representation we therefore, need to impose certain restrictions to impose certain gauge requirements. One of the gauge requirements that we impose is the called the Lorentz gauge.

In the Lorentz gauge what we require is that the expression within brackets here within the curly brackets here in the green box at the bottom of this slide in this should vanish. That is the requirement of the Lorentz gauge.

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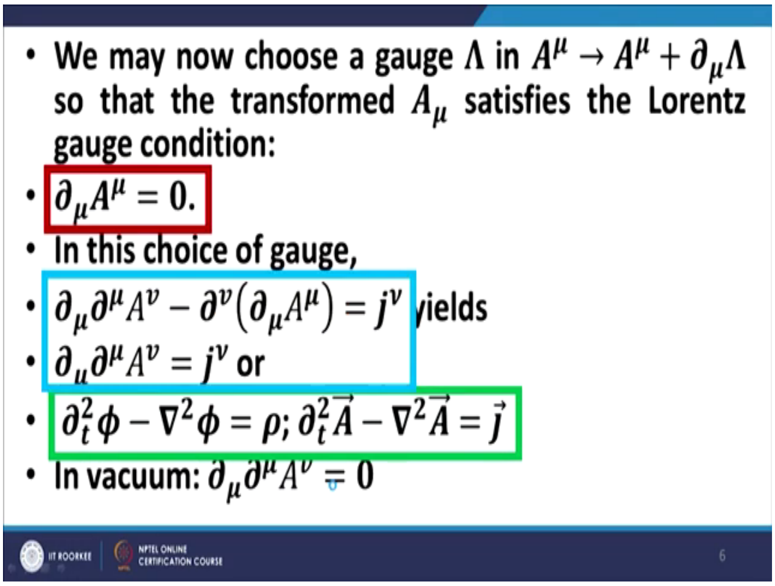

Let us see what it implies. If we have this requirement which is here in the red box here what we end up with is the expression that is in the green box at the bottom of the slide after simple algebraic manipulations. And, for the vacuum we will have an expression of the form box of A nu is equal to 0.

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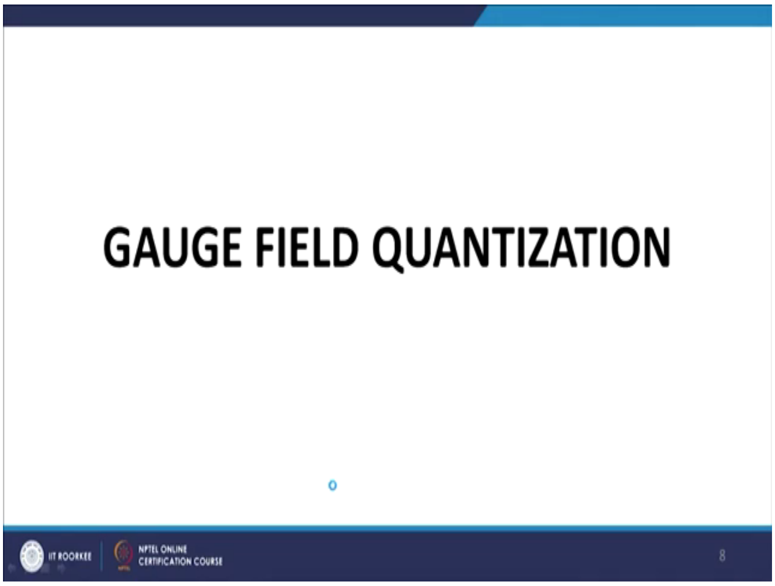

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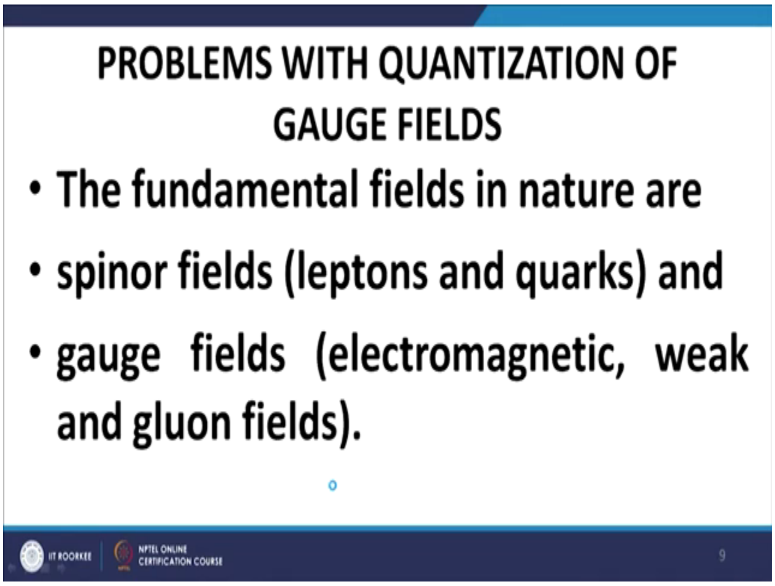

Now, we talk about now the fundamental fields in nature are this spinor fields that are leptons and quarks and gauge fields that we are talking about just now and also gluon fields.

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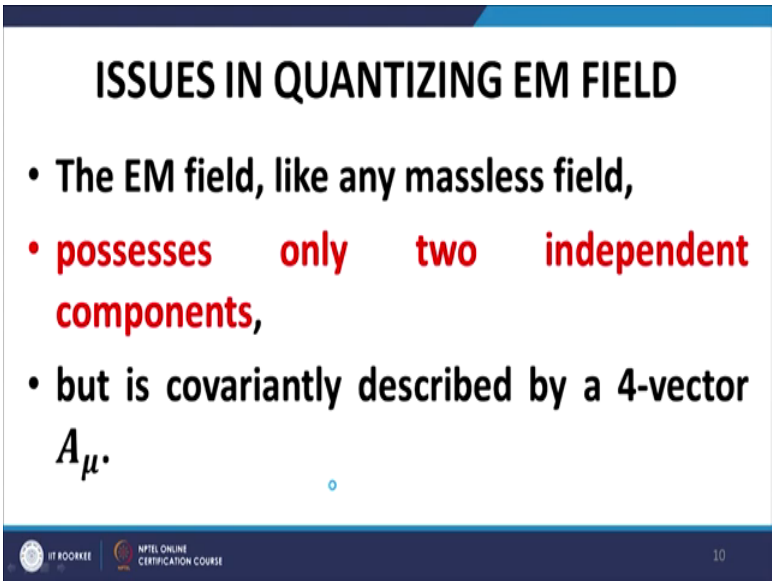

But, in the case of electromagnetic field we have only two independent components in contrast to four independent components which are required for a covariant representation in 4-dimensional space time for a single particle. So, that being the case we are faced in a quandary which in some sense also makes us presents fields which also manifests itself in the gauge freedom that we just talked about.

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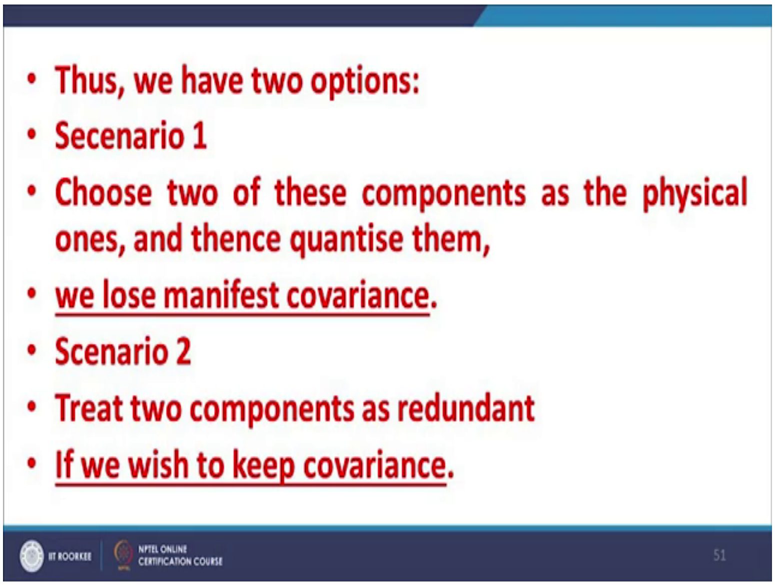

So, we really have two choices. We have the choice of choosing two of these components as the physical components and then quantize those two components, but in this case we lose Lorentz covariance. And, on the other hand, the other choice that we have is that we treat the two components as redundant and we retain Lorentz covariance.

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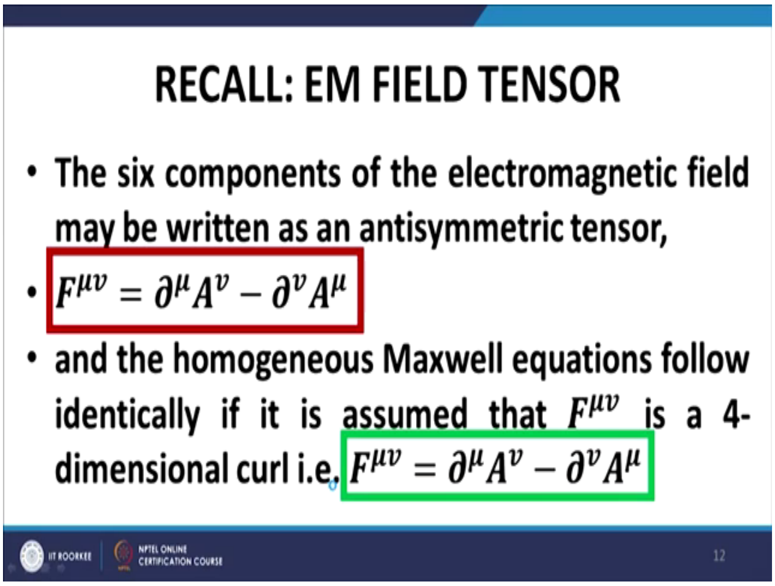

Let us do the first approach. So, to recall then the electromagnetic field tensor we are defined in terms of the expression that is given in the red box here. And the 4-dimensional curl.

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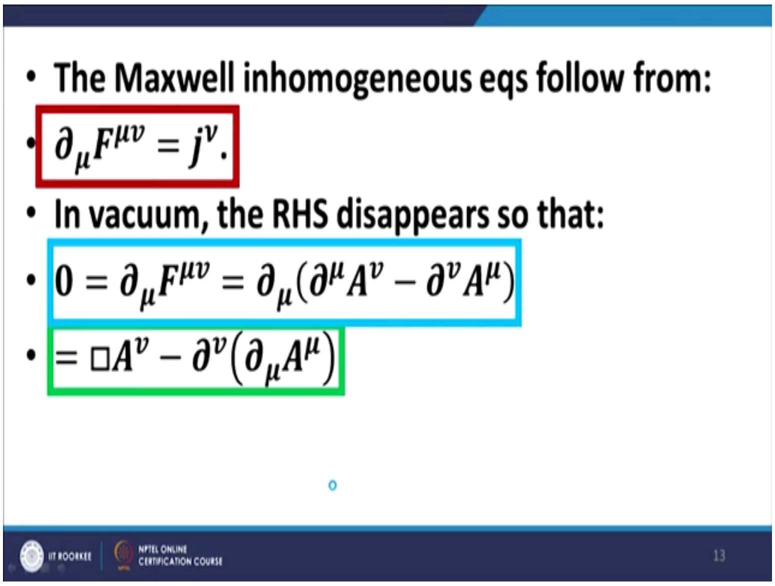

And, as I mentioned the d mu of the or the derivative of the first 4-dimensional electromagnetic field tensor is equal to the j nu, the components of the current. And, this encompasses the inhomogeneous equations and the homogeneous equations are automatically satisfied from the property of the electromagnetic field tensor being a 4-dimensional curl.

In vacuum what happens? Right hand side disappears and we have the expression which is given in the green box at the bottom of your slide which we have already discussed.

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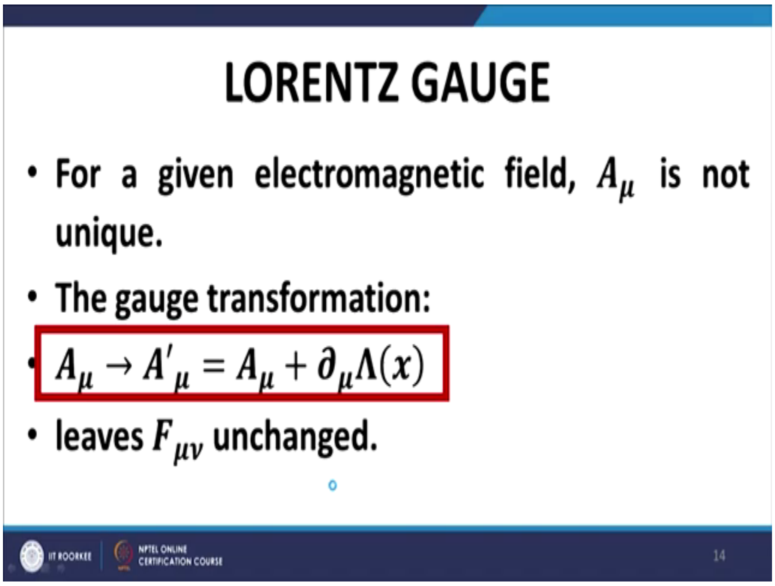

For the Lorentz gauge we write the Lorentz gauge in the form and the form given in the red box here we have seen that this expression leaves the electromagnetic field tensor unchanged.

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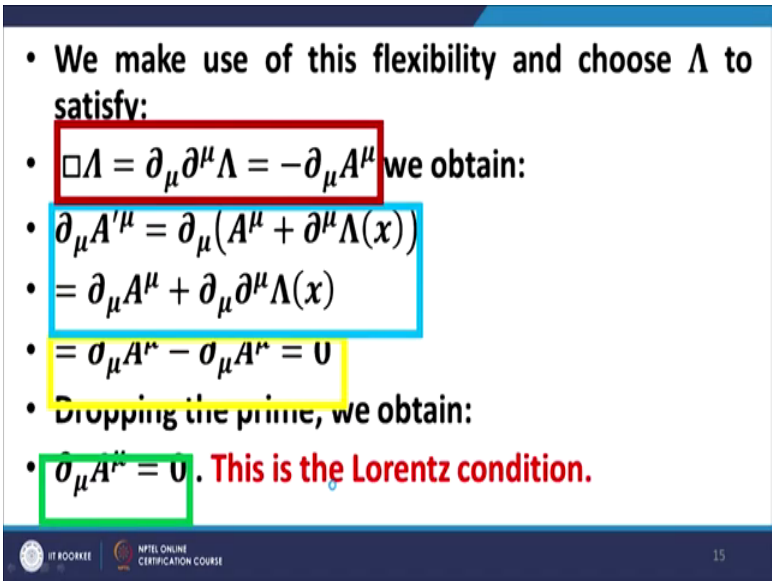

We impose the condition given in the red box here and what we end up with is the for the Lorentz gauge if we impose this restriction that is given in the red box here, then on simplification what we find is that the derivative of A dash mu that is the transformed A mu takes the form which is given here in the blue box.

And if you simplify this using the expression in the red box here what we find is that the we have d mu A dash mu is equal to 0. Now, if you drop the primes we get the expression for the Lorentz gauge that we want to impose that is in d mu of A mu is equal to 0. So, this is the Lorentz gauge or the Lorentz condition that we want to impose that we impose in our analysis.

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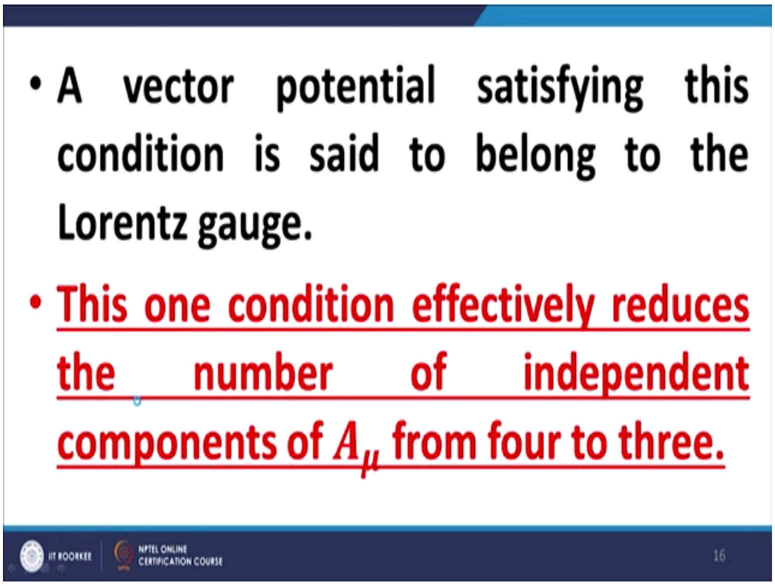

So, a vector potential satisfying this condition that is d mu of A mu is equal to 0 is said to belong to the Lorentz gauge. This one condition, this one restriction reduces the number of independent components from four to three, but we are still not unique.

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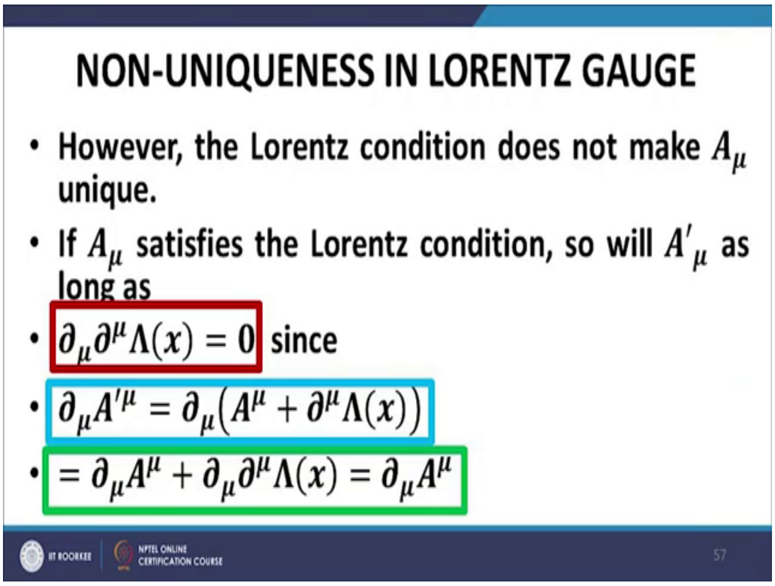

Why we are still not unique is because if any scalar functions satisfies the requirement given in the red box here, the box of eta x is equal to 0, then what we find is that what we find is that the condition or the Lorentz condition is automatically satisfied. Therefore, this gives us another there still as some non uniqueness remaining even after the imposition of the Lorentz gauge.

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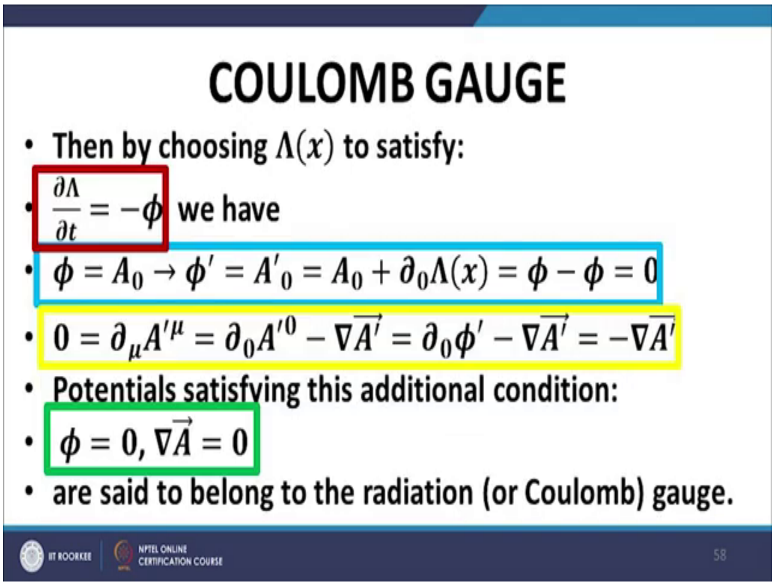

For this purpose we impose the Coulomb gauge to make the definition unique. How we do it? We impose the following requirement which is given in the red box here the time derivative of this scalar function is equal to minus phi and when we do that what we get is that phi goes to phi dash and what we get for A A dash 0 is equal to 0. And, for A for the spatial part of A we get minus delta del A that is the divergence of A 3 divergence of A is equal to 0.

In other words the Coulomb gauge is specified by the expression in the green box that is phi is equal to 0 and the divergence of A the spatial A is equal to 0. These two together make the Coulomb gauge that meansnote they are not relativistically covariant.

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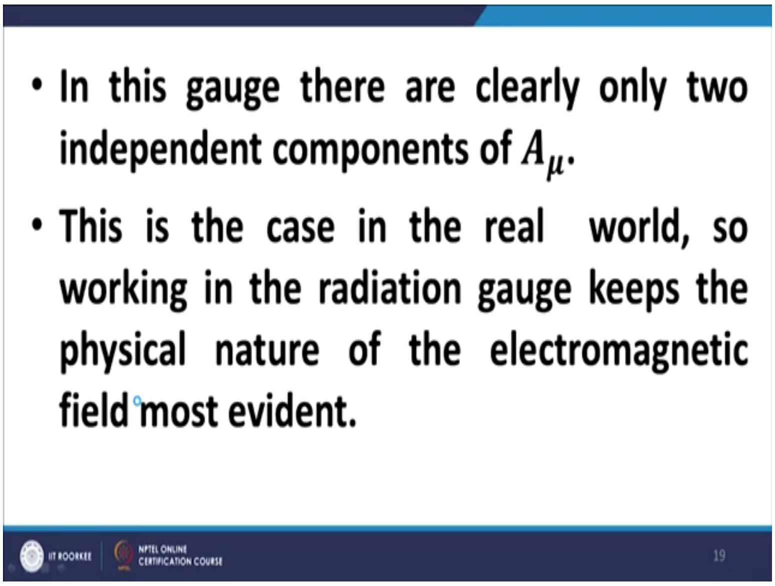

So, we are now left with only two independent components of A mu. And this is the case of the real world where we work, so, working in the Coulomb gauge or this is also called the radiation gauge we retain the physical nature of the electromagnetic field.

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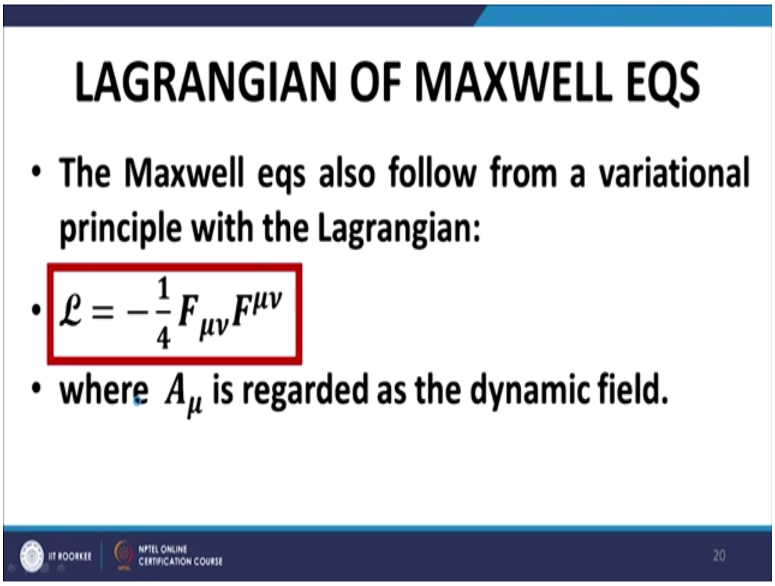

Now, the Lagrangian of the Maxwell equation we shall be dealing with it when we talk about the path integral is given by the expression in the red box here, but we will be coming back to it.

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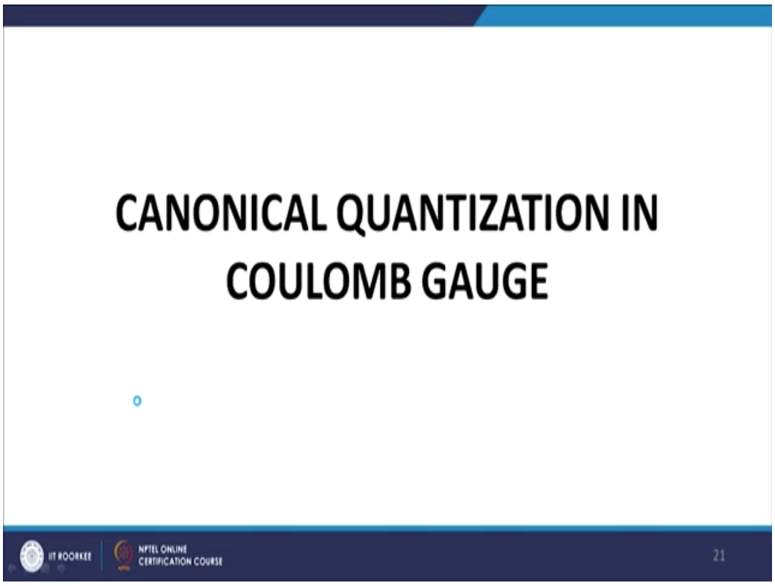

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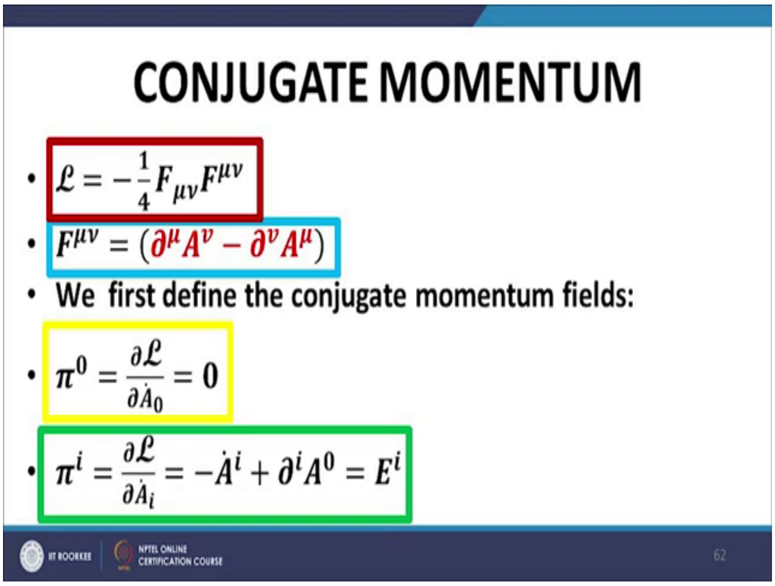

The canonical quantization in the coulomb gauge we start with the Lagrangian this which is given in the red box here from which by variational principle we can derive the Lagrangian equations.

And, we define the conjugate momenta in terms of the expression that is given in the yellow box and the expression in the green box the time component of the conjugate momenta vanishes and this spatial component of the conjugate momenta represent the electric fields.

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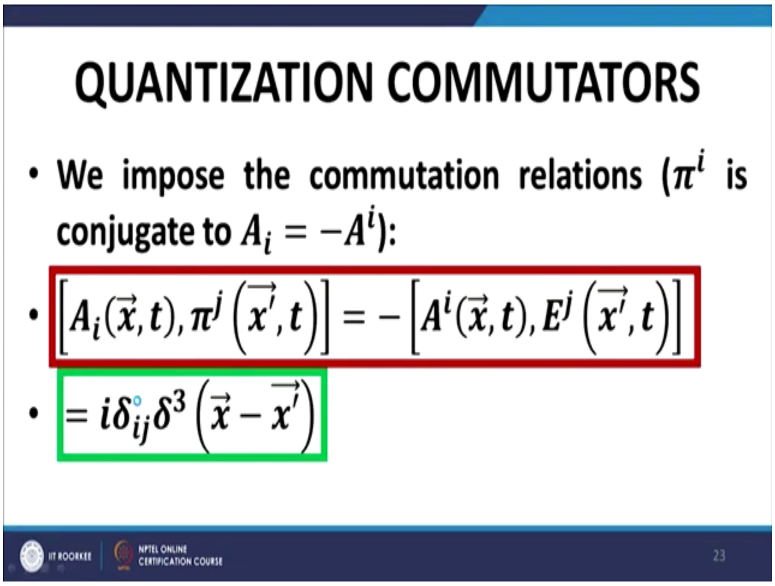

And there therefore, the quantization commutator that we desire or we attempt or we postulate is given by the expression here in this slide which is self explanatory. So, let us carry on.

Although there is a small problem here in the sense, that here the indices if you look at the second commutator the indices are upstairs whereas, i and j indices are upstairs whereas, on the right hand side of the equation the i and j indices are downstairs.

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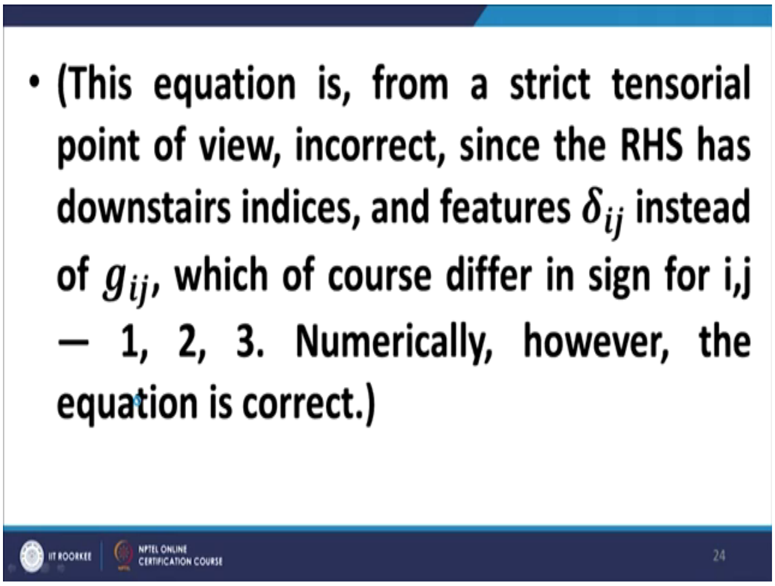

And, secondly, the appearance of data idea instead of the flat metric eta ij or gij has also make certain relativistically incorrect, but numerically of course, it turns out to be correct.

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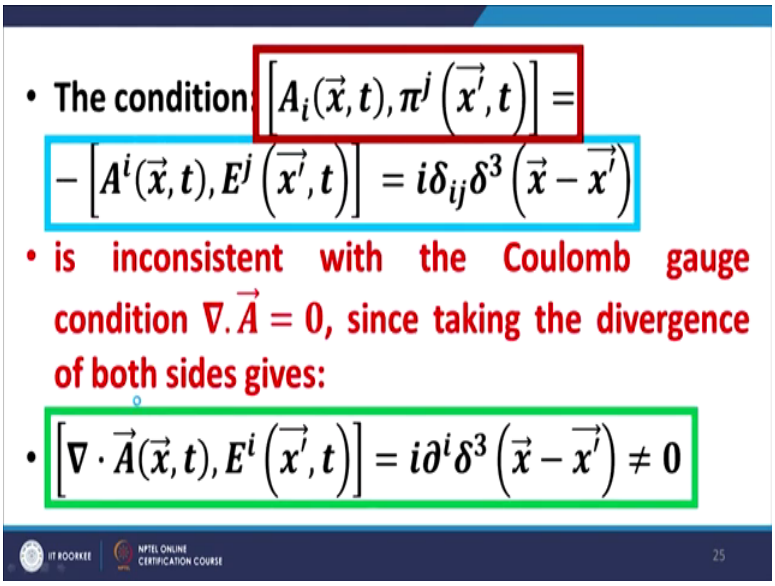

However, when we look at this these commutators in the context of the Coulomb gauge we find that they are inconsistent with the Coulomb gauge because if I take divergence of both sides divergence of delta A is equal to 0 divergence of the spatial A is equal to 0 as you will recall in the case of a column gauge. So, if I take divergence of both sides what I get is that the commutator is unequal to 0 which is not compatible with the Coulomb gauge.

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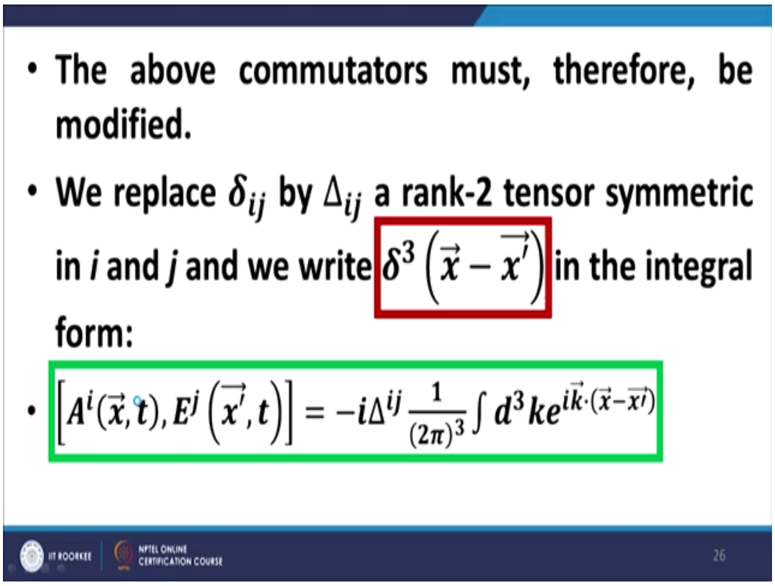

And, therefore, it this commutator is required to be modified to modify the commutator we write delta ij by capital delta ij which is which we know postulate as a second rank tensor symmetric in i and j and we also write the delta function in 3-dimensions in terms of the Fourier representation as shown in the green box here.

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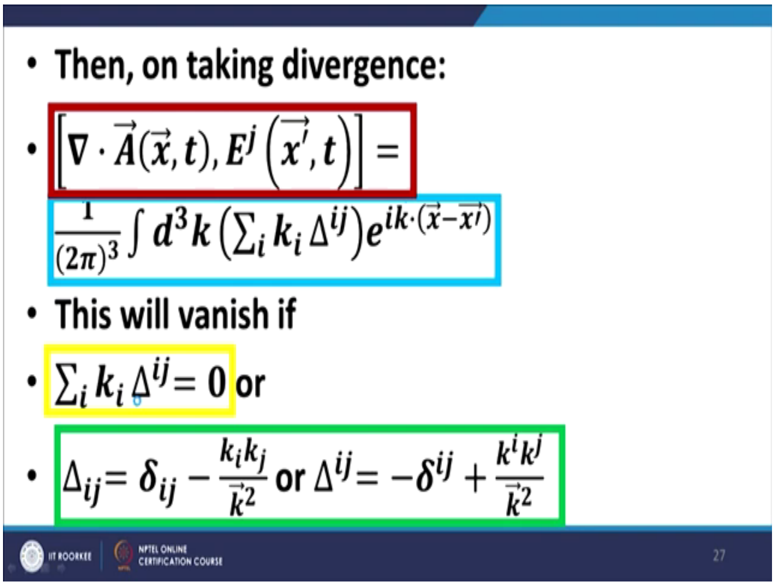

When we take divergences now the we find that the divergences would vanish when the condition imposed in the yellow box is satisfied. And, the condition imposed in the yellow box would be satisfied when the condition imposed in the green box is satisfied that is when capital delta ij is equal to minus small delta ij plus k i k j upon k squared k spatial squared.

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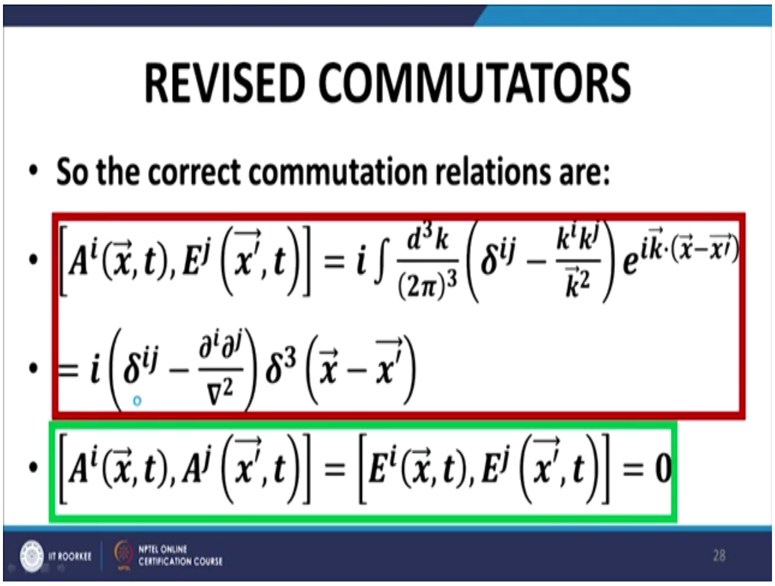

So, the revised commutator now take the form which is given in the slide the upper red box covers the commutators between A i and E j that is the E j represents the conjugate momenta. And, A i represents the field variables and the and the others are of course, the commutators between similar quantities.

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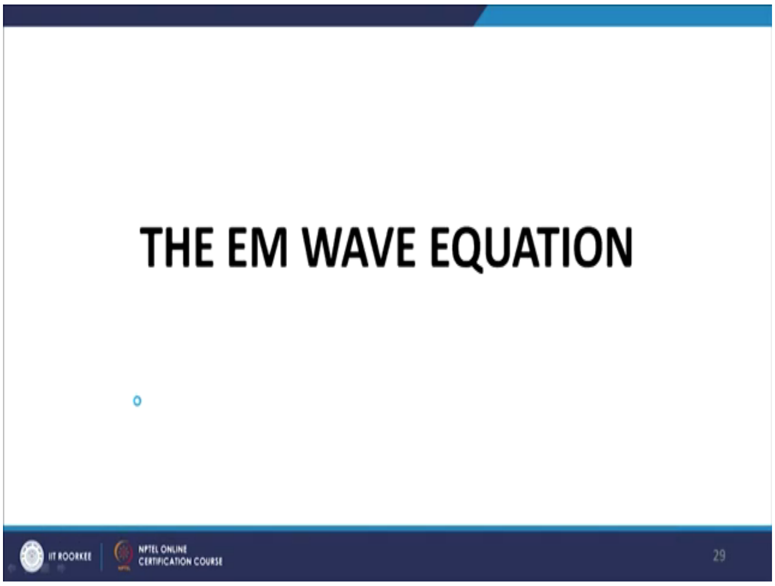

Now, we come to the electromagnetic wave equation.

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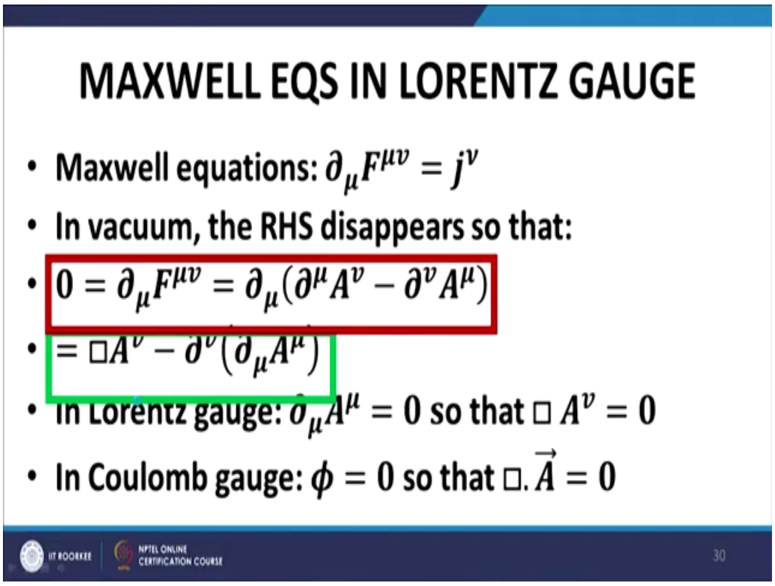

The Maxwell equations as you recall is given by the expression given in the first expression in terms of the electromagnetic field tensor. In vacuum, of course, no sources so, the right hand side vanishes no currents are there so, the right hand side vanishes. And, in the Lorentz gauge the term in the round brackets and the green box also vanishes. So, what we are left with is a box of a nu is equal to 0 furthermore in the Coulomb gauge the time component of A also vanishes phi is equal to 0.

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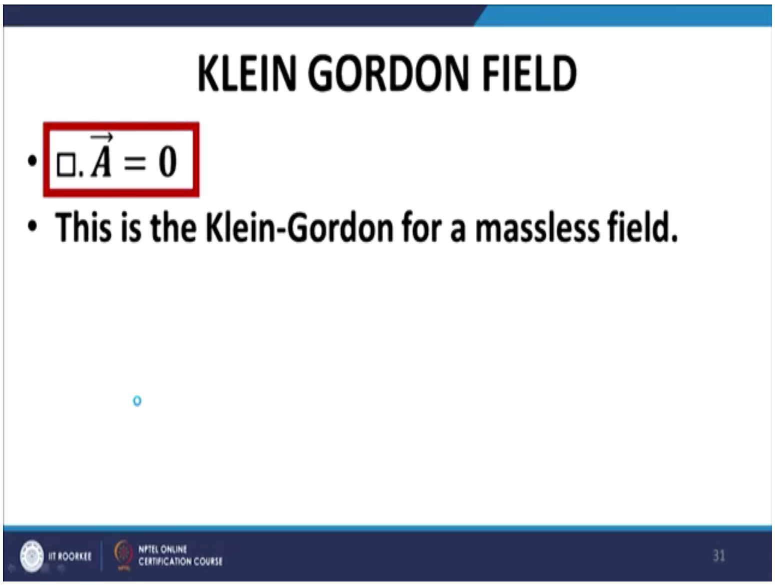

And therefore, we have the box of the spatial A is equal to 0. This is nothing, but similar absolutely similar to the Klein Gordon equation for the massless field.

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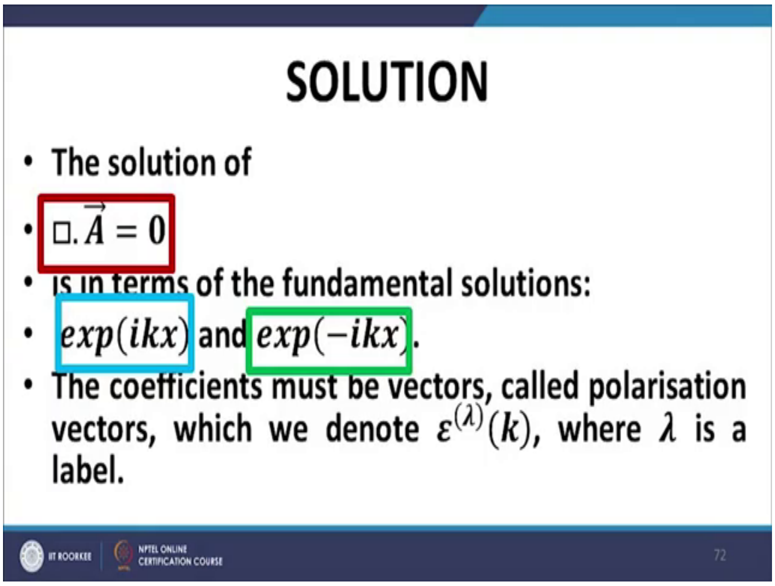

The solution of this expression is given in terms of the fundamental solutions. Exponential wave vector solutions – exponential ikx and exponential minus ikx and these are called and the coefficients of course, must be vectors called polarisation vectors and we denote them by epsilon superscript lambda k, where lambda is a label.

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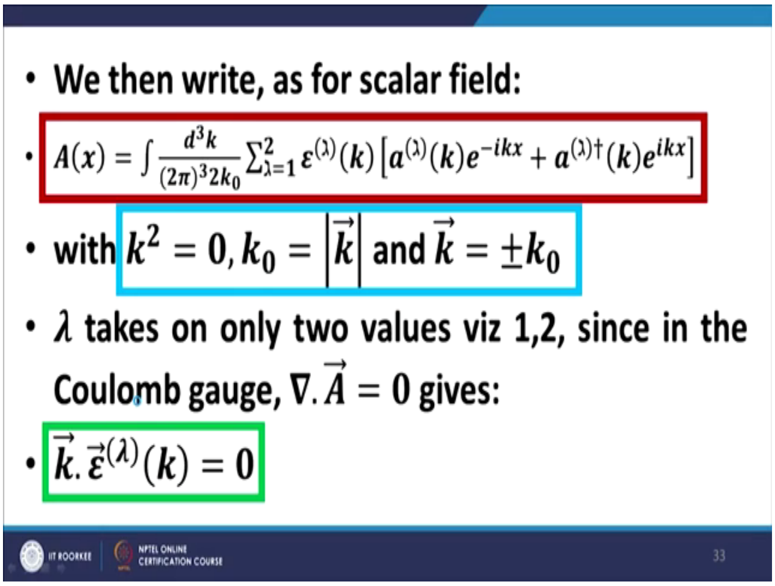

Now, we therefore, we write as in the case of scalar fields we write A x as the expansion in terms of the creation and annihilation operators in the as in the case of the Klein Gordon equation for the scalar field as shown in the red box here.

In this case, because m mass is 0 so, k square will be equal to 0, k 0 will be equal to mod of k and k will be equal to plus minus of k 0 that is a spatial k will be equal to plus minus k 0. And, lambda can take only two values 1 and 2 since in the Coulomb gauge the divergence of a spatial A gives 0 and that gives the condition that required in the green box with k can because k can take only two values therefore, lambda can also take only two values here.

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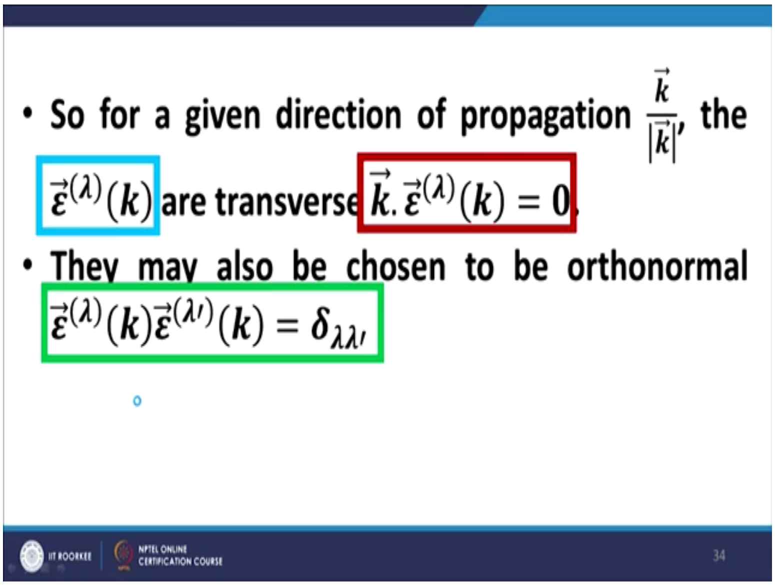

That is compatible with the fact that the photons or the massless spin one particles travel with the speed of light and have only two independent components.

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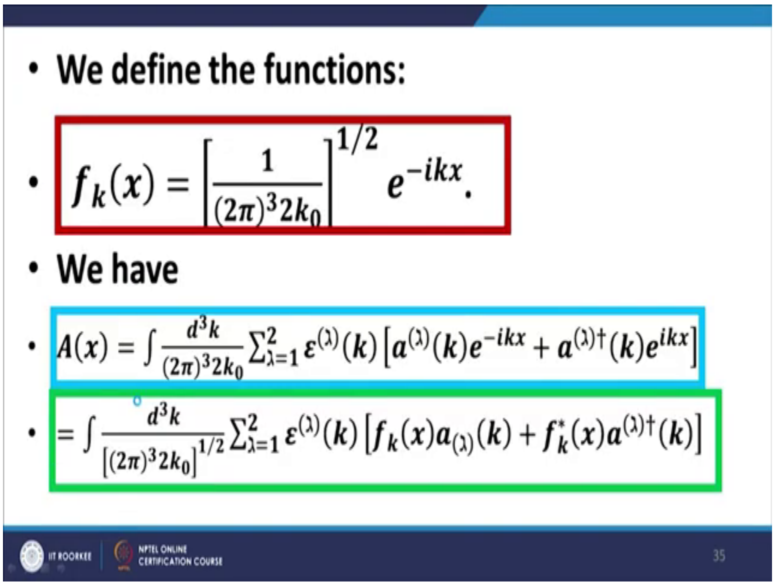

Now, if we define the to get to the get the expressions for the creation and annihilation operators we can define functions in terms of the expressions given in the red box here in which a x can be represented in the expansion given in the green box at the bottom of the slide.

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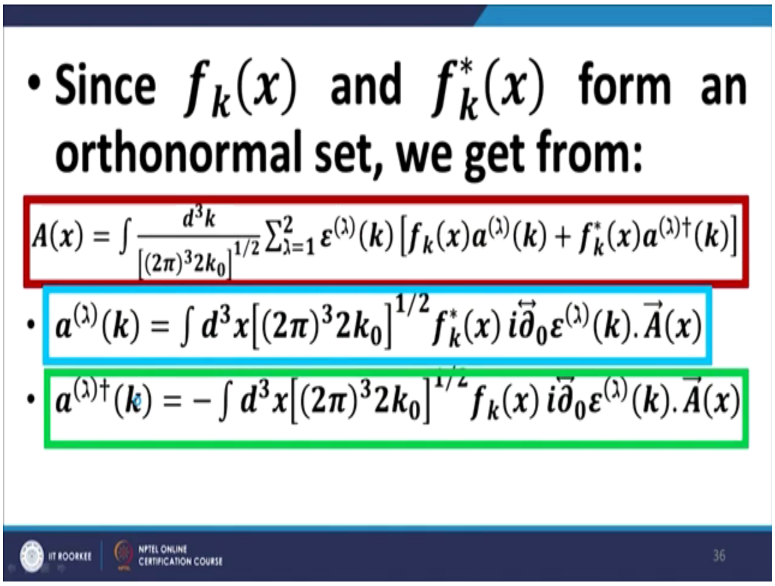

And, because these functions f k x these are there in the red box here are orthogonal orthonormal so, we can recover straight away the expressions for the creation and annihilation operators has the expression given in the blue box and the green box at the bottom of the slide.

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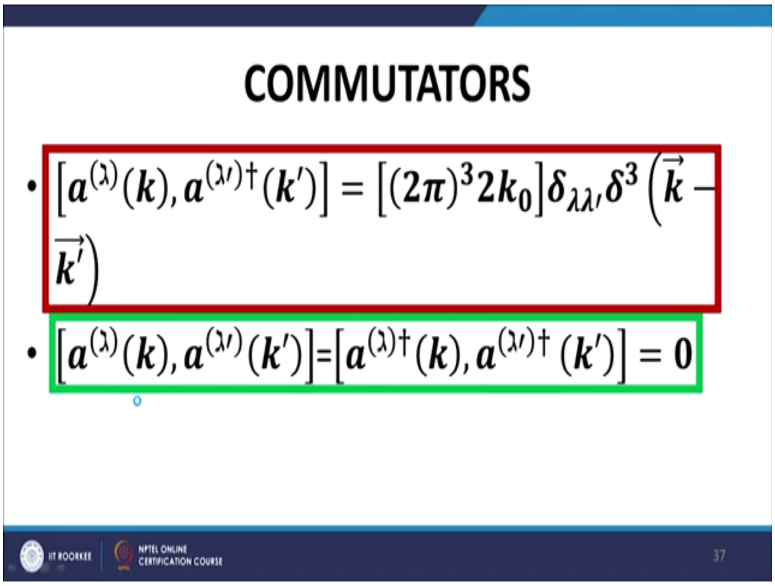

And, the commutators between the creation and annihilation operators are can be easily worked out as straightforward and I given in the this slide.

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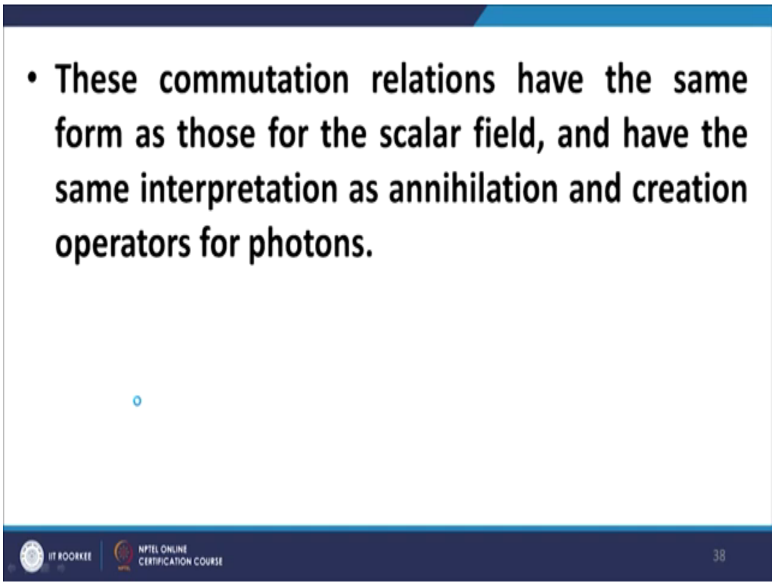

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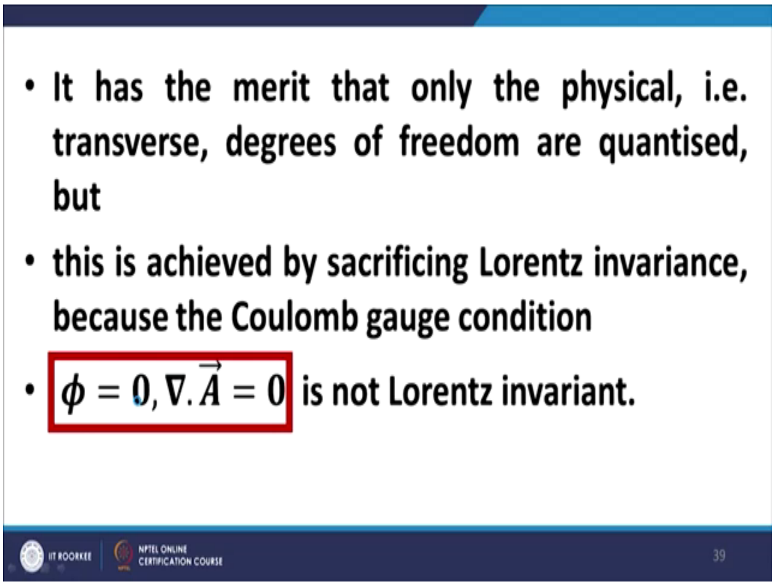

To reiterate the Coulomb gauge which is shown in the red box here is not Lorentz covariant and hence in a sense we lose Lorentz covariant while working in the Coulomb gauge.

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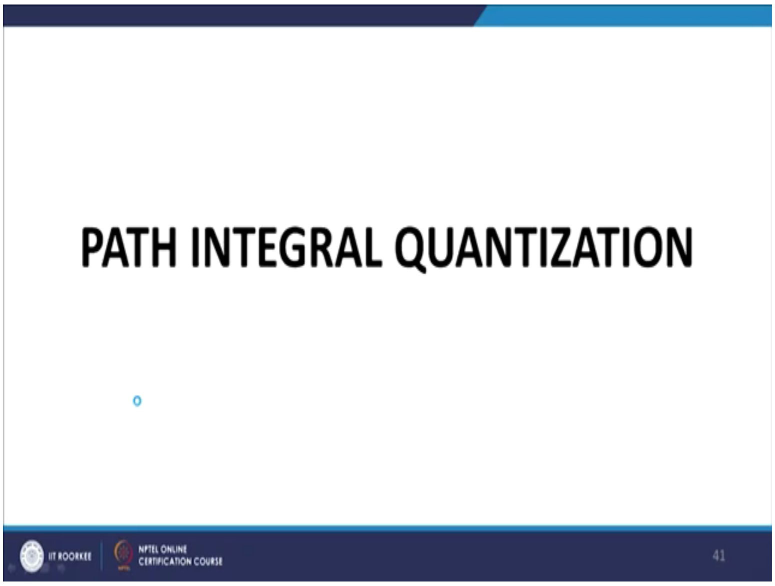

Now, we come to the most important topic here the path integral quantization.

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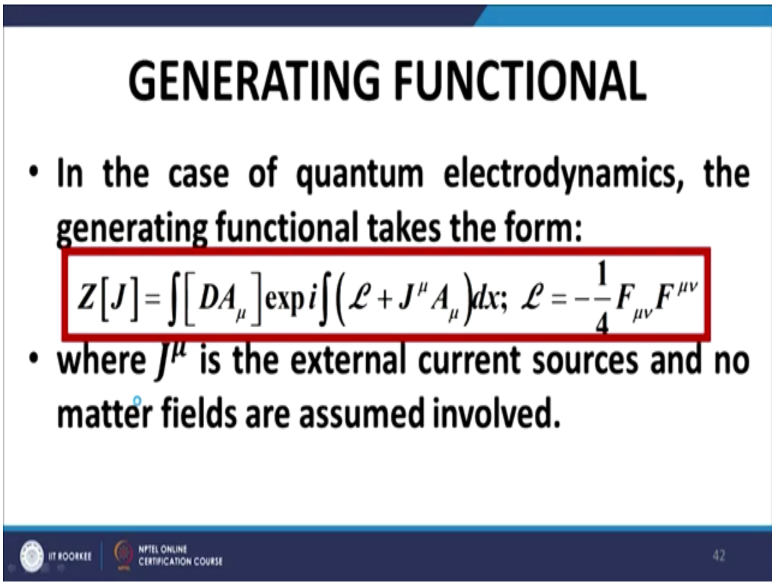

The generating functional in the case of quantum electrodynamics that is the electrodynamics of the electromagnetic field will take the form that is given here where L is the Lagrangian script; L is the Lagrangian which is given by the expression in the right side of the red box and J mu are the respective sources, A mu are the field variables and this Lagrangian or this generating functional has been written in consonants or in line with the expressions with similar expressions written for the scalar fields. And, the particle and the free particle in the quantum mechanical case.

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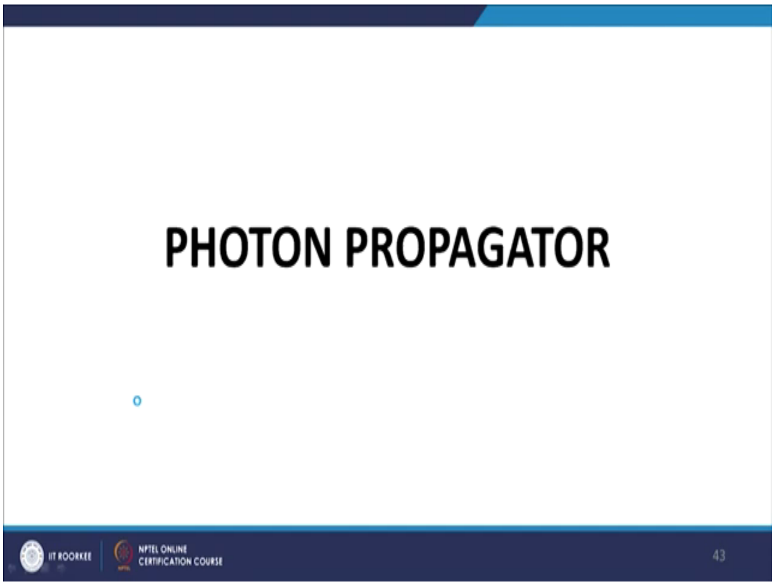

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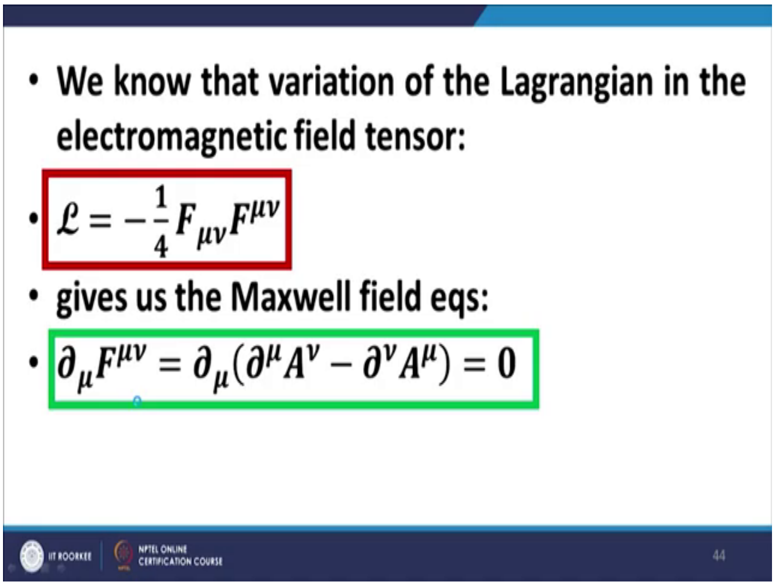

Now, the variation of the Lagrangian in the electromagnetic field tensor is the Lagrangian is this and if you do the variation of this Lagrangian, we can recover the Maxwell field equations which are given here. And, these Maxwell field equations, can also be represented as we have discussed again and again in terms of the electromagnetic field tensor, in this form ;in the form given in the green box here at the bottom of the slide.

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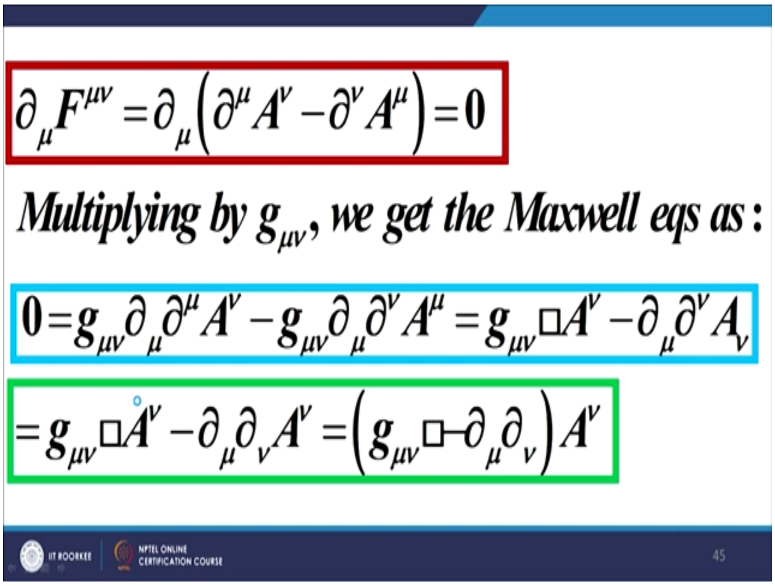

Now, starting from this expression that we brought over from the previous slide if we multiply by g mu nu the metric the flat metric in this case we get please note throughout we are representing the flat metric by g mu nu although in many references and they use eta mu nu. So, but here we are using g mu nu as the flat metric and we are using the convention of plus minus minus and minus.

Now, multiplying by g mu nu we get the Maxwell equations as the expression that is given in the blue box and which can be written in the form which is given in the green box here. Simply some algebraic simplifications multiplying by g mu nu throughout and doing some algebraic simplifications gives us the expression that is given in the green box here at the bottom of this slide; some manipulations of indices, nothing more here.

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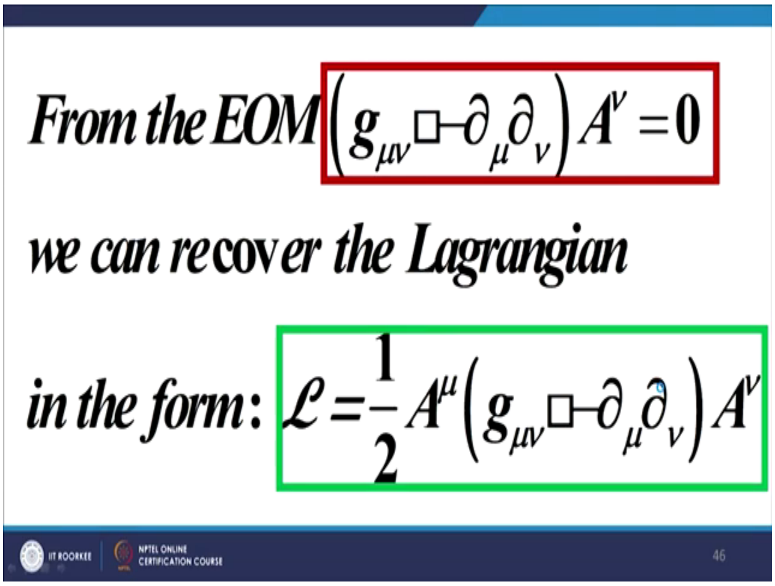

Now, from this set of Maxwell equations which I got from the previous slide, we can recover the Lagrangian in the form which is given in the green box at the bottom of the slide.

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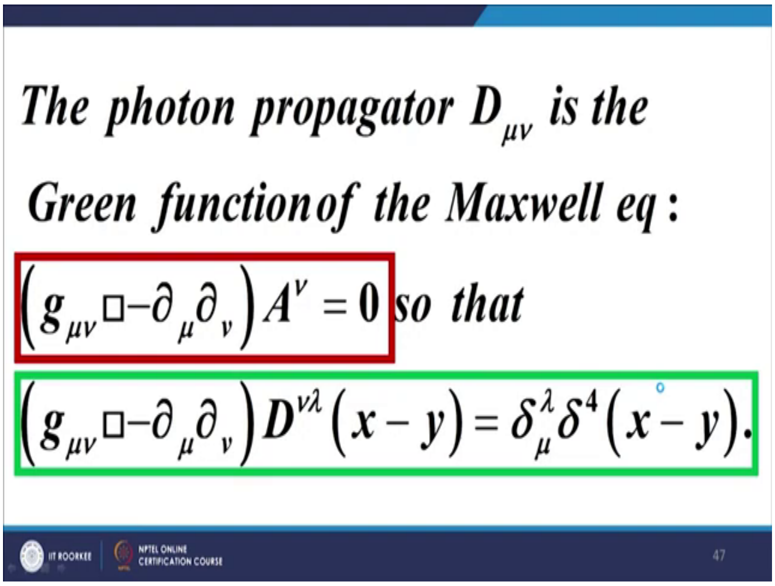

Now, now let us look at the photon propagator; what is the behavior of the photon propagator, what happens to the photon propagator corresponding to the Maxwell equation that is given by the expression here in this green box here.

Now, corresponding to the corresponding to the Maxwell equation that we had in the previous slide that is this one in the green box here the photon propagator should be the green function of the Maxwell equation. And, therefore, we must have the expression here in the green box must be satisfied because this is the Maxwell equation in the red box and my photon propagator D mu nu must be the green function of this particular equation. And, that leads us to the expression that is given in the green box at the bottom of the slide.

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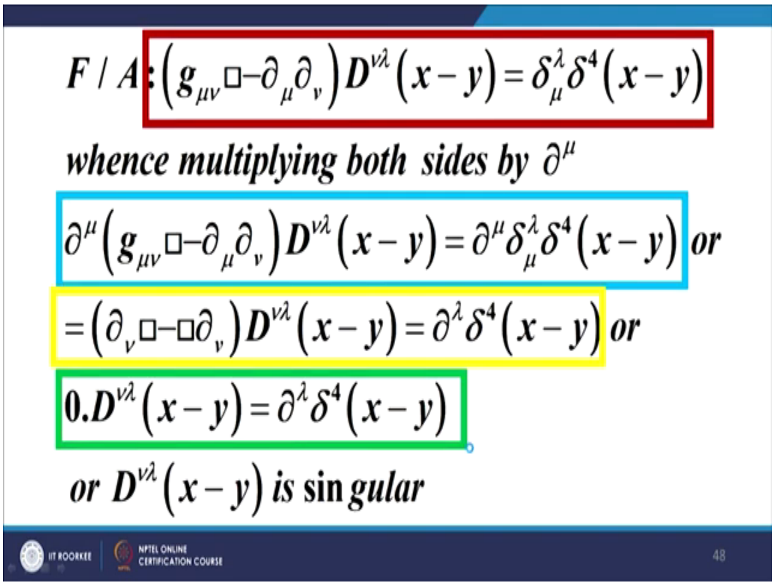

Now, let us explore what happens here. If we multiply both sides by D of nu and mu or we differentiate force as by this way, take the four derivative of both sides. What we get is that again with some manipulations of indices very some careful manipulation of indices, you find that our propagator turns out to be singular because the left hand side coefficient vanishes. The coefficient of the propagator vanishes and although the right hand side does not vanish and that means, that the propagator is singular.

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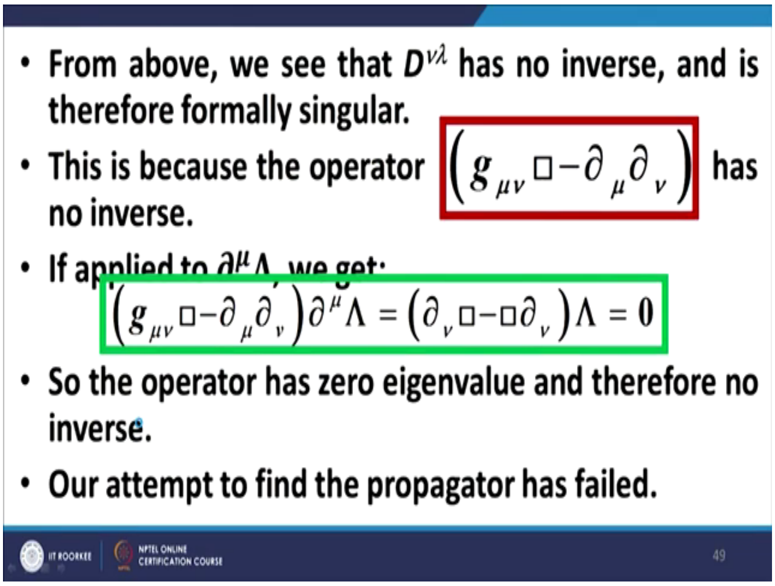

And, this is why does this happen because this operator and that is given in the red box here has no inverse. And, this because this operator has no inverse we have to adopt a certain detour to arrive at the expression for the propagator. We multiply by del mu eta. On multiplying by del mu eta we get 0 which shows which in fact, shows further that the operator has zero eigenvalue and therefore, has no inverse.

So, it is clearly it is explicitly shown by the expression in the green box that this operator here in the red box has zero eigenvalue and hence does not have any and does not have any inverse. So, what do we do?

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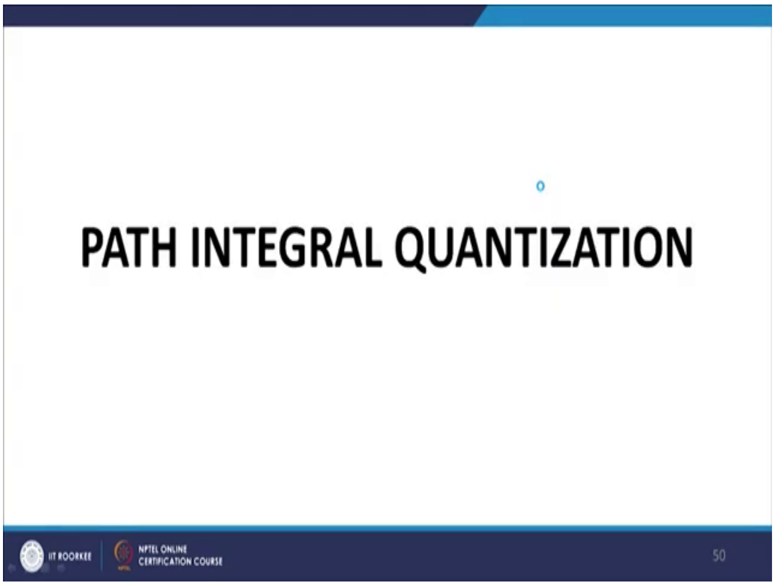

Now, why is this happening first and then what do we do?

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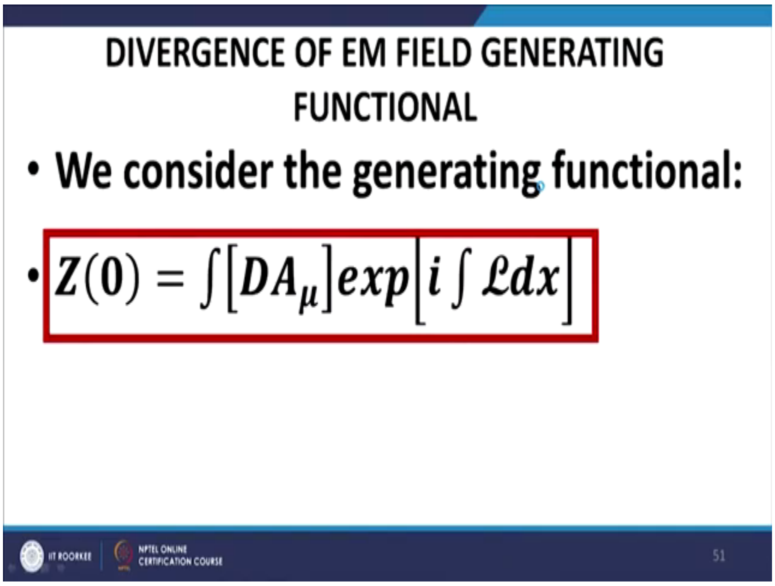

We have the expression for Z 0 that s the normalizer as the expression given in the red box.

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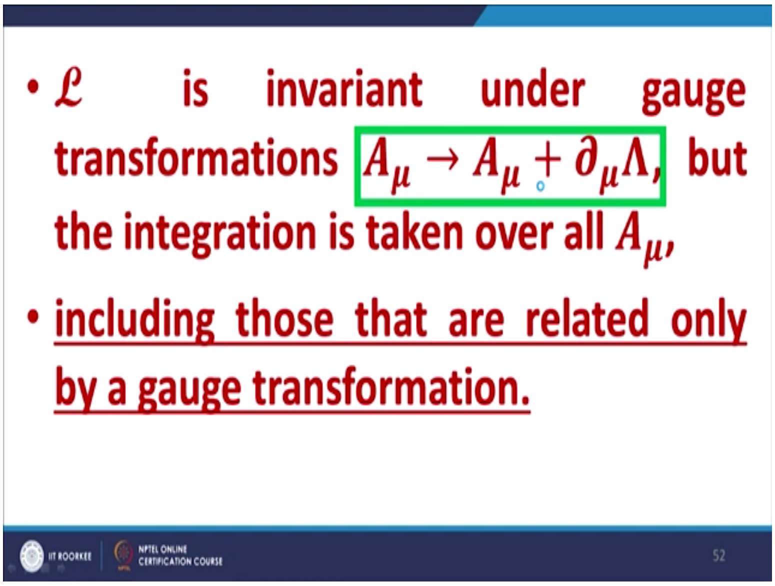

Now, the important thing is that as we mentioned earlier the entire theory is invariant under gauge transformations of the form that is given in the green box. Now, but when we are working out this integration this integration is taken over all of A mu including those including those which are related to each other by this gauge transformation given in the green box.

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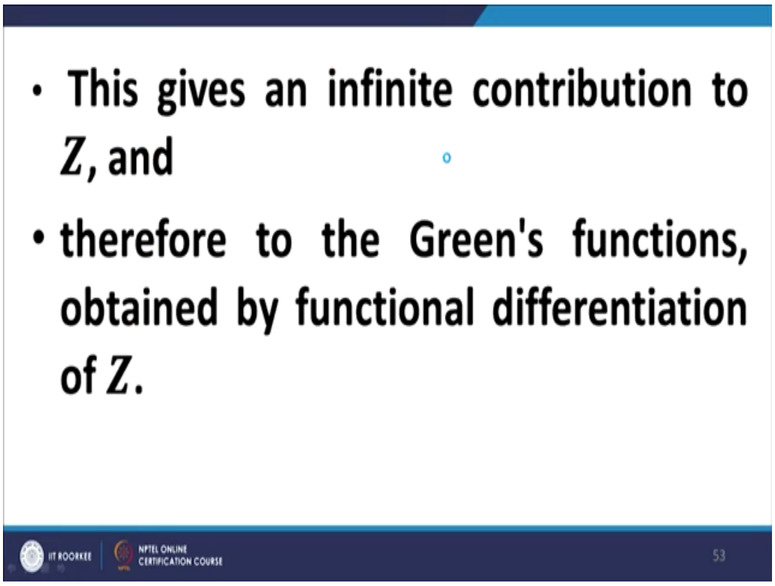

In other words, there is some sense of over counting going on and as a result of which the contribution to Z is infinite because particular potential would be counted again and again because of the gauge freedom that is allowed between or in defining a potential A. And, therefore, the in the this particular problem will give an infinite contribution to Z and as a result the green functions that could be obtained from this particular path integral or the generating functional will also be infinite.

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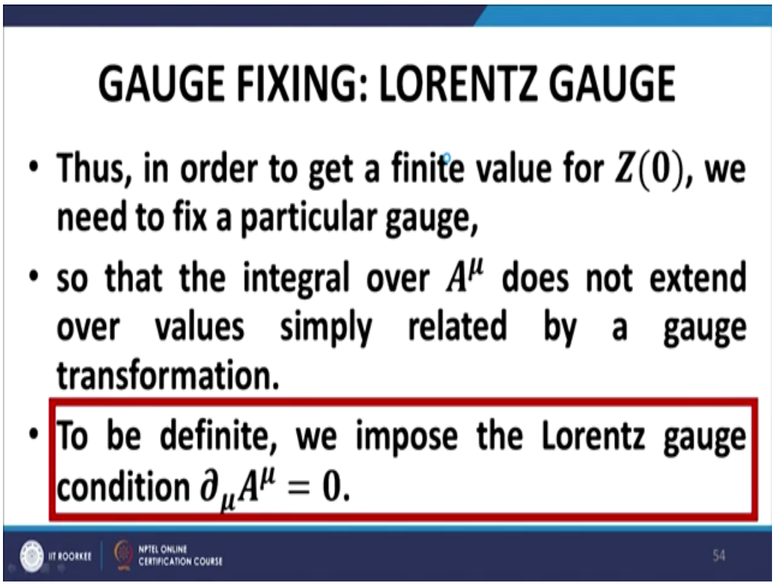

So, we need to do gauge fixing here as we did in the in earlier case we need to do a gauge fixing before we do the path integral quantization. As we discussed in the gauge fixing that is normally done as the Lorentz gauge and followed by the Coulomb gauge. So, what we do is we impose the Lorentz condition given in the red box here.

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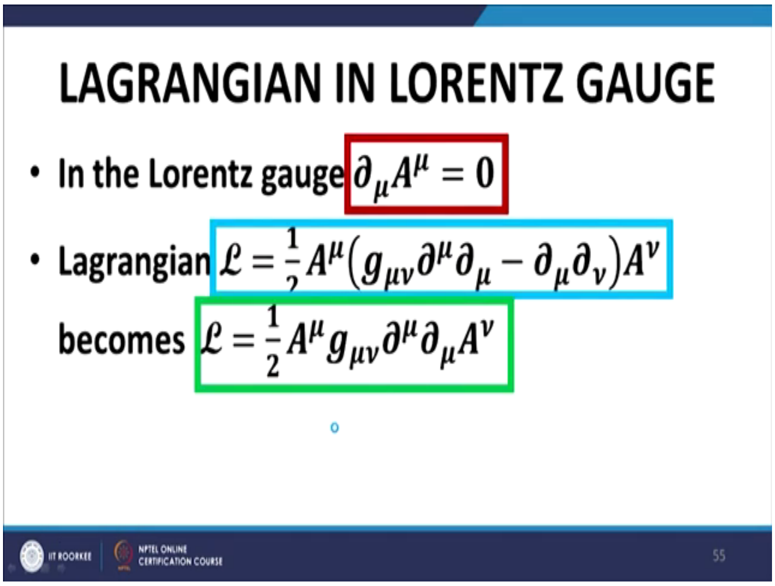

And, now when we impose this Lorentz gauge when we imposes this Lorentz can condition given in the red box here the Lagrangian becomes the second term in the Lagrangian if you look at it the second term in the Lagrangian vanishes and we can write the Lagrangian in the form which is given in the expression in the green box here.

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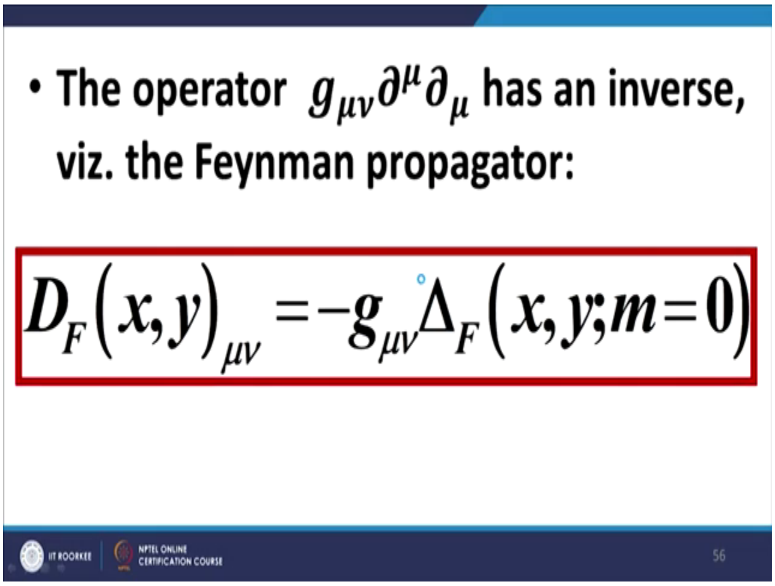

Now, the operator in contrast to the earlier situation if you look at this operator g mu nu d mu d mu or g mu mu box has an inverse and it and the universe is given by the Feynman propagator. You can see that in the red box here at the in the slide.

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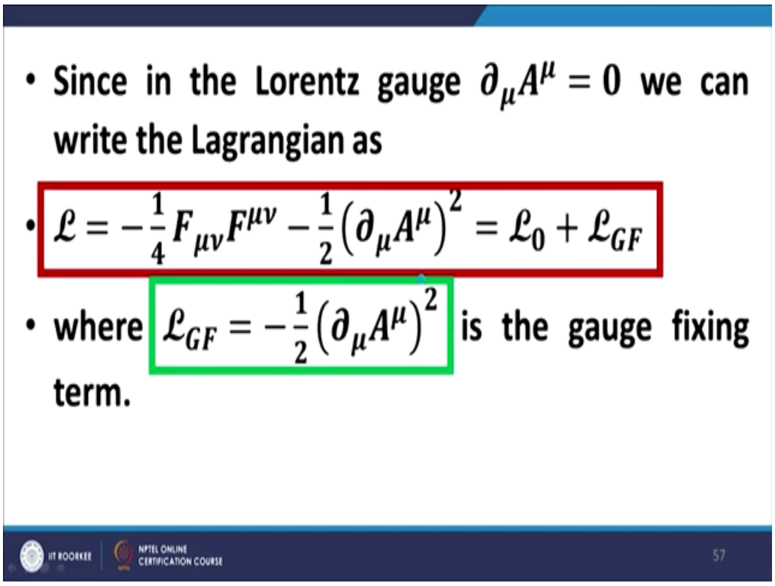

So, by imposing the Lorentz gauge we can write the Lagrangian this form and we now have a nonsingular propagator. And, furthermore because this expression in the round brackets is equal to 0 in the Lorentz gauge, the Lagrangian can be modified and it can be written in the form that is given in the red box here where the first term is L 0 L script 0 which is the Lagrangian of the free field.

And, the second L script GF the subscript GF represents gauge fixing in other words G L script GF represent the gauge fixing term in the Lagrangian.

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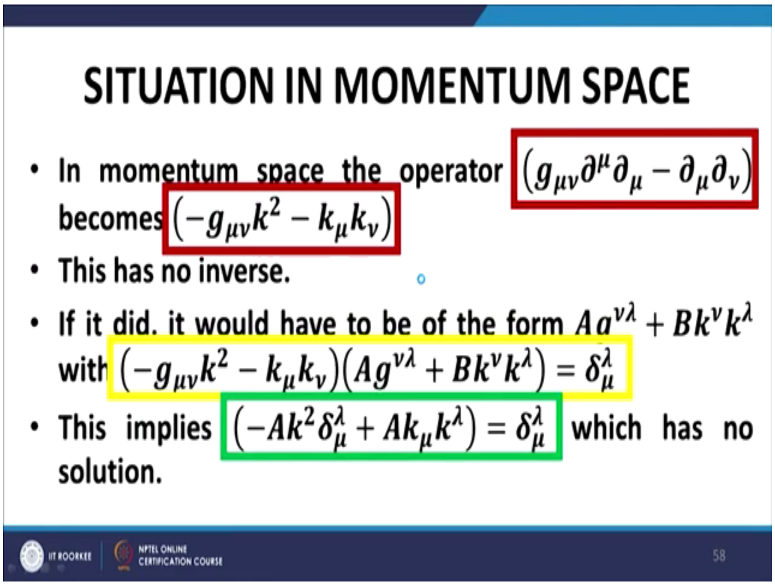

In momentum space also we have a similar catastrophic, similar problem in the momentum space this becomes the expression that is in the red box here. And, again this expression is no inverse in the momentum space and that is established in the in this slide by because if there were to be exist an inverse it would be of the form a capital A into A v mu nu plus B k nu k lambda.

And, on to taking the products of this inverse and the original operator the we should have delta on the right hand side but, this equation has no solution. So, the net outcome of this is that the expression that the expression that is there in the red box does not have a inverse.

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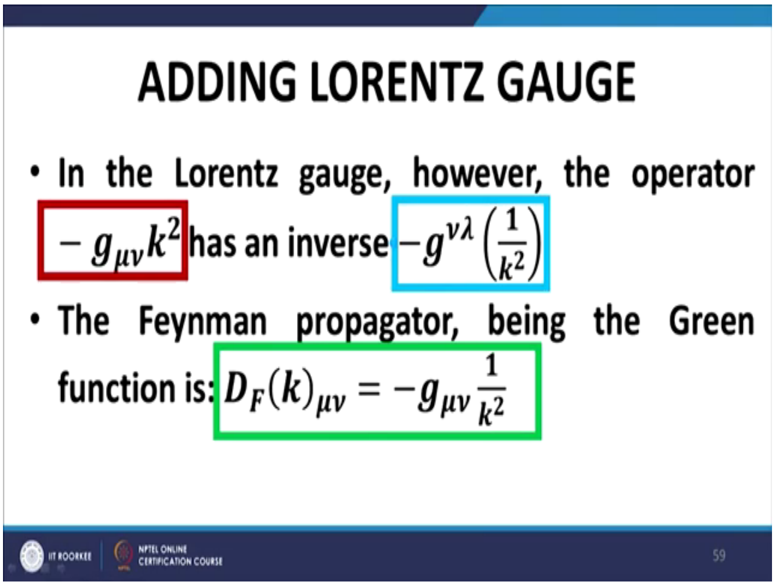

In the Lorentz gauge the operator minus g mu nu k square that is the first term if you can eliminate the second term by the by introducing the Lorentz gauge the first term as the in the inverse given in the blue box. And the Feynman propagator being the green function is given is represented by the expression that is given in the green box here.

Of course, we can generalize what we have done earlier we may add to the Lagrangian an arbitrary amount of the gauge fixing Lagrangian instead of having L gauge fixing we can have a coefficient a scalar coefficient attached to it and we can write the Lagrangian in the form in a generalized form which is given in the red box here.

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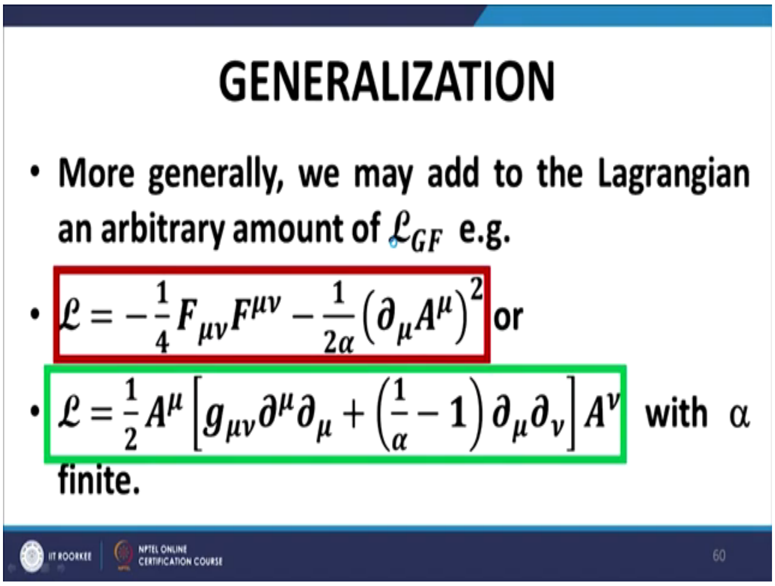

And, which gives us the expression given in the which is equivalent to the expression given in the green box here where alpha is a finite expression.

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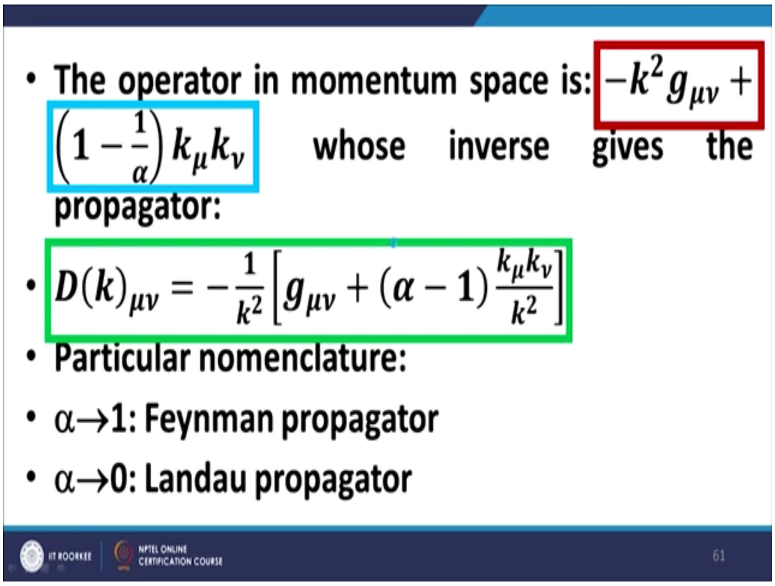

The corresponding expression in the operator in momentum space is the expression that is given in the red and green box here red and blue box here and the inverse is given by the expression given in the green box here.

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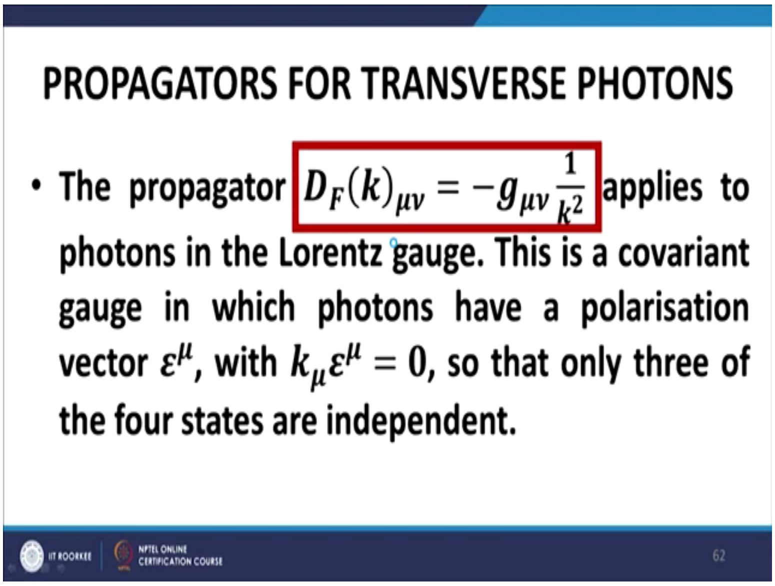

Now, we come to the transverse photons. Now, this propagate we have not; we have not introduced the Coulomb gauge yet, now let us talk about the Coulomb gauge.

The propagator given by this expression which was which is obtained after putting in the or after introducing the Lorentz gauge this is a now the Lorentz gauge is a covariant gauge where the polarization vector, epsilon mu has the scalar this scalar product between the wave vector and the polarization vector equals 0. And therefore, only three states out of the four are independent.

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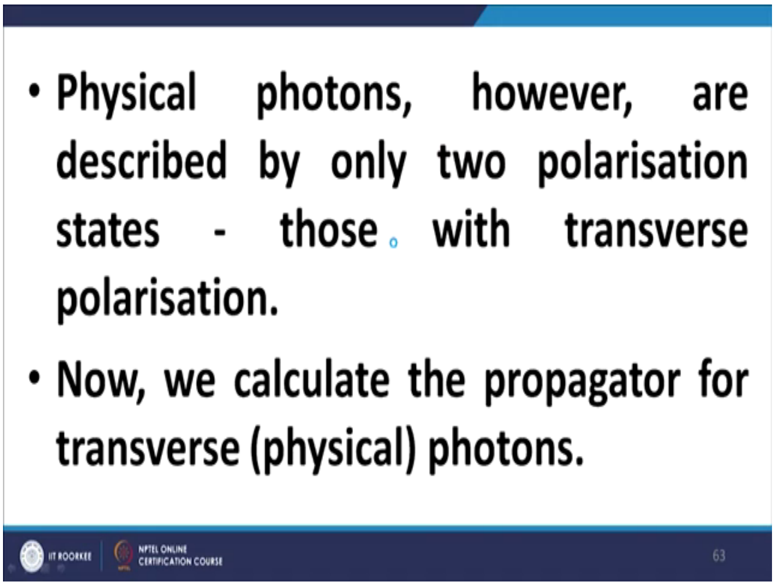

But, as we know that there are only two independent states of the photons and therefore, we are still a little bit away from getting a physically viable expression for the propagated representative of physical photons.

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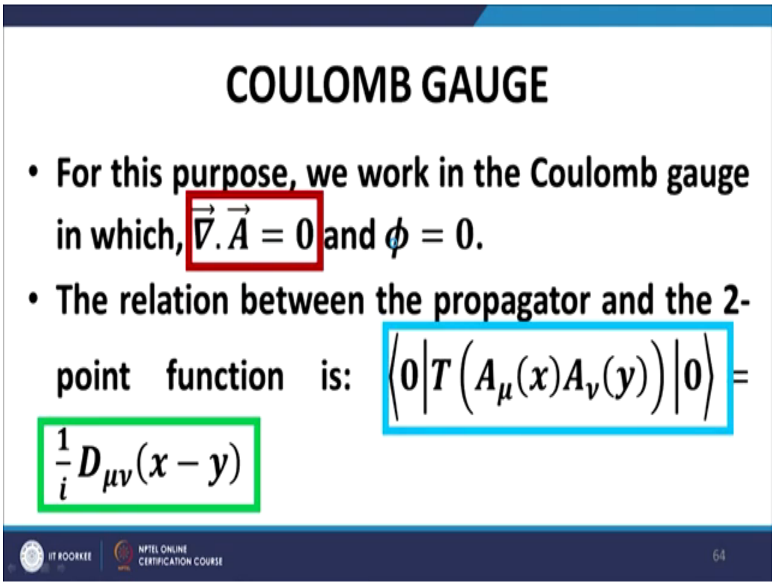

For this we introduced the coulomb gauge. As we discussed earlier the coulomb gauge represents the is represented by the time component of the 4-vector potential being zero and the divergence of the spatial component of the 4-vector being zero. And, the relation between the propagator and the now the relation between the propagator and the 2-point function is given by this expression.

This is the 2-point and time ordered product of the field variables you remember in the electromagnetic field the as are taken at the field variables. So, there is the expression in the blue box is the time ordered product of the field variables and on the right hand side we have the expression for the propagator.

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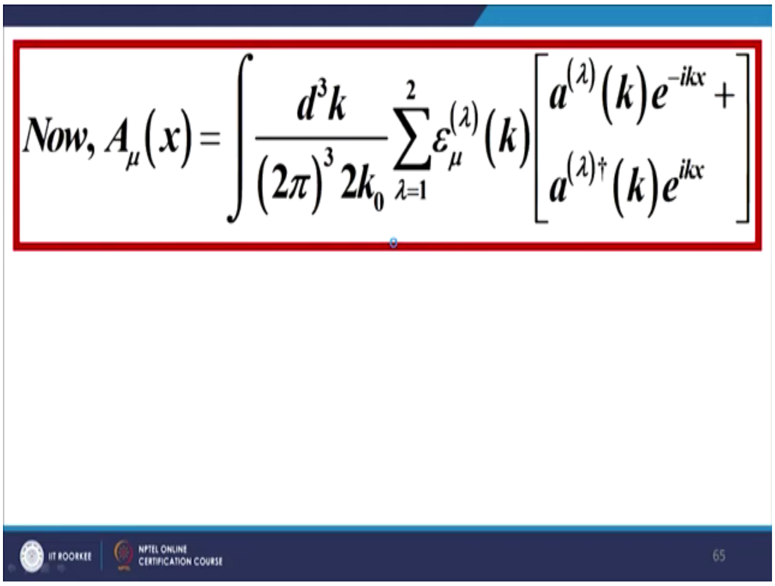

Now, you would recall that we had obtained the expression for A mu of x earlier in this lecture in the form that is given in the red box here and it is in terms of the wave free waves or e to the power ikx and e to the power minus ikx with the polarization vectors.

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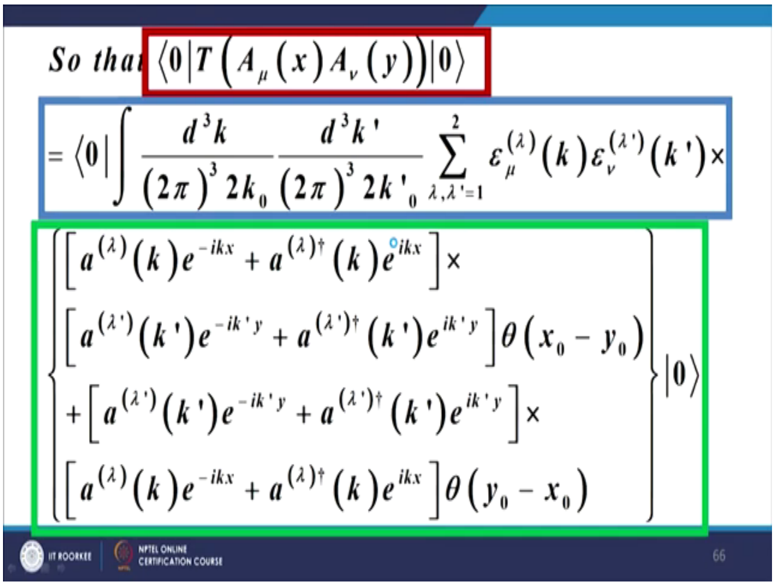

So, what do we have? We have on the one hand when we introduce these expressions in the expression for the time ordered product when we introduce the expressions for A mu x and A nu y when we introduce these expressions in the form which is given in the slide in the red box here in the slide and we take note of the time ordering represented by the theta functions the (Refer Time: 29:14) functions what we get is the expression in the green box here.

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$$=\langle 0|\int \frac{d^3k\,d^3k'}{(2\pi)^6\,2k_0\,2k'_0}\sum_{\lambda,\lambda'=1}^{2}\varepsilon_{\mu}^{(\lambda)}(k)\,\varepsilon_{\nu}^{(\lambda')}(k')$$

$$\times\left[\begin{array}{l} a^{(\lambda)}(k)\,a^{(\lambda')\dagger}(k')\,e^{i(k'y-kx)}\,\theta(x_0-y_0) \\ +a^{(\lambda')}(k')\,a^{(\lambda)\dagger}(k)\,e^{i(kx-k'y)}\,\theta(y_0-x_0) \end{array}\right]|0\rangle$$

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So, this is simplified and what we get is the expression in the in this whole slide. This entire slide is the simplification of what we had in the previous slide in the green box here at the slide.

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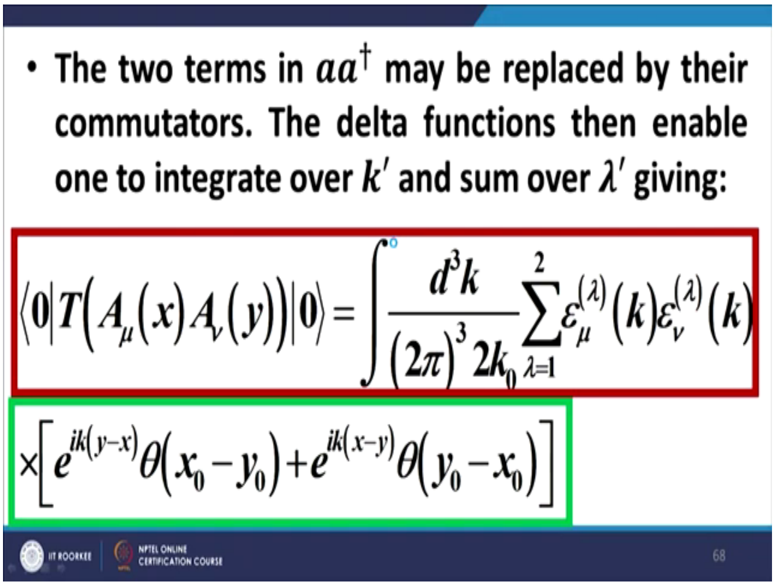

Now, the terms in aa dagger, aa dagger can be replaced by their commutators and then delta functions can be introduced. And, delta functions after introductions they can be integrated over and once they are integrated over and then there are summed over lambda.

So, there are a number of steps involved here the terms involving a is a dagger you replace them by the commutators, and replacing them by the commutators will involve introduction of delta functions; integrate over these delta functions and then sum over lambda and the expression that we get is the expression that is here in the slide.

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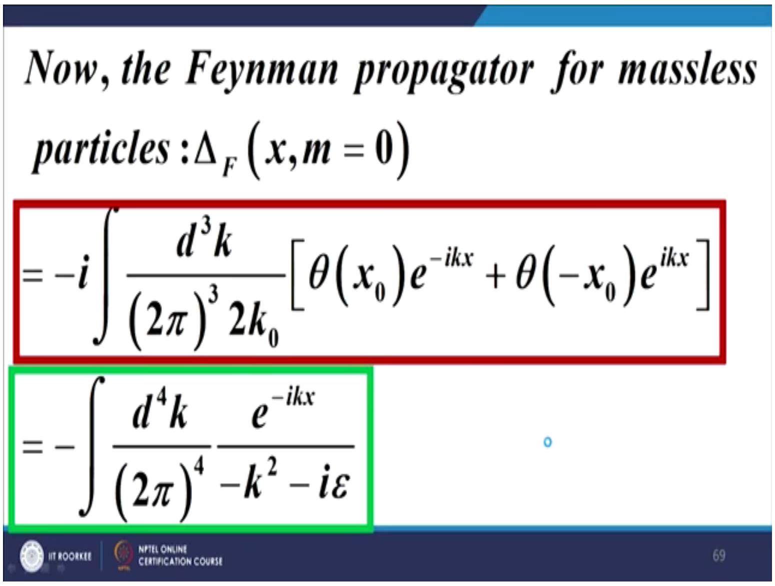

The Feynman now the Feynman propagator for the mass less particles has the expression which is given here for y equal to 0 and a given x it is given by the expression in the green box right at the bottom of the slide. So, we now have two expressions connecting the time ordered product and Feynman propagator.

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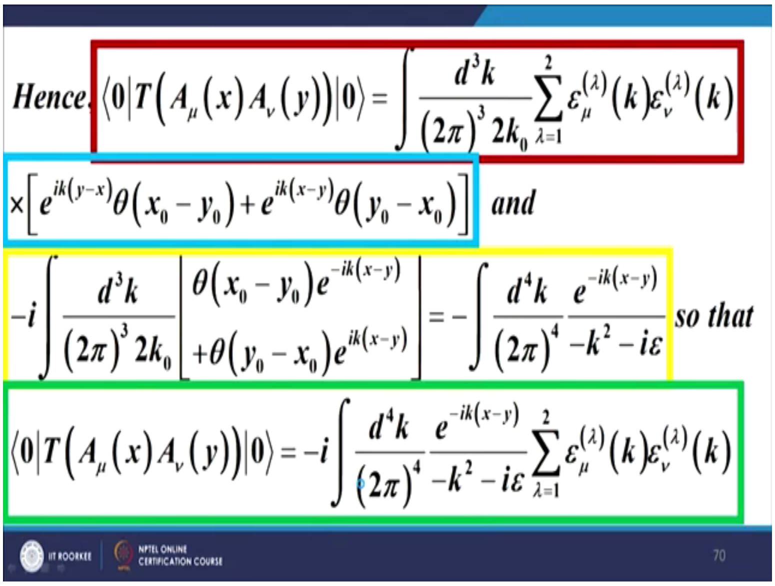

Both of them are here on the slide. On the one hand the time ordered product has been represented in the form given in the first equation. In the second hand the right hand side of the time order product has is related to the expression given in the yellow box here through the through the Feynman propagators expression.

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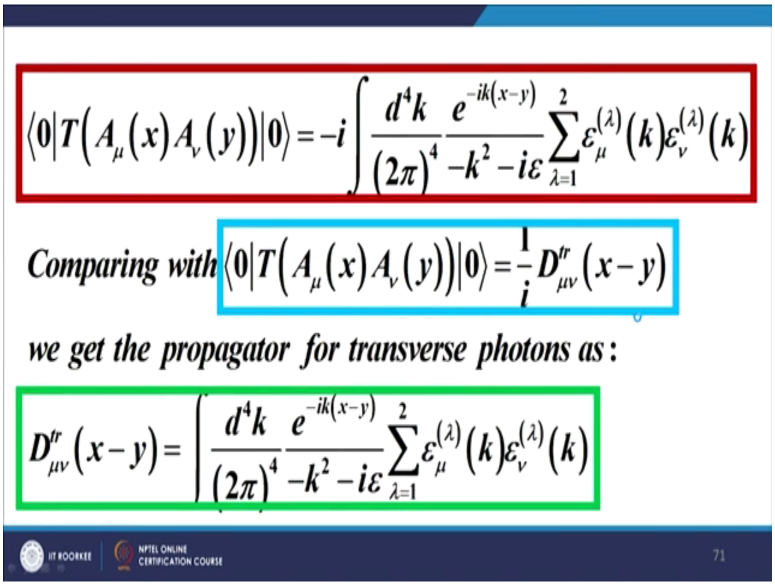

So, combining the two and simplifying and using the explicit expression for the Feynman propagator we will get the expression that is given in the green box at the bottom of the slide for the for the propagator for transverse photons.

So, with that we conclude this segment relating to the path integral formalism of quantum mechanics and quantum field theory. From here on, we will be taking up the final segment of this course that will relate to the financial applications of the path integral formalism.

Thank you.