Path Integral Methods in Physics & Finance
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Lecture – 51
Gauge Fields (1)

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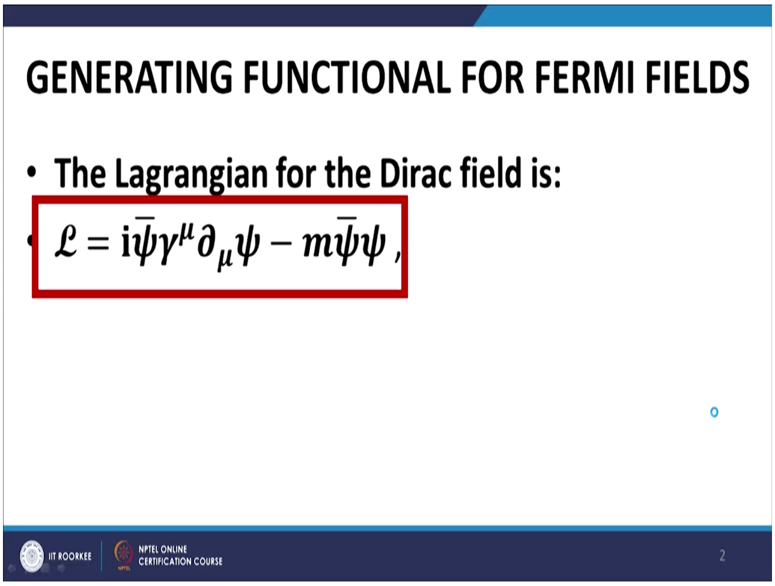

Welcome back. In the last lecture, I started discussing about the Dirac Fermi Path Integral formalism. So, let us continue from where we left off in the last lecture. We started with the Lagrangian for the Dirac field as the expression that is given here in the red box.

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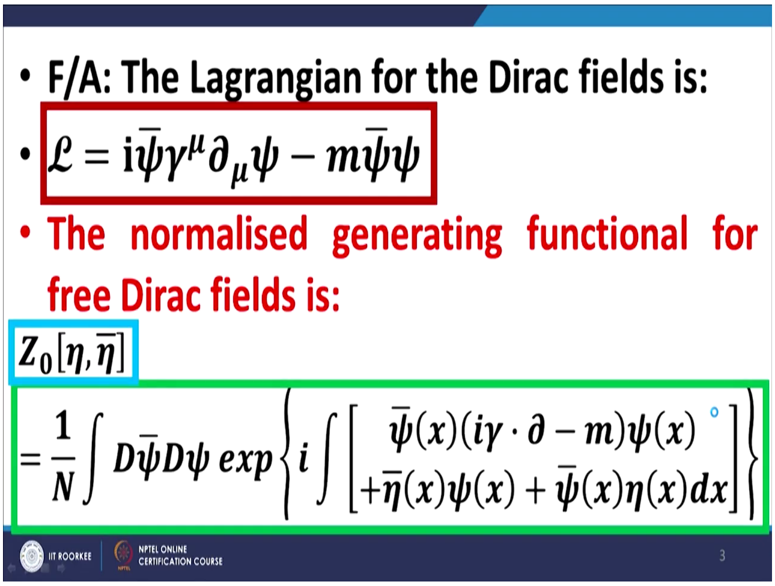

And on that basis on the basis of this Lagrangian we derive we obtained the path integral the generating functional for the Green functions full Green functions for the free Dirac field as the expression that we have in the green box at the bottom of this slide. This is the normalized generating function with the normalization factor being represented by 1 by n.

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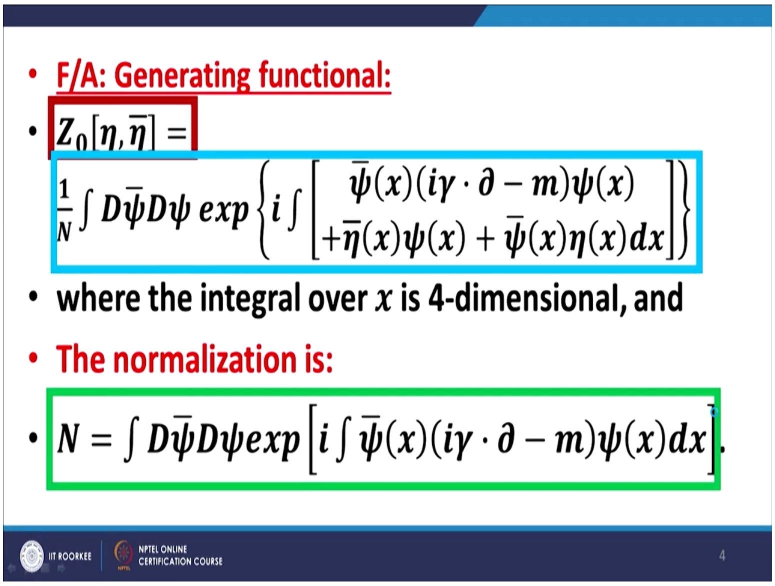

And the normalization is represented by the expression that is given in the green box at the bottom of your slide. It is obviously obtained by omitting the source terms.

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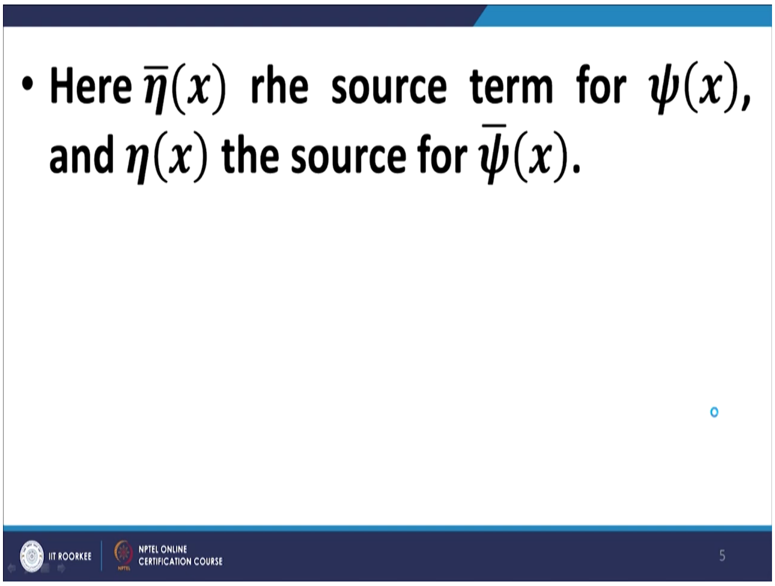

And in this particular path integral the generating functional eta bar x is the source term corresponding to psi x and eta x is the source term corresponding to psi bar of x.

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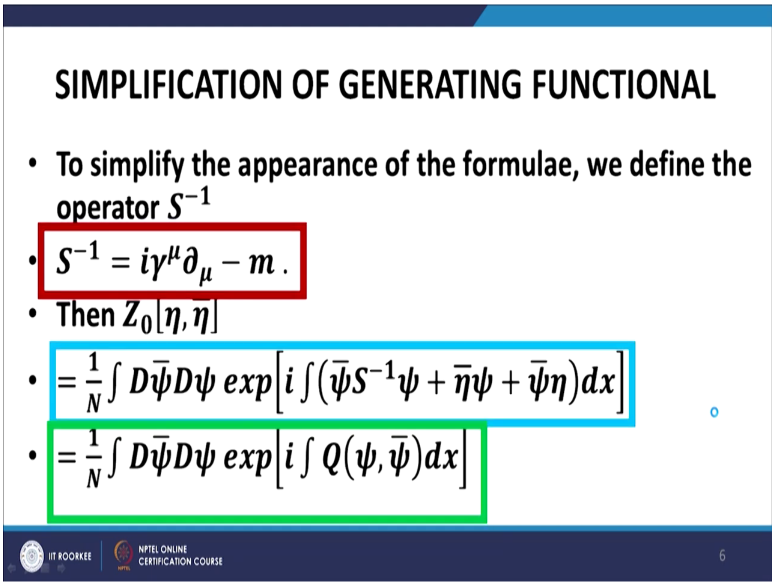

Now, we do the simplification of the generating functional. We have got the definition of the generating functional. Our objective now is to arrive at a simplified version of the generating functional. For this, we introduce the matrix S inverse which is given by the expression in the red box and in terms of this matrix S inverse the generating functional takes the form given in the blue box in the middle of your slide. And, we abbreviate the expression in the round brackets as Q of psi and psi bar.

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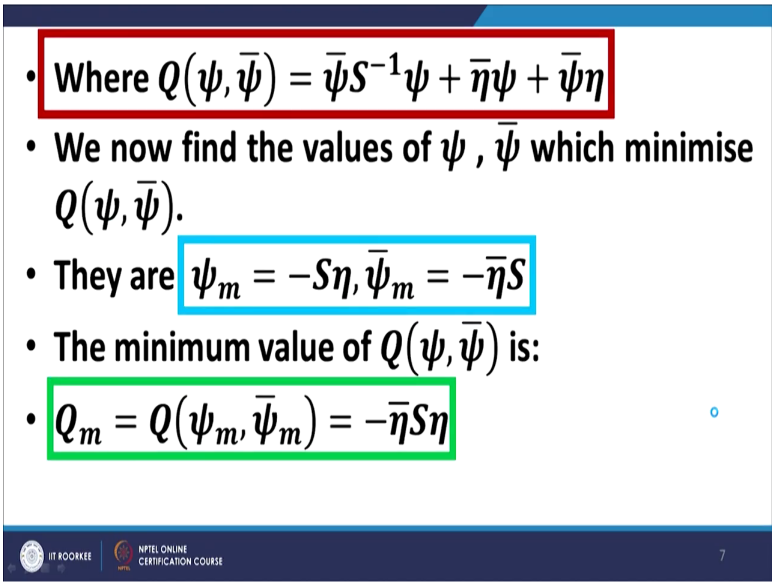

Where q psi and psi bar are represented by the expression, that is given in the red box here. Now, what we do is we find those values of psi and psi bar which minimize the value of Q. In other words, we minimize Q subject to or with respect to psi and with respect to psi bar.

It is quite a very elementary exercise and the results that we get is that psi minimum is equal to minus S eta and psi bar minimum is equal to minus S bar minus eta bar S. And the value of the minimum Q that is Q m is equal to minus eta bar S eta. This is can be done can be obtained by taking the derivatives and equating them to 0 with respect to the derivatives of Q with respect to psi and psi bar and then equating them to 0 we get this result straight away.

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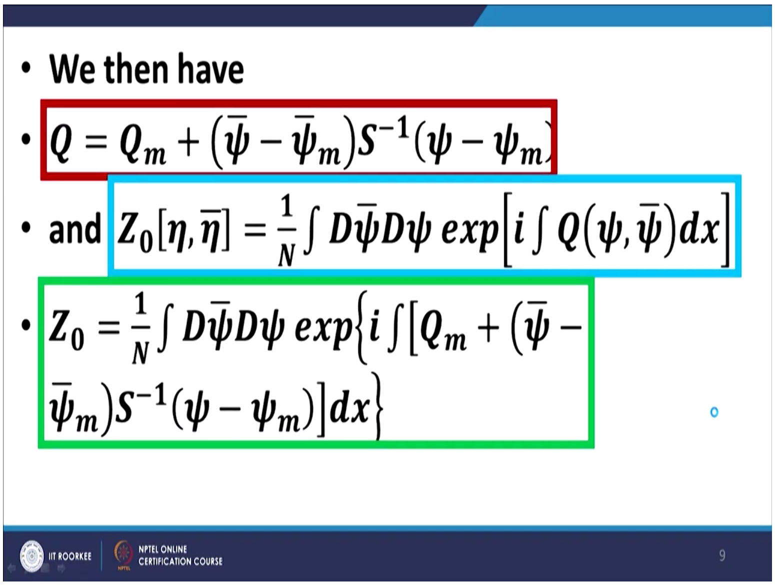

Now, we expand Q in around Q m and expanding Q around Q m we get the expression that is given in the red box at the top of the slide. And, in terms of which when we write the generating functional we get the expression that is here in the green box at the bottom of this slide.

Now, please note this point here and that the Q m term is coming out separately and the other terms the psi and psi bar dependent terms are coming separately in this particular expression for the generating functional. And, also note the fact that as we have just shown Q m is independent of psi and psi bar. So, this enables us to simplify this expression as we shall see in the next slide.

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- $$Z_0 = \frac{1}{N}\int D\bar{\psi}D\psi\, exp\left\{i\int[Q_m + (\bar{\psi} - \bar{\psi}_m)S^{-1}(\psi - \psi_m)]dx\right\}$$
- $$= \frac{1}{N}exp\left(i\int Q_m dx\right)\int D\bar{\psi}D\psi\, exp\left\{i\int\left[(\bar{\psi} - \bar{\psi}_m)S^{-1}(\psi - \psi_m)\right]dx\right\}$$

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When we do the simplification what we can do is we take this expression in the exponential of i integral Q m. We can take this outside the integral sign. Why? Because as I mentioned just now Q m is independent of psi bar and psi which are the integration variables and therefore, Q m being independent of the integration variables we can take this term involving Q m outside the integral.

And the expression becomes what we have in the green box at the bottom of your slide please note and this exponential i integral Q m dx now appears as a prefactor to the path integral.

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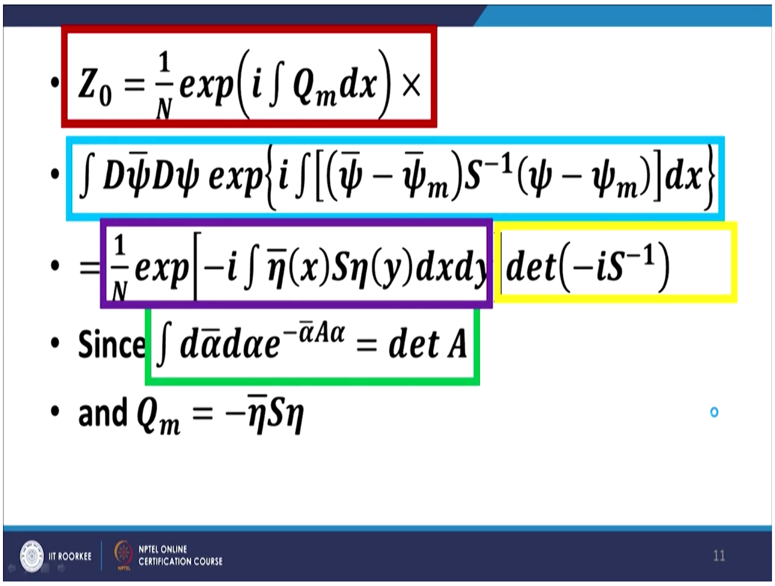

Now, our next exercise is to right. Now the there are two simplifications involved here, 2 steps involved here. The first step we write Q m in terms of it is value that we obtained earlier minus eta bar S eta. You will recall that is the value that we obtained when we minimized Q with respect to psi and psi bar.

These were the values of Q m which represent the minimum value that is the value of Q m and that we have retained in the blue square here and the second part is the simplification of the path integral. Using this property that is here in the green box that we derived in the earlier lecture, by using this property we are able to simplify this path integral and this path integral takes the simple form of determinant of minus i S inverse.

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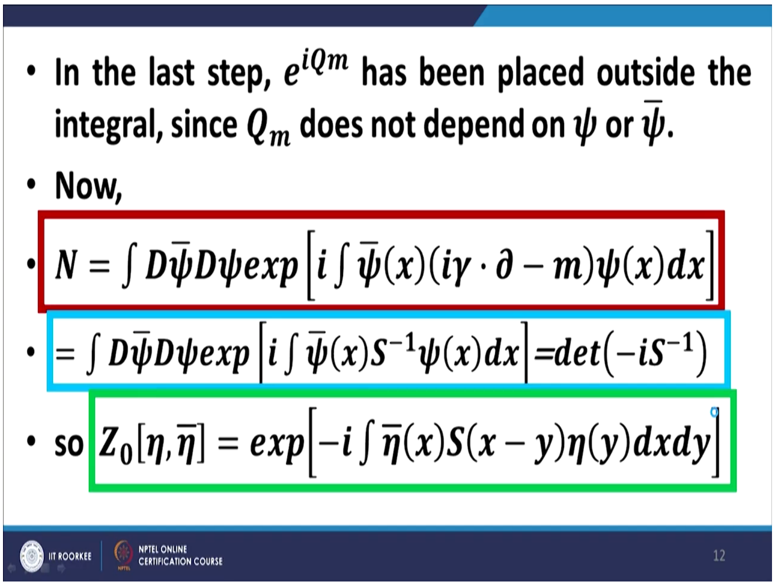

So, we now have now we look at the normalization. In the normalization the, the source terms are absent. Otherwise situation remains more or less the same and because the source terms are absent we can write this in the form of psi bar S inverse psi as the term in the exponential.

And, when you use the same formula that we had used earlier let me go back to the previous slide this particular formula for Q m minus. Sorry, this formula that is there in the green box here integral D alpha bar D alpha exponential of minus alpha bar e alpha is equal to determinant A. When I use this formula and substitute in this expression which is here in the blue box here the expression that i get is determinant minus i S inverse. So, now we have got two results.

We have got one result n is equal to determinant minus i S inverse and the other result that we have here is in the earlier slide and that has given as this expression which is here in the purple and the yellow slides together purple and yellow boxes together. So, using these two expressions if you see the factor of determinant minus i S inverse is common to both the expressions.

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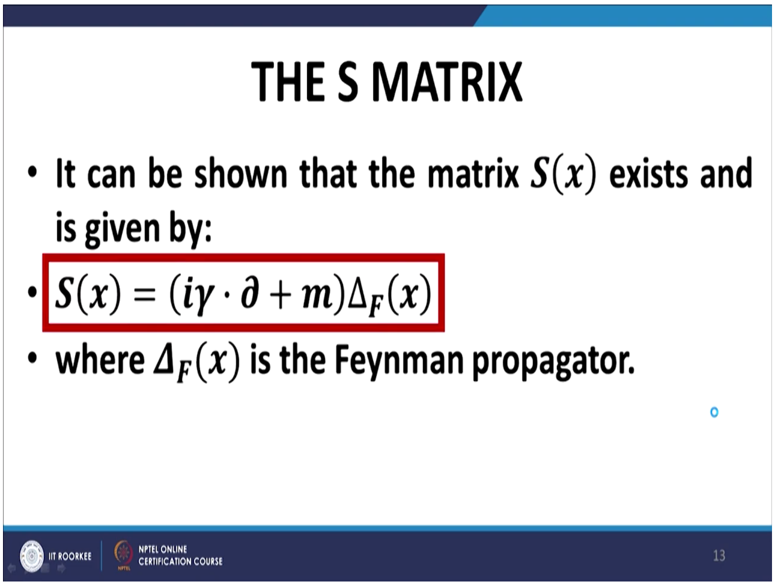

And, that gives us when this expression is eliminated because this appears in the numerator and in the denominator. What we are left with is the generating functional takes the very simple form which is given in the green box right at the bottom of your slide.

The normalization and the normalization factor and the same determinant expression that appears in the numerator they cancel out and whatever remains in the numerator represents our generating functional for the pre Dirac field. Now, let us see let us explore the S matrix the S inverse matrix that we have introduced for as a simplification tool so far. Let us explore that particular matrix; we write the S matrix no not the S inverse matrix. This is the S matrix we write the S matrix in the form which is in the red box here.

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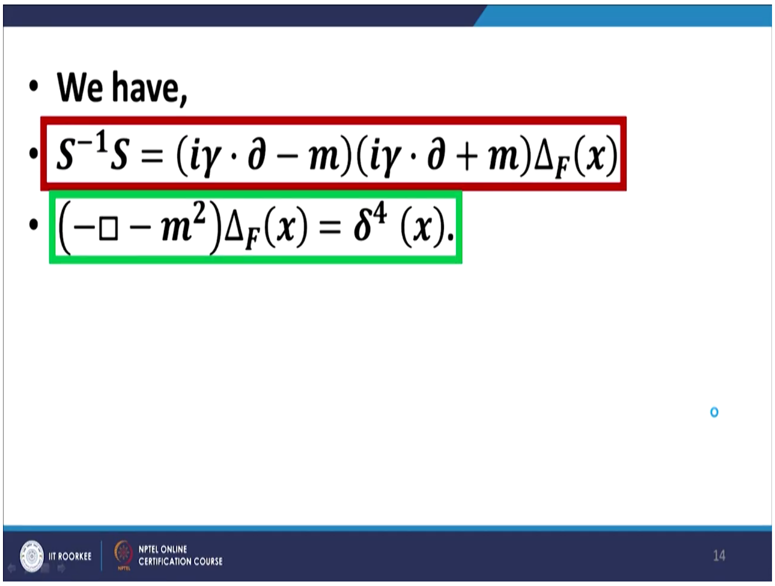

And this is nothing, but the Klein Gordon operator or the negative of the Klein Gordon operator.

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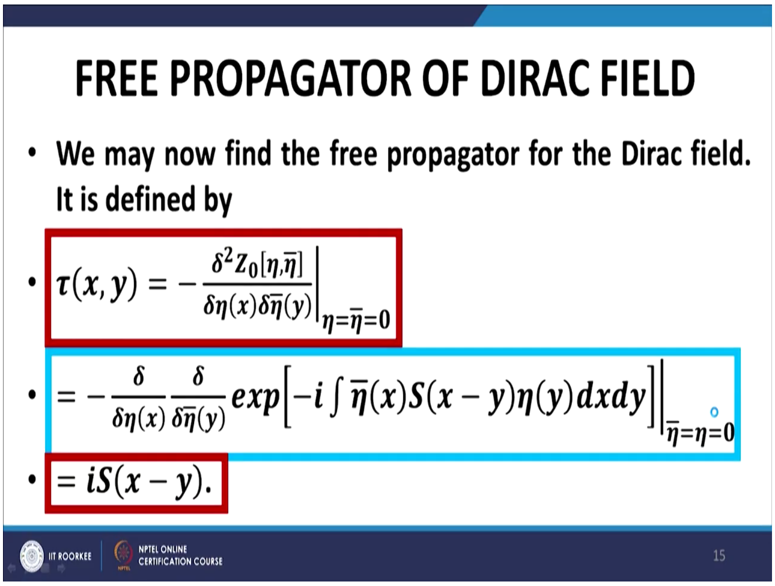

So, what we infer here is that the S matrix does exist and the S matrix is given by this expression which enables us to arrive at the S inverse matrix which is also which has been used in the previous calculations. In other words, the existence of the matrix S inverse has been established by this expression. By using by postulating this expression for the S matrix we work out the expression for S inverse S and thereby we find that S inverse S gives us the delta function.

Now, we look at the free propagator of the Dirac field. We worked out the generating functional to get the free propagators. We simply take the functional derivatives and for the 2 point propagator or the 2 point function we work out the expression which is given in the red box here and what we find is it is given by i of S x minus y. Recall S is given by this expression here in the red box which we just discussed.

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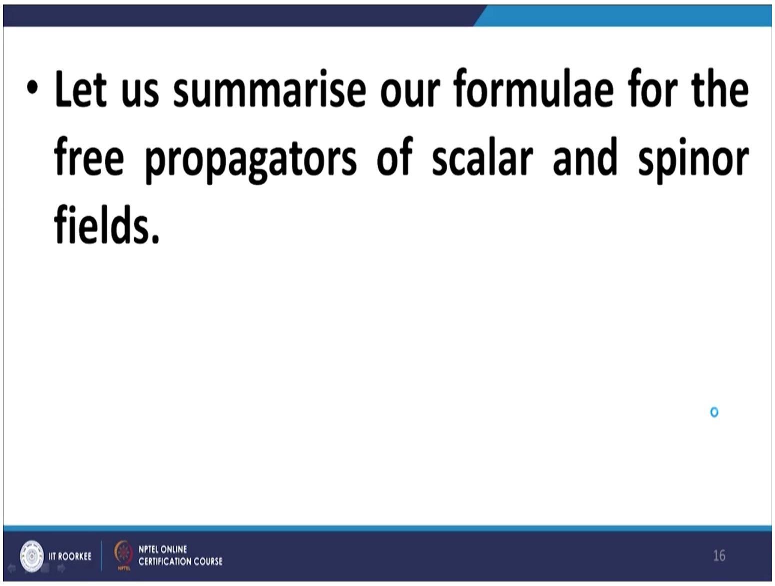

So, let us now summarize our formula for the free propagators of the scalar field that is the Klein Gordon field and the spinor field that is the Fermi Dirac fields.

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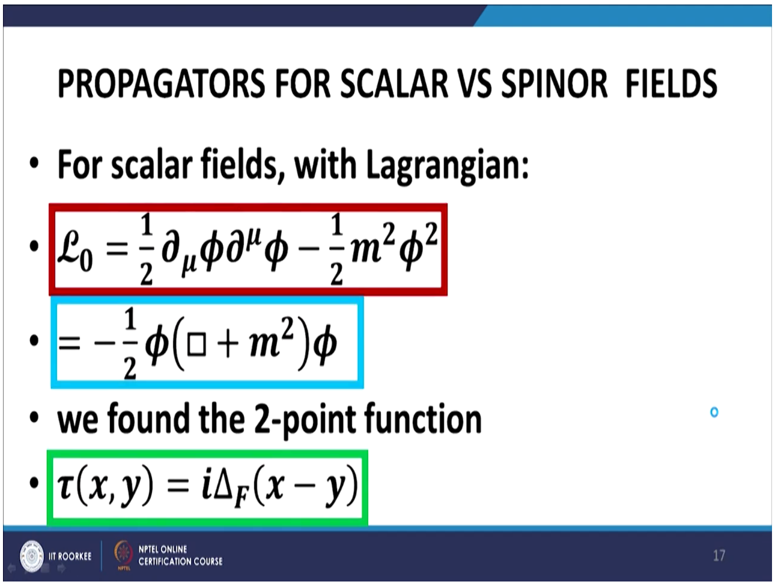

For the scalar fields, we have the Lagrangian which is given in the red box here and which is equivalent to the Lagrangian which is given in the blue box and that yields a 2 point function i delta F x minus y where delta F is the Feynman propagator.

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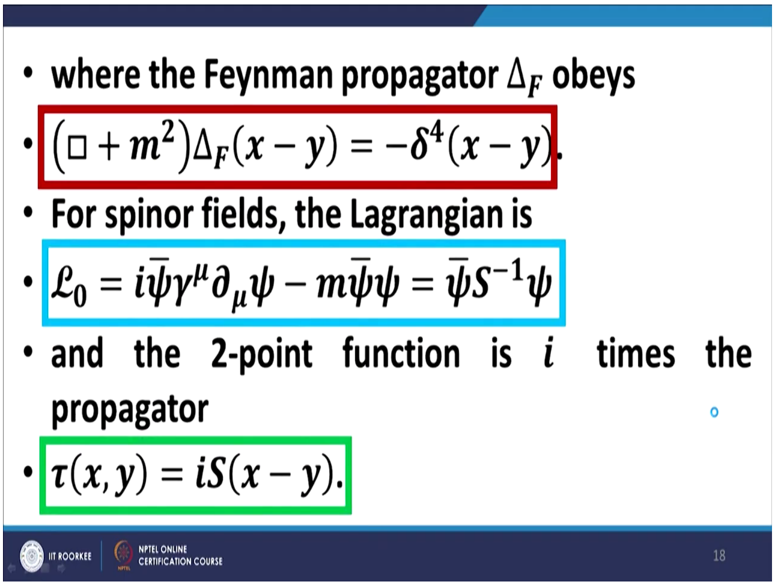

And delta F satisfies the condition that we just talked about box plus m square. This is the Klien Gordon operator delta F x minus y is equal to minus delta 4 x minus y 4 dimensional delta x minus y. For this spinor fields, the Lagrangian we have started with is the expression that is given here in the blue box that takes the form psi bar S inverse psi i. Recall the definition of S inverse and in this case the 2 point function is found to be i times S of x minus y.

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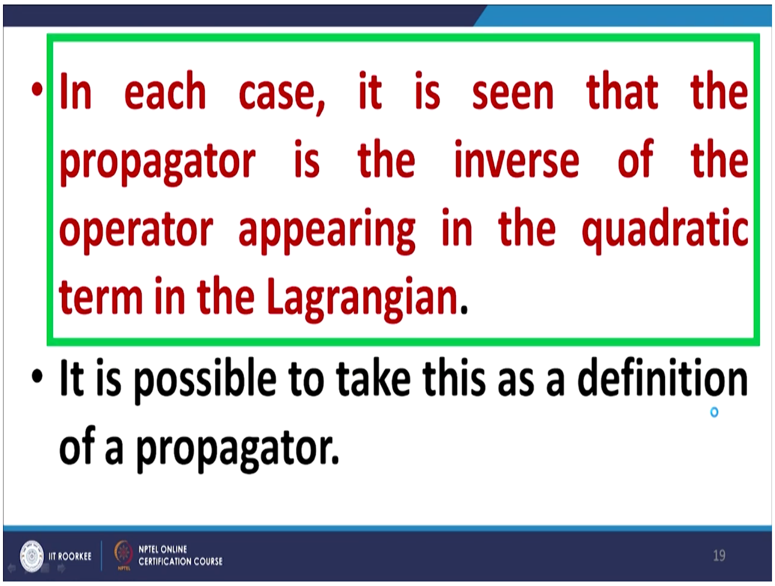

So, this certain common properties here; in each case, it is seen that the propagator is the inverse of the operator appearing in the quadratic term in the Lagrangian. And in fact this particular expression can be used as a definition for the propagator as well to reiterate the propagator is the inverse of the quadratic term in the Lagrangian.

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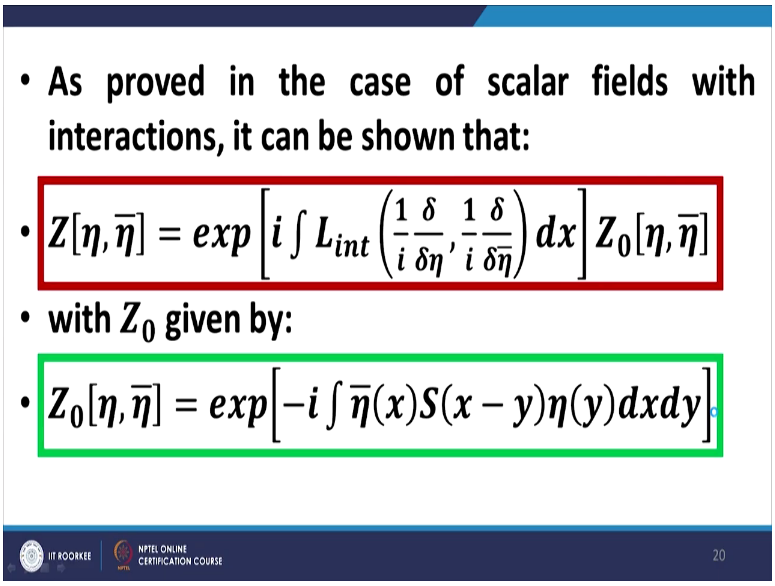

And for this was for the free Dirac field for the free for the interacting Dirac fields we can use this relationship which we have derived earlier in the context of the scalar fields with interactions.

The same expression literally holds in the context of the interacting Dirac fields and we have the version in the context of the Dirac field given in the red box here the middle of this slide, where Z 0 is the free field generating functional which we have just derived to be the expression given in the green box here at the bottom of this slide.

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Now, we come to the gauge fields. In the context of gauge fields, the two important fields are the electromagnetic field and the yang mills field I should be focusing my attention because of positive of time on the electromagnetic field. So, let us discuss and let us start the discussion of the electromagnetic field with the Maxwell equations.

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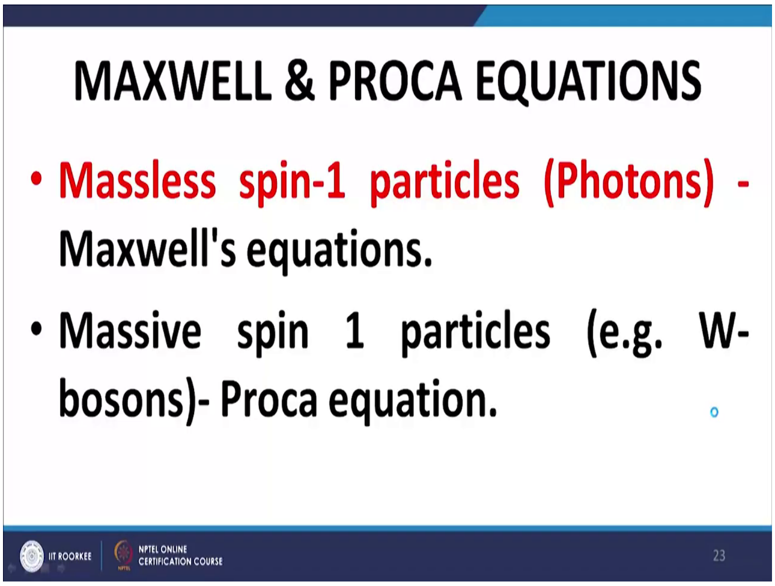

In fact, we have got the Maxwell equations and the Proca equations both relate to spin-1 particles. Maxwell equations govern the dynamics of the massless spin-1 particles that is photons whereas, the massive spin-1 particles are like the W-bosons are given by the Proca equations.

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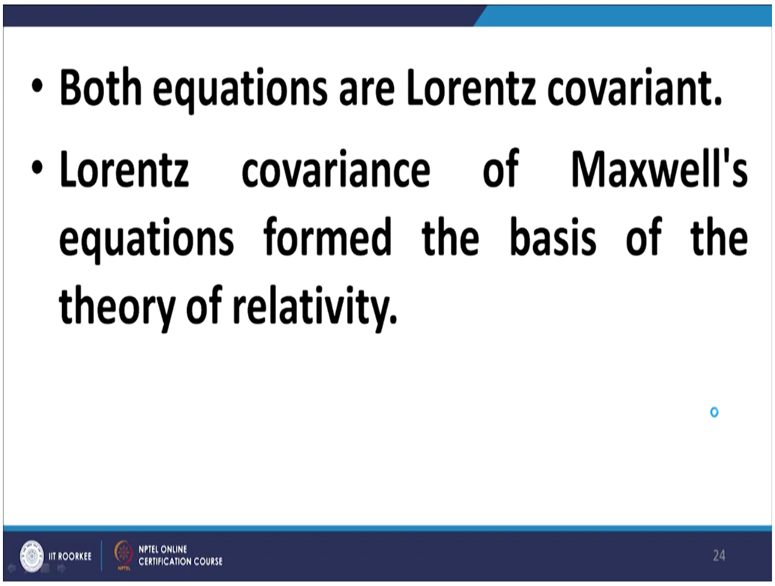

We shall be focusing on the Maxwell equations. Both equations are Lorentz covariant and the in fact it was the covariance form of the Maxwell equations that paved the way towards for Einstein to arrive at the equation for general relativity.

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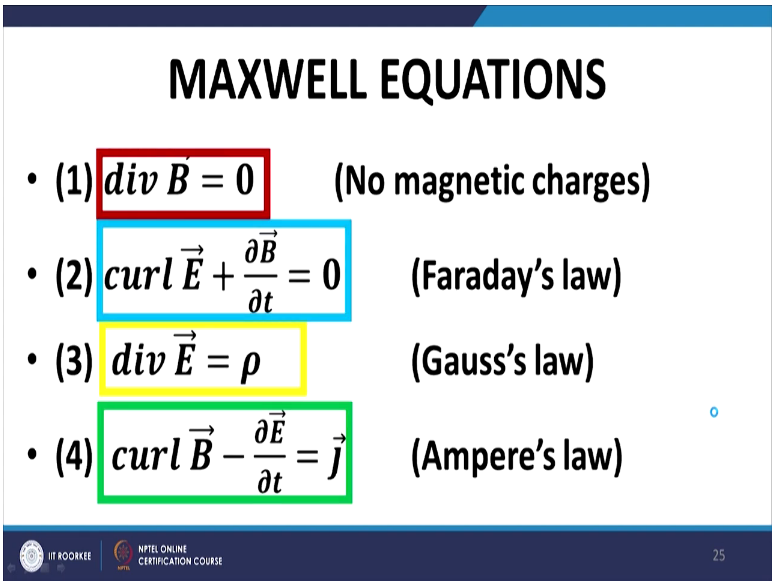

So, Maxwell equations we start with these are the 4 Maxwell equations and these there relevance or there physical interpretation we have on the next slide. The first is divergence of the magnetic field is 0, curl of the electric field plus the time rate of change of the magnetic field is 0. The divergence of the electric field is equal to the electric charge and finally, the curl of the magnetic field minus the rate of change of the electric field with respect to time is equal to the current.

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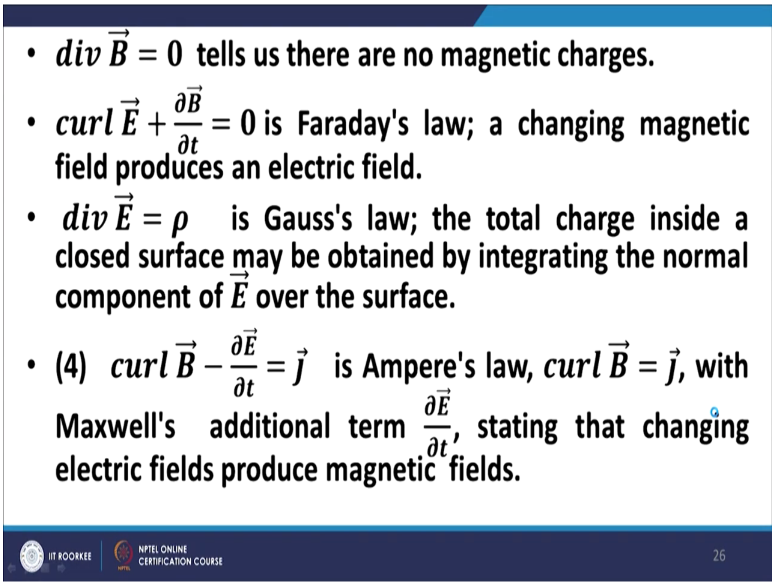

So, the first one as I mentioned is divergence B is equal to 0 and divergence of the magnetic field is 0. This is simply the mathematical representation of the fact that there are no magnetic charges. Then we have curl E plus the time rate of change of magnetic field is equal to 0.

This is Faraday's laws of electromagnetism. A changing magnetic field produces an electric field ah. Then we have divergence E is equal to rho. This is Gauss's law the total charge inside a closed surface can be obtained by integrating the normal component of E over the surface.

And then we finally have the fourth equation, the curl of B minus the time rate of change of the electric field is equal to the current. This is Ampere's law of course, with the modified Maxwell term which say says that the changing electric field produces magnetic fields.

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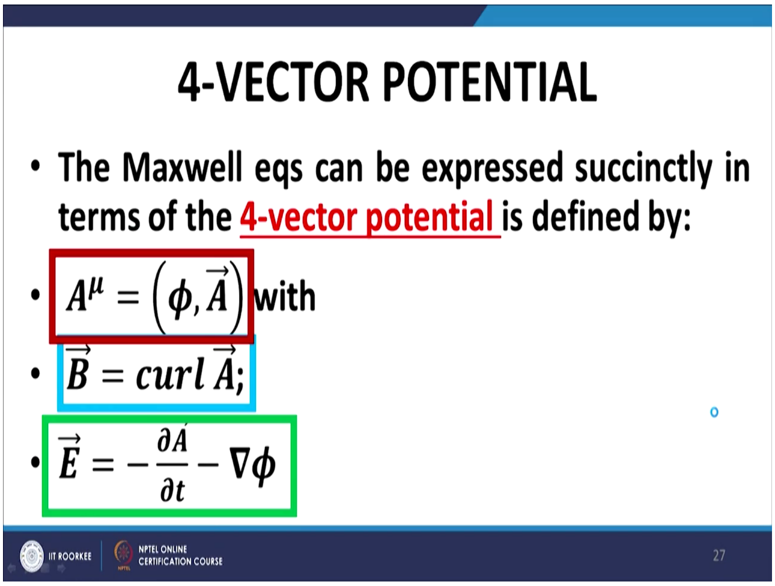

So, these are the physical interpretation of the 4 Maxwell equations. Now, the these 4 Maxwell equations can be put succinctly by introducing a 4 vector potential A which is defined as the scalar potential phi and the vector potential or the 3 vector potential let us call it A. And the magnetic field can be represented as the curl of the 3 vector A or and the electric field can be represented as the negative of the time rate of change of the 3 vector a potential A minus the gradient of the scalar potential phi.

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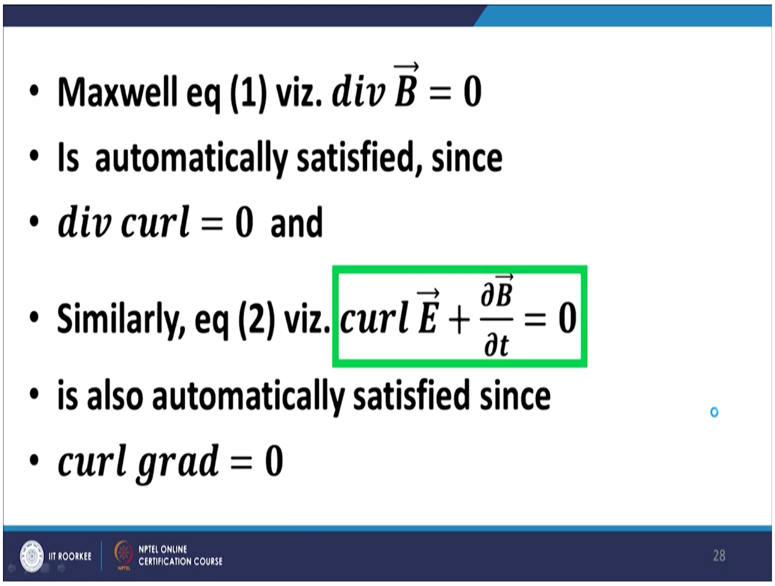

So, now the important thing is we have got 4 Maxwell equations and of this in the first two are homogeneous, the last two are in homogeneous and it is interesting that by introducing this 4 vector potential A or A mu. We have the first two equations are automatically satisfied since the divergence of a curl is always 0 and similarly the curl of a gradient is always 0.

So, the equation 1 which is divergence B is equal to 0 and remember we are writing B as curl of A. So, it is divergence curl of A. So, that is always 0 because it is the divergence of A curl. So, the first equation is automatically satisfied by the definition of A of B in terms of A and similarly the second equation homogeneous equation is also satisfied because the curl of a gradient is 0.

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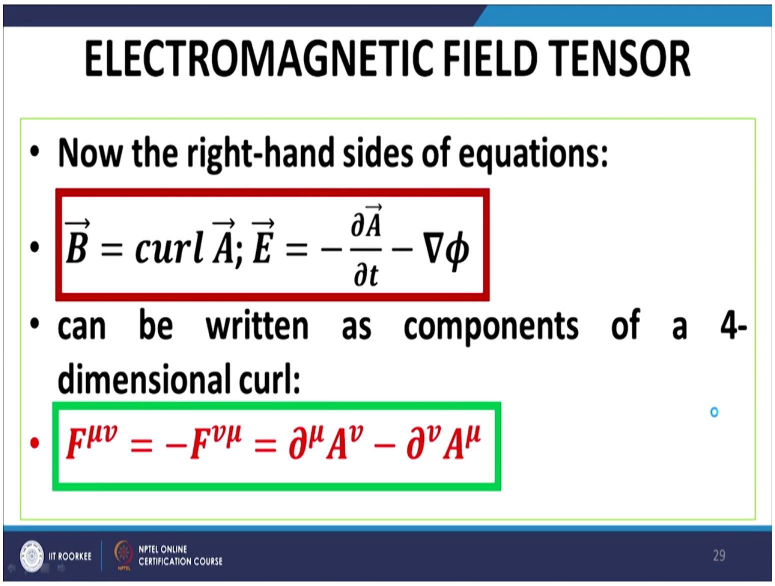

Now, further compactness can be introduced into the notation by introducing the concept of the electromagnetic field tensor. For that purpose we look at the right hand sides of the first 2 equations E is equal to curl A and B. I am sorry B is equal to curl of the special component A or the 3 vector A and E the electric field vector is equal to minus the time rate of change of the 3 vector A minus the gradient of i.

These can be written as components of a 4 dimensional curl which we call F mu nu and we define F mu nu which is called the electro electromagnetic field tensor whose properties we will discuss in the sequel. And that we represent as the curl of as the 4 dimensional curl given by the expression in the green box at the bottom of your slide.

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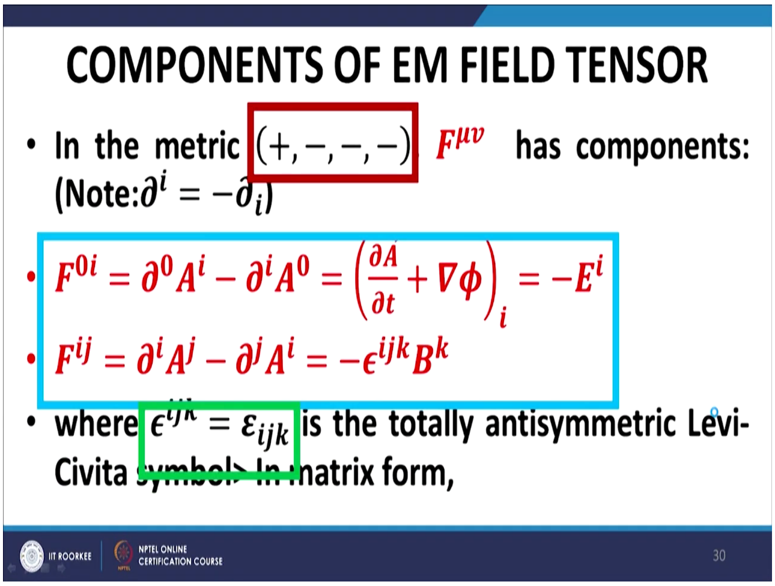

Now, in the metric plus minus minus minus that we have been following throughout F mu nu that is the electromagnetic field tensor it has the components given by as you can see here there is straightforward exercise. If you look at F 0 i the F 0 i th component of F mu nu turns out to be minus E i and F ij component of your i and j are 1, 2 and 3 respectively turn out to be equal to minus epsilon ijk B k where epsilon ijk is the 3 dimensional totally antisymmetric Levi Civita tensor.

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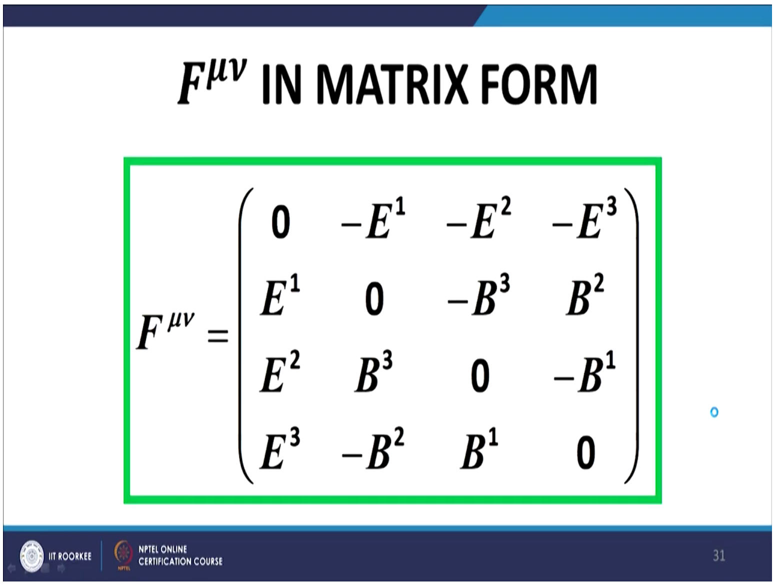

The explicit expression for the electromagnetic tensor F mu nu is given in the matrix form in this slide. The first row and the first column in fact represent the components of the electric field and the other and the sub matrix the remaining sub matrix represents the components of the magnetic field. Please note all the diagonal elements as 0 because the tensor is anti symmetric. So, the diagonal elements have to be 0 and of course because of antisymmetricity F mu nu is equal to minus F mu nu.

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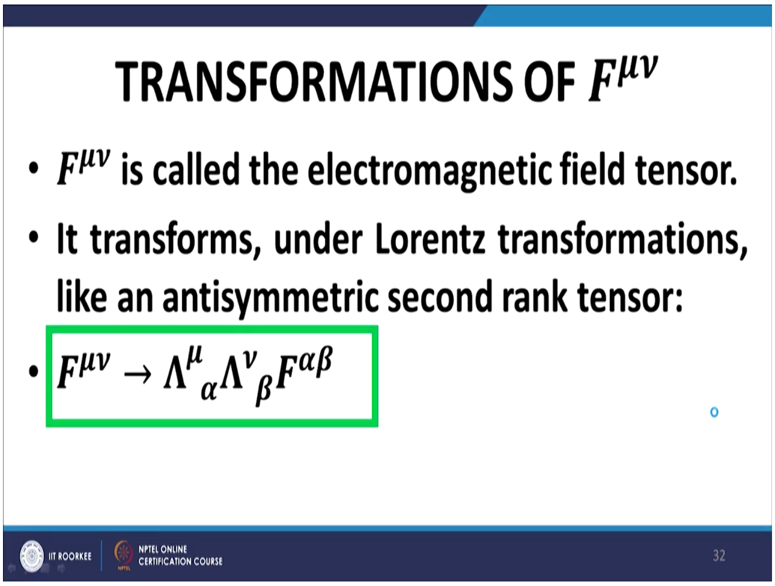

With the transformation of F mu nu under Lorentz transformations when it transforms as an anti-symmetric second rank tensor. Obviously, it has 2 Lorentz indices mu and nu and therefore, it transfers as an anti symmetric second rank tensor and its transformation equation is given in the green box at the bottom of this slide.

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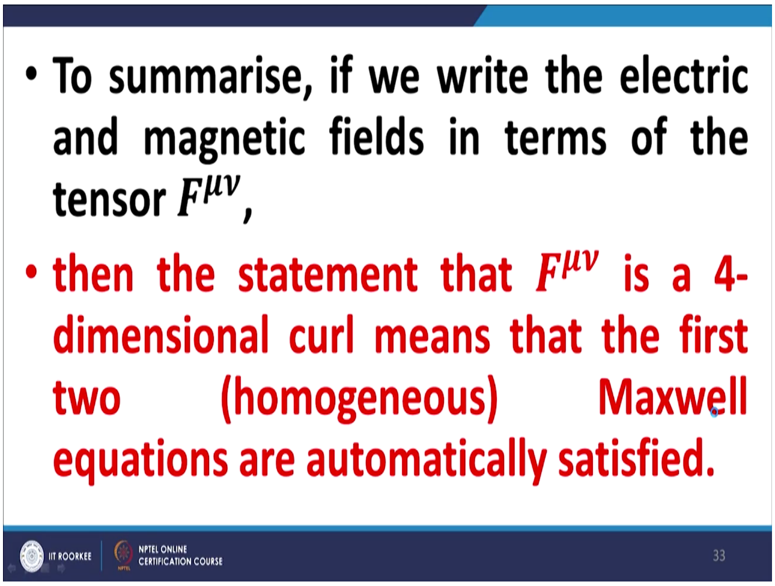

So, now, the important thing is that if we write the electric and magnetic fields in terms of F mu nu then the very definition of F mu nu as the 4 dimensional curl implies that the first two homogeneous equations or first two homogeneous Maxwell equations are automatically satisfied. First two homogeneous Maxwell equations are automatically satisfied by virtue of the definitions of F mu nu as a 4 dimensional curl. Now, it go to the Inhomogeneous Maxwell equations.

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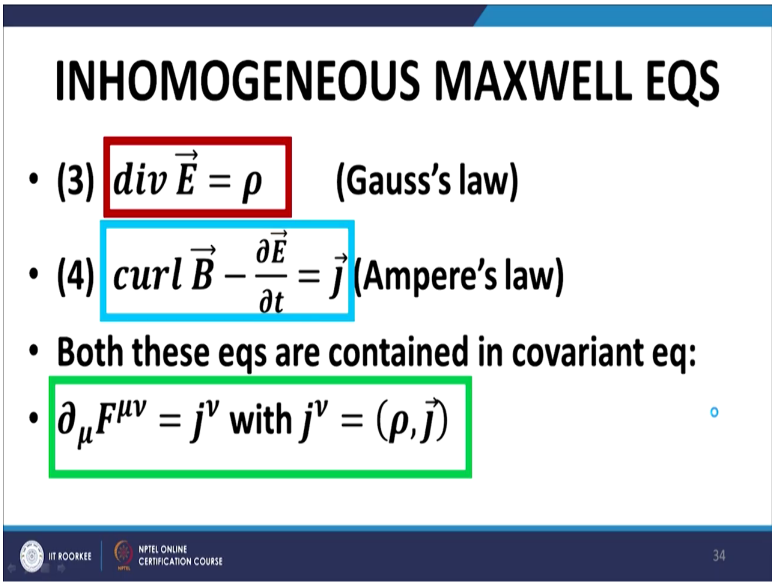

We talk about the Inhomogeneous Maxwell equation. What are they? We have divergence of E is equal to rho and we have curl B the electric field magnetic field I am sorry curl B minus the time rate of change of the electric field is equal to the current. Now, it can be shown as you should see that both these equations are contained in the covariant equation and del of mu F mu nu is equal to j nu where j nu has the time component or the 0 th component as the charge and the special components has the current.

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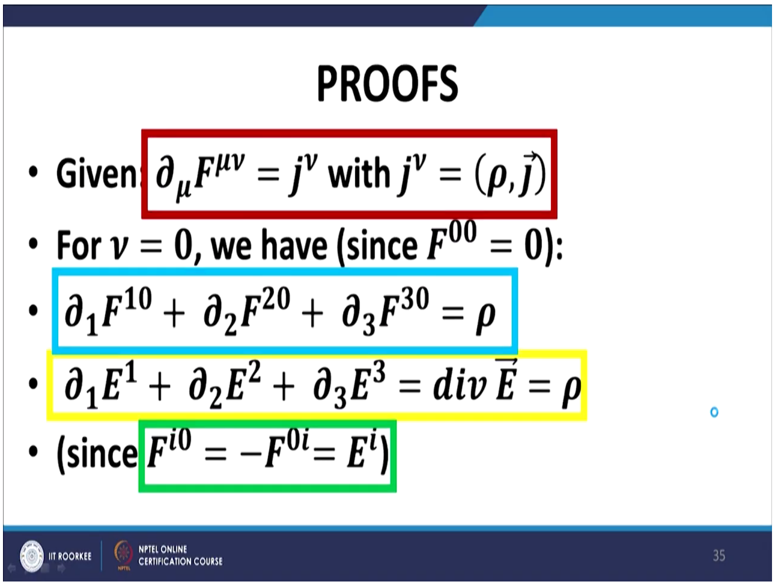

So, this is the expression is easily verified. If from the very first principles in fact for nu equal to 0 we have since F 0 0 is equal to 0 because F is antisymmetric and therefore, we have del 1 F 1 0 plus del 2 F 2 0 plus del 3 F 3 0 is equal to rho. Now, F 1 0 is nothing, but E 1, F 2 0 is nothing, but E 2 and F 3 0 is nothing, but E 3. Therefore, we have the divergence of E the electric field is equal to rho which is nothing, but the third Maxwell equation.

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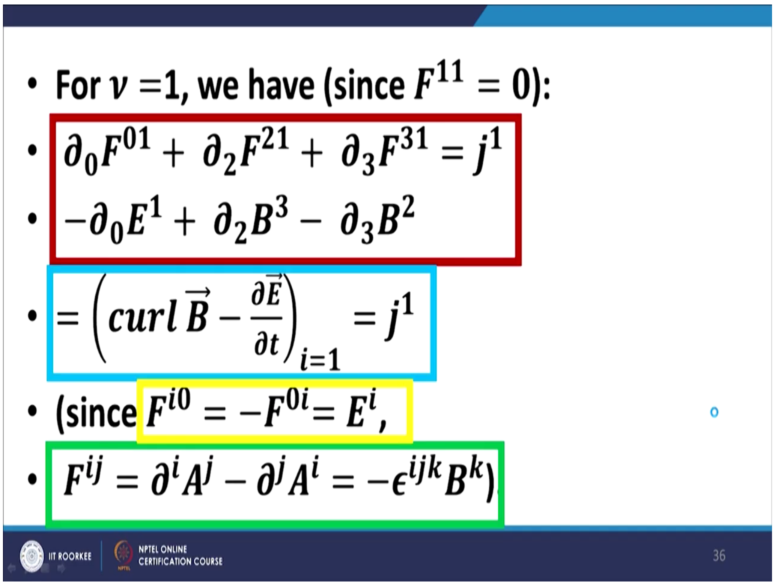

And similarly for example, if we want to establish the fourth equation we write nu equal to 1. So, that we have F 11 equal to 0. About the rest of the components we have del d 0 F 01 plus d 2 F 21 plus d 3 F 31 is equal to j 1. If you put the respective values of 0 of F mu nu in terms of the electric and magnetic field components we have minus d 0 E 1 plus d 2 B 3 minus d 3 B 2 because of because F 01 is equal to minus E 1, F 21 is equal to B 3 and F 31 is equal to minus B 2.

So, we substitute these values and what we get is nothing, but the first component i equal to 1th component that is the first special component of curl B minus the time rate of change of the electric field. This first special component and that is equal to on the right hand side j 1 which is what the fourth Maxwell equation mandates.

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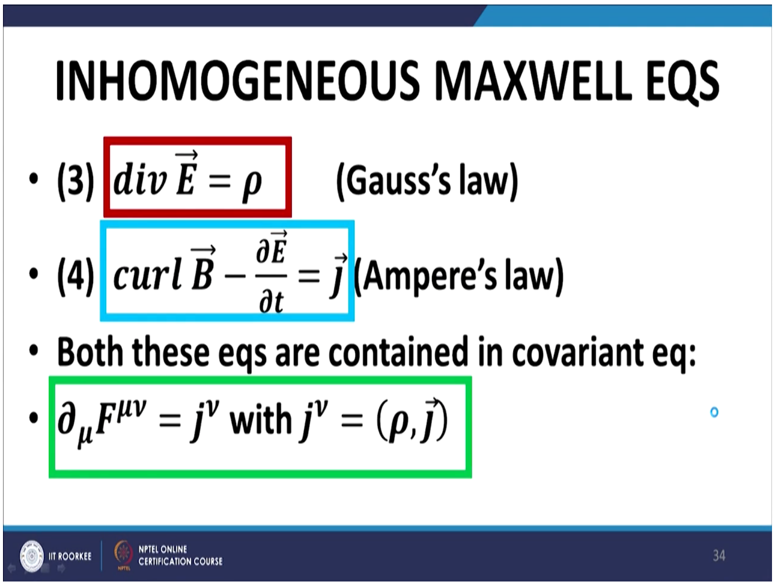

So, the third and the fourth Maxwell equations can be covariantly expressed in the form of the expression that is given here in the green box at the bottom of this slide.

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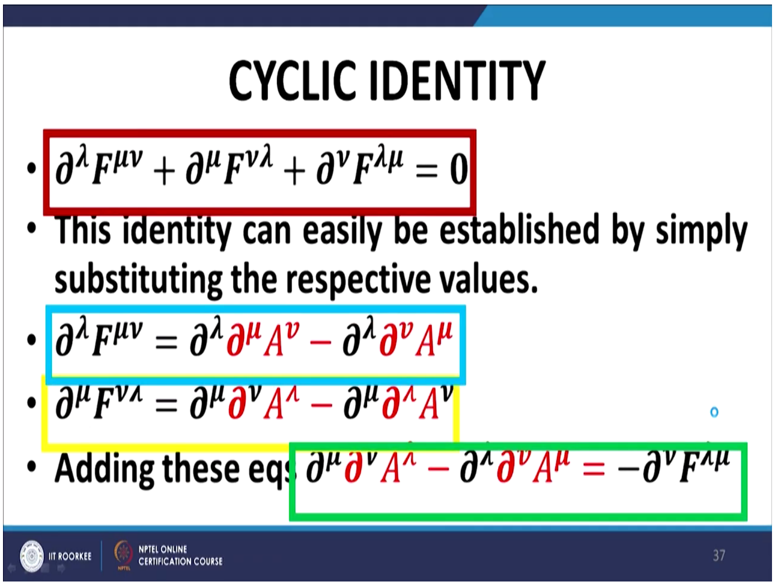

And this cyclic identity for the electromagnetic tensor is easily verified. It is by substituting the respective values respective components by substituting F mu nu as the 4 dimensional curl. You simply substitute the definition of F mu nu and by simplification as simple algebraic simplification we can establish the cyclic identity of F mu nu.

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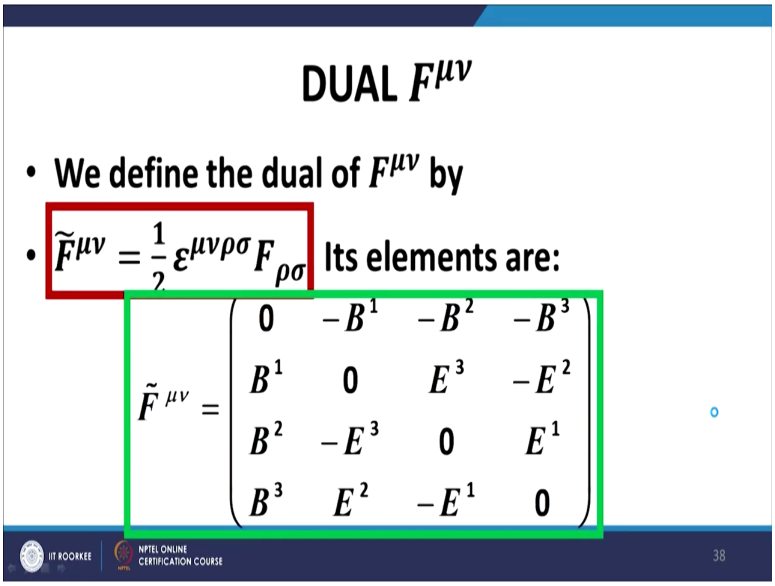

Now, we look at the dual F of mu nu. The dual F of mu nu is defined as the F tilde of mu nu. F tilde mu nu is defined as 1 by 2 epsilon mu nu rho sigma of F rho sigma where epsilon mu nu rho sigma is the 4 dimensional Levi Civita tensor totally antisymmetric. And the components of F tilde mu nu in terms of the electric and magnetic fields are given in the green box in the bottom of this slide.

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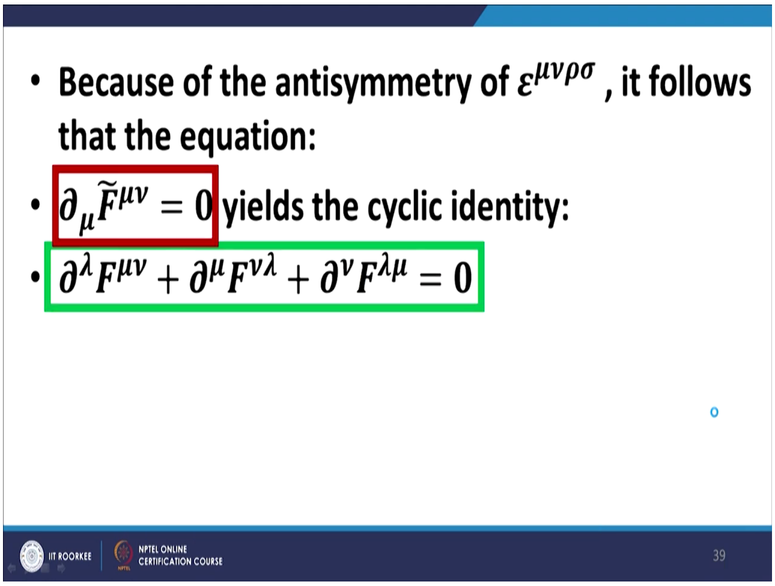

And because of the antisymmetry of this Levi Civita tensor it follows that the equation d mu of F tilde mu nu yields a cyclic identity which is given in the green box at the bottom of the slide.

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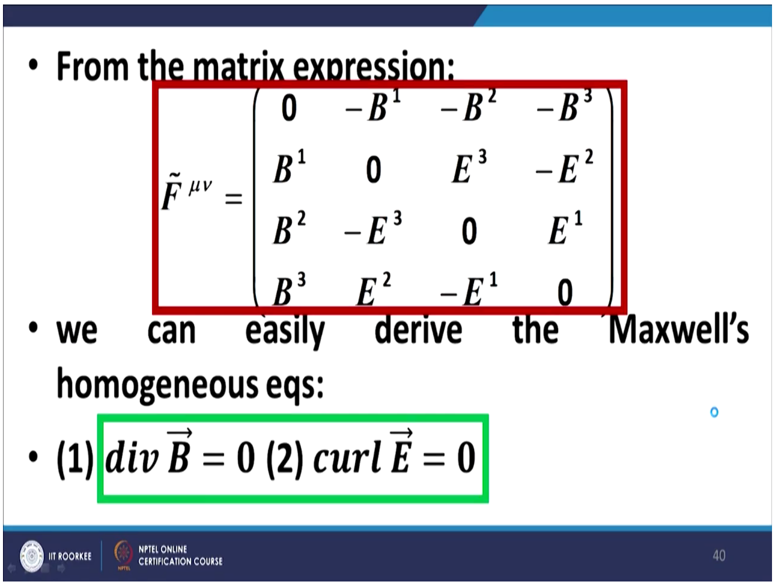

Now, from the matrix expression for F tilde mu nu that is the dual electromagnetic tensor we can derive the first and second equations quite straight away. In fact, we can literally read out the first and second equations of the Maxwell set and therefore, we have right from the dual electromagnetic tensor divergence of B is equal to 0 and curl of E is equal to 0.

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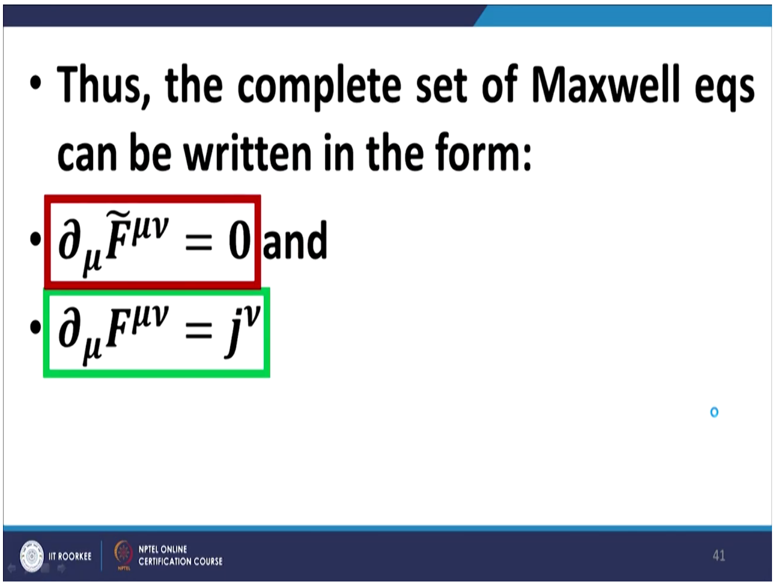

So, the complete set of Maxwell equations can be written in terms of the electromagnetic tensor f mu nu and it is dual F tilde mu nu in the form that d of covariant derivative of F tilde mu nu is equal to 0 and d mu of F mu nu is equal to j nu. These two equations together the equation in the red box and the equation in the green box together comprise or imply the entire set of Maxwell equations.

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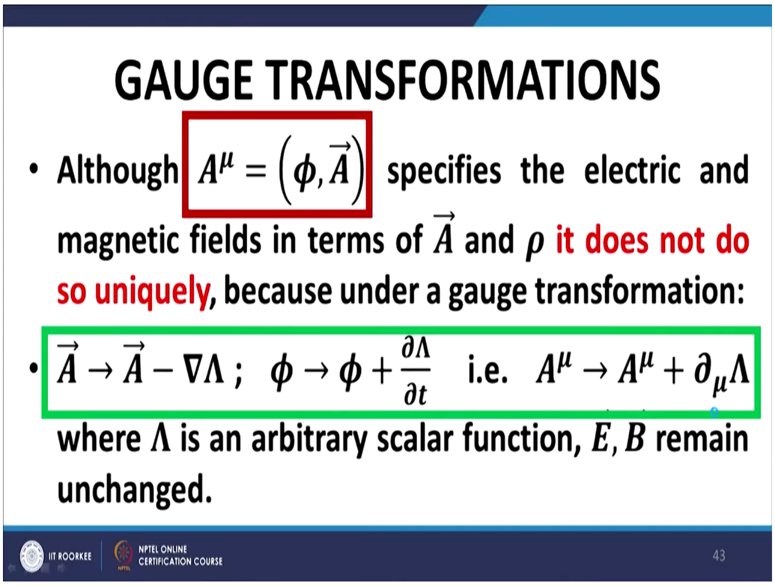

Now, we come to the issue of gauge transformations. Now, we have got to understand a fundamental thing which is special to the this electromagnetic fields. We have introduced the concept of 4 potential A mu as phi and the special A vector are. But, the point is that when we are making this shift or representing the combined electric and magnetic fields by this 4 vector phi and A especially the specification is not unique.

In other words, it does not the A mu that we define here is not unique and we can in fact find a complete set or a number of A mu's which are connected by a gauge transformations involving a scalar function such that each of it each of this satisfies the requirement of imposed or the constraint imposed on A mu the by the Maxwell equations.

So, let us try to understand this. If we introduce or if you change our components our given components of a mu which are phi and A. Especially if it change special A to special A minus the in gradient of eta and phi to phi plus the time rate of change of eta where eta is any arbitrary scalar function.

We find that; we find that our specification for a or the Maxwell equations do not change, the electromagnetic field tensor does not change. In other words, putting it covariantly if I write A mu as A mu plus d mu eta this 2 are equivalent.

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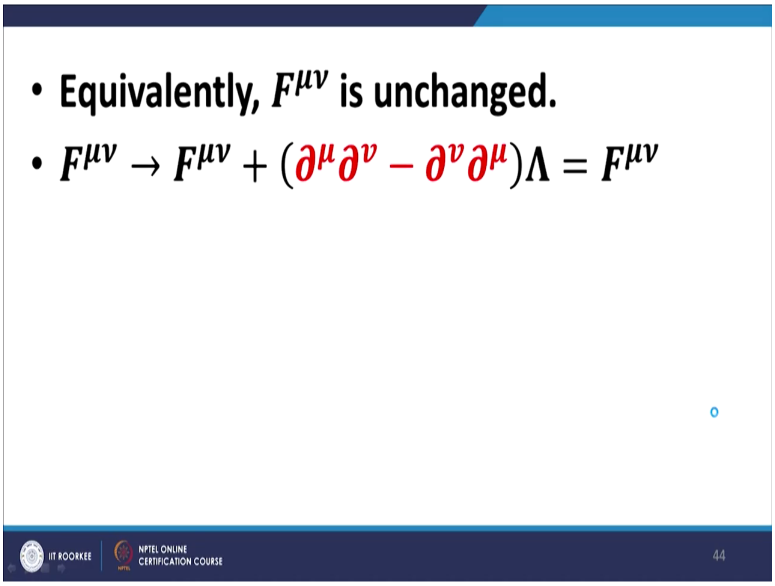

Then we find that the electromagnetic tensor does not change.

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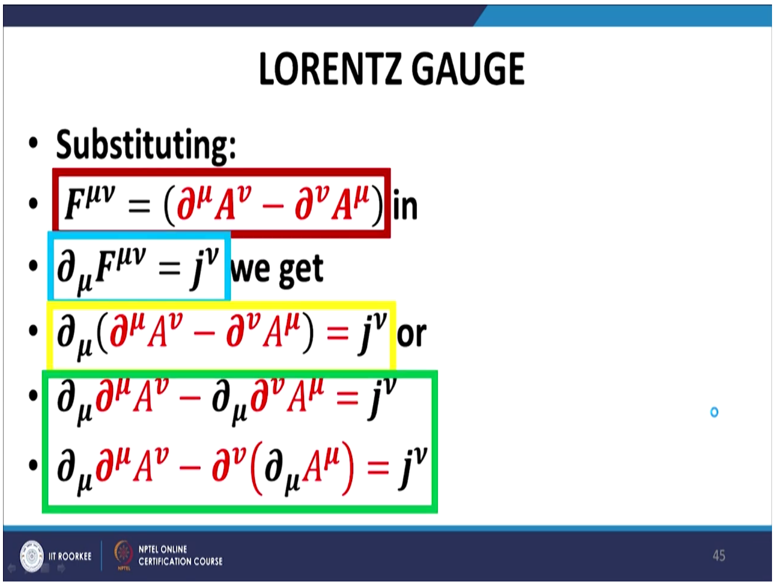

Therefore, in order that we are able to specify our unique representation of the Maxwell's equations in terms of F mu nu or in terms of A mu we have to impose certain gauge restrictions. And on the basis of introducing those gauge restrictions we then are able to establish a unique representation of the Maxwell equation. So, what we do is let us try to understand. This is the definition of F mu nu which is given here in the red box we started with that and these are the Maxwell equations which is in the blue box here in the expression that we have are the Maxwell equations.

Now, if I write if I take the derivative of F mu nu equal to if I write F mu nu in the explicit form here what I get is the expression which is given in the green box here. The derivatives commute with each other and therefore, we can write it in the form of the second equation in the green box. The first equation can be written in the form of; in the form of the second equation in the green box because the derivatives commute with each other.

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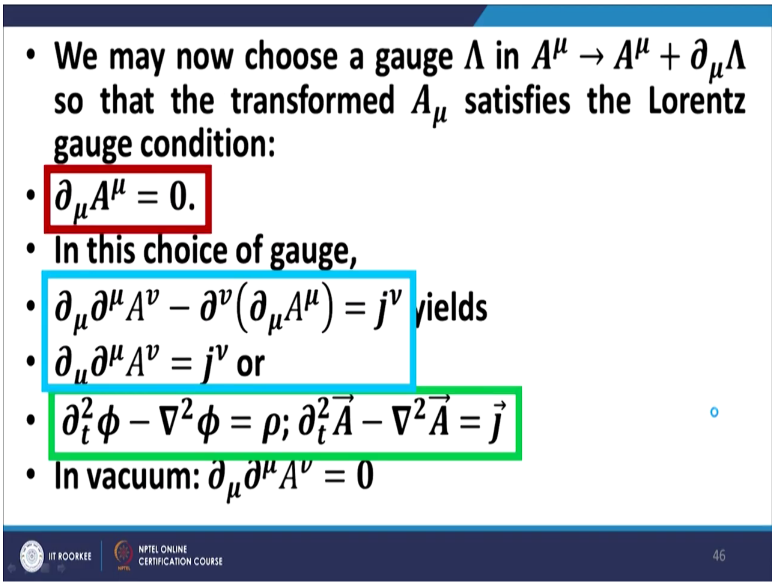

Now, we choose the gauge which is in because you see we have got the freedom. We have got the freedom given by this expression A mu goes to A mu plus d mu of eta and we can choose any eta which you we desire because the Maxwell equations will be satisfied.

So, we choose a particular A mu or a particular eta as we shall see which satisfies a certain gauge condition and that gauge condition is represented by the expression in the red box here. And in this if I choose the expression or the restriction please note this is an imposed restriction.

If I choose this restriction given in the red box here, then what I get from the previous slide is that the second term vanishes. The second term vanishes and I get the box of A nu is equal to j nu or expressing them explicitly I get the expression which is here in the green box at the bottom of this slide. Of course, in vacuum because we have no sources we have no currents it simplifies to the expression. The right hand side goes to 0 and we get the simple equation box of A nu is equal to 0 right.

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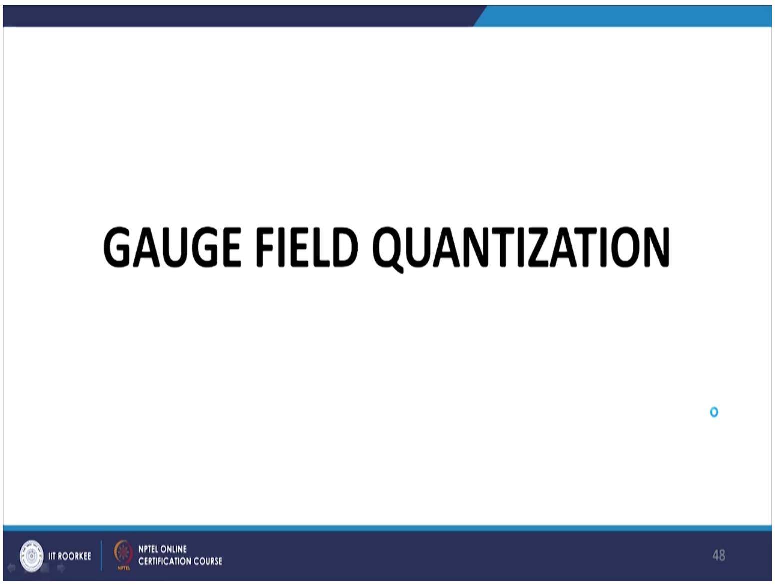

So, from here we will take a after the break we will take in gauge field quantization and then we will work towards the path integral quantization of this electromagnetic field.

Thank you.