Path Integral Methods in Physics & Finance
Prof. J. P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee

Lecture – 05
Probability, Gaussian Distribution & Integration

Welcome back. Let us continue from where we left off in the last lecture, but before that a quick recap of the important points and a correction as well.

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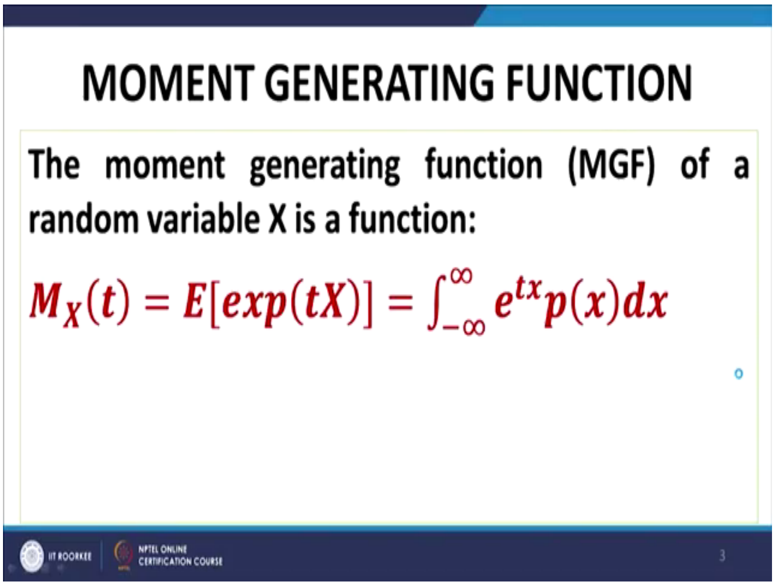

I define the moment generating function as the expected value of exponential t X.

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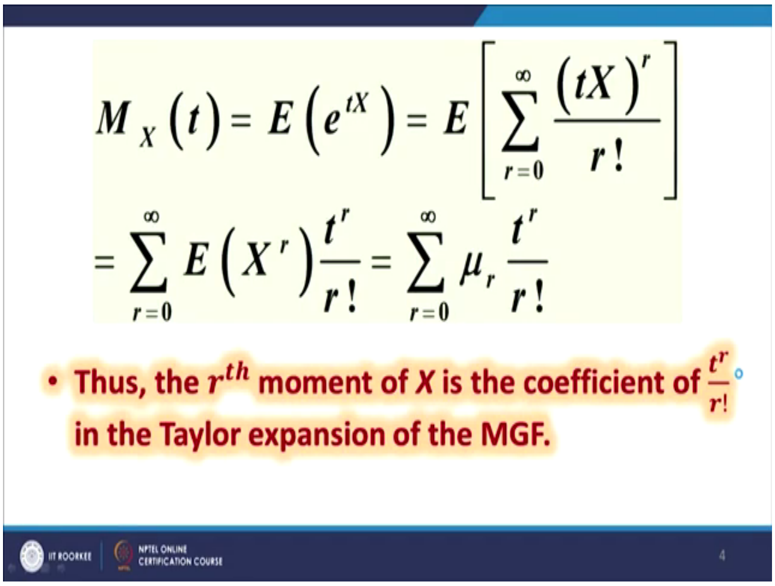

And then we are arrived at the relationship between the moment generating function and the various movements. The by expanding the exponential as the Taylor series, exponential series we arrive at summation of mu r, where mu r is the rth moment about the origin t to the power r upon r factorial. Therefore, t to the power r upon r factorial gives us the coefficient of t to the power r upon r factorial gives us the rth moment.

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$$\frac{d^r}{dt^r} M_X(t)\Big|_{t=0} = \frac{d^r}{dt^r}\left[E\left(e^{tX}\right)\right]\Big|_{t=0}$$

$$= \frac{d^r}{dt^r}\left\{\sum_{r=0}^{\infty} E\left(X^r\right)\frac{t^r}{r!}\right\}\Bigg|_{t=0} = E\left(X^r\right)$$

Another approach to that is by taking derivatives. You take the rth derivative of the moment generating function with respect to t and then put t equal to 0 and you are simply get the value of x of the rth moment about the origin. So, that is another approach either of the two yield the same result of course.

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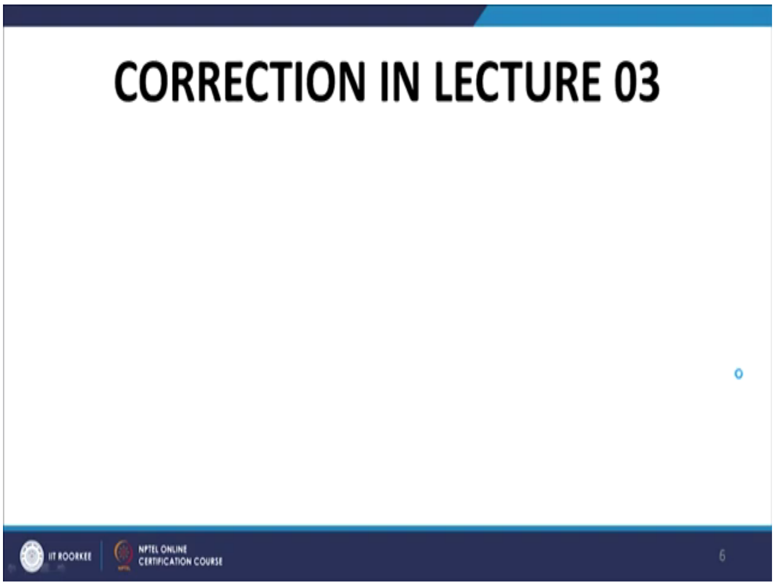

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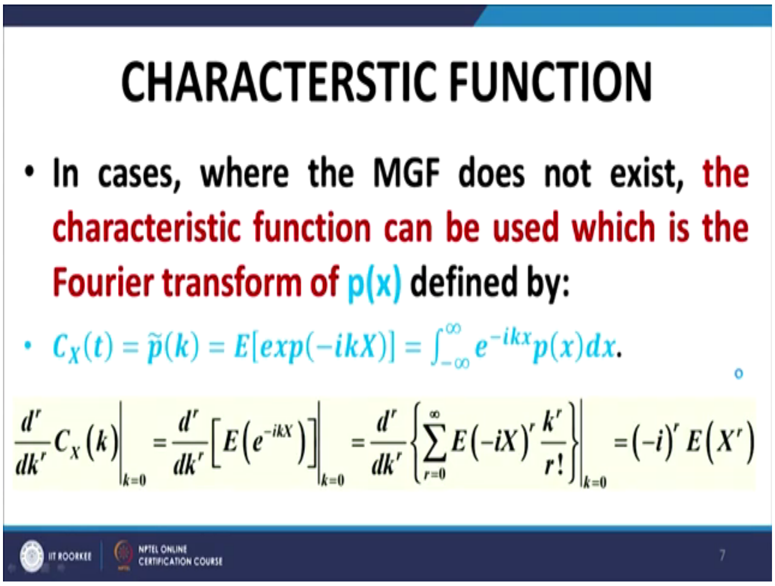

Then there is a correction that mentioned in lecture 3. I defined the characteristic function as the Fourier transform of the moment generating function. It is not actually the Fourier transform of the moment generating function, it is the Fourier transform of p X; the probability density function.

As you can see from the in this slide in the blue color, the equation in the blue color C X t is p tilde k p tilde k, which is the expected value of exponential minus i k x and which can be written as the Fourier transform of p of X.

So, that was the correction and the characteristic function performs a role very similar to the role performed by the moment generating function. And, we can recover the various movements in more or less the similar manner by taking derivatives of appropriate orders.

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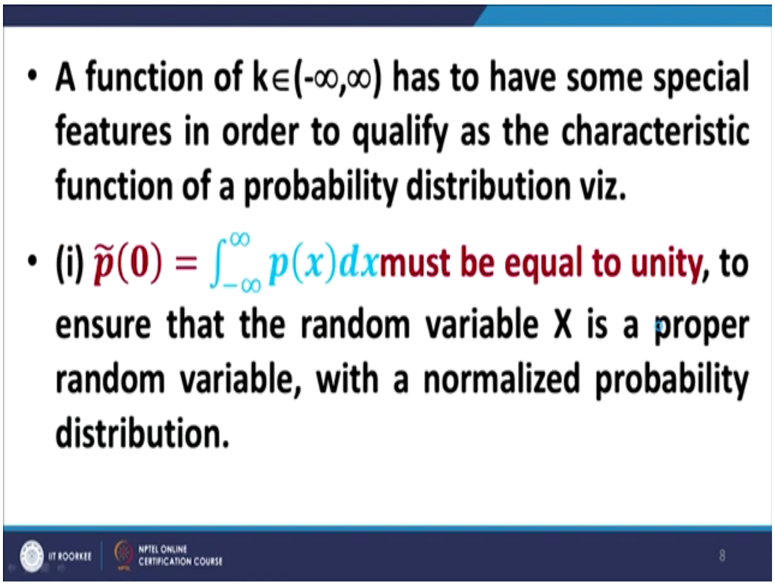

The two important properties that the characteristic function must satisfy is p tilde of 0 which will be equal to when you put k equal to 0 there in the definition or expression for the characteristic function that will be equal to the integral of p x dx over the entire range of values of x and that must be equal to 1.

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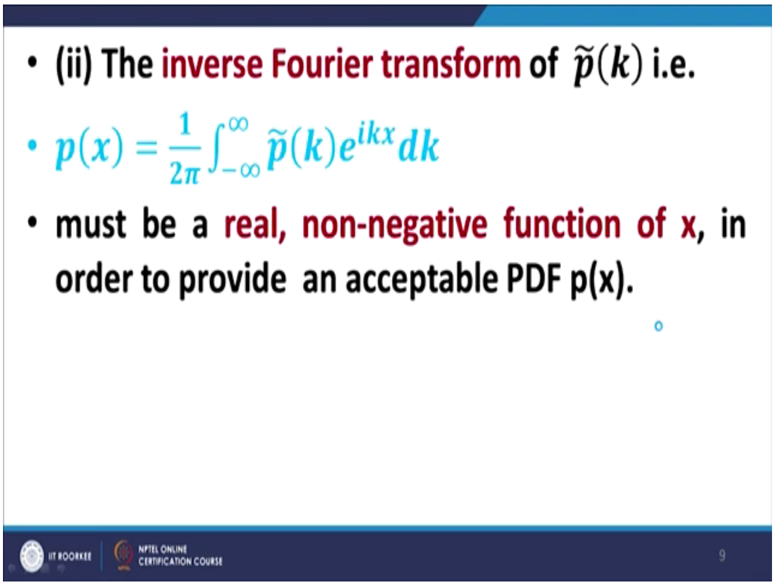

So, we must have p tilde of 0 equal to 1 and secondly, because p x is going to be defined in terms of the inverse Fourier transform of p tilde k, we require that the inverse Fourier transform of the characteristic function must be real non-negative function of x, so that it defines a probability density function. So, these are two fundamental properties that the characteristic function must satisfy.

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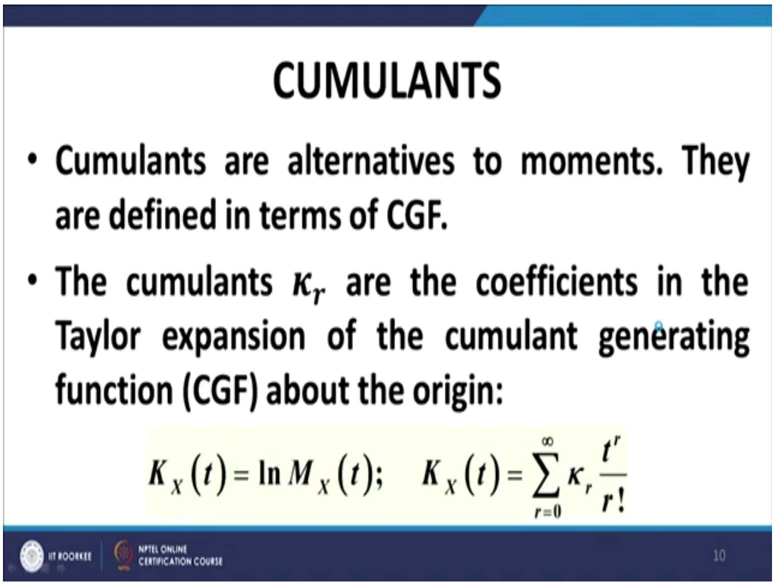

Then we moved over to the cumulants. We defined the cumulants as the coefficients of a power series expansion of the cumulant generating function, where the cumulant generating function is defined as the natural log of the moment generating function that you can see in this particular equation.

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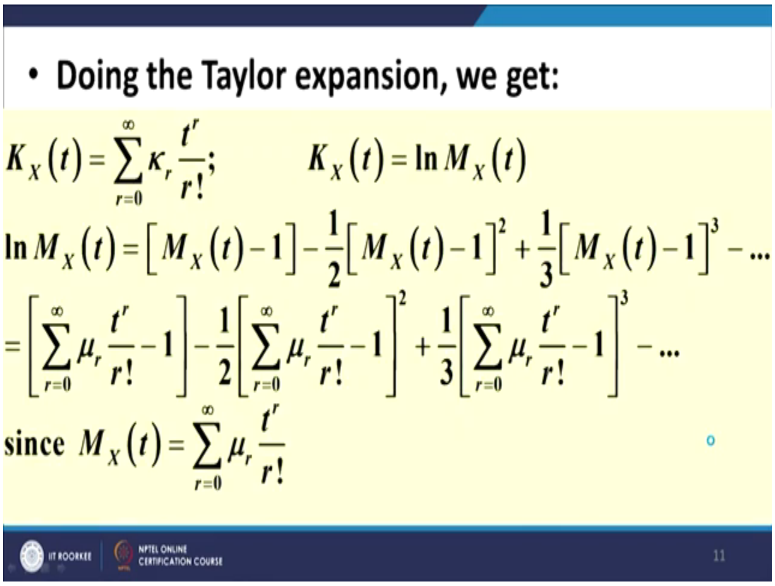

Then by using this relationship between cumulant generating function and the moment generating function, the latter the former being the natural logarithm of the latter we can arrive at a relation between the cumulants which are coefficients of the power series in t and the movements as obtained from the moment generating function which can be recovered through a power series expansion of t. And, then you equate coefficients of appropriate powers of t and you arrive a relation between kappa and mu.

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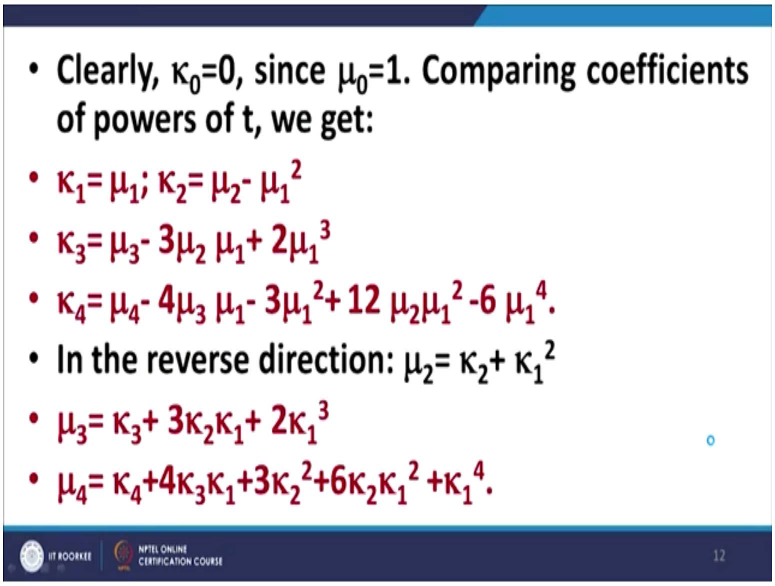

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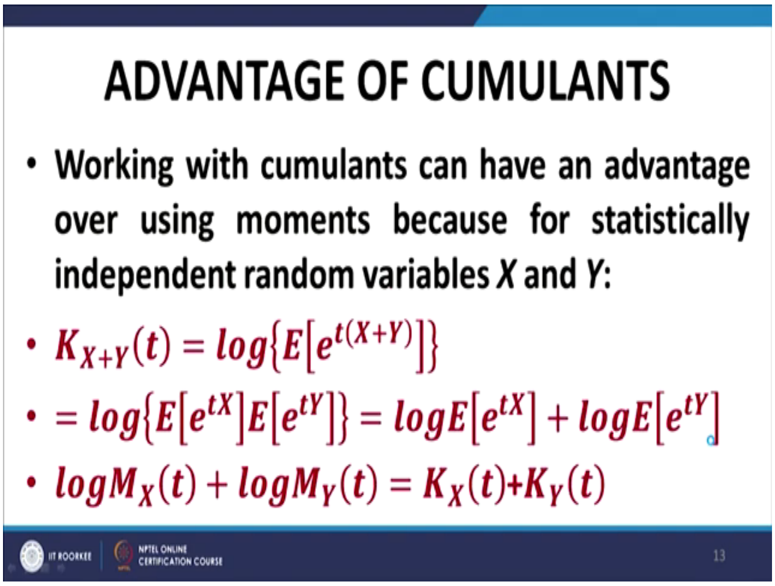

These are some instances of how the relations turn out to be. Advantages of cumulants over the moment generating function: well, if you have two random variables X and Y and they both are statistically independent of each other. Then it turns out that the cumulant generating function of the sum of those two variables is equal to the sum of the cumulant generating functions of the two variables that is K of X plus Y is equal to K of X plus k of Y.

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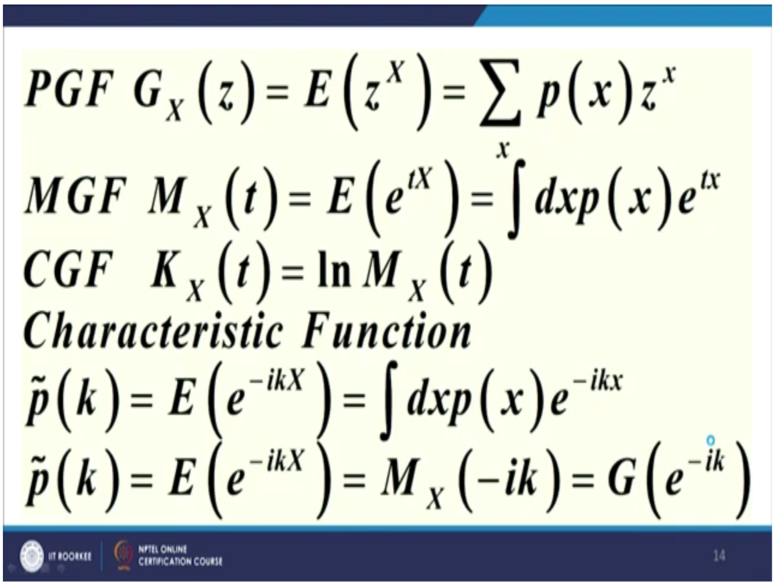

So, this is a summary of all the important generating functions that we have talked about: the probability generating function, the moment generating, functions, cumulant generating functions and the characteristic functions.

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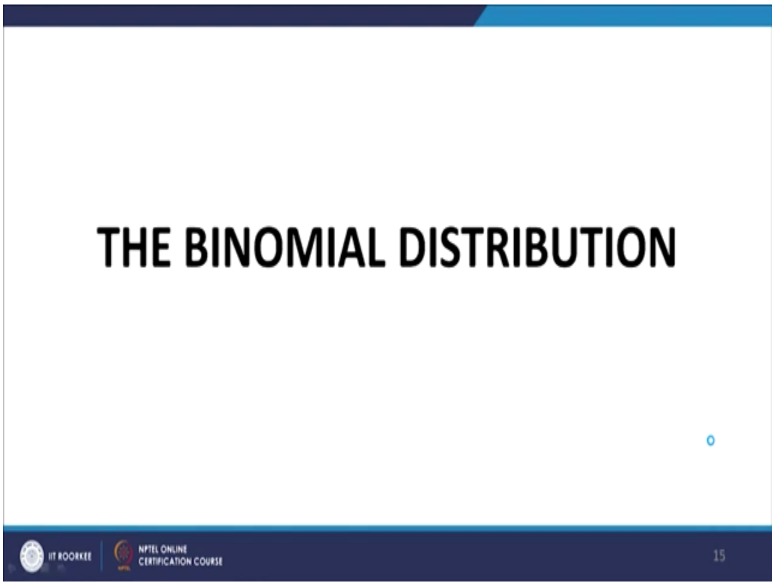

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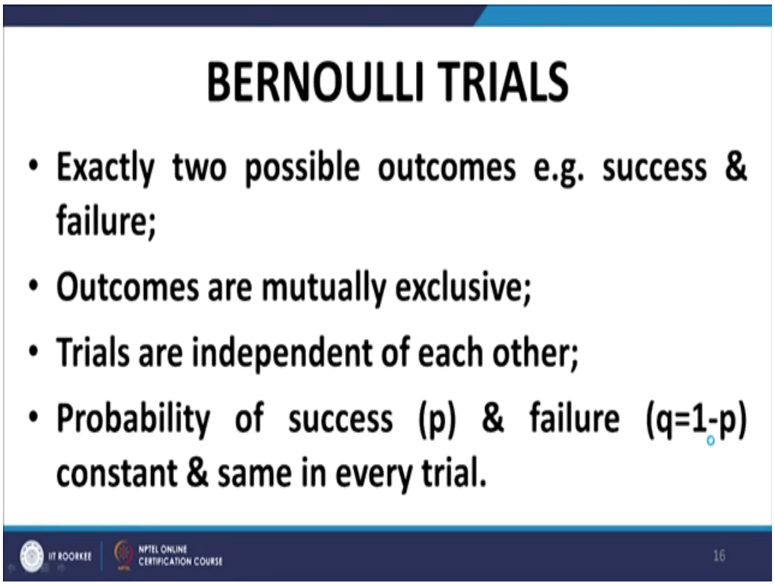

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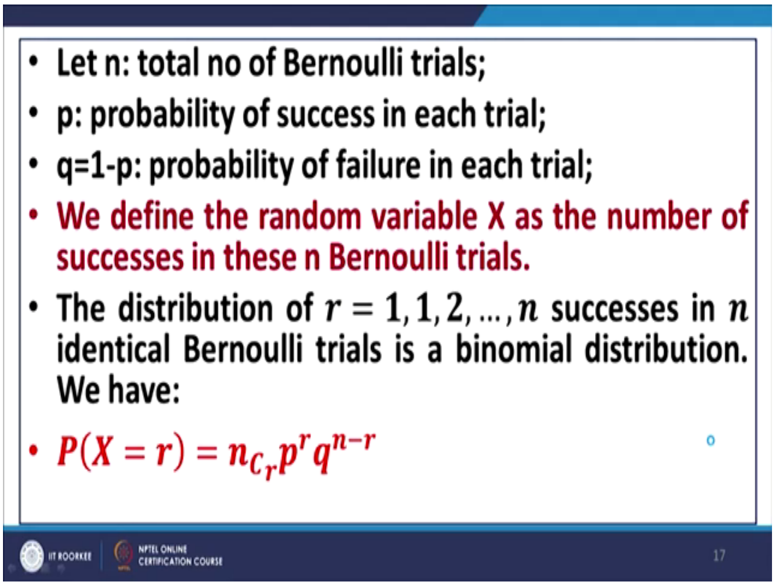

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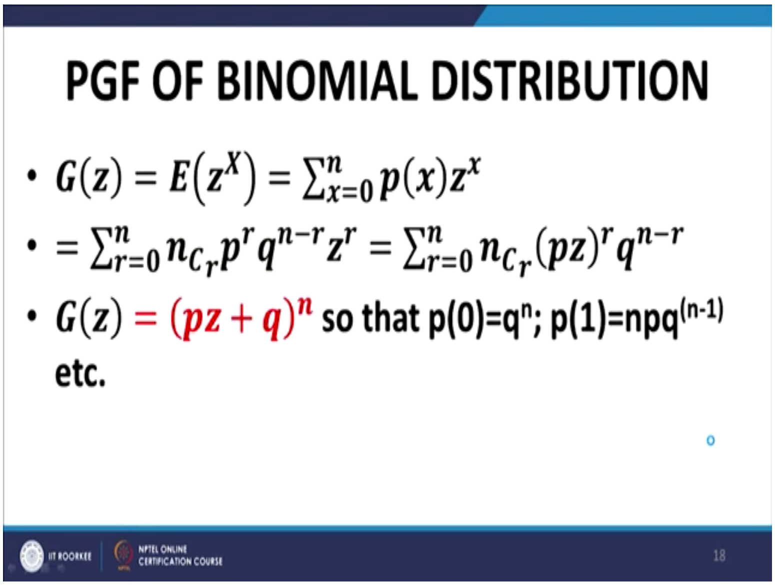

Then we talked briefly about binomial distributions. Binomial distributions is defined by the probability mass function P X equal to r is equal to n C r p to the power r q to the power n minus r. The probability generating function G of z is given by p z plus q to the power n.

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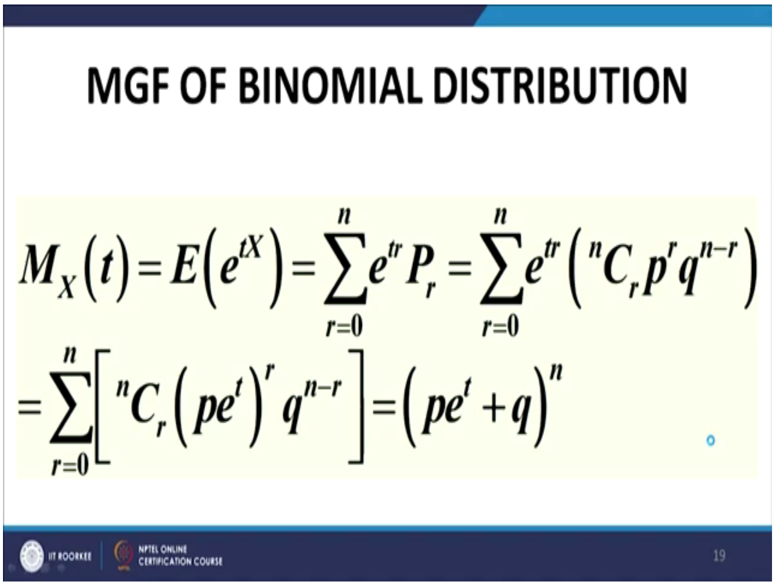

The moment generating function turns out to be p e to the power t plus q to the power n.

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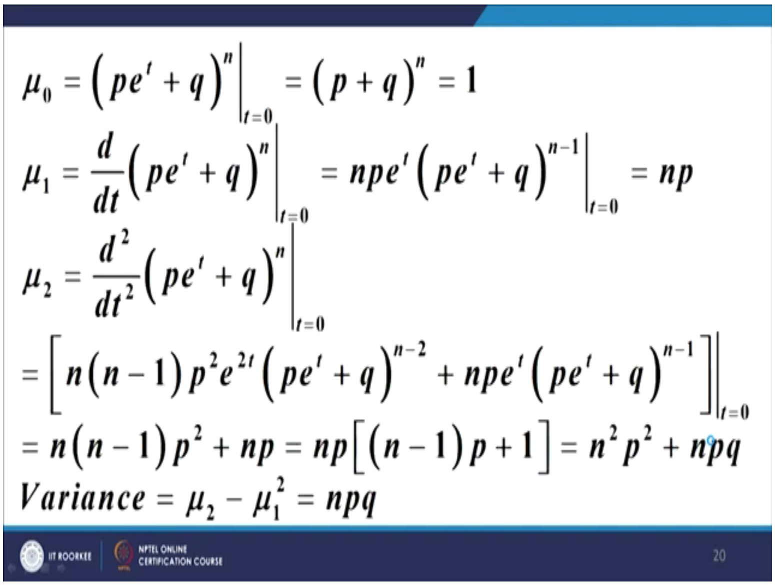

And the various movements can be obtained as I mentioned earlier by taking derivatives and putting t equal to 0.

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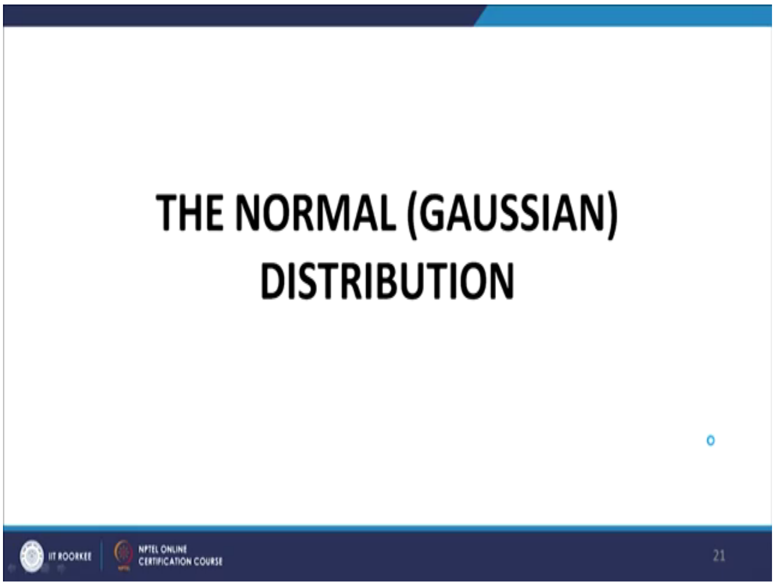

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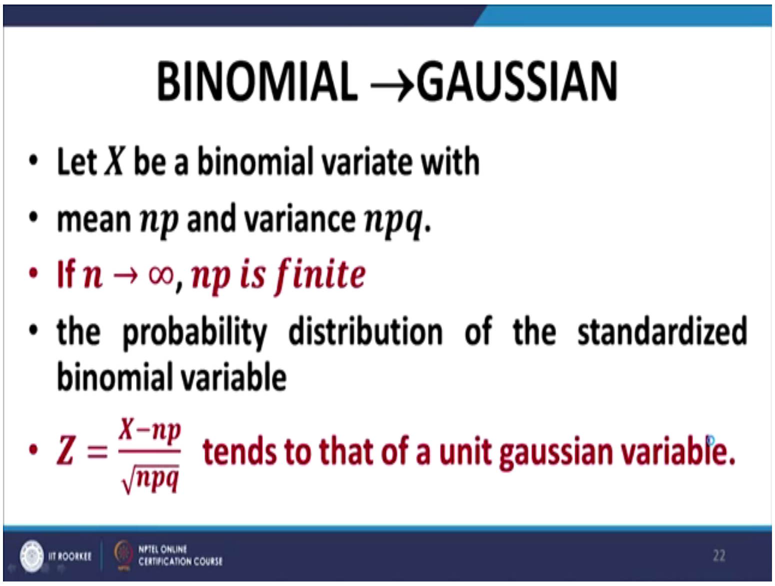

Then we moved over to the normal distribution. The normal distribution as I mentioned is the limiting case of binomial distributions, where n the number of Bernoulli trials tends to infinity; however, n p and n p q both remain finite.

So, that these are the 3 conditions under which the binomial distribution progresses to a continuous distribution which is also called the Gaussian distribution over the normal distribution.

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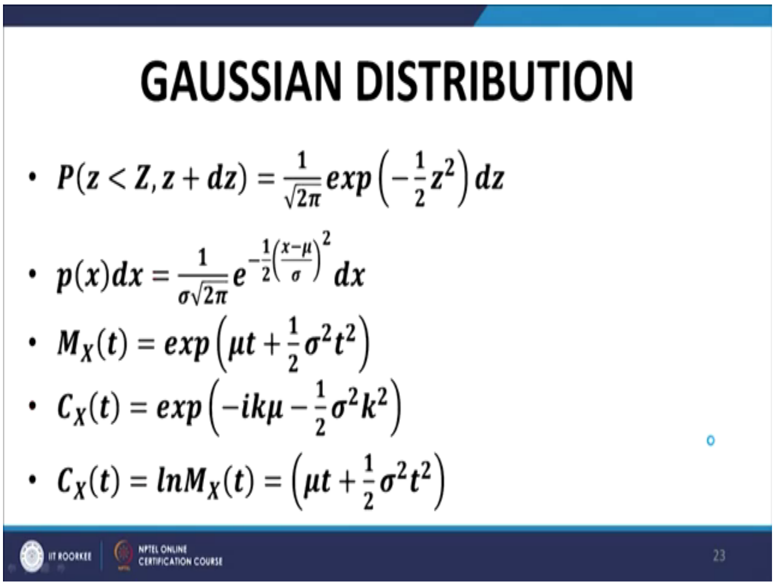

To arrive at the probability density function of the Gaussian distribution we made use of the Stirling formula for large factorials because n is tending to infinity. So, when you talk about nCr and so on, we are talking about large factorials which case have to be approximated and that is done through this Stirling approximation.

The probability density function of the standard Gaussian distribution is given by this first expression which is 1 upon root 2 pi exponential of minus 1 upon 2 z square. This is the probability density function of the standard normal variate. What is the standard normal variate? This standard normal variate is a normal variate having mean of 0 and a variance of 1.

The general normal variate has slightly more involved pdf which is given by this second expression. The moment generating function, we worked out last time for the normal distribution. The Gaussian distribution turns out to be the exponential mu t plus 1 by 2 sigma square t square.

The characteristic function is simply read, replace t by minus i k you get the characteristic function. And the cumulant function is the log of the moment generating function which is equal to mu t plus 1 by 2 sigma square t square.

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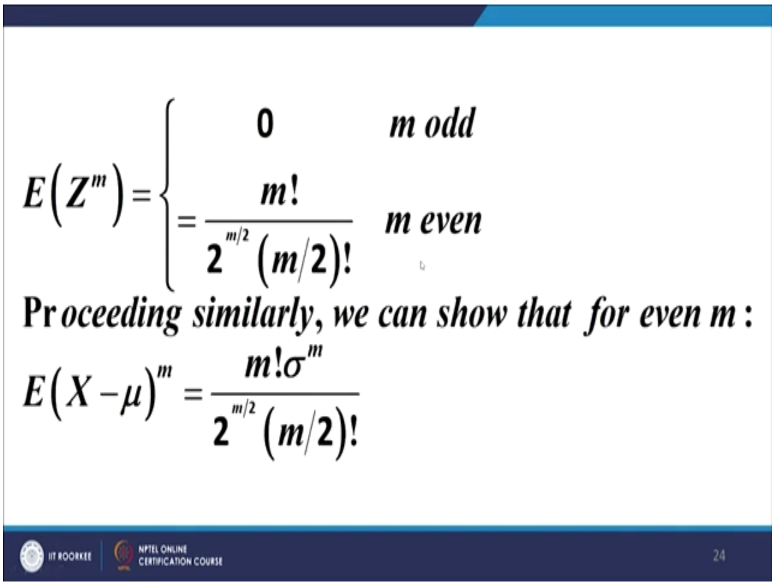

Then we worked out the various other moments of the normal distribution, the standard normal distribution. We found that because of the symmetry of the normal distribution about the origin and in fact, the Y axis we find that we find that the odd moments when they vanish and the even moments are given by this formula of m factorial divided by 2 to the power m by 2 into m by 2 factorial. The odd moments all the odd moments of the Gaussian distribution vanish even moments are given by this expression. This is for the standard normal distribution standard Gaussian distribution.

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$$M_X(t) = E\left(e^{tX}\right) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx - \frac{1}{2\sigma^2}(x-\mu)^2}\,dx$$

$$Now,\ tx - \frac{1}{2\sigma^2}(x-\mu)^2 = \frac{2\sigma^2 tx - (x-\mu)^2}{2\sigma^2}$$

$$= \frac{2\sigma^2 tx - x^2 - \mu^2 + 2\mu x}{2\sigma^2} = \frac{-x^2 - \mu^2 + 2x\left(\mu + \sigma^2 t\right)}{2\sigma^2}$$

$$= -\frac{\left[x - \left(\mu + \sigma^2 t\right)\right]^2 + 2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2}$$

$$= \mu t + \frac{1}{2}\sigma^2 t^2 - \frac{\left[x - \left(\mu + \sigma^2 t\right)\right]^2}{2\sigma^2}$$

How we arrived at these expressions? For example, the moment generating function we arrived at by working out this integral, in working out this integral we made converted the exponent of the exponent appearing in the integral as a perfect square.

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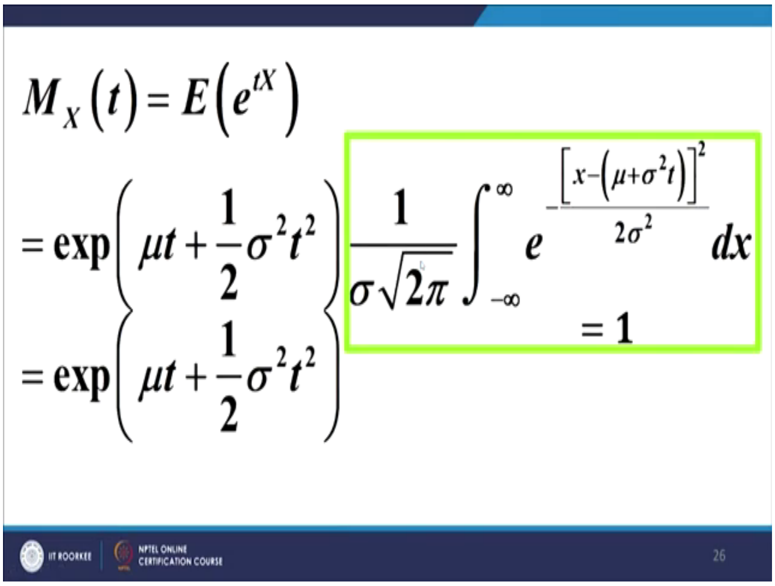

And then if you look at this carefully, if you look at this expression within the box this is a shifted normal distribution. You are simply shifted the origin by a certain length certain distance. Otherwise the normal distribution remains unchanged, the limits remains unchanged and therefore, this whole expression is the value of pdf integrated over minus infinity to infinity which has to be equal to 1.

So, this whole expression within the green box works out to 1 and we are left with exponential mu t plus 1 by 2 sigma square t square which is the moment generating function of the normal distribution with a mean of mu and a variance of sigma squared.

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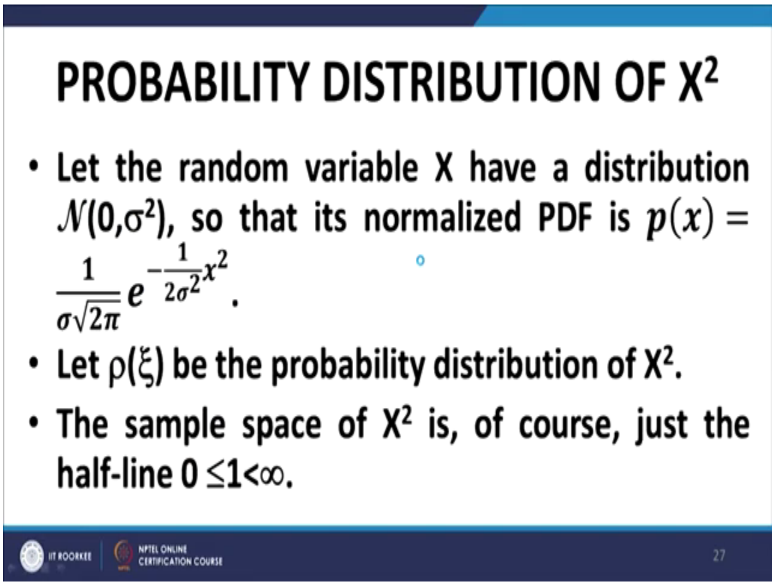

Then let it know now let us take a problem. Let us work out the probability distribution of X square, where X square is has a normal distribution with a mean of 0 and a variance of sigma square. Then its probability density function is given by this expression: p x is equal to 1 upon sigma root 2 pi e to the power minus 1 upon 2 sigma square into x squared.

Let us assume we need to work out the probability distribution of X squared. Let us say assume that xi is equal to X square, xi is equal to X square; xi is the random variable which represents X square then we need to work out the probability density function of xi. Let us call it rho of xi.

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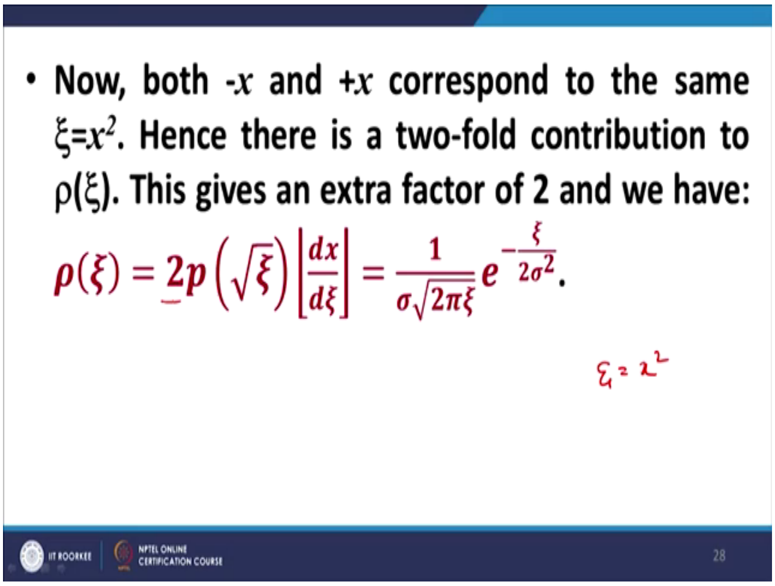

Clearly the sample space of xi will lie from 0 to plus infinity be being a square and being a real number it will not have any values, it will not take any values between minus infinity and 0 it will have values all values of xi will lie in the interval 0 to infinity which constitutes its sample space.

Now, both you know the important thing is you see what are we saying? We are saying xi is equal to x squared; that means, minus x and plus x both are going to yield the same value of xi. So, naturally whatever the probability of xi is going to be is going to be and it is symmetric. It is symmetric for minus x and plus x.

Therefore, if I select a particular value of xi then that value can be achieved by x taking the value of minus x and x taking the value of plus x. For example, if I want a certain specific value of xi then the ways in which that value of xi can be achieved is by taking the value x taking the value minus x and x taking the value plus x.

Therefore, the probability will have to be doubled because the same xi can be achieved in two ways and therefore, we have this factor of 2. Then the rest is quite simple. We need to work out p xi; xi is nothing, but under root of xi is nothing, but x square. Therefore, x is nothing, but xi under root say xi. So, instead of x we substitute under root xi and we introduced this dx upo d dz to change the variables from x to xi and we end up with this simple probability density function for xi right.

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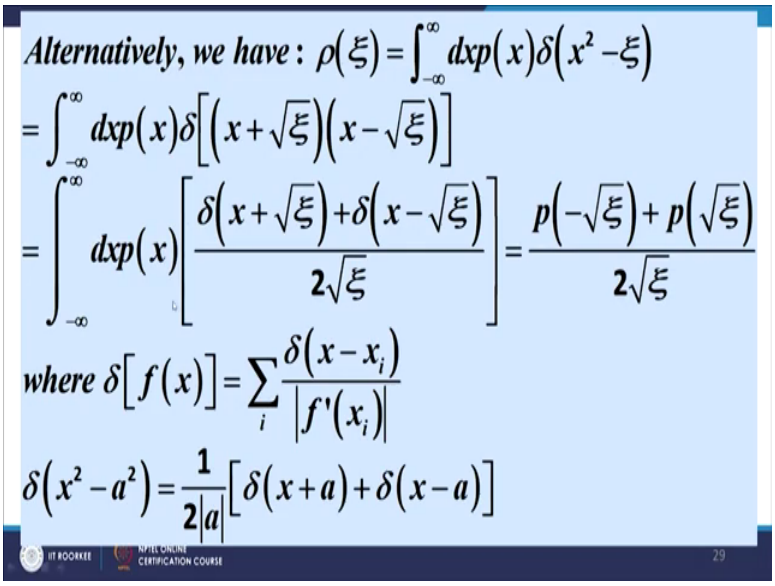

Now, and this can be achieved in a different way in a more explicit manner. The probability density function of xi rho of xi will be you see now xi is equal to x square, but we want a certain value of xi which is represented by this particular value xi and this xi we need all those values of x such that we get this value of xi waited by the respective probabilities.

All the values of x that end up with giving a value of xi, but waited with the respective probabilities subject to the condition subject to the condition that x square must be equal to xi. That is our requiring we cannot take any arbitrary value of x and try to find out the combine it with another value of x to find to arrive at the value of xi. The combination rule is that x square must be equal to xi.

How to incorporate this x square equal to xi condition? This constraint in determining the probability density function is by introducing the direct delta function; delta of x square minus xi. This automatically will ensure that only those values of x are selected only those values are x are selected when we do this integral only those values of x are selected such that x square is equal to xi.

So, we in cooperate this condition by the delta function and this delta function can be written in this form. x delta of x plus root xi into x minus root xi and using the this property of delta functions I can write them in the form of 2 summed upto 2 delta functions. Delta x plus root xi plus delta x minus root xi divided by 2 root xi.

So, when I do the integration due to the delta function and we end up with this particular expression; p of minus root xi plus p of plus root xi divided by 2 root xi.

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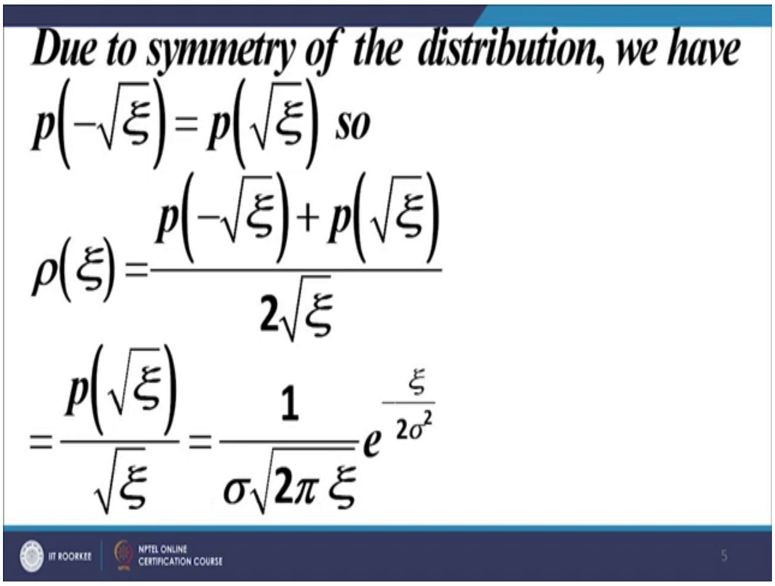

Because p of minus root xi and p of plus root xi are the same because of the symmetry of the distribution, this can be written as 2 p of root xi which is and this 2 and 2 cancel out, we are left with p of root xi upon root xi which when we substitute the value in the pdf of x equal to root xi we put then we get this expression for the probability density function of xi, which is exactly the same as we got earlier by a more intuitive and a shorter reasoning.

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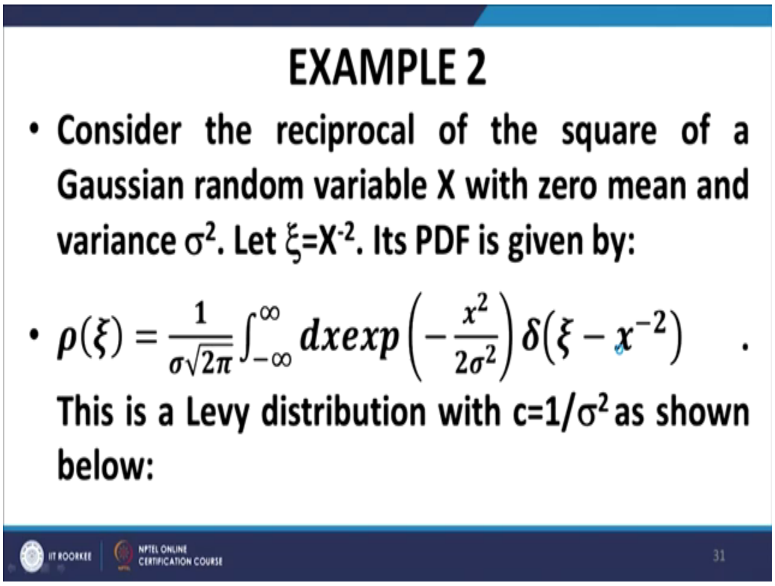

Let us there is another example that I have put here. In this case we were we worked out with the procedure is more or same. In this case we are talking about a variable X which is again a randomly distributed has a Gaussian distribution with a mean of 0 and a variance of sigma square. We try to work out the pdf of xi. Xi is given by X to the power minus 2. We have just done an expression where xi was equal to X squared, now we work out the distribution of xi equal to 1 upon X square.

So, again we apply the same approach. Rho of xi will be equal to this expression. Now in this case again we introduced a delta function to ensure that this constraint is met. Xi is equal to X to the power minus 2 that constraint, we have incorporated by feeding in a delta function xi minus x to the power minus 2 and integrating over x the rest is the pdf of the of the variable x which is a normal distribution with a mean of 0 and a variance of sigma square.

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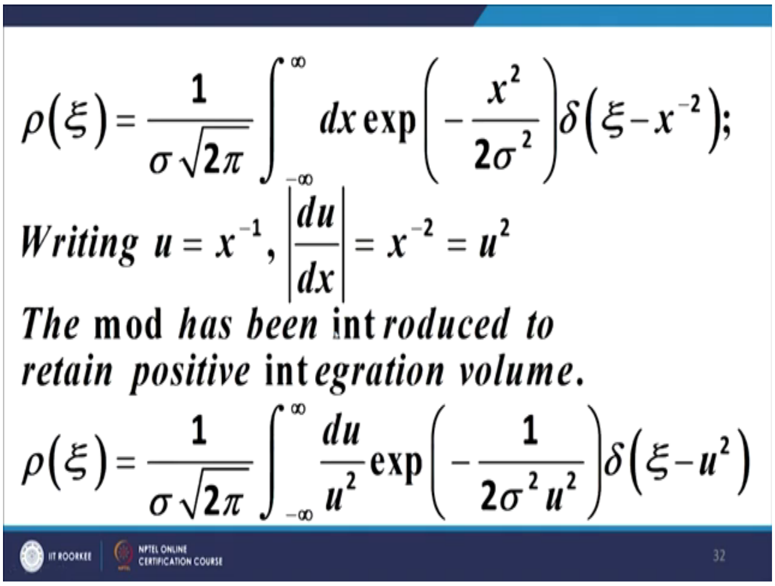

Again we simplify in the same way. We write u is equal to x to the power minus 1 x inverse and we have mode of du upon dx is equal to x to the power minus 2 that is equal to u square. We use mode here because we want to retain the positivity of the integration volume. So, we use the mode of du by dx rather than the expression for du by dx with this sign.

The rest is once you incorporate change the variable from x to u, you get this expression du upon u square exponential of this whole thing into; now the delta function in terms of u becomes delta xi minus u square which is precisely of the form that we had in the earlier problem.

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$$\rho(\xi)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{du}{u^2}\exp\left(-\frac{1}{2\sigma^2u^2}\right)\delta\left(\xi-u^2\right)$$

$$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{du}{u^2}\exp\left(-\frac{1}{2\sigma^2u^2}\right)\delta\left[\left(\sqrt{\xi}-u\right)\left(\sqrt{\xi}+u\right)\right]$$

$$=\frac{1}{\sigma\sqrt{2\pi}}\left(\frac{1}{2\xi^{1/2}}\right)\int_{-\infty}^{\infty}\frac{du}{u^2}\exp\left(-\frac{1}{2\sigma^2u^2}\right)\delta\left[\left(\xi^{1/2}-u\right)+\left(\xi^{1/2}+u\right)\right]$$

$$=\frac{1}{\sigma\sqrt{2\pi}}\left(\frac{1}{2\xi^{1/2}}\right)\left[\frac{2}{\xi}\exp\left(-\frac{1}{2\sigma^2\xi}\right)\right]=\frac{1}{\sigma\sqrt{2\pi}}\left(\frac{1}{\xi^{3/2}}\right)\exp\left(-\frac{1}{2\sigma^2\xi}\right)$$

So, we simplify this delta function xi minus u square. Put it as a product of xi minus under root xi minus u into a under root xi plus u and then we split it up into a sum of 2 delta functions delta of xi under root xi minus u plus delta of under root xi plus u. Of course, there will be an additional factor coming in the denominator; 1 upon 2 xi to the power 1 by 2, that is under root xi.

We carry out the integration making use of these delta functions. It becomes quite simple. Simply replace the u by xi under root xi plus minus as the case may be in the case of the 2 delta functions.

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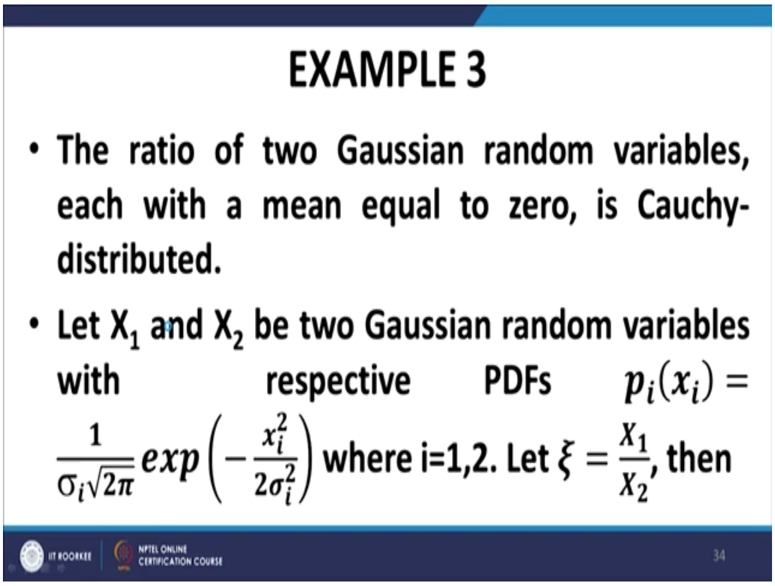

And again we end up with a result which is again because of the symmetry we end up with this result: 1 upon sigma root 2 pi 1 upon xi to the power 3 by 2 exponential minus 1 upon 2 sigma square xi. So, this is the probability distribution of xi equal to 1 upon x square and this is a level distribution.

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$$\rho(\xi)=\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\,p_1(x_1,t)\,p_2(x_2,t)\,\delta\left(\xi-\frac{x_1}{x_2}\right)$$

$$=\frac{1}{2\pi\,\sigma_1\sigma_2}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\,e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}\right)}\delta\left(\xi-\frac{x_1}{x_2}\right)$$

$$=\frac{1}{2\pi\,\sigma_1\sigma_2}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\,e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}\right)}\delta\left[x_1\left(\frac{\xi}{x_1}-\frac{1}{x_2}\right)\right]$$

$$=\frac{1}{2\pi\,\sigma_1\sigma_2}\int_{-\infty}^{\infty}\frac{1}{|x_1|}dx_1\int_{-\infty}^{\infty}dx_2\,e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}\right)}\delta\left(\frac{\xi}{x_1}-\frac{1}{x_2}\right)$$

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Setting $u = \frac{1}{x_2}$ *so that* $du = -\frac{1}{x_2^2}dx_2 = -u^2 dx_2$

$$\rho(\xi) = -\frac{1}{2\pi\,\sigma_1\sigma_2}\int_{-\infty}^{\infty}\frac{1}{|x_1|}dx_1\int_{-\infty}^{\infty}\frac{du}{u^2}e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2}+\frac{1}{\sigma_2^2u^2}\right)}\delta\left(\frac{\xi}{x_1}-u\right)$$

$$= -\frac{1}{2\pi\,\sigma_1\sigma_2}\int_{-\infty}^{\infty}\frac{1}{|x_1|}\frac{x_1^2}{\xi^2}e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_1^2}{\sigma_2^2\xi^2}\right)}dx_1 = -\frac{1}{2\pi\,\sigma_1\sigma_2\xi^2}\int_{-\infty}^{\infty}|x_1|e^{-\frac{1}{2}(ax_1)^2}dx_1$$

where $a^2 = \left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2\xi^2}\right)$.

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Setting $ax_1 = z,\ dz = a\,dx_1,\ \rho(\xi) = -\frac{1}{2\pi\,\sigma_1\sigma_2\xi^2|a|a}\int_{-\infty}^{\infty}|z|e^{-\frac{1}{2}z^2}\,dz$

$$= -\frac{2}{2\pi\,\sigma_1\sigma_2\xi^2|a|a}\int_{0}^{\infty} z e^{-\frac{1}{2}z^2}\,dz = -\frac{2}{2\pi\,\sigma_1\sigma_2\xi^2|a|a}\int_{0}^{\infty} e^{-t}\,dt$$

$$= \frac{1}{\pi\,\sigma_1\sigma_2\xi^2|a|a} = \frac{\sigma_1\sigma_2}{\pi\left(\sigma_1^2+\sigma_2^2\xi^2\right)} = \frac{\sigma_1/\sigma_2}{\pi\left(\sigma_1^2/\sigma_2^2+\xi^2\right)} = \frac{\lambda}{\pi\left(\lambda^2+\xi^2\right)}$$

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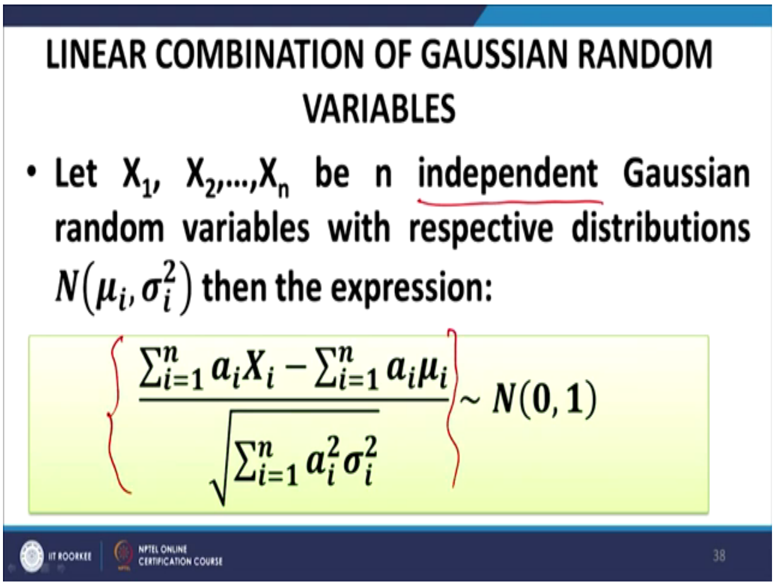

Then we have another example here. Ratio of two Gaussian random variables; I leave it as an exercise. The procedure is more or less the same as you have done in the case of the two examples. It is solved in the presentation and those interested can work it out for themselves that will give more practice work absolutely on similar lines to the two examples that I have discussed just now.

Now, we talk about a linear combination of Gaussian random variables or let us start with a finite case. A finite linear combination of Gaussian random variables X 1, X 2, X 3, up to X n each of them is independent of the other.

This is very important condition. The random variables are Gaussian variables are independent and let us assume that they are respective means and variants are given by mu i and sigma i square that is x i is normally distributed with a mean of mu i and a variants of sigma i squared and it is independent of all the other x j, where j is an equal to i.

We need to find the distribution of this expression and establish that this expression is also normally distributed with a mean of 0 and a standard deviation or a variance of 1 which is it is distributed as a standard normal variant this whole expression.

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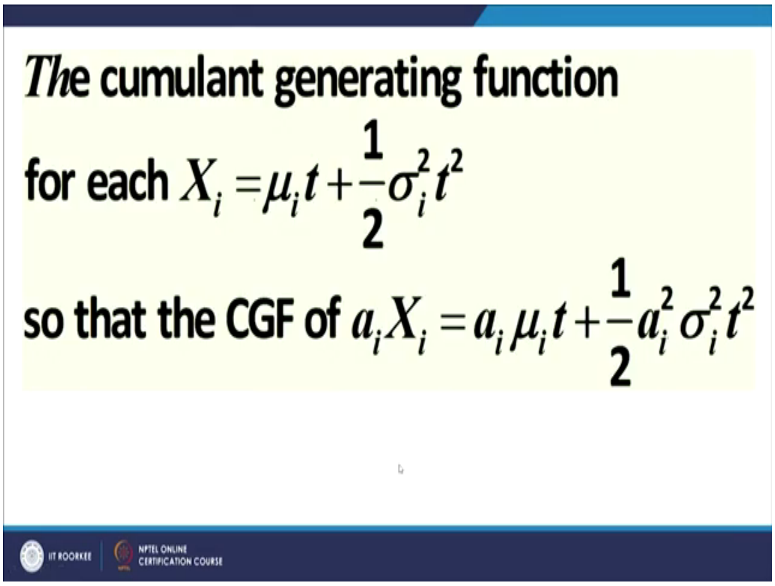

So, let us work it out. It is quite simple. Now, the cumulative we can use the cumulant generating function. In fact, you can use we use that particular property of cumulant generating function that I had eluded to earlier that is the cumulant generating function of a sum of independent random variables is equal to the sum of the cumulant generating functions of various random variable.

So, let us define let us take any of this random variables X i, it has the cumulant generating functions mu i t plus 1 by 2 sigma i square t square. And because a is deterministic all the coefficients a are deterministic their real numbers without any randomness and therefore, they do not have any variants what you they do not have any varinats they are constants.

In fact, and therefore, we also the cumulant generating functions of a i X i is given by a i mu i t is simply a scaling factor actually. So, a i mu i t plus a I square sigma i square into t square.

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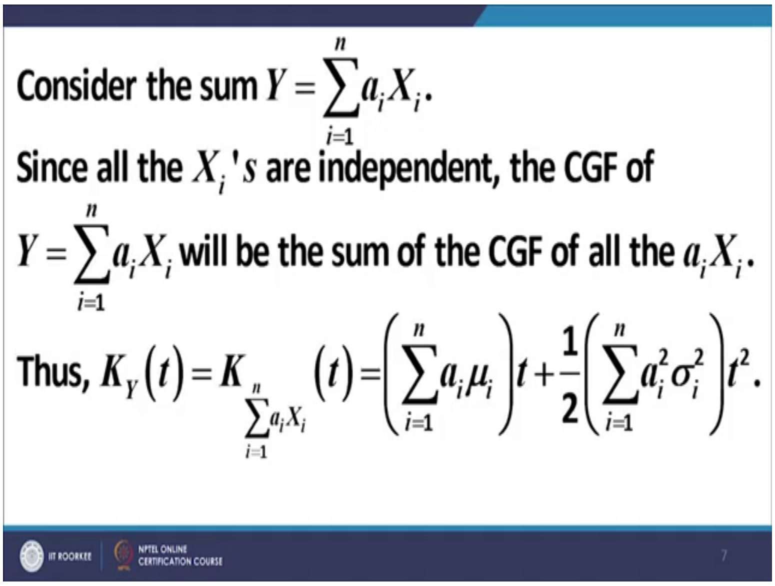

Now, we consider the sum of the linear combination of the n random variables that we have a 1 X 1 plus a 2 X 2 plus a 3 X 3 up to a n X n. Let us call it Y. Then because all the random variables are independent, the cumulant generating functions of Y will be equal to the sum of the cumulant generating functions of each of these terms; a 1 X 1 a 2 X 2 a 3 X 3 and so on.

In other words, the cumulant generating function of Y is given by the cumulant generating functions of sigma a i X i, where so that will be equal to sigma a i mu i into t plus 1 by 2 sigma a i square sigma i square into t square because this is Y is given by this expression.

So, because all these terms a 1 X 1 is independent of a 2 X 2 is independent of a 3 X 3, therefore, the cumulant generating function of Y is equal to the sum of cumulant generating functions of all these terms.

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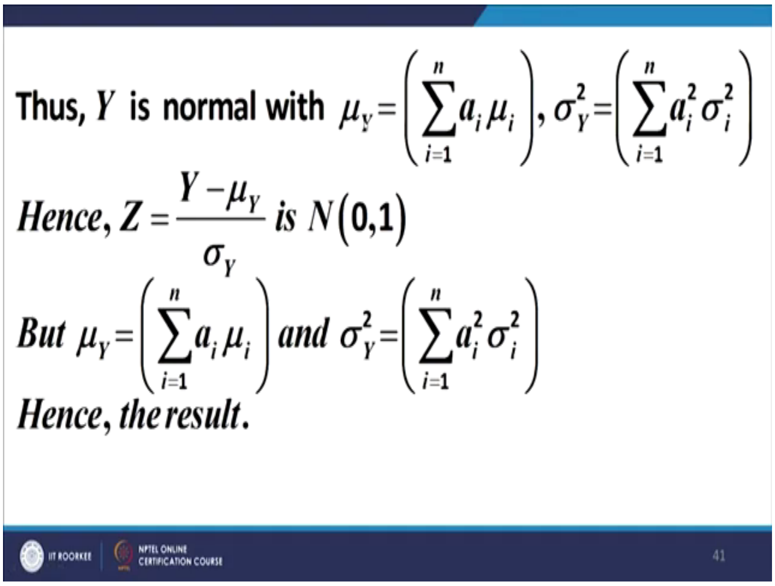

Now, what does it mean? It means that Y is normally distributed with a mean of this expression and a variants of this expression right. I repeat. Y what is Y? Recall that. Y is equal to this expression. This is Y.

Y is normally distributed with a mean of this and a variance of this and that implies that if I define another variable Z as Y minus mu of Y upon sigma of Y and that would be normally distributed as a standard normal variant standard Gaussian variant and that is precise the what we had to prove. If you substitute Y equal to that expression mu Y equal to this expression sigma Y square equal to this expression you precisely get what we wanted to establish.

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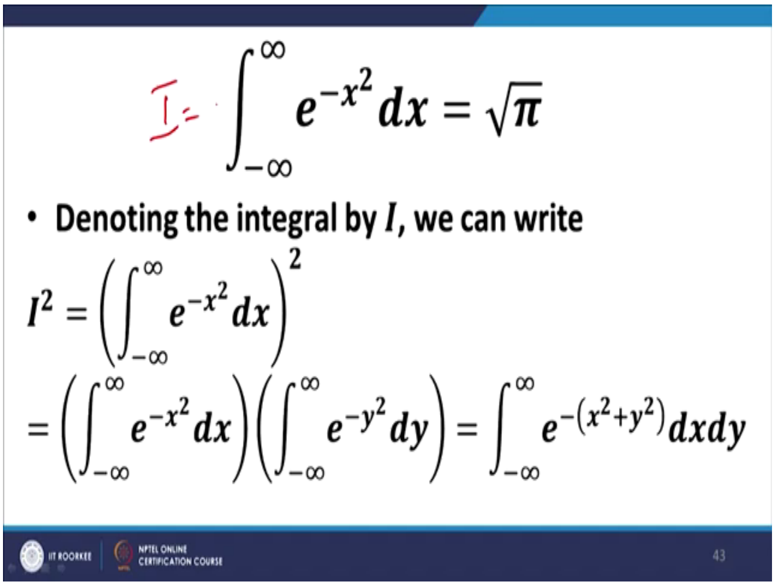

Now, we talk the Gaussian integration that is a very important that forms the premise of almost all path integrals. You see there are two fundamental ways of solving path integrals in quantum mechanics and quantum field theory.

Either you use a perturbation expansion or where they where possible where you end up with some kind of a Gaussian integral you use this solution by Gaussian integral. So, this form this integration of Gaussian or integrals which can be converted to some form of Gaussian integration forms the cornerstone of the theory of path integrals in quantum mechanics in quantum field theory.

So, let us look at this particular case very carefully. We start with a very simple case. We start with the integration of exponential of minus x square dx within the limits minus infinity to plus infinity. It is a classic integral. What we do is we write this integral the product of this integral with itself twice over.

So, what we get is let us call this integral as I. Let us call it I. Then we have I squared is equal to this integral square and which can be written as integral of e to the minus x square dx integral of e to the power minus y square dx and which can be written as integral of e to the power minus x square plus y square dx dy.

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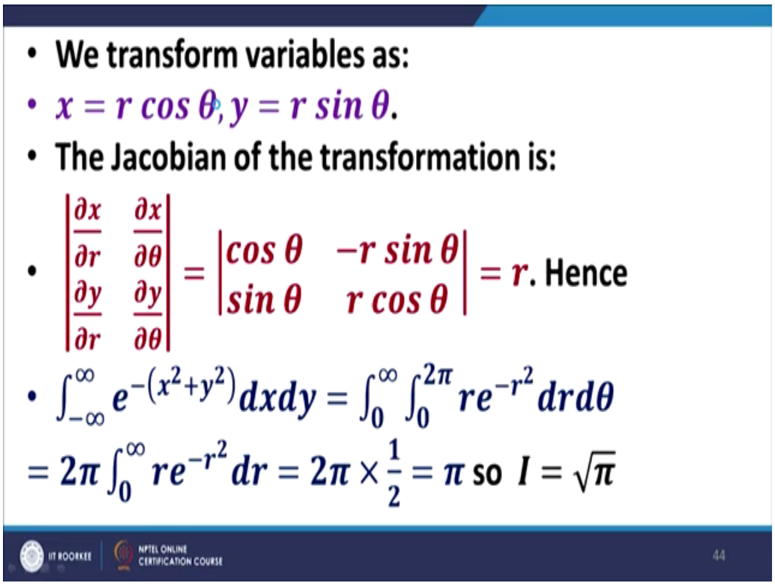

Now, let us do a transformation of variables. We from the Cartesian coordinates to polar coordinates in two dimensional space; x is equal to r cos theta, y is equal to r sin theta. Jacobian as you can see and indeed is very well known the Jacobian turns out to be r and therefore, our integral that is I square remember. I square is equal to integral 0 to 2 pi d theta and then integral 0 to infinity r e to the power minus r square.

r e to r is the component is the part that arises from the Jacobian; e to the power minus r square is nothing, but minus x square plus y square by substituting r cos theta equal to x and r sin theta equal to y and dr d theta are expressed in terms of dx dy by making use of the Jacobian of this transformation.

When you seems now this is elementary simply the integral over theta can be carried out to give 2 pi and the integral over r gives us 1 upon 2 and we end up with pi. Therefore, pi square therefore, I square is equal to pi or I is equal to root pi. So, this is quite a simple integral, but its very very important.

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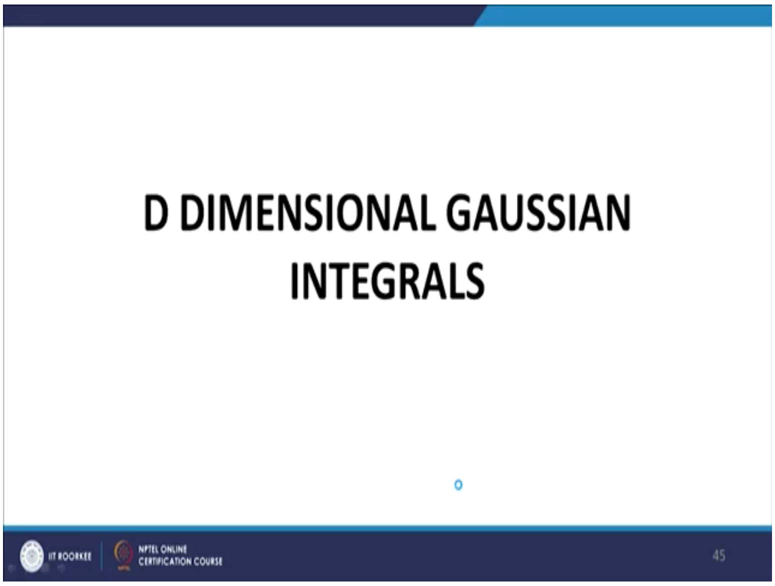

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$$(1)\int d^D y \exp\left(-\frac{1}{2}y^T Ay\right)=(2\pi)^{D/2}\exp\left(-\frac{1}{2}Tr\ln A\right)$$

$$(2)\int d^D y \exp\left(-\frac{1}{2}y^T Ay+\rho^T y\right)=(2\pi)^{D/2}e^{-\frac{1}{2}Tr\ln A}e^{\frac{1}{2}\rho^T A^{-1}\rho}$$

$$(3)\int d^D y y_{k_1}\ldots y_{k_n}\exp\left(-\frac{1}{2}y^T Ay\right)$$

$$=(2\pi)^{D/2}e^{-\frac{1}{2}Tr\ln A}\left(A^{-1}_{k_1k_2}\ldots A^{-1}_{k_{n-1}k_n}+permut.\right)$$

This is an integral one dimensional space. Now, we come to integration in D dimensional spaces where we have to deal with matrices. There are three fundamental properties that we need to handle here. The first is the equivalent of the integral that we have just done where we had numbers to contain with here we have matrices to content with, y is the column is a column vector and A is a real symmetric matrix and we need to establish these results, rho is also a column vector.

So, these are 3 results which I proposed to explain because not only as a results important the methodology that is followed also gives us a lot of insights that we are going to use in the future.

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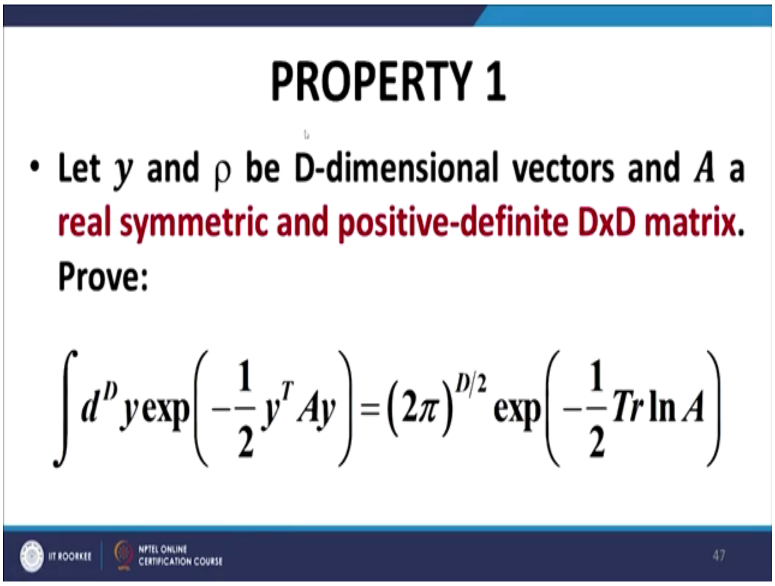

So, let us start with property 1. This is property 1. Recall the first one is the equivalence of what we have just done in one dimensions. So, y and rho are D dimensional vectors and A is a real symmetric, positive definite three properties; real symmetric each of these will have a certain relevance as you will see gradually. Real symmetric, positive definite, D cross D matrix we need to prove this.

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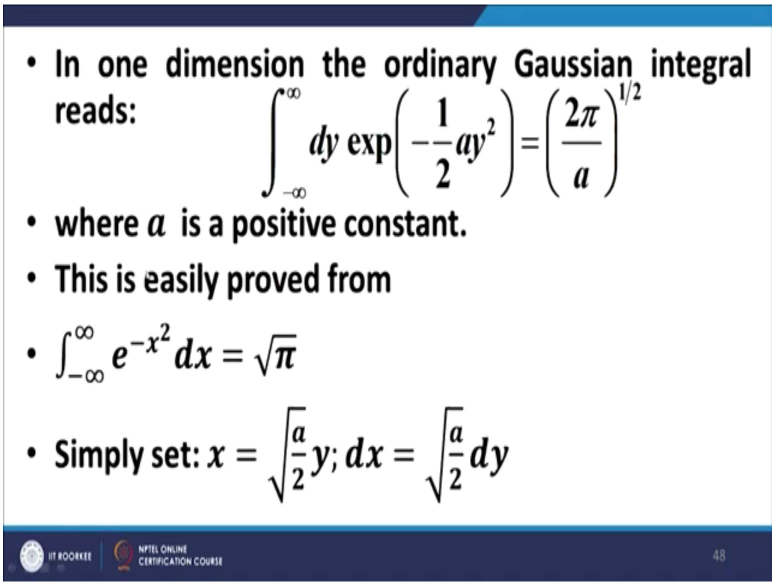

Now, in one dimension you see what we proved just now? We proved just now the that integral of e to the power minus x square is equal to root pi; this is what we have proved just now. If we simply sit if we simply make these changes it a change of variables x equal to under root a by 2 into y dx; obviously, becomes under root a by 2 dy. If you make this substitutions you end up with this expression quite straightforward.

It is quite elementary, you simply make a substitute a of variables and you get this result. So, this is as far as the one dimensional case is concerned.

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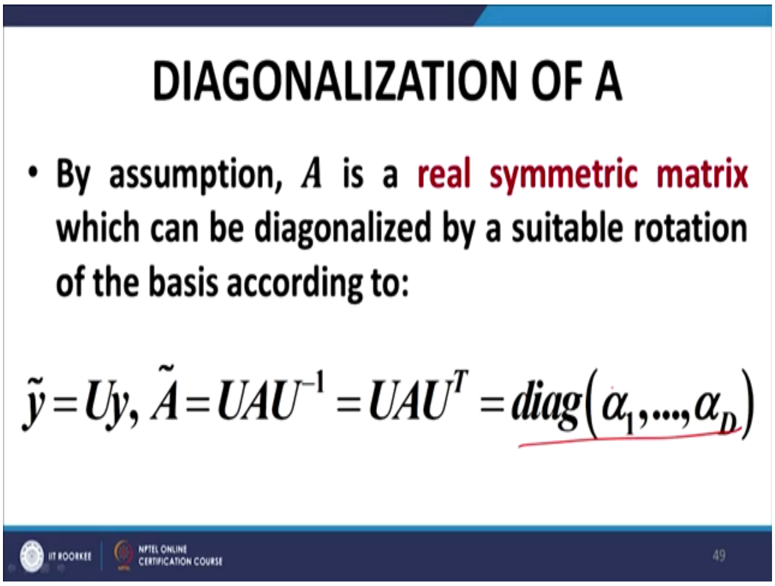

Now, the now because A is real symmetric matrix therefore, we can diagonal and diagonalized by a orthogonal by a rotation of the basis. In a by a rotation of the basis we can convert A to a diagonal matrix and the elements of those diagonal matrices are nothing, but the eigenvalues of the matrix A.

So, that is how we proceed the first step is to diagonalise our matrix A, which is done by a orthogonal transformation or rotation transformation of the basis let us call it U. So, the transformation operates on the various on the matrix and the various column vectors in the following way which is given here.

And we end up with a diagonal matrix which is given by this expression where alpha 1, alpha 2 and upto alpha D are the eigen values of the matrix A.

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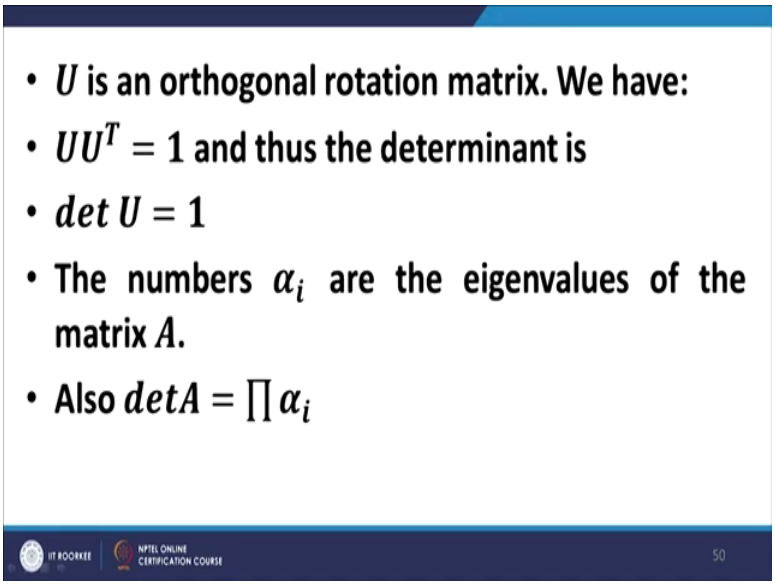

Now, because U is an orthogonal rotation matrix UU transpose is equal to 1 and immediate corollary of this is the determinant of U is also equal to 1. And the number a I are the eigenvalues and therefore, the determinant of A, the determinant of A is the product of these eigenvalues it is the product of alpha i. So, these are fundamental constituents which results which are obtained from matrix theory, very elementary results.

First is you have a rotation matrix. You then U into U transpose were gives you the identity matrix or the U transpose is the inverse of U and then that determinant of U is equal to 1, for every rotation matrix because the length of the vector needs to be unchanged because of a rotation due to a rotation. Then the diagonal matrix as real symmetric matrix can be diagonalized and the diagonal matrix the elements of the diagonal matrix are the eigenvalues. And the product of the eigenvalues is equal to the determinant.

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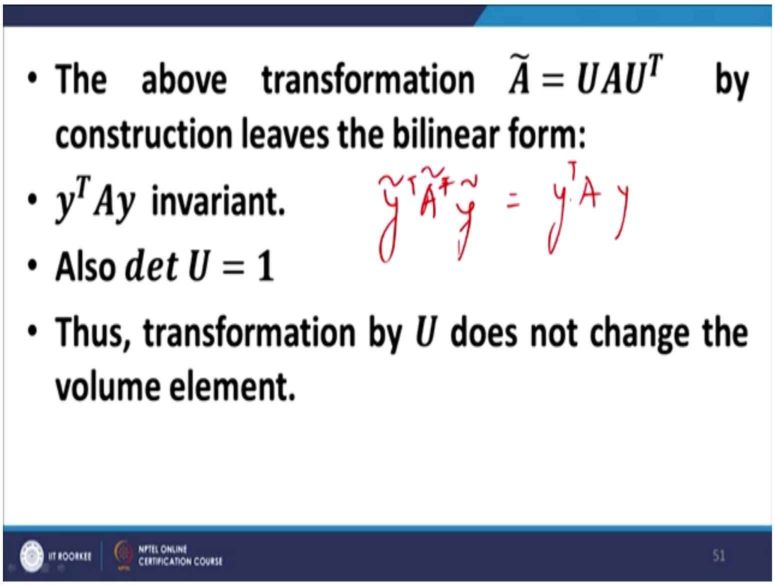

So, these are certain properties that we borrow from matrix theory. Now, the important thing is this transformation A tilde from A that is obtained through a rotation is does not change the bilinear form y transpose A y.

You simplicity substitute the value of y tilde transpose, substitute the value of A and substitute the value of y tilde. In other words you simply work it out y tilde transpose A transpose A tilde transpose sorry A tilde and a y tilde and you find that this is equal to y A y transpose A y.

It is quite simple. Just put the values. The U transpose will cancel out giving you the identity matrix and this is what you will get right.

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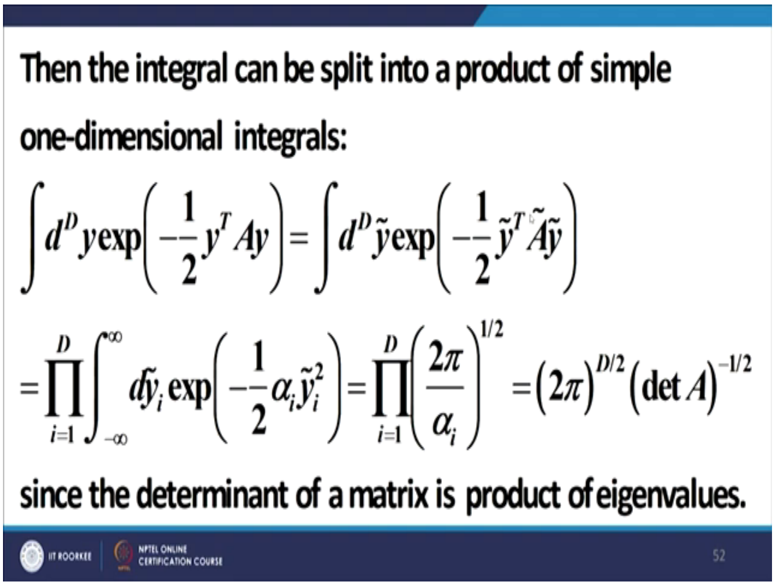

So, now the important thing comes. Now, let us look at this integral. This expression this bilinear form y transpose Ay as I mentioned earlier is not affected is invariant under the transformation which is we are comes under the transformation which is precipitated by U. And therefore, the and the second thing is; second thing is because determinant U is equal to 1, therefore, the transformation does not change the volume element either.

So, two things are important. y transpose Ay is invariant under the transformation and number 2 that the integration element the integration volume element is also invariant under the transformation. So, we can simply write our given integral in the form of the transposed by linear element y tilde t A tilde y tilde and the integration element d D y can be written as d D y tilde.

Now, the important thing is this is a diagonal matrix. A tilde is the diagonal matrix and because A tilde is the diagonal matrix this simplification immediately arises because the only elements of A tilde are the diagonal elements which are the eigenvalues of the matrix.

So, we can convert this integral to a product of we can separate this integral into a d integrals each corresponding to one eigenvalue of the matrix A tilde, we will continue from here.

Thank you.