**Path Integral Methods in Physics & Finance**
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**Lecture – 49**
**SDE for Field Theory in Minkowski Space**

Welcome back. So, in the last lecture we talked about the causality structure of the Feynman propagator in the Klein Gordon field and we also discussed the Schwinger Dyson equation. Let us quickly review the two problems and then we will move on to the path integral formulation for the firm ions.

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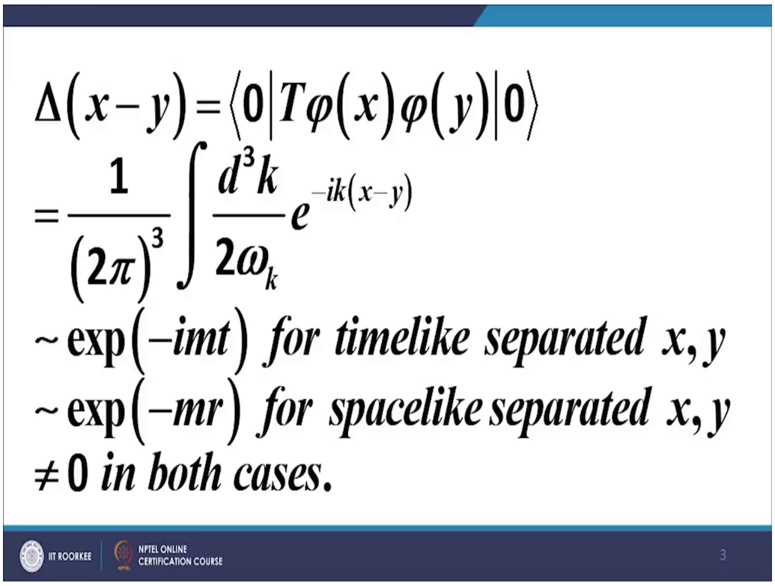


So, the causality for the Feynman propagator what we did was; we obtained the expression for the time order product in terms of the Feynman propagator and we found that the behaviour of the propagator approximated that the behaviuor of a exponential minus m r, it approximated the behaviour of exponential minus m r for space like separation between x and y.

It was clear that the propagated does not vanish in both cases the propagator has a nonzero value in both cases, even at even for space like separations between x and y.

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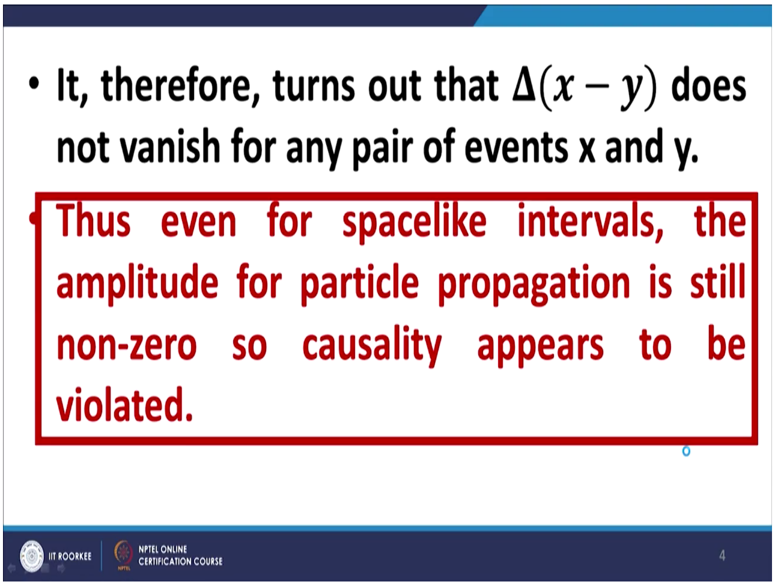


So, this was an apparent violation of causality. We decided to investigate it further and for that purpose we obtained the commutators between the field operators at the points x and y.

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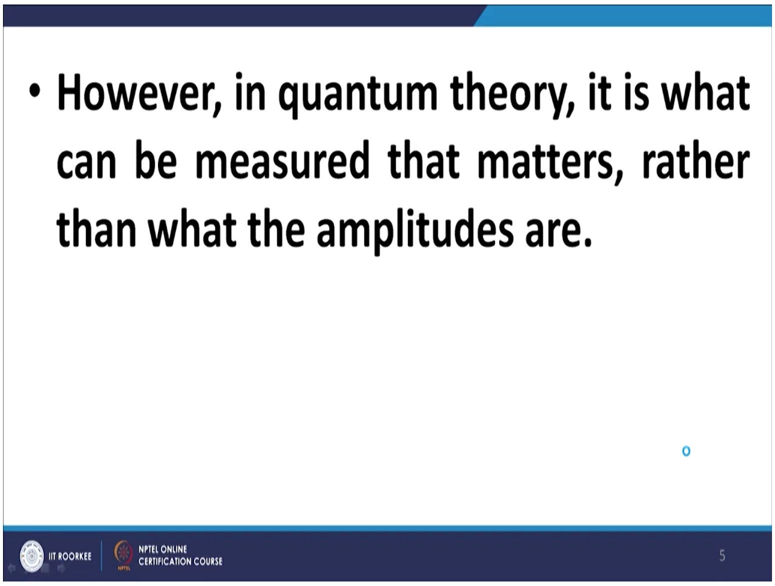


So, because in essence the in the context of quantum theory, it is not, it is the quantities that are can be measured, the attributes that can be measured that that matter rather than what the amplitudes are, the amplitudes can afford to be nonzero provided, the measurement is compatible with the causality structure.

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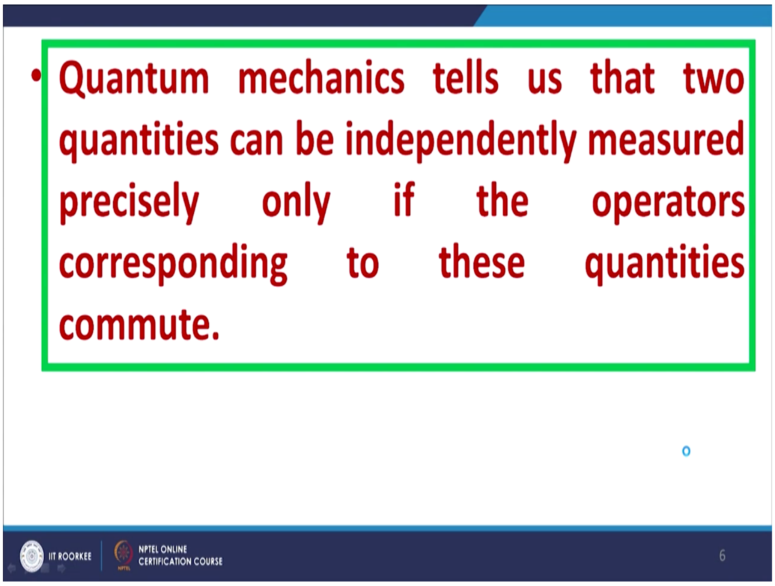


So, that is what we proceeded to investigate by obtaining the commutators of the field operators at the points x and y and what did we find? Let us see.

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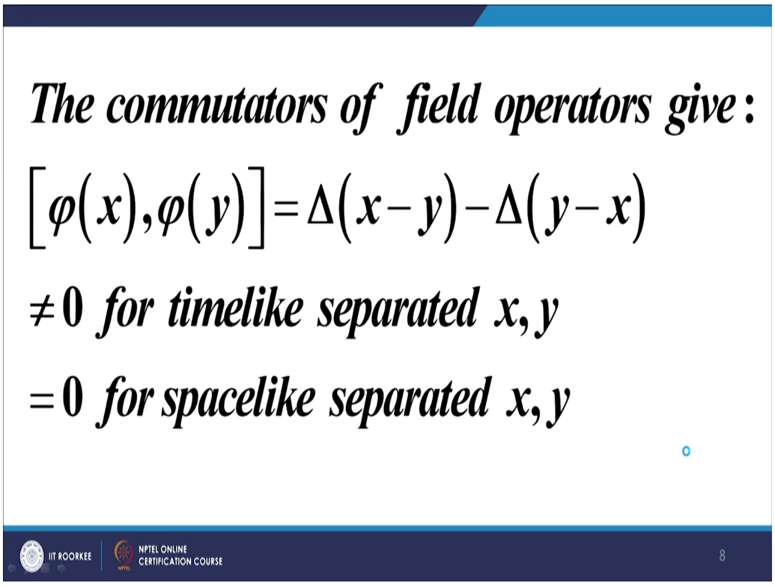


What we found was that the commutators between x and y for time like separations does not vanish; however, when we worked out the commutator between phi x and phi y for space like separation we found that it vanish. The fact that in the is the commutator between this field operator that x and y for space like separation vanishes implies what? It implies that simultaneous measurements at the points of the field operators at the point x and point y can be made and; that means, in other words the measurement at one point does not influence or cannot influence the measurement at the other point provided the two points are space like separated and that incidentally is compatible with the causality structure that we end which is to proof.

So, at the end of the day what we conclude is that the quantum structure or the field structure presented through the Feynman propagators or the path integral formalism is compatible, at least for the Klein Gordon field is compatible with the causality structure and we searched in special directivity.

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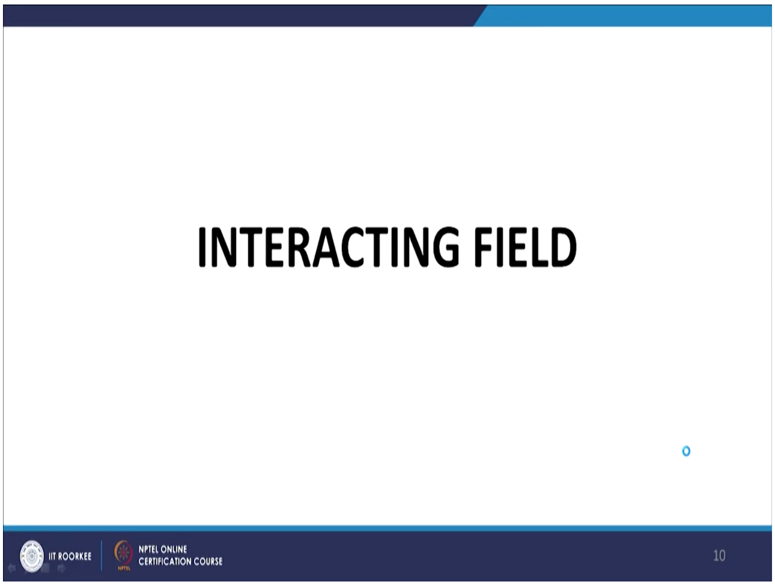

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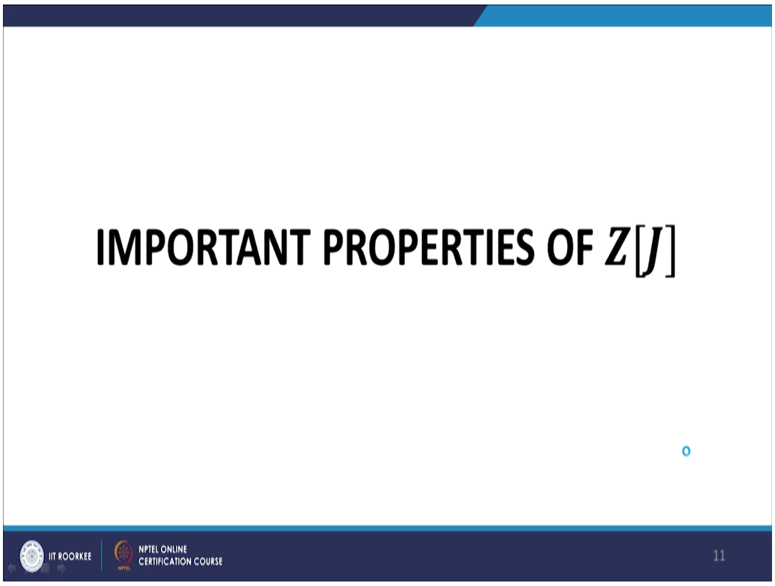


Now, then thereafter, we started talking about the interacting field and we obtained the expression for the generating functional for the full green functions of the of Z J and for the interacting field interacting Klein Gordon field.

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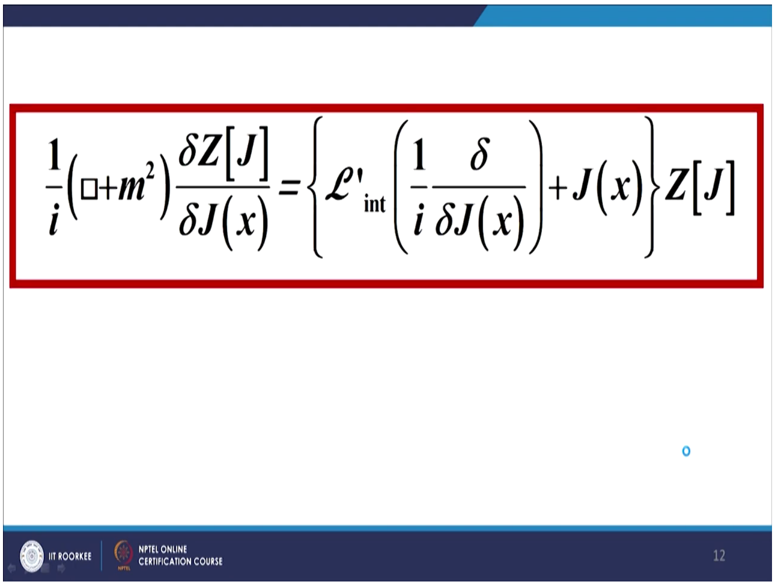


And then we obtained certain the certain properties of this Z J the one of the first property that we obtained is given in the red box here.

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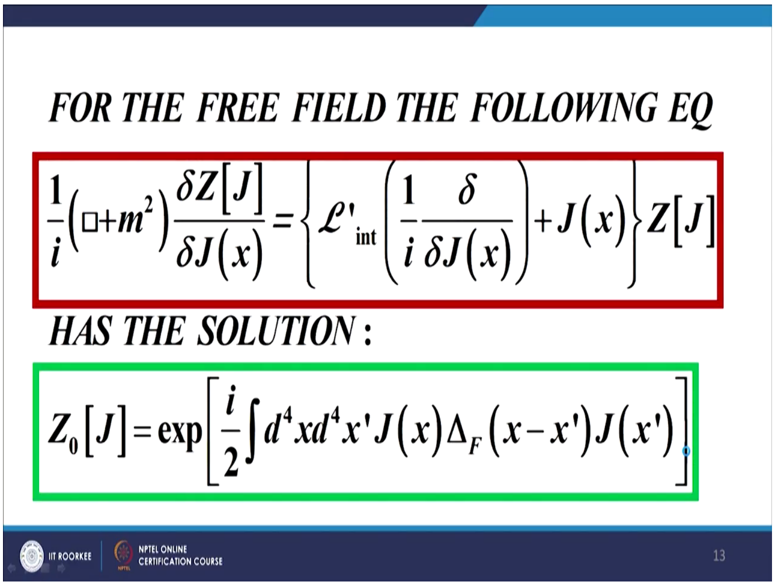


At the and the second property is given here for this is for the free field; for the free field, we have the solution for the generating functional for the full green functions given in the green box at the bottom of your slide.

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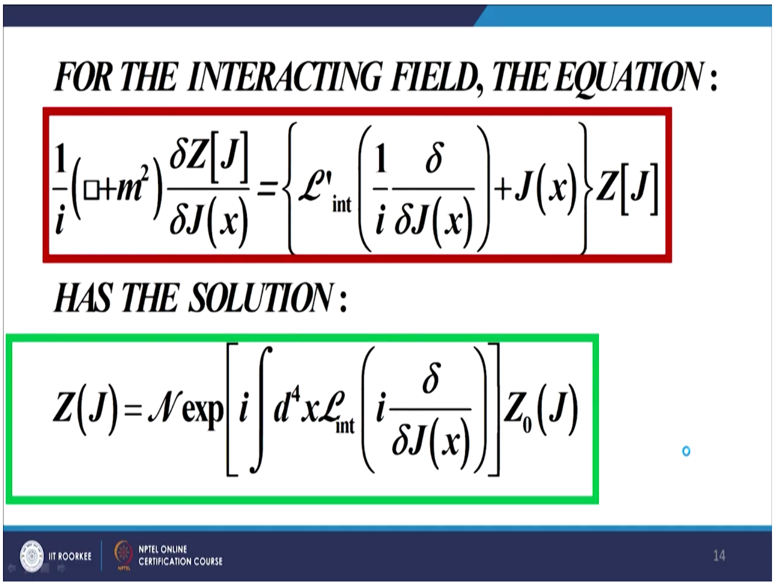


And for the interacting field the solution takes the form which is given in the green box in this particular slide at the bottom. And thereafter we moved on to the Schwinger Dyson equations.

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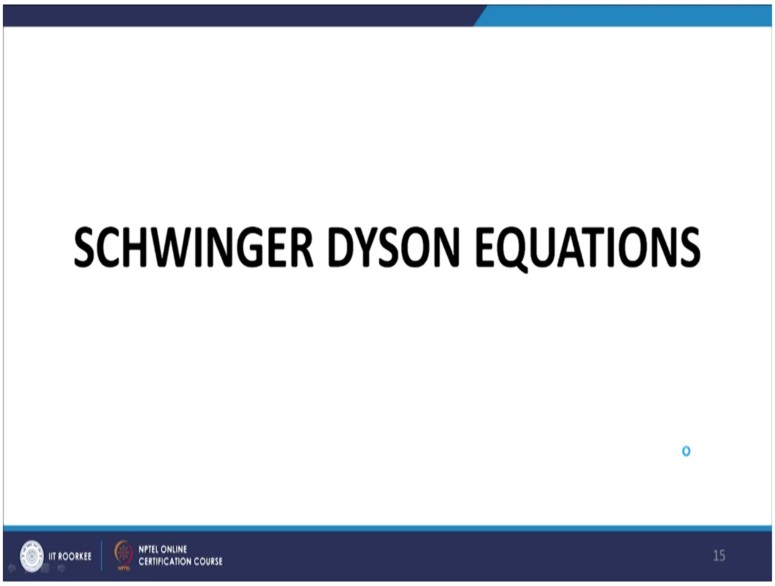

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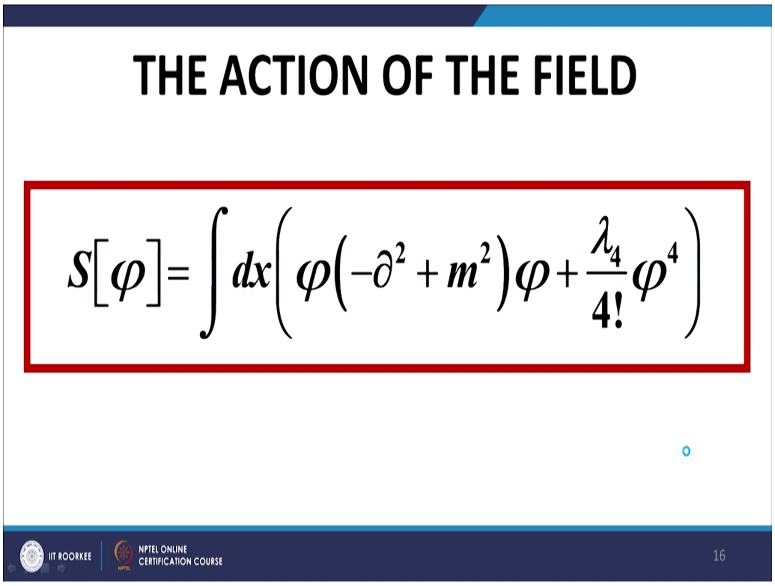


We first defined various terms to recapitulate we define the action of the 5 to the power 4 field as the expression that is given in the red box in this slide in terms of the field variable.

Now, the important thing before we go further I emphasize the fact that these files that appear in the action here and that will subsequently appear in the generating functional as well are c numbers or classical numbers they are not operators. That is an important thing and therefore, the action is a function of functions and that is what is called a functional.

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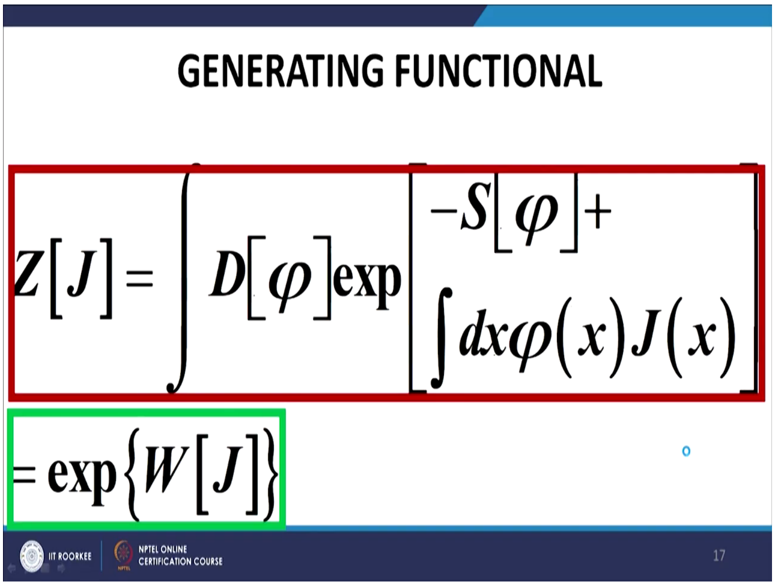


The generating functional as I was just mentioning a movement back is given by this expression again this all these phi's appearing here or field or classical variables and they may be functions, they may be numbers, but they are not quantum operators.

That is an important point that will arise in due course when we talk more about the in Dirac fields. And W J is the exponent, is the logarithm of the generating functional for the for the full green functions and W J represents the generating function for the connected green functions and we can represent W J as the logarithm of Z J.

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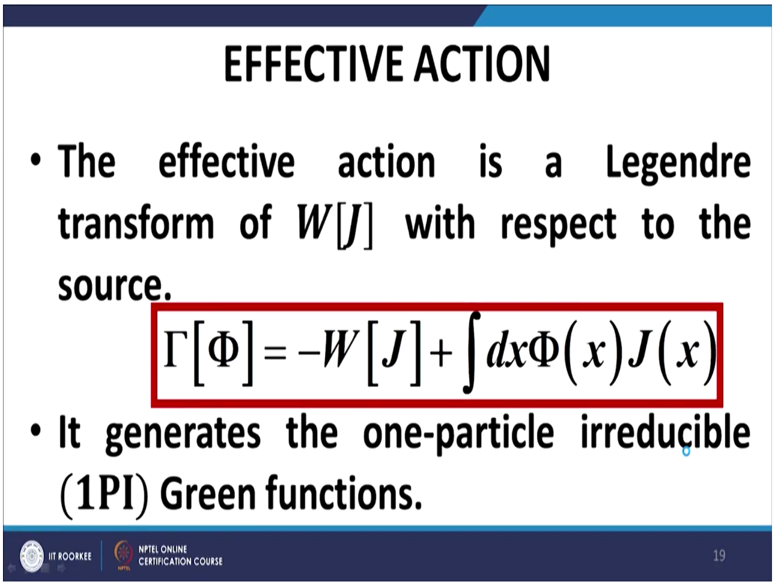


The effective action; the effective action is defined as the Legendre transform of W J with respect to the source J and it is given by the expression that is here in the red box, in the middle of your slide and it generates the one particle irreducible representation or irreducible green functions.

So, we have Z J which represents or which is the generating function for the full green functions, we have W J which is the generating function for the connected green functions and then we have gamma phi the effective action which is the generating function for the one particle irreducible represent irreducible green functions of the theory. Field functions as derivative of W J this is the definition in fact of the field function.

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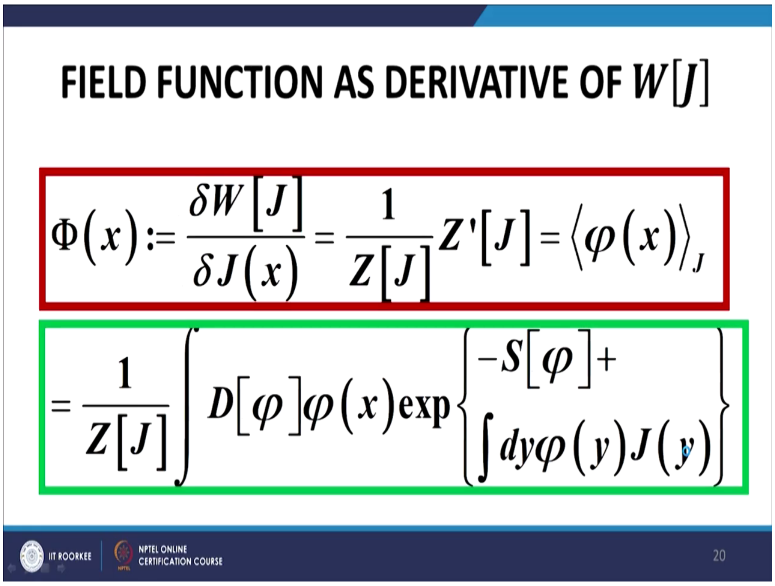


The field function is the functional derivative of the of W J. What is W J? W J is the generating function for the connected green functions and this incidentally works out to the average value of the field operator in the presence of sources.

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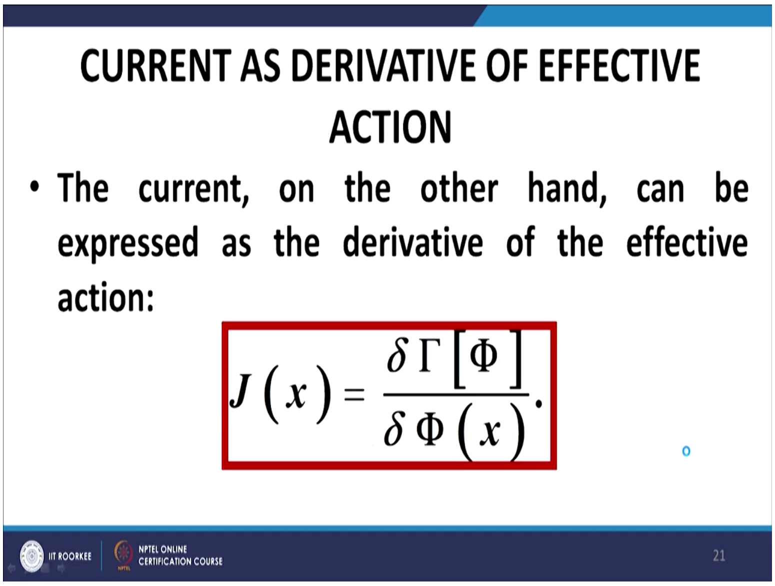


And the current can be represented as a derivative as the functional derivative of the effective action. This also follows directly from the definition of the effective action.

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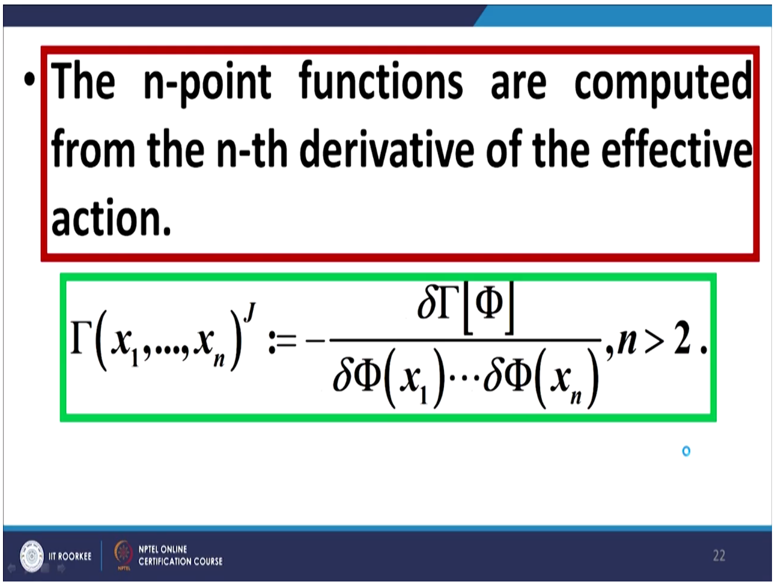


The n point functions are computed from the n th derivative of the effective action. These are the vertices in the presence of sources without J Jth Schwing set equal to 0 without J Jth Schwing set equal to 0 and these are given by the various n point various n th functional derivatives of the effective actions with respect to the various phi phi 1 phi 2 and phi n, where phi is are the are the in the field functions.

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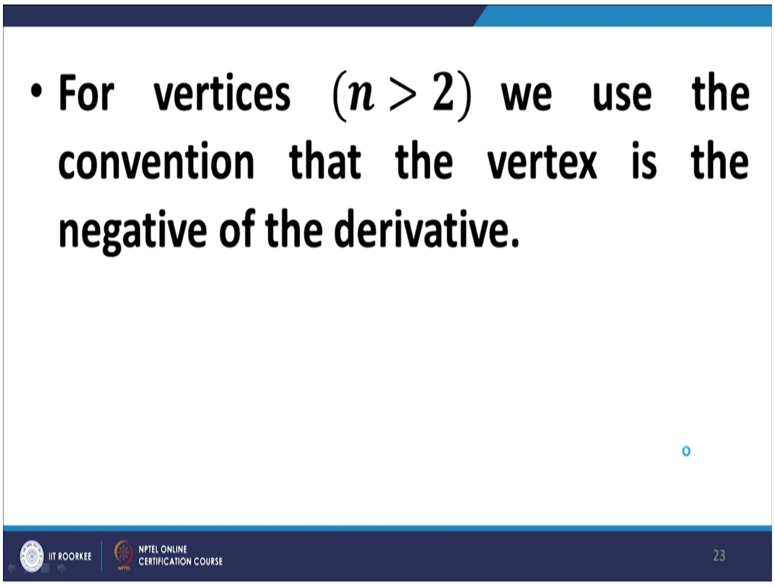


And importantly, we use the convention that the vertex is the negative of the derivative that is in to facilitate the compatibility or convenience of the science. We do not have to change it the plus minus again and again. As you also see in the example that follows this exposition.

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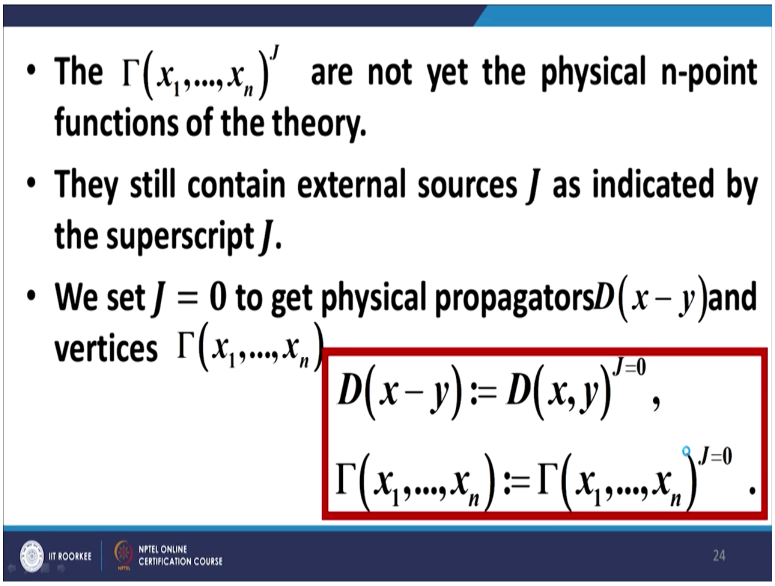


The this is a very important slide, if you see when we have the superscript J and in fact, the expressions that we have obtained here in the previous slide the expressions for the vertices that we have obtained here and the expressions for the propagators also that we obtained in the similar manner are still not the physical propagators, they are propagators where J is not yet set to 0.

In order to obtain the physical propagators here we need to set J equal to 0 after doing all the calculations, after working out the derivatives we get functions of J and then in those functions of J we put J equal to 0 and the expressions that we then get are the physical propagators and the physical vertices and the function D x y.

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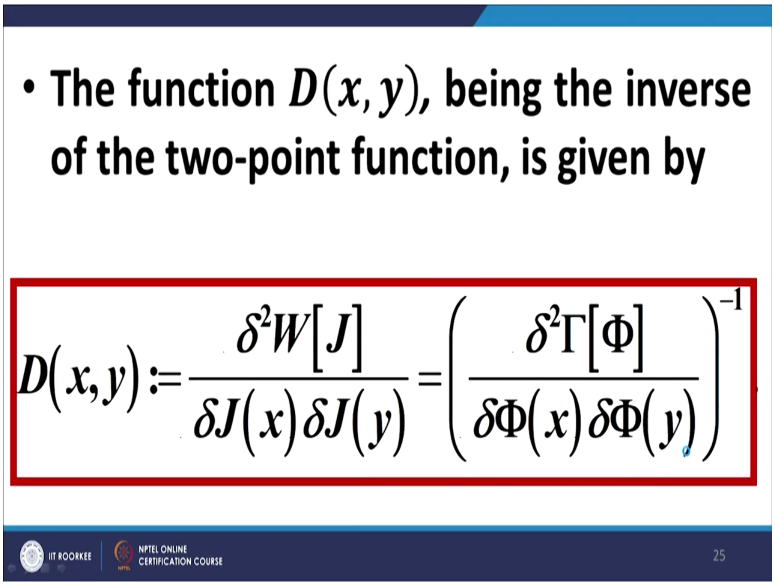


Now, this is in presence of J, please note that this is still not without J Schwing set equal to 0 is given by the expression in the red box here. It immediately follows from the definition of the field function phi; field function phi is the functional derivative of W J with respect to J x differentiating it again will give me the propagated D x y with J not yet set equal to 0 and then we said J equal to 0 and we arrive at the expression for the physical propagator.

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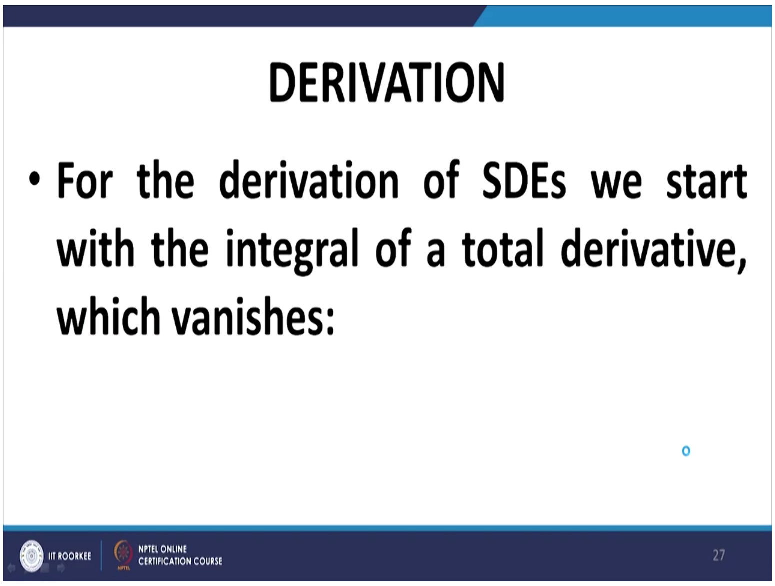

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$$0=\int D[\varphi]\frac{\delta}{\delta\,\varphi}\exp\left\{-S+\int dy\varphi(y)J(y)\right\}$$

$$=\int D[\varphi]\left(-\frac{\delta S}{\delta\,\varphi(x)}+J(x)\right)\exp\left\{-S+\int dy\varphi(y)J(y)\right\}$$

$$=\left(-\frac{\delta S}{\delta\,\varphi(x)}\bigg|_{\varphi(x')=\delta/\delta J(x')}+J(x)\right)Z[J]=0.$$

Now, we look at briefly go back to the derivation recap the derivation of Schwinger Dyson equation, this forms the cornerstone of the theory. So, just let us quickly recap this. We start with the; we start with the total integral of a total derivative which vanishes, because of the surface terms not contributing to the integral. And therefore, the integral as a whole vanishes and we get the expression that is there in the red box at the top of few slide.

When we do the; when we do the functional differentiation we extract the term that is given in the round brackets and the rest of the term remains at as it is. And if we set now, comes the important point the, now at this point, at the point when we are in the blue box the expression that is in the curly, in the round brackets it is the function of y x the field variables. And the integration is also with respect to the path integration with respect to the field variable so; obviously, we cannot take this expression outside the integral.

However, we note the fact that if we substitute the functional derivative delta by delta J x instead of phi x, we delta first functional derivative of J x acting on what is to the right pulls back a factor of phi into this integral. So, we can in a sense we can replace this phi x here with phi x equal to delta upon delta J x.

The two will do an equivalent job and now, this expression within the round brackets becomes independent of phi and therefore, we can pull it outside the integral. This trick has been employed several times in the previous exposition and we can take it outside the bracket and what we are left with now is nothing, but Z J. So, in the ultimate when we do all this maneuvers, what we end up is the expression that is given in the green box at the bottom of your slide.

Phi x is replaced by delta the functional derivative with respect to J x, because it pulls down phi from when it acts on Z J, when this delta upon delta J x the functional derivative with respective J x x and Z J, it pulls down a phi and so it is equivalent to the term within the round brackets and the blue box and, but would being independent of phi now, this can be pulled outside the integral and we have Z J as the integral and this expression in the round bracket goes outside the integral.

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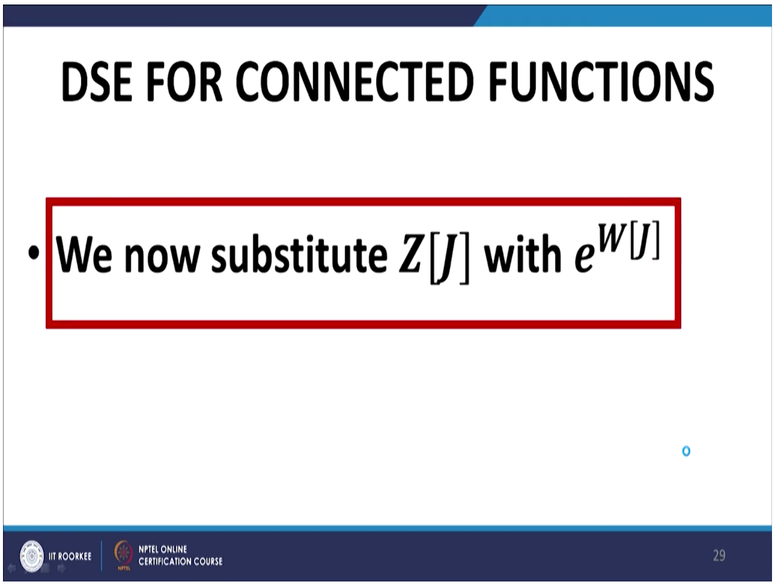


For the connected green functions with substitute Z J by e to the power W J; as I mentioned the generating function for the connected green function is the logarithm of the generating function for the full green functions at J.

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$$\text{We use}: e^{-W[J]}\left(\frac{\delta}{\delta J(x)}\right)\left(e^{W[J]}f\right)$$

$$= e^{-W[J]}\left[e^{W[J]}\frac{\delta W[J]}{\delta J(x)} + e^{W[J]}\frac{\delta}{\delta J(x)}\right]f$$

$$= \left\{\frac{\delta W[J]}{\delta J(x)} + \frac{\delta}{\delta J(x)}\right\}f.$$

And we use the this property which is given in the red box at the top of the slide if you walk through this it is quite elementary, quite straightforward, simple functional differentiation here. And, when we do the functional differentiation of the product e to the power W J into an arbitrary function f what we find is that the expression, the expression when operating on f gives rise to the expression within the curly brackets operating on f.

So, we have e to the power minus W J functional derivative with respect to J x, e to the power W J is equal to the expression that is given in the green box right at the bottom of your slide.

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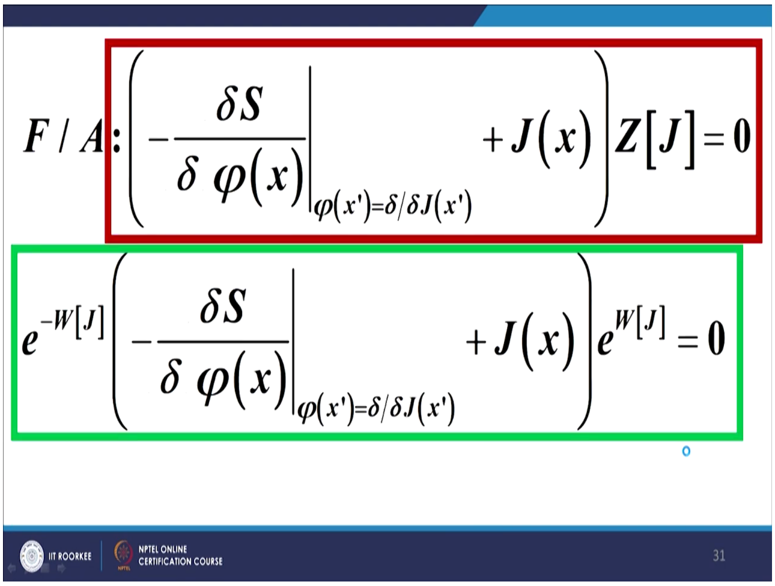


We write the our Schwinger Dyson equation that we are obtained earlier, for Z J e to the power minus W J on one side and e to the power W J on the other side.

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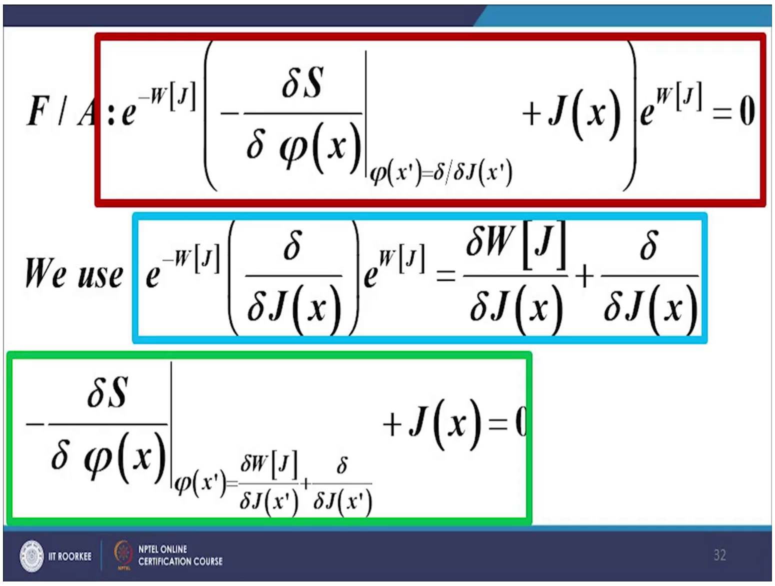


When this x on the functional derivative with respect to J x what we make use of the expression that is in the previous slide, which is here and we can write it in this form which is given here in the green box right at the bottom of your slide. Is he, let us try to understand this is bit more carefully.

If u look at this expression and the top and the expression in the top red box it is the functional derivative of S with respect to phi x and then phi x been replaced by delta of delta upon delta J x x dash. So, in other words S is a function of phi x. We do the functional, S is a functional of phi x, we take his functional derivative with respect to phi x which will again be a functional of phi x and then we substitute instead of phi x in that functional in the derivative we substitute this as delta J x.

So, what we will get is essentially a functional of delta J x and delta upon delta J x we will get here when we replace phi x in the derivative of the action by and delta upon delta J x. Now, each this delta upon delta J x is when they are acted upon the two sides on the left hand side by e to the power minus W J and acted upon on the right hand side by e to the power W J, will return quantities or will return expressions that are given in the blue box in the second, in the blue box in the middle of the slide.

And we in other words what will happen is when each term which has this factor here, in the expression which is given in the upper curly bracket, red box curly bracket is acted on by e to the power minus J and e to the power plus W J we will get a factor which is on the right hand side of this in the blue box instead of delta upon delta J x

In other words, everywhere delta upon delta J x will be replaced by this expression on the right hand side and that is precisely what is happening here and that is what is shown in the green box and the bottom of the slide. So, this is the; this is the expression that we have for the Schwinger Dyson equation for W J.

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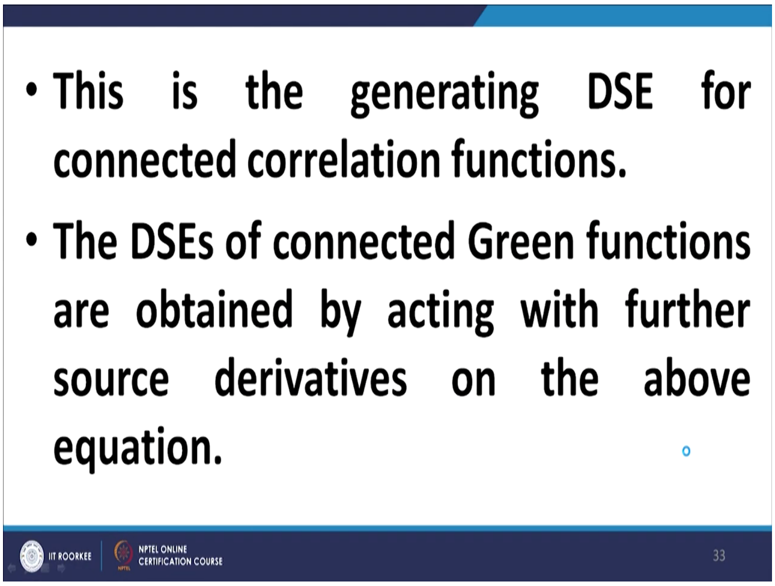


And this is that what is W J? Remember, W J is the generating function for the connected green functions connected correlation functions.

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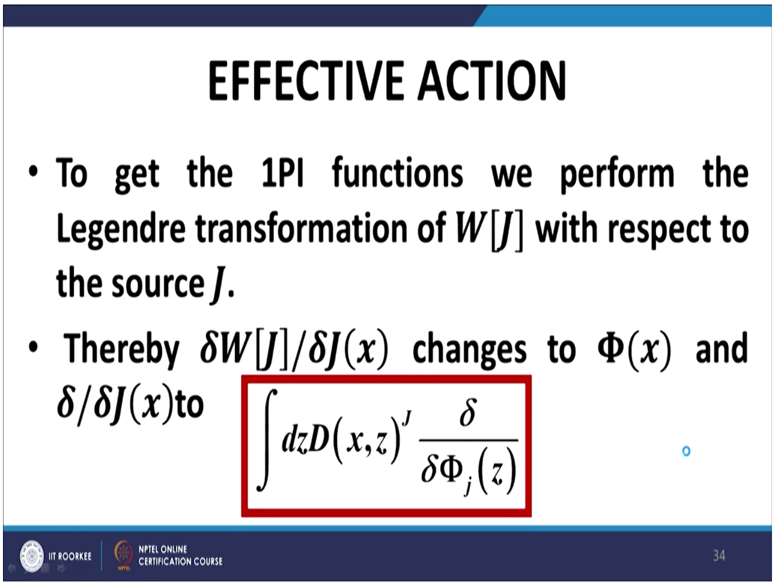


Now, we work out the effective action recall delta W J upon delta J x is nothing, but the is nothing, but the field function.

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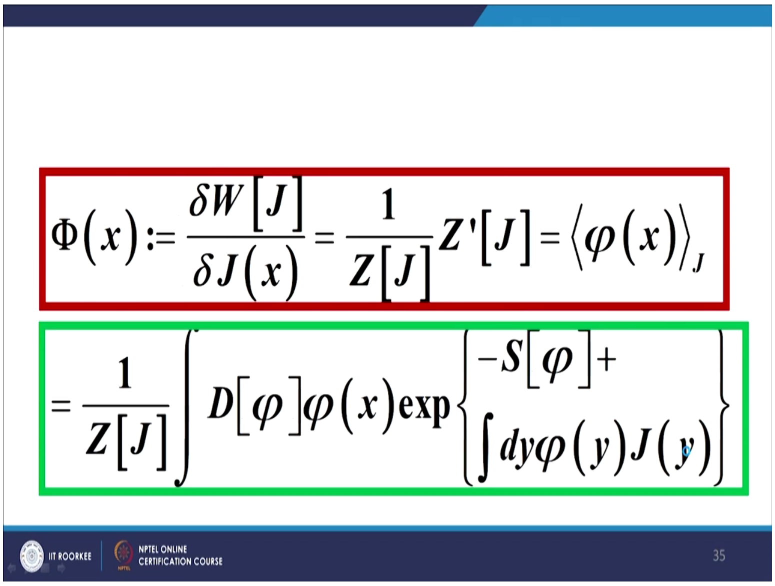


This is in fact, the definition of the field function and as far as the second part is concerned and let us go back a minute. See there are two terms here, in the subscript in delta and the functional derivative of W J with respect to J x dash that is one term that that becomes phi x dash and the second term will come back to the functional derivative with respect to J x dash.

Let us go, let us now address this particular term the first term is straightforward delta W J upon delta J x dash is by definition phi x dash phi capital Phi extract the field function.

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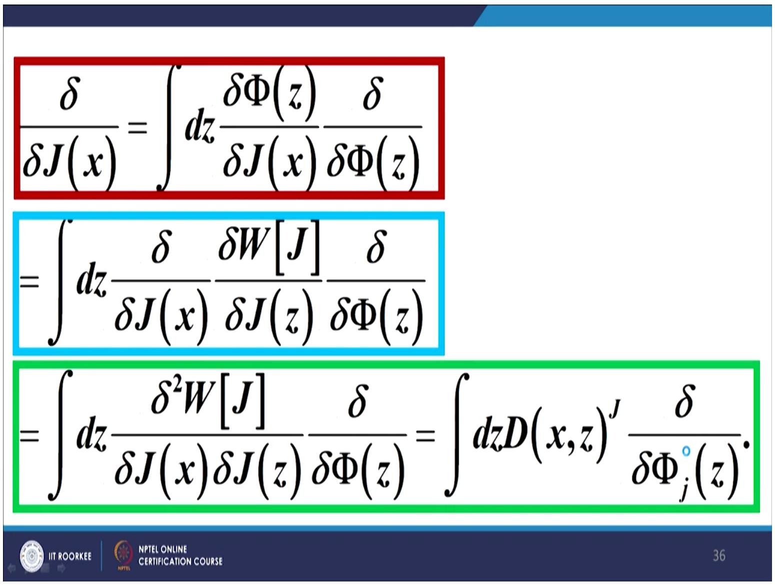


Yes, now about the second term, if you look at the second term if you look at this derivation its quite straightforward and we what we end up with is the expression that is given the right hand side on in the green box where we have simply use nothing, but the definitions; we have written the functional derivative with respect to J x as the functional derivative with respect to phi z and thereafter we have defined phi z in terms of W J upon delta J z that is phi z that appears in the numerator here.

Delta upon delta J x is carried forward to the blue box unchanged, phi z is written as delta W J upon delta J z that is by definition of the field function and the third term is also unchanged. So that, leads us to the left hand side of the expression in the green box and if we can identify the second derivative W J with respect to J x and j z as nothing but, the propagator x, z in the presence of sources.

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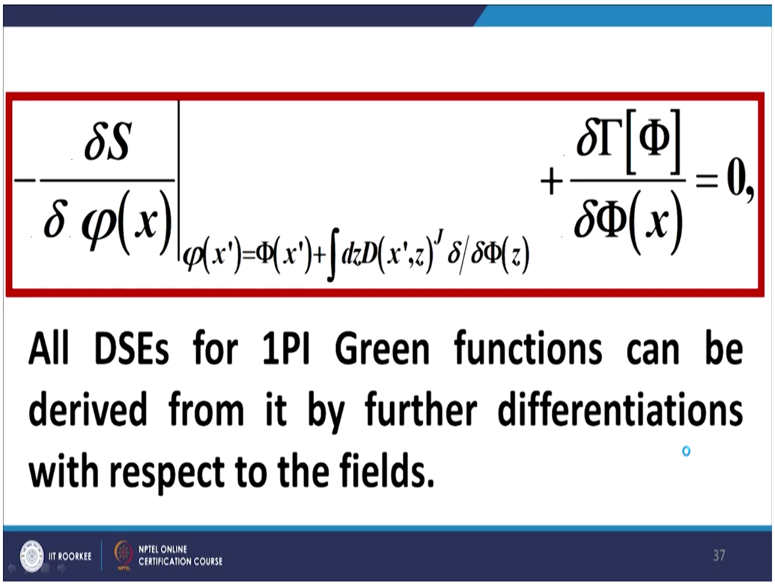


Therefore, we make these substitutions we make the, let us go back again. Yes, in this expression as I mentioned the first term becomes the field function the second the expression for the second term we also arrived at in terms of the propagator and as the expression in the right hand side of the green box of this slide.

This particular slide and we make both these substitutions in the expression for the W z and we arrive at the expression for the if effective action as the expression that is given here in the red box. So, this is the effective action and this gives us the generating function for the 1 pi green functions or the irreducible green functions for the theory.

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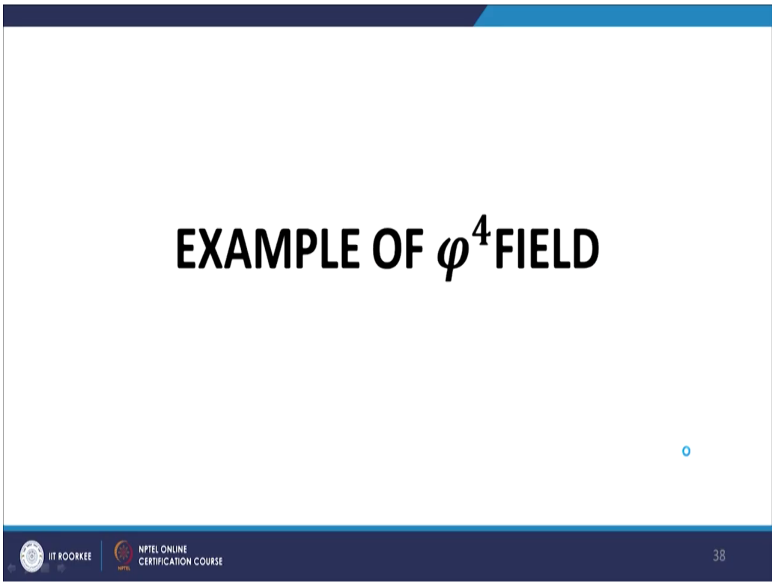


Now, we take on this example, example of what we have discussed. So, far let us illustrate it with the example of the phi 4 field.

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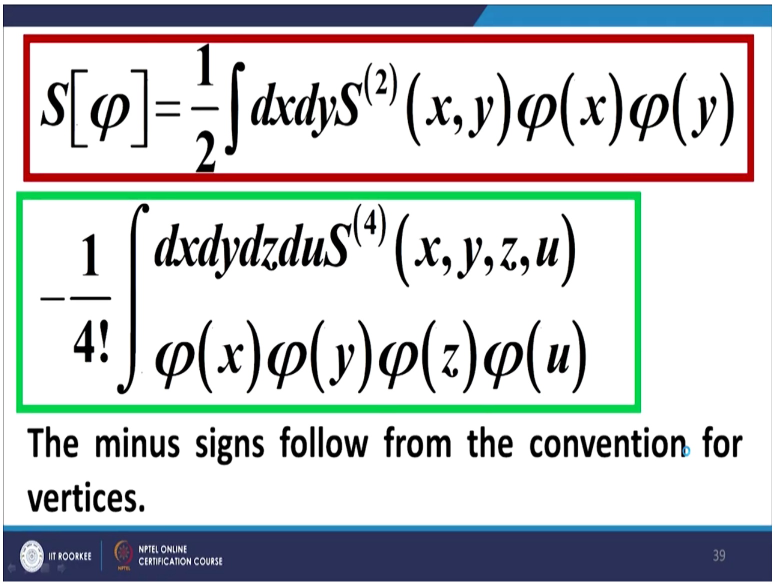


We will quickly run through it. The action is defined by the expression that is in the red and green box. The expression in the red box is the bare action and the expression in the green box represents the interaction term. So, this is straightforward.

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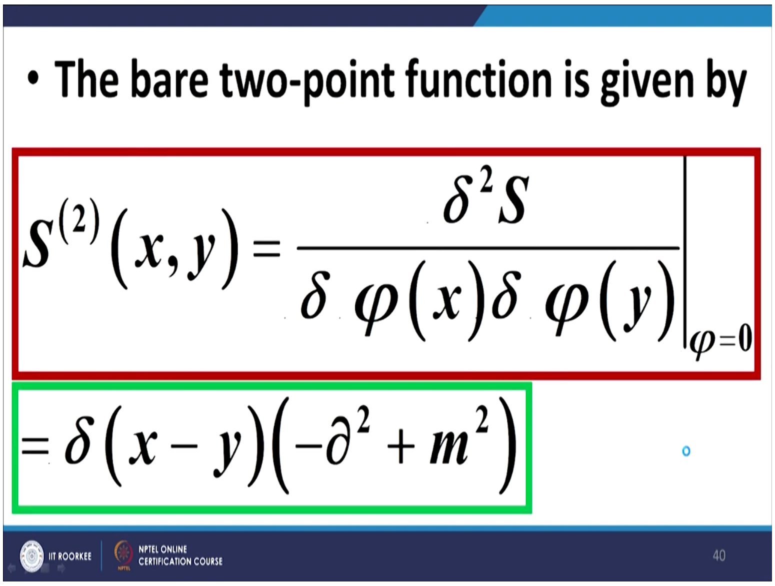


And, the bare two point function is given by the second derivative of the for second functional derivative of the action with respect to x and with respect to y and then putting phi equal to 0 and that gives us the expression when we introduce the Klein Gordon free field operator that is given by the expression given in the green box at the bottom of the slide.

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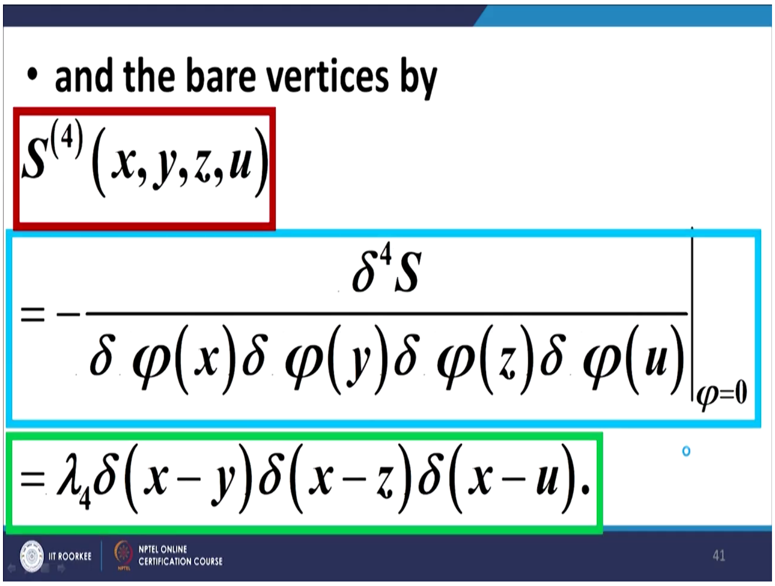


And the bare vertices, if you work out the bare vertices the expression that you get is again at the bottom of the slide in the green box. It comprises of three delta functions delta x minus y delta x minus z delta x minus u multiplied by lambda 4 which is the coupling constant.

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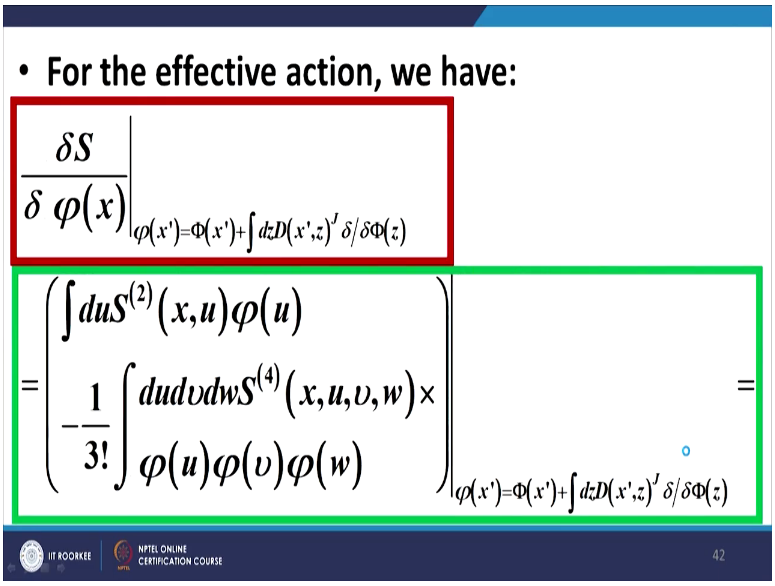


Now, we come to the effective action the expression for the Schwinger Dyson equation for the effective action as I mentioned just few minutes back is the expression that is given in the red box here, expressed in terms of, explicitly in terms of the action that we are considering for the phi 4 field, it takes the form that is given in the green box. And, the first term is the bare action term and the second term represents, the first term represents the derivative of the bare action term and the second term represents the derivative of the interaction term functional derivative.

And now, we have to make the substitution. In this expression we need to make the substitutions given by the suffix here, right at the bottom of right hand corner of the slide we have to make the substitutions for the field variables in terms of the effective action and the propagators and that is what we do in the next slide.

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$$= \int du S^{(2)}(x,u)\Phi(u) - \frac{1}{3!}\int du\,d\upsilon\,dw\,S^{(4)}(x,u,\upsilon,w) \times \left(\Phi(u) + \int dz D(u,z)^{J}\,\delta/\delta\Phi(z)\right) \times \left(\Phi(\upsilon) + \int dy D(\upsilon,y)^{J}\,\delta/\delta\Phi(y)\right)\Phi(w)$$

And in that is precisely what is done here, we have substituted the expressions here phi x dash equal to capital phi x dash plus integral of the propagator times the delta upon the functional derivative with respect to the capital phi z we are substituted this expression for phi x dash. Wherever it occurs in the green box here and we get the expression that is here in the green box.

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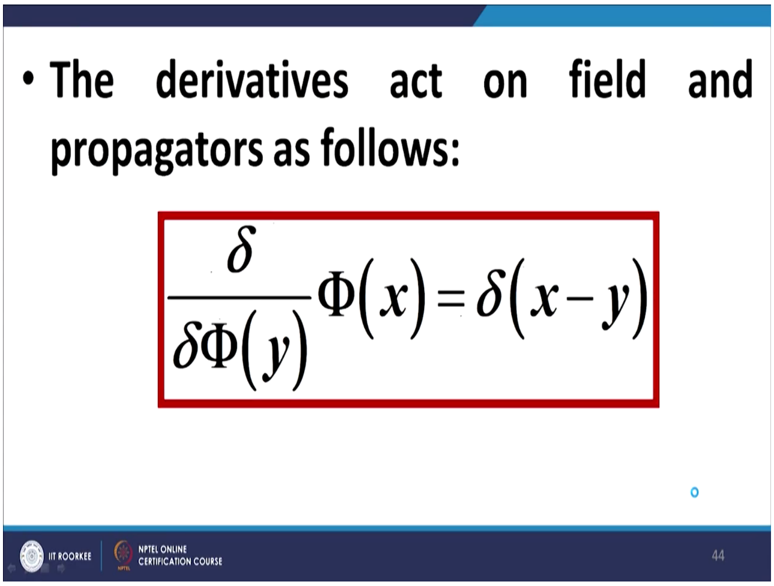


Now, the derivatives this they act on the propagators and the field functions and the propagators their action on the field functions and the propagators is given in terms of the following for the field in the functional derivative with respect to the field function of another field function yields delta function that is straightforward. I repeat the and the functional derivative of capital Phi x the field function at x with respect to the field function at phi yields the Dirac delta function.

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$$\frac{\delta}{\delta\Phi(x)}D(y,z)^{J}=\frac{\delta}{\delta\Phi(x)}\left(\frac{\delta^{2}\Gamma[\Phi]}{\delta\Phi(y)\delta\Phi(z)}\right)^{-1}$$

$$=-\int dz_{1}dz_{2}\left(\frac{\delta^{2}\Gamma[\Phi]}{\delta\Phi(y)\delta\Phi(z_{1})}\right)^{-1}\left(\frac{\delta^{3}\Gamma[\Phi]}{\delta\Phi(z_{1})\delta\Phi(x)\delta\Phi(z_{2})}\right)\left(\frac{\delta^{2}\Gamma[\Phi]}{\delta\Phi(z_{2})\delta\Phi(z)}\right)^{-1}$$

$$=\int dz_{1}dz_{2}D(y,z_{1})^{J}\Gamma(z_{1},x,z_{2})^{J}D(z_{2},z)^{J}$$

And the functional derivative with respect to x of the propagator y, z in the presence of sources of J is slightly more complex. It gives us the expression which when simplified yields the expression which is at the bottom of the slide in the green box here, which consists of the two propagators and the vertex.

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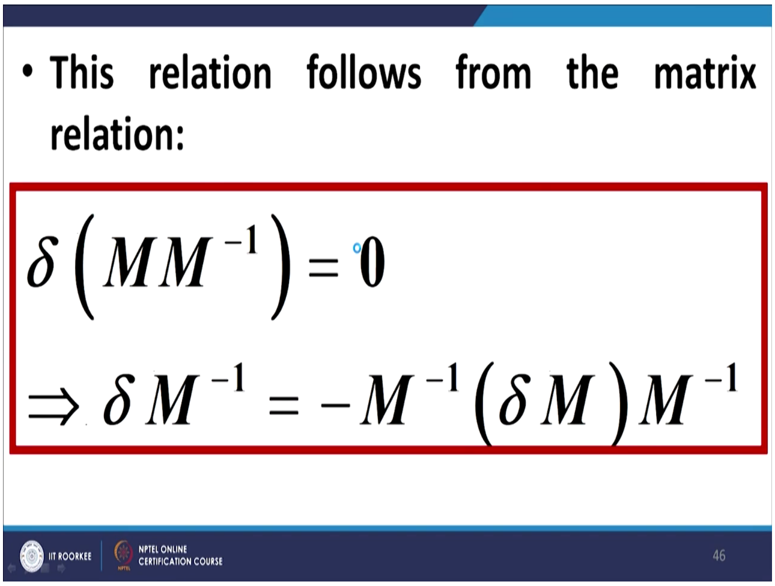


And in arriving in the second step, in from moving from the red box to the blue box expression; from the expression in the red box to the expression in the blue box we have used this property delta for vertices, for matrices, I am sorry for matrices; delta M M inverse is equal to 0 if this implies delta m inverse is equal to minus M inverse delta M M inverse.

This is the property that we have used in arriving at the previous slide which you can see here in this blue box we have this is M inverse, this is delta M and the first term is M inverse, the second term is delta M and the third term is again M inverse.

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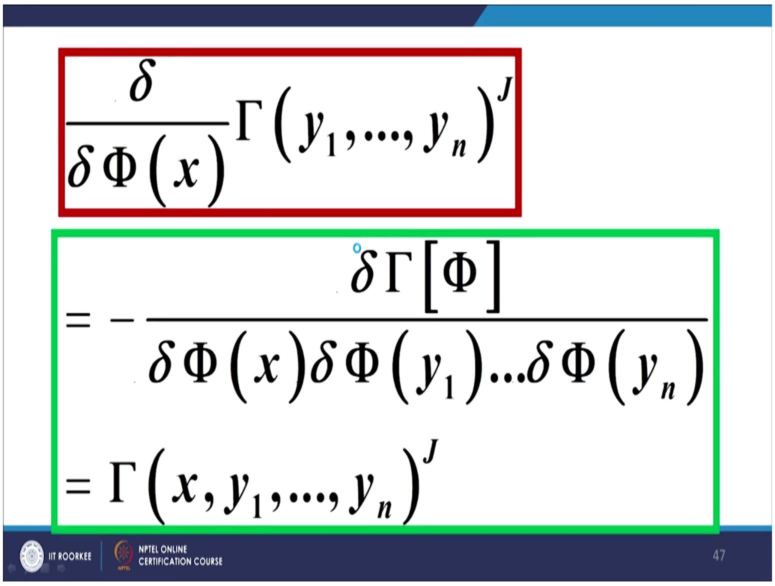


As far as the action of the functional derivatives of the with respect to the field functions on the vertices the if you, if this field functions act on, if this field in the functional derivatives with respect to the field function act on an n point vertex we get an n plus 1 point vertex by the functional differentiation.

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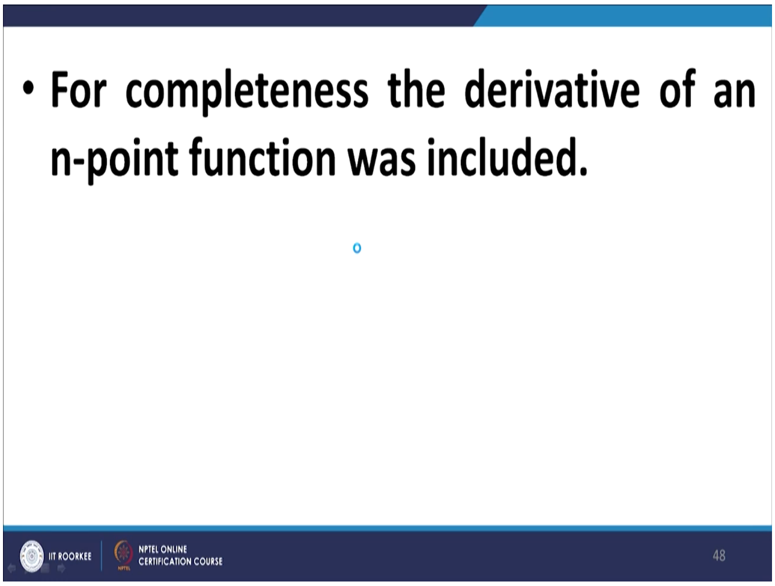

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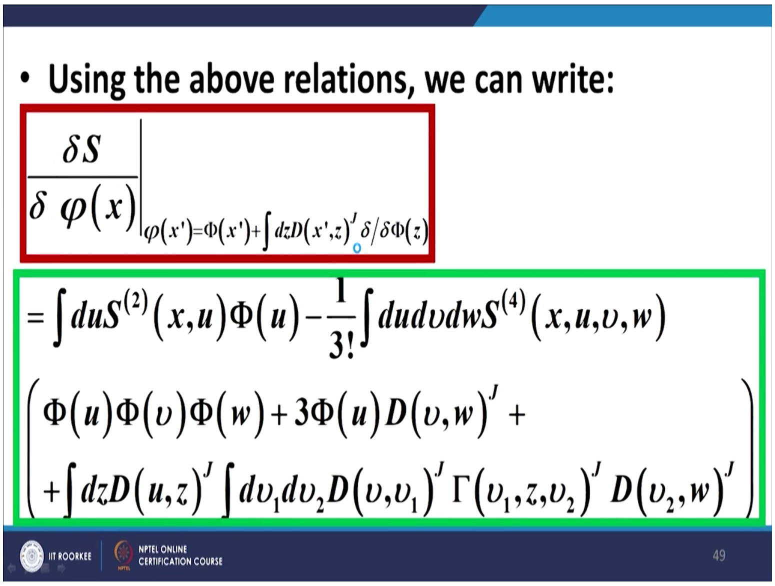


So, these are the properties and using this or this results we can write down the functional derivative of our action. The action that we have assumed the phi 4 action in terms of the expression, that is given in the green box at the bottom of your slide. The expression is; obviously, quite involved what I will do is; I will elaborate this computation of this expression in the notes that I will provide together with this set of lectures.

So, you can have access to that and stepwise computation of the various, particularly in the in this particular lecture there are lot of extensive calculations which are difficult to explain on the power point. So, I will put it in the notes and provide them together with the with this lectures and the power points.

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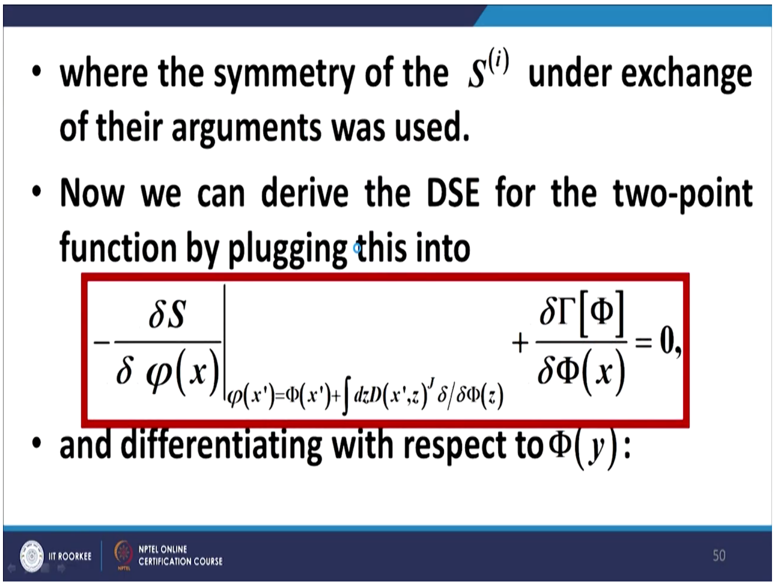


So, having worked out the first term here in the red box; having worked out the first term here in the red box, we can now work out this Schwinger Dyson equation for the field function and that takes the form which is in the full expression takes the form given in the red box. The expression for the effective action here is the is there and therefore, we can work out the effective action from this particular expression and by differentiating this step by step we can arrive at various two point functions.

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$$
\begin{aligned}
&\Gamma(x,y) = S^{(2)}(x,y) - \\
&-\frac{1}{2}\int du d\upsilon S^{(4)}(x,y,u,\upsilon)\Phi(u)\Phi(\upsilon) - \frac{1}{2}\int du d\upsilon S^{(4)}(x,y,u,\upsilon) D(u,\upsilon)^{J} - \\
&-\frac{1}{2}\int du d\upsilon dw S^{(4)}(x,u,\upsilon,w)\Phi(u)\int d\upsilon_1 d\upsilon_2 D(\upsilon,\upsilon_1)^{J}\Gamma(\upsilon_1,z,\upsilon_2)^{J} D(\upsilon_2,w)^{J} - \\
&-\frac{1}{3!}\int du d\upsilon dw S^{(4)}(x,u,\upsilon,w)\int dz D(u,z)^{J}\int d\upsilon_1 d\upsilon_2 D(\upsilon,\upsilon_1)^{J}\Gamma(y,\upsilon_1,z,\upsilon_2)^{J} D(\upsilon_2,w)^{J} - \\
&-\frac{1}{2}\int du d\upsilon dw S^{(4)}(x,u,\upsilon,w)\int dz dz_1 dz_2 D(u,z_1)^{J}\Gamma(z_1,y,z_2)^{J} D(z_2,z)^{J} \times \\
&\times\int d\upsilon_1 d\upsilon_2 D(\upsilon,\upsilon_1)^{J}\Gamma(\upsilon_1,z,\upsilon_2)^{J} D(\upsilon_2,w)^{J}.
\end{aligned}
$$





This is given here; an example of two point functions is given here in this particular computation. As you see this is in this left hand side expression we have already obtained. The right hand side expression is also can be obtained straight away, because we the gamma phi is nothing, but the effective action. So, and gamma phi upon this delta phi x is nothing, but J.

So, we can obtain this expression which is here in the red box and by differentiating this with respect to phi y. We need to write J in terms of this expression in terms of the effective action, because the differentiation is to be done with respect to capital phi y and therefore, we need to represent J as the function of phi y and that is precisely what is done in this equation using the effective action and on taking various derivatives, further derivatives we can arrive at two point functions and so on.

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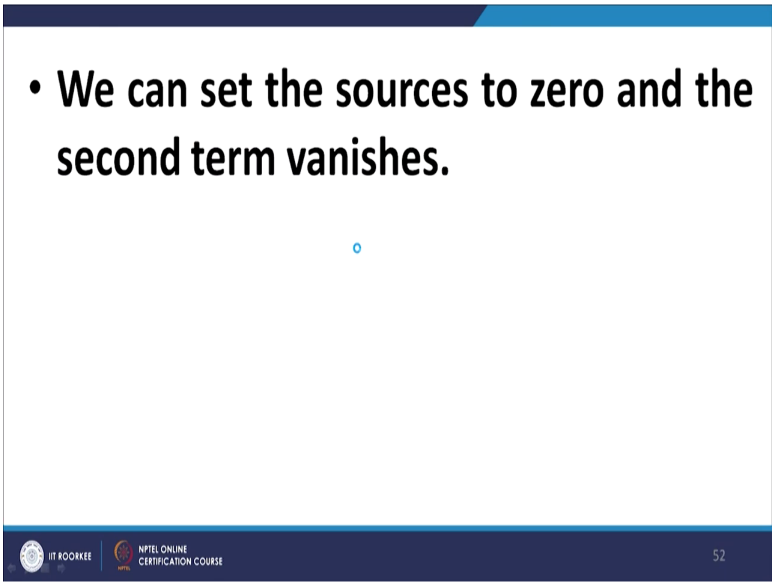

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So, this is an example this is an illustration of how the process operates for the simple to point, simple case of the phi 4 field, in the context of the Klein Gordon field. After the break we will start the Fermi Dirac field and obtain the propagator for the Fermi Dirac field using the path integral formalism.

Thank you.