Path Integral Methods in Physics & Finance
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Lecture - 48
Causality, SDE in Minkowski Space

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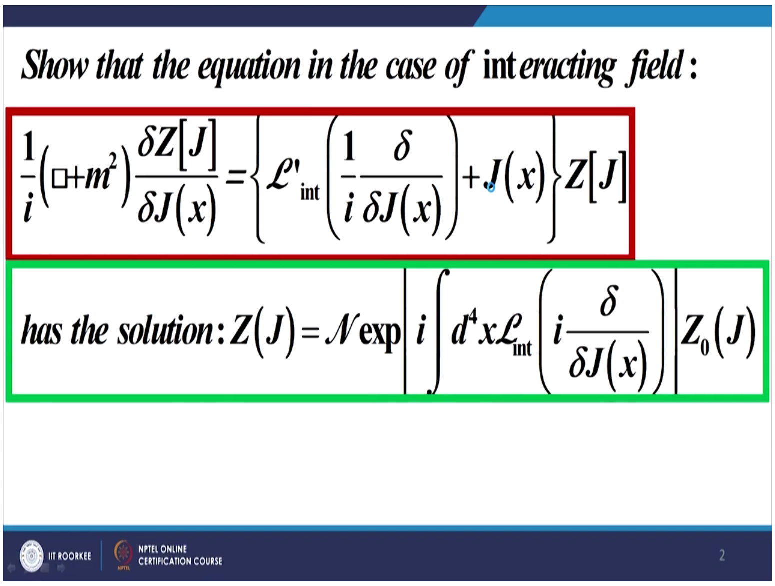

Welcome back. So, let us continue from where we left off. The third property of the generating function is given in the red box at the top of the slide. And the this is the expression that was given for the generating function and this has the solution and given in the green box of the slide; that is what we have to establish.

We have to establish the solution for the equation that is given in the red box at the top of the slide. The top of the slide contains the equation and the green box contains the solution, we have to establish the validity of the solution.

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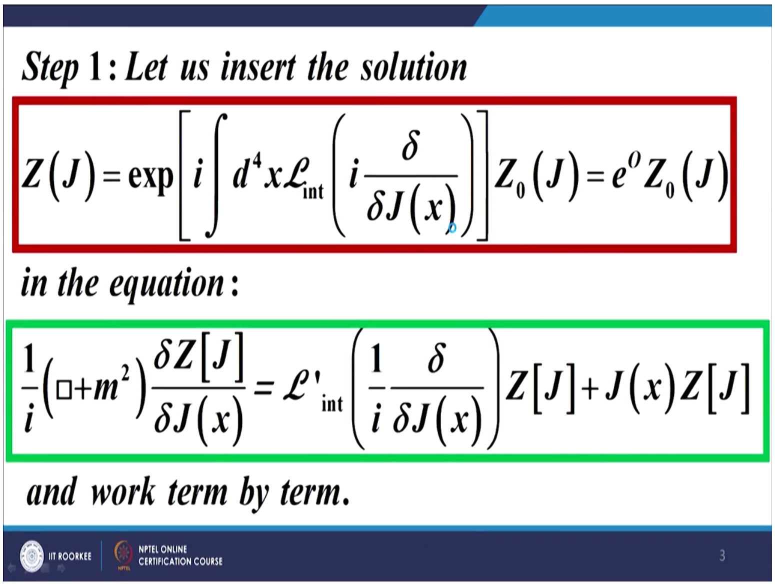

So, we start by inserting the solution into the equation that is given to us. On inserting the solution, this solution; which is in the red box at the top of your slide which is in fact, here also in the in this expression. And we substitute this; in the green box which was the equation that is given to us; that we are that we have in fact, derived earlier and please note as you will see in the next slide here.

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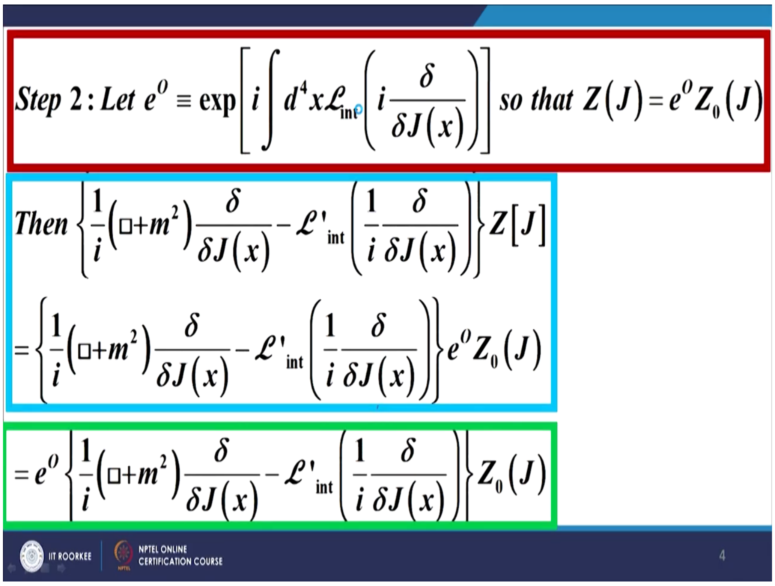

We introduce an expression o as the operator which is equal to i integral this whole thing. So, e of o is equal to e of exponential of the expression that is given in the square bracket and in the red box in the top of the slide.

In other words, we have substituted e o as exponential of the square bracket that is there in the top of your slide. So, we can write Z J as e o Z 0 J; this will clearly originates from the expression that we have for, that we have for Z J.

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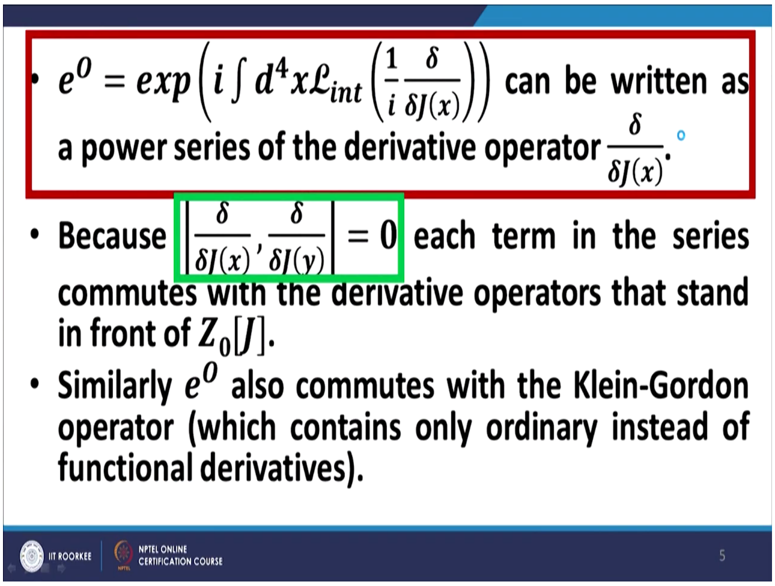

Z J is equal to this expression into Z 0 J we have already established it; in the previous lecture a number of time. So, Z J is equal to e of o e o into Z 0 J. Then we have on operating with the Klein Gordon operator box plus m square into del by del J x minus L integral this into Z J will give us what? It will give us e. In other words we are starting with the left hand side now.

If you look at this let us go back; this is we are now starting with the left hand side of this equation box plus m square del Z J upon del J x del Z J upon that is precisely what we have of course, we were. Now introduced system minus L dash integral this expression into Z J; this is been obtained by substituting the solution into the original equation. As we go back in the previous slide. This is equal to this expression, so if I take this to the left hand side I will get this expression.

Now let us look at what we get from here. Z J as I mentioned earlier as is shown in the red box given at the top of your slide is nothing, but e o Z 0 J we write it as e o Z 0 J. And then we argue at a later point immediately after this that; e o commutes with whatever is there in the curly brackets. And, therefore but it can be taken to the other side of the curly bracket and can be written in the form that is given in the green box at the bottom of your slide.

Now to establish we need to establish that e o commutes with the terms that are appearing in front of it; in front of it in the expression given in the green box in the in this expression in the curly brackets in the blue box at the right hand side of the blue box at the middle of the slide. So, e o is equal to this expression by definition that is what we have substituted.

Now, we can you see the interaction Lagrangian can be written as a power series of the derivative operator of the functional derivative of J x; has a power series of the functional derivative of J x. Now the commutator of the functional derivatives J x and J y at commutators the functional derivative at point x and point y do commute with each other. And; that means, what? That means, that every turn in this exponential commutes with the functional derivative of J x.

Let us go back now; so what we have established now is that this e of o e of o commutes with del of del J x that is one part we have established. Now as far as the Klein Gordon operator is concerned it consists of ordinary derivative and therefore, e of o also commutes with it is e of o also commutes with the Klein Gordon operator that is also. So, it commutes with this term del of functional derivative of J x and it also commutes with box plus m square.

And remember we have written L interaction that is the interaction Lagrangian as a power series in the derivative operator. Therefore, L dash therefore, L dash of the interaction and they are the derivative of the interaction Lagrangian will also be a power series in the functional derivatives of J x. And therefore, e of o will also commute with this term the derivative of the interaction Lgrangian.

So, it in it commutes with all the terms in the curly brackets and therefore, it can be taken to the left hand side of the equation.

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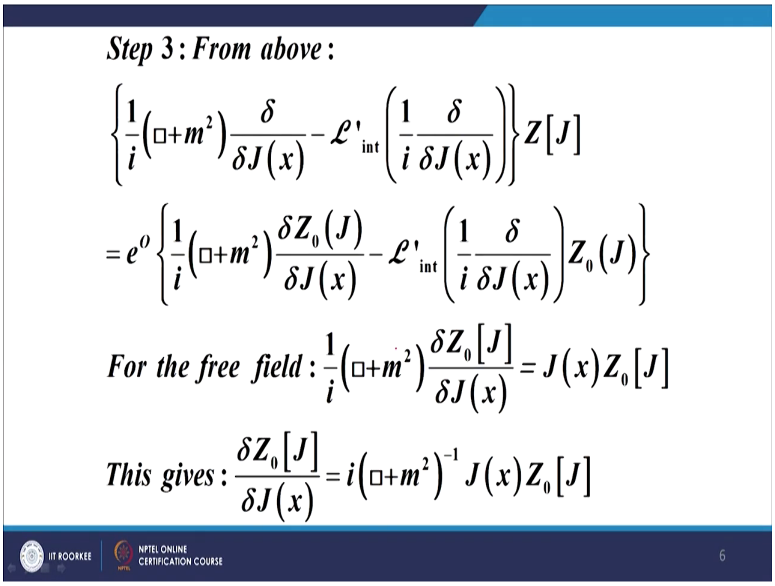

So, that is where we have now. And now for the free field what do we get? For the free field we have from the previous property that we have derived the expression that is given here; 1 upon i box plus m square del of Z 0 upon del J x is equal to J x Z 0 J.

So, we substitute this in the first term on the curly brackets in the previous equation. So, what do we get when you substitute from this after transposing the Klein Gordon operator to the right hand side as its inverse?

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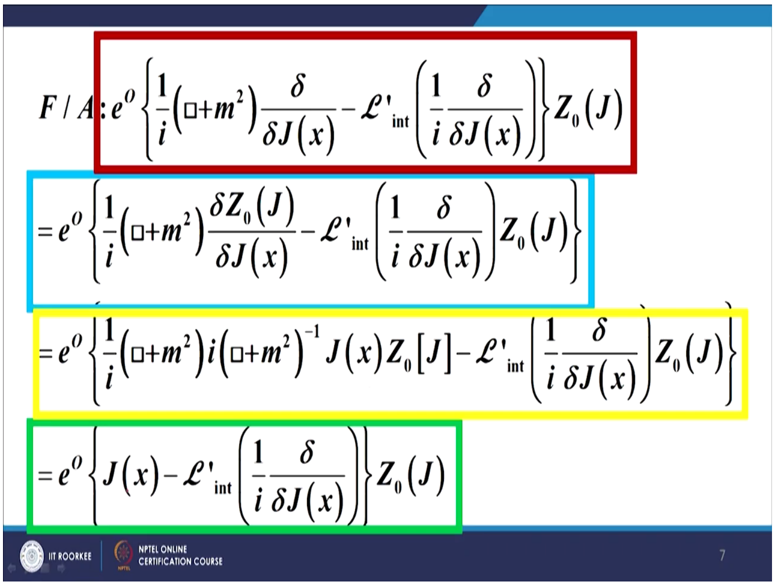

When you substitute this here what we get is; the expression that is given in the yellow box. And this expression in the yellow box now, gives us J x this in the box plus m square and it is inverse cancel each other gives us the delta function give us the unit function. And, J x remains as it is Z 0 J remains as it is Z 0 J is taken common and taken outside the integral.

The second term remains unchanged and e 0 at the beginning at the left hand side is also unchanged. So, therefore, what we have simply done is, we have used the expression for the free field here for the free field here the expression that is here for the free field. And we have using this expression for the free field we have substituted for del Z 0 J upon del J in terms of the inverse of the Klein Gordon operator.

And that inverse of the Klein Gordon operator cancels the Klein Gordon operator here giving us only J x; as the first term within the curly brackets. And the second term we have not disturbed so far.

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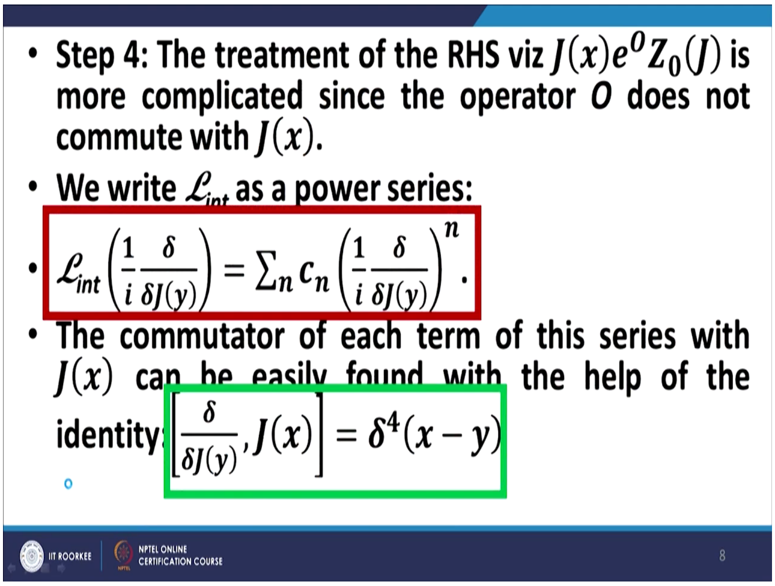

Now, let us see what we happen; what happens as far as the; as far as this second term is concerned. As mentioned earlier we write L into interaction as a power series in terms of the functional derivatives as some of the power series in some of the functional derivatives as the expression in the red box.

Now, the commutator of each term of the series with J x can be found with by using this identity. This identity is straightforward it is easy elementary to prove. And this identity in the green box here will be used for commuting higher for commuting the higher commutator of this expression.

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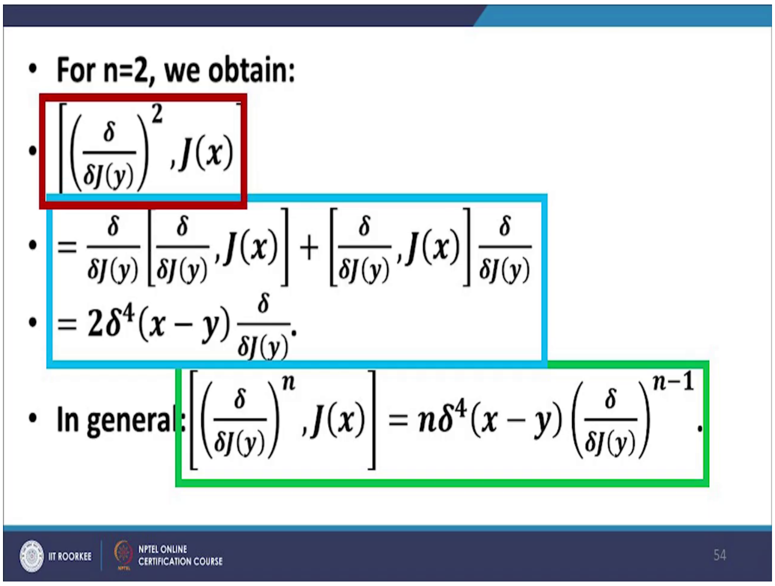

For example, we have for the second competitor n equal to 2 we have the expression in the red box gives us on simplification the expression in the blue box. And, in general what we have is the expression here for the nth commutator or the nth power of the functional derivative commuting with J x gives us the expression that is given in here; in the right hand side of equation of in the green box right at the bottom of your slide.

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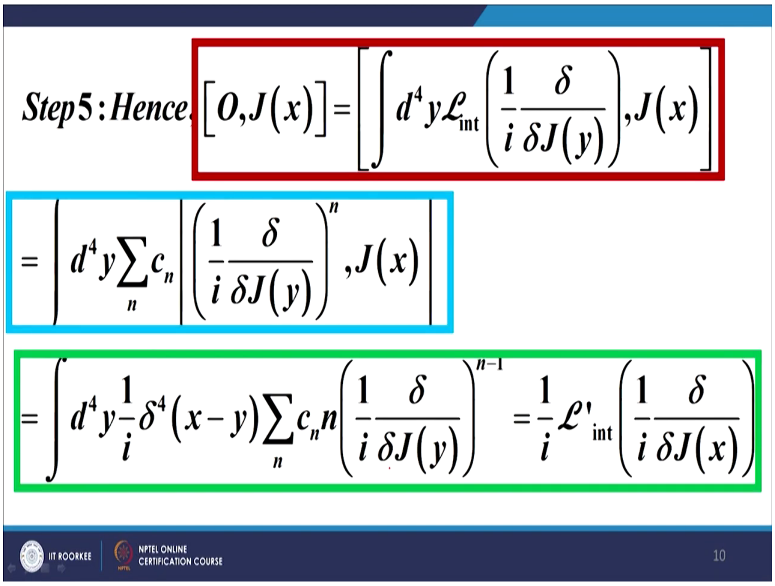

So, from this what do we get? We get the commutator of o and J x now o is written as this you see e to the power o was written as e to the power this expression e to the power integral d for y L integral this expression. So, o was equal to this expression which is here and the left hand side expression in the commutator J is retained as it is.

Now we have written L integral as a power series in the form which is shown in the blue box in the middle equation on the slide. And now we have the commutator of this expression this expression within the round bracket with J x as giving us the delta function here together with 1 power less that is n minus 1 which is sigma n.

And if you integrate it over y the delta function goes away with the substitution y is equal to x. And we also have the expression which is within the summation if you look carefully the expression within the summation is nothing, but the first derivative of L interaction.

If you right, let me reiterate it; if you write any interaction in the form of the power series which is shown in the blue box here. Then the expression that you get in the green box here is nothing, but the first derivative of L interaction.

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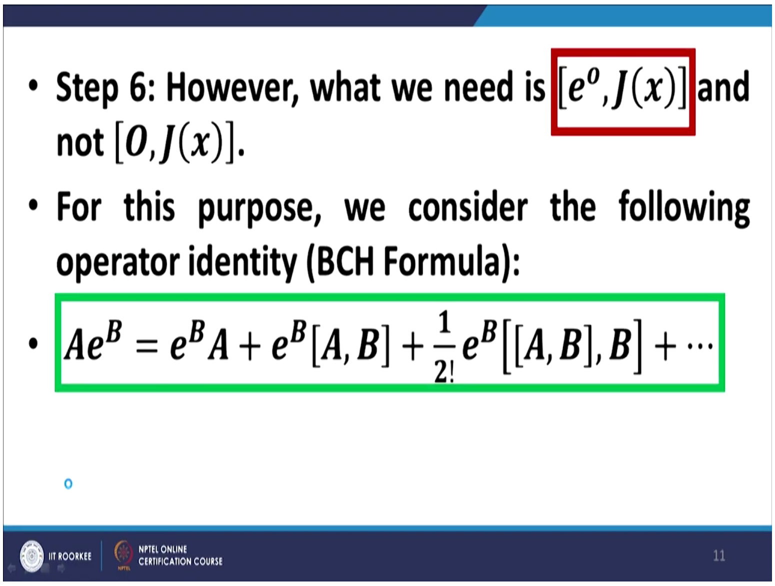

But our requirement if you look carefully is e to the power o J x if you look at it here. Let me go back; e to the power o J x that is what we need we do not need o J x the we do not need the commutator between o and J x. We need the commutator between e to the power o and J x. So, we still have some work to do.

Now while we consider or we make use of the Baker Campbell Hausdorff formula which is written in the form given in the green box at the bottom of your slide the Baker Campbell Hausdorff formula.

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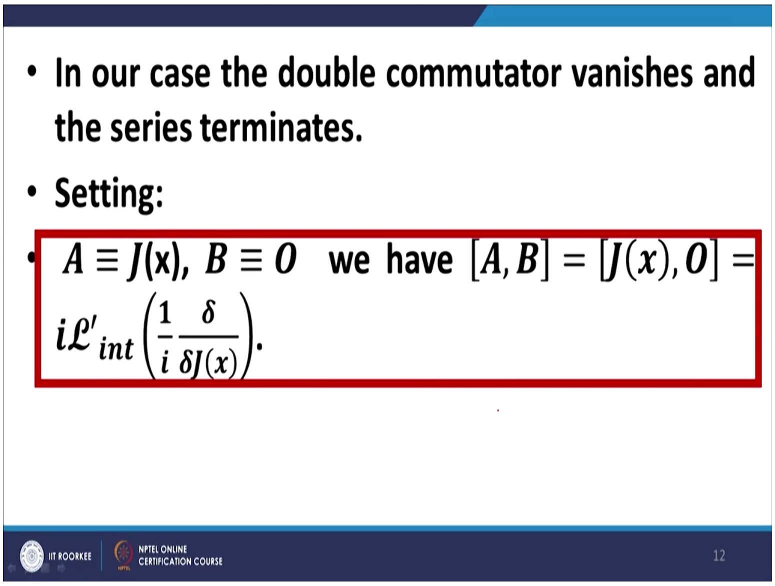

Now the important thing here is that in our case the double commutator vanishes and therefore, the series terminates if we write A equal to J x and B equal to o. We have A B is equal to J x comma o which we have already found as equal to i L dash please note it is not L it is L dash we found it here and let me go back. Yeah this is the expression o J x the commutator between o J x is equal to 1 upon i this expression.

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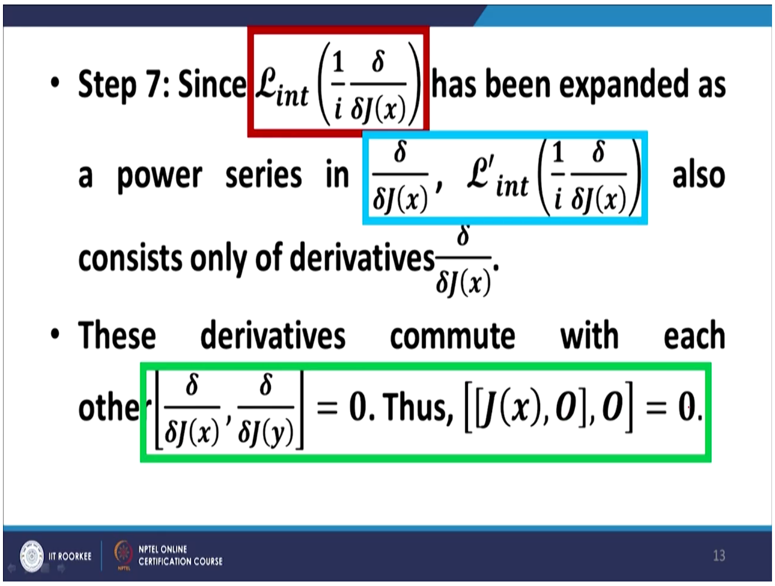

Therefore, the commutator between J x o is minus of this which gives us this expression. Now, since L interaction is represented as a power series in J x therefore, L dash interaction is also a consists only of derivatives of J x the power series of J x of derivatives of J x it consists of power series of derivatives of J x.

Let me repeat L interaction consists of powers has been expanded as power series in derivatives of J x cum functional derivatives of J x. Therefore, L dash interaction will also consist of power series of derivatives or functional derivatives of the form del upon del J x.

Now, these derivatives commute with each other therefore, and J x comma o is equal to L dash J. So, L dash J and O; because o consists of only derivative of J x L dash J x consists of derivatives of J x therefore, they commute with each other and we get 0. So, the series terminates.

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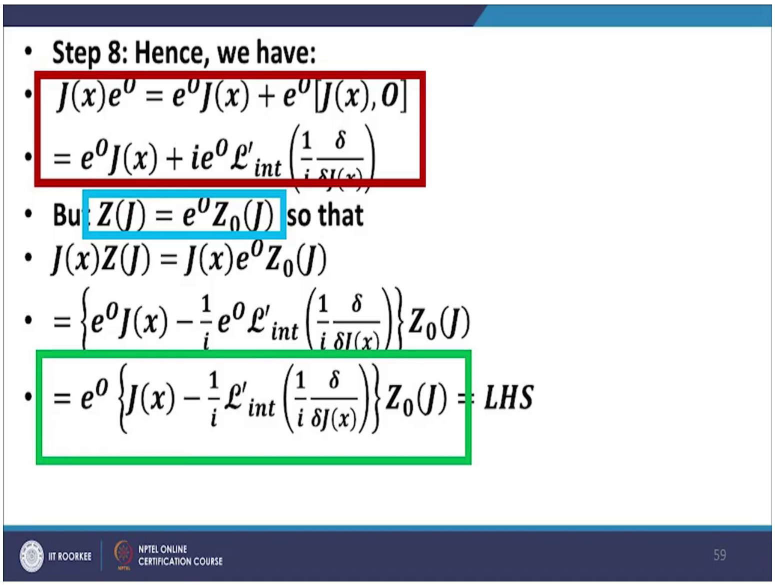

And therefore, what we have is J x e to the power o is equal to e to the power o J x plus e to the power J x comma o which is equal to e to the power J x plus i e to the power o L dash integral where we have substituted the value of this commutator here.

Now, if we multiply throughout by Z 0 because we want Z j; so J x e to the power now J x. We want J x Z x so we multiply throughout by Z 0 J; multiplying throughout by Z 0 J we get the result which we were required to prove.

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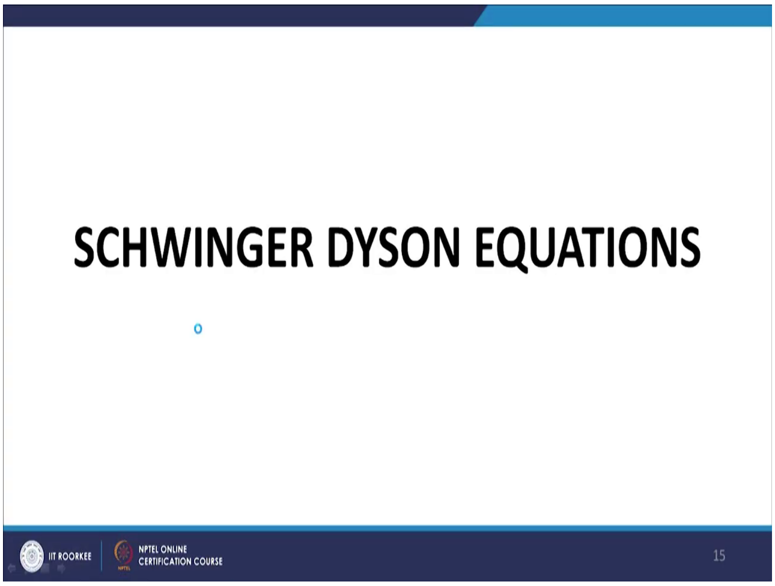

Now, we come to Schwinger Dyson equations Schwinger Dyson equations in the context of the phi 4 field in the Minkowski space.

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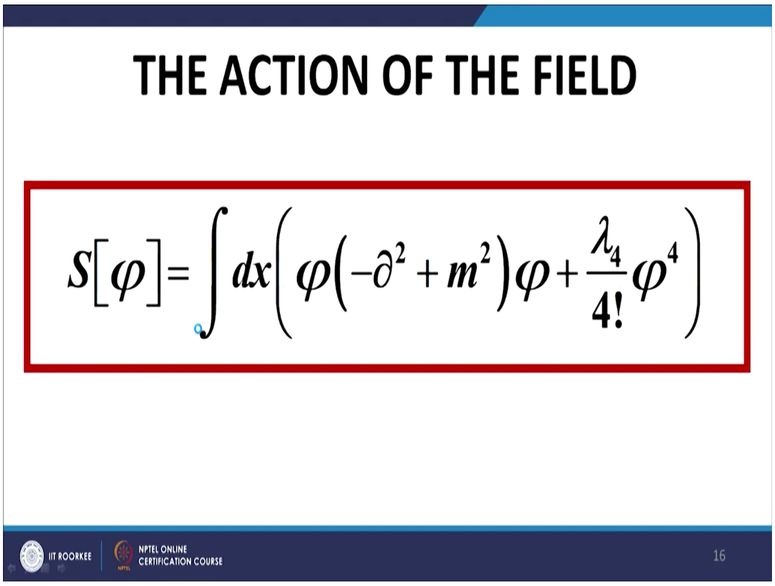

So, the action of the field can be written in this form which is in front of you we have already discussed that. After a integration by parts it can be put in the form that is here in the in the red box. And the coupling constant is been retained as lambda 4; we are talking about the phi 4 field. So, we have a phi 4 term in the action.

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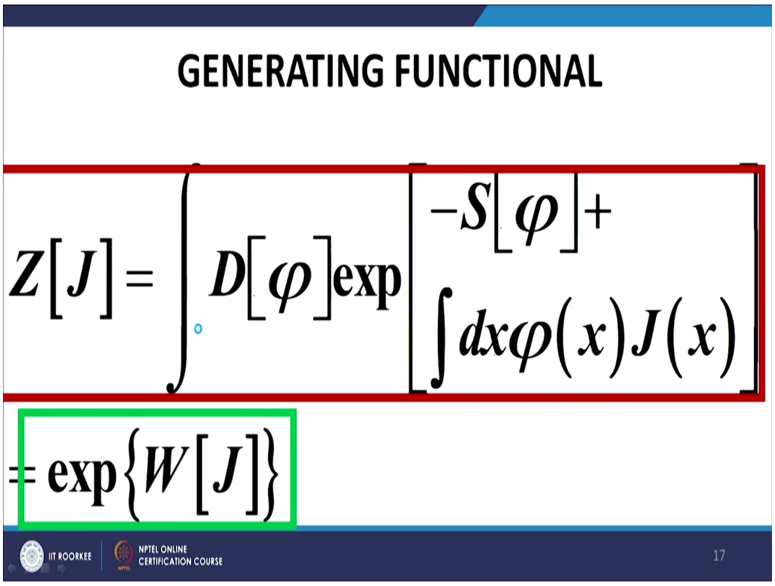

The generating functional then takes the form that is given in the red box in this slide.

And that can be written as exponential W J where; Z J is the generating function of all Green functions and W J is the generating function for the connected Green functions and J x is the source of the field.

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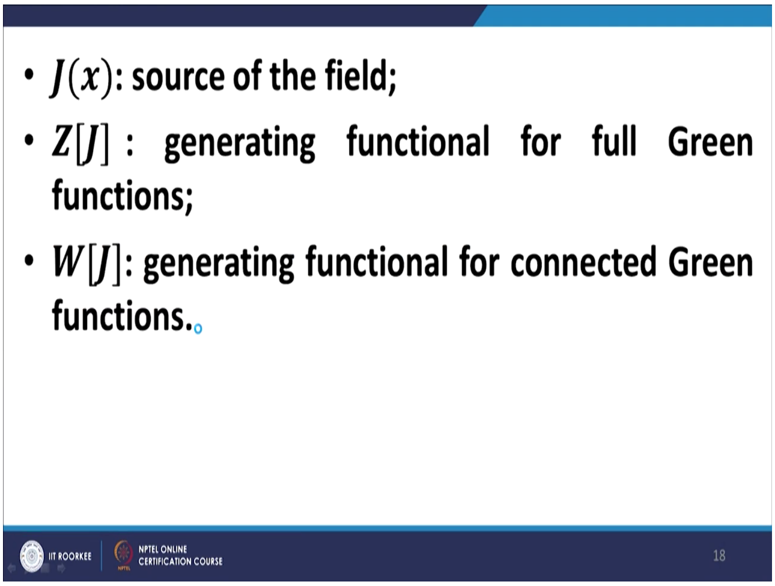

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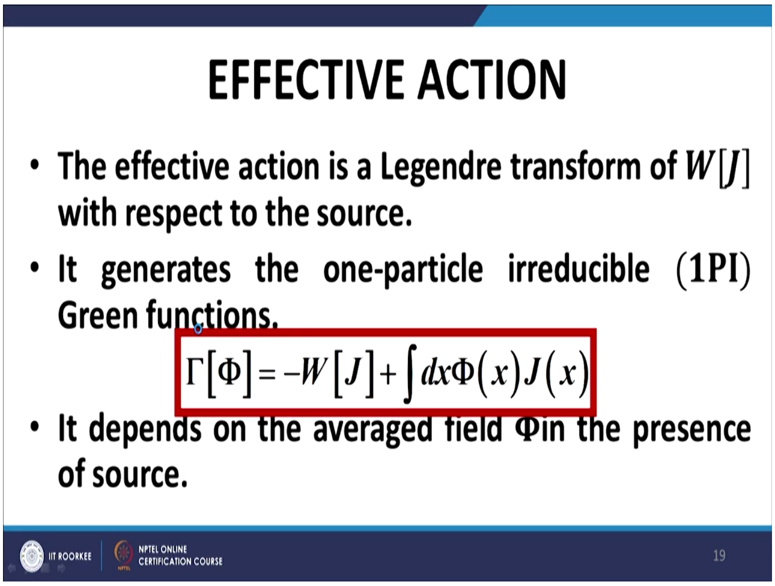

The effective action is given by the Legendre transformation of W J. And it can be written in the form that is given in the red box in the slide and it depends on the average field capital phi in the presence of the source J.

And in particular the effective action generates the Green functions. One particle irreducible Green functions 1 P I Green functions which are a subset of the connected Green functions.

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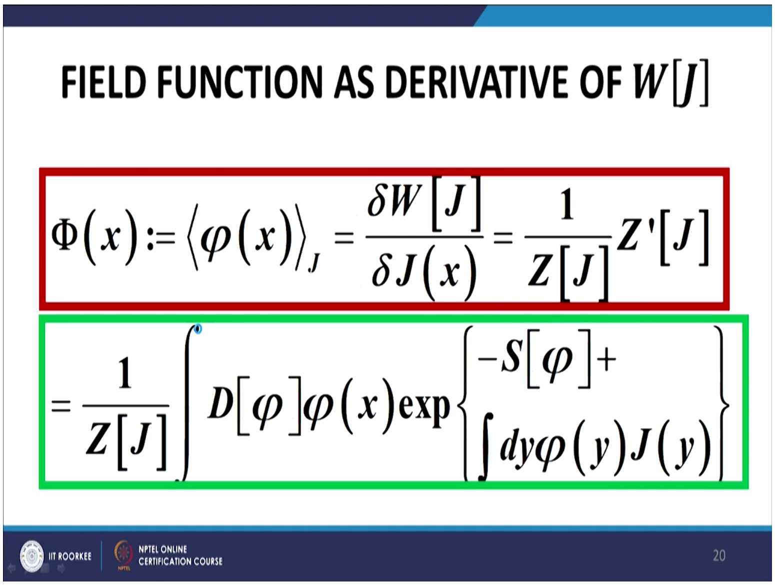

And the field functions is derivative of W J and can be written in the form that is shown in the red box at the top of the slide. And that can be simplified and put in the form of the green box at the bottom of the slide.

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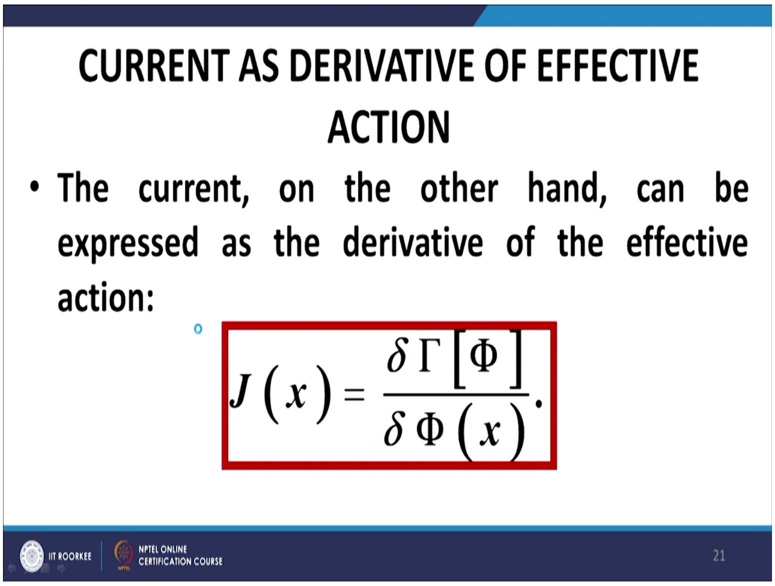

And the current is also a derivative of the effective action. Look at this the field function is a derivative of W J. And please note this W J with respect to J x field function is the function of the of the W J that is the generating function for the connected Green functions with respect to J x. And the derivative of the effective action with respect to phi J capital phi J which is the field function gives us J x.

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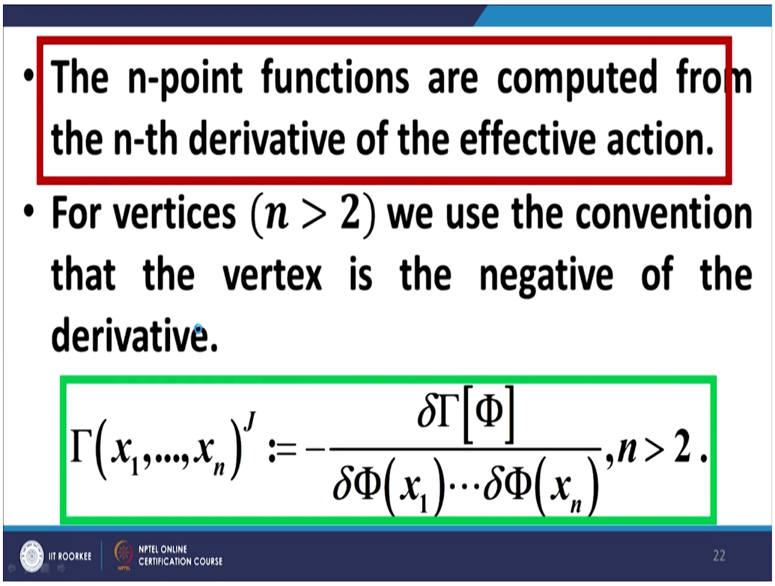

The n point functions are computed from the nth derivative of the effective action and the vertices are also computed in the same way. The convention that we use is that the vertex is the negative of the derivative as shown in the green box at the bottom of your slide.

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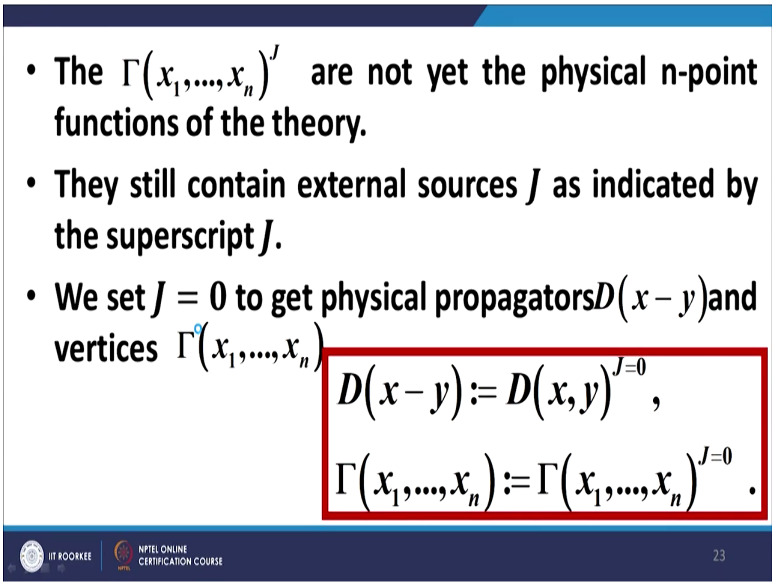

And the vertices are how I not get the physical vertices, after working out this vertices we need to substitute after working out the expressions for the vertices including the J's. Including the sources external sources we need to put the external sources equal to 0 to get the physical vertices and the propagators. And that therefore, the physical vertices and propagators that are relating to the field are given by the expressions in the red box at the bottom of this slide.

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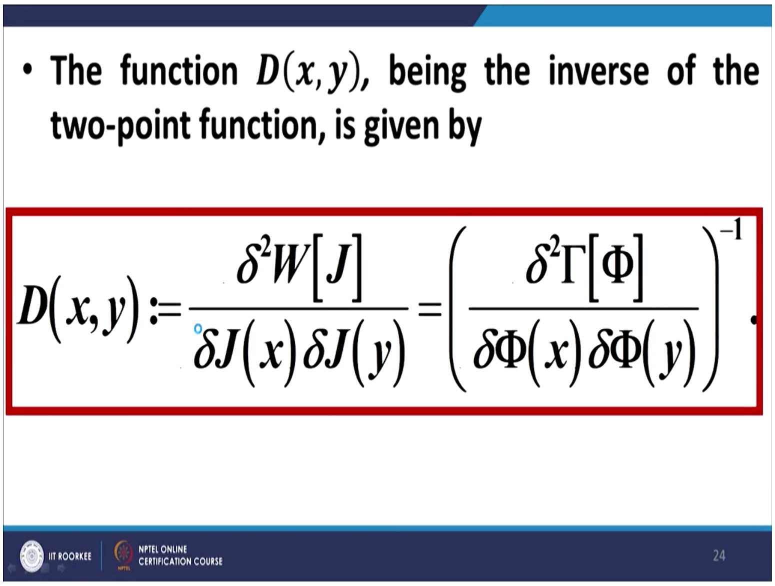

And because D x y is the inverse of; the two point function. We have the expression which is given in the red box at in this slide.

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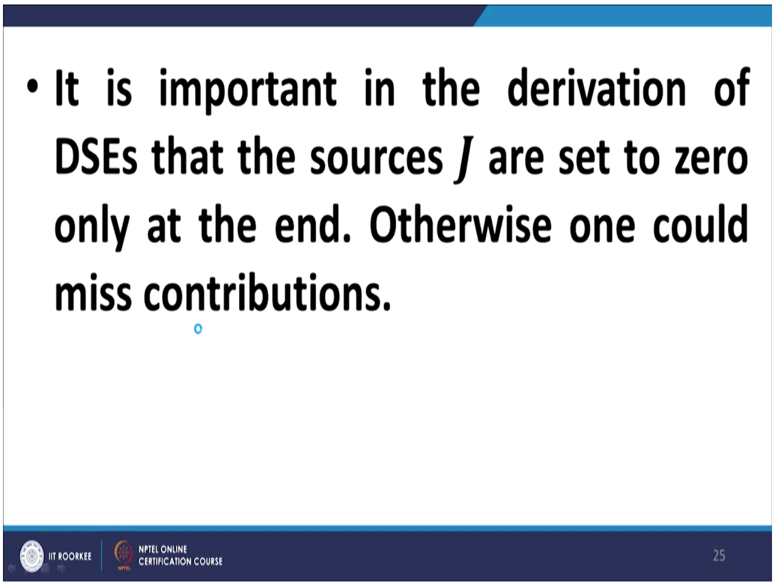

So, this is an important caveat as I mentioned we need to include the sources when we do the workings. And after we get all the expressions including the sources J we then put the sources J equal to zero and then arrive at the contributions or arrive at the expression for the propagators and the vertices.

In other words, let me repeat first of all we work out the expressions for this propagators and vertices in the presence of sources, then put the sources is equal to 0. And arrive at the physical expressions for these propagators and the vertices.

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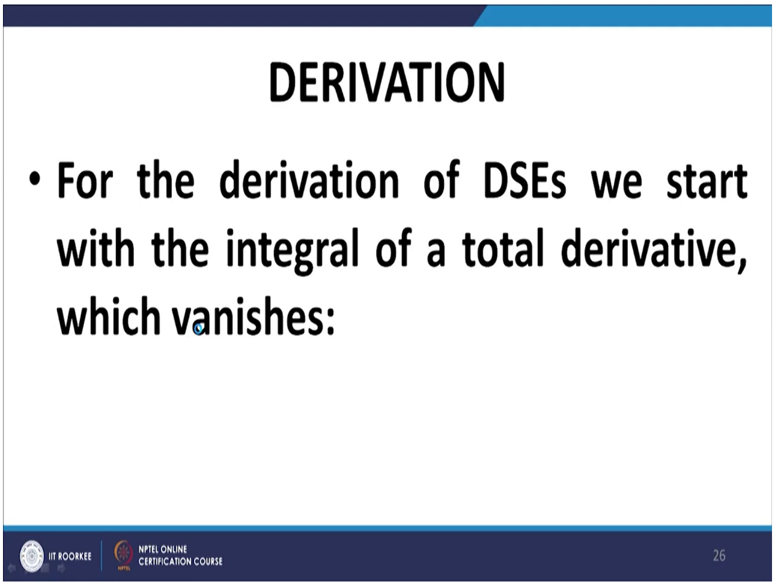

Now, let us look at the derivation skeletal, derivation of the Schwinger Dyson equation or Dyson Schwinger equation. We start with the total derivative of the form which; is given here in the red box.

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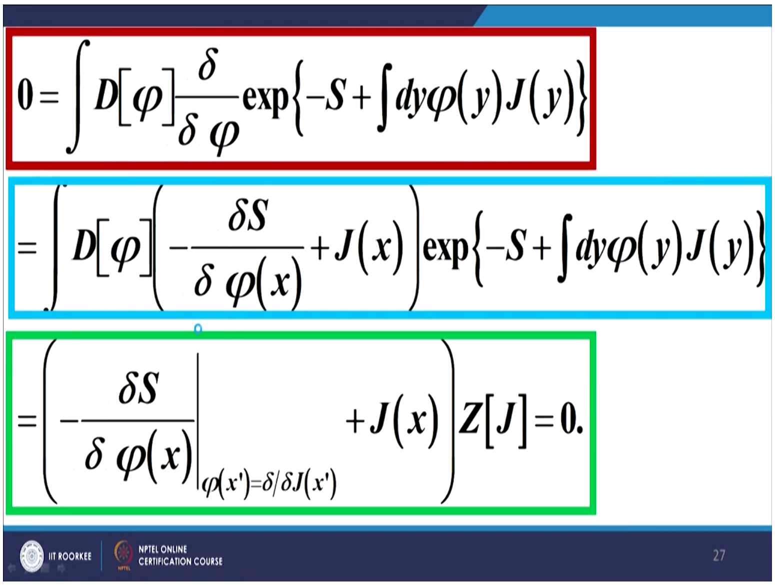

Obviously, this being a total derivative it vanishes. And when we do this derivative as we have done earlier in the context of the 0 point the field theory in 0 dimensions; I am sorry field theory in 0 dimensions that the approach is almost exactly parallel.

And in fact, that was one of the advantages of constructive field theory in 0 dimensions. The modus operandi is almost parallel exactly and look at although the calculations naturally become more cumbersome more tedious anyway.

So, when you do this derivative functional derivative with respect to phi you get the expression within the curly brackets and the rest of it remains as it is. And, this gives us the expression which is given in the green box which you recall coincides with the expression; similar expression which you obtained in the case of the 0 dimensional field theory.

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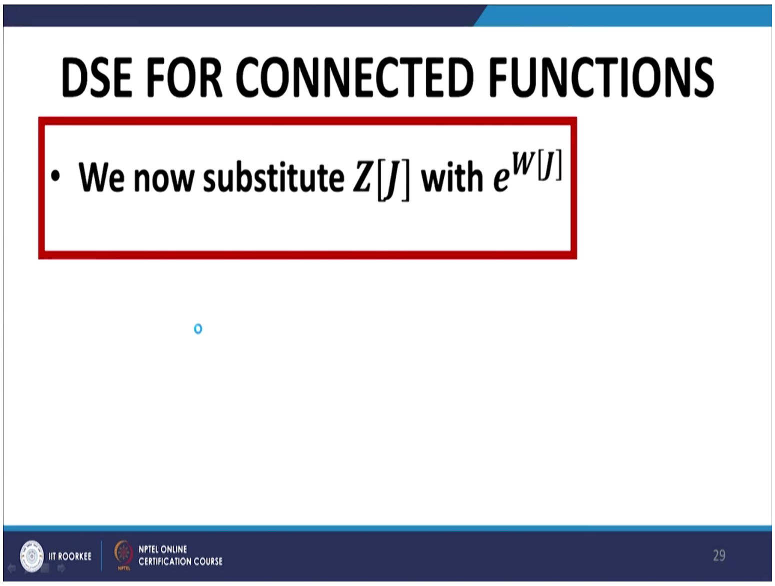

Then we now substitute Z J with e to the power W J to obtain the expression for the W J.

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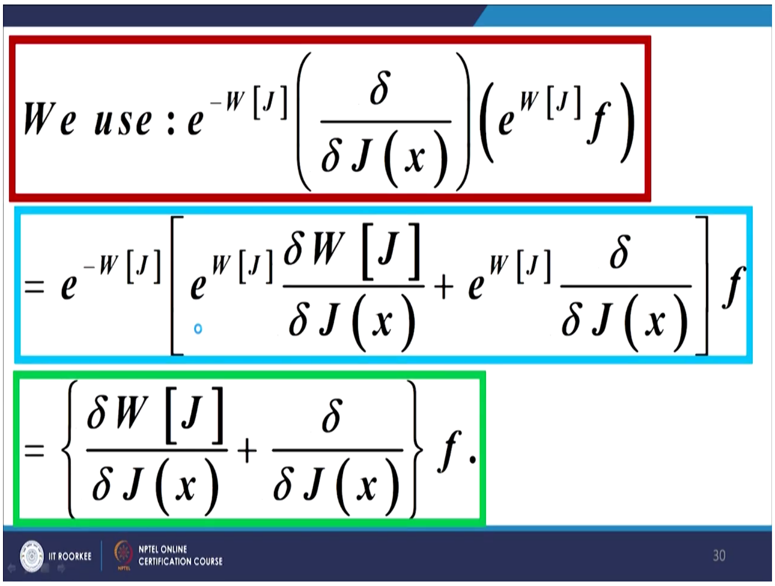

And we make use of this identity. This identity is quite straightforward if you look at this the functional derivative of J x functional derivative with respect to J x of e to the power W J into arbitrary function f. Arbitrary function f e to the power W J into an arbitrary function f is given by the expression given in the blue box here.

When you simplify this expression what we find is and it can be written in the form that is in the green box here. When we simplify this whole expression the e to the power minus W J cancels with e to the power W J. And that leads us to the expression given in the green box right at the bottom of your slide.

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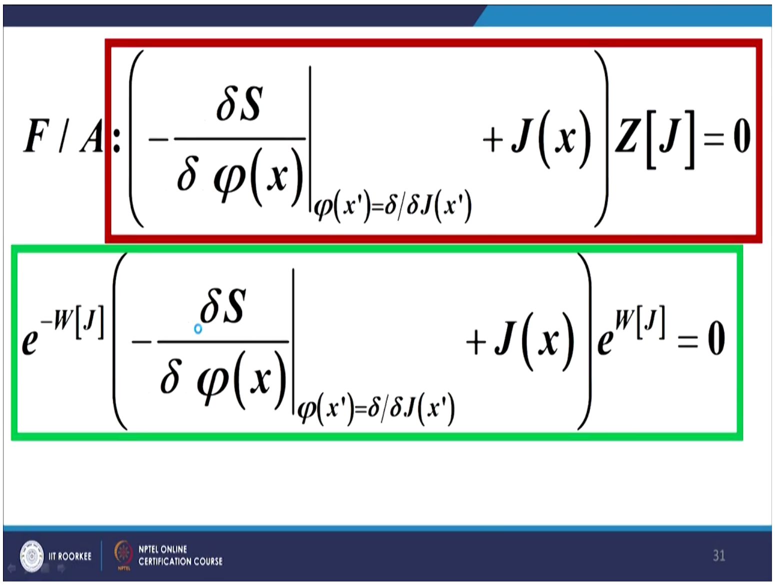

So, this is this is the expression that we had this is the expression that we had for the generator generating function; the Schwinger Dyson equation for the generating function.

And this expression enables us to write the expression for the for the expression for the connected green function generating function for the connected Green functions, in the form that is given in the green box at the bottom of your slide.

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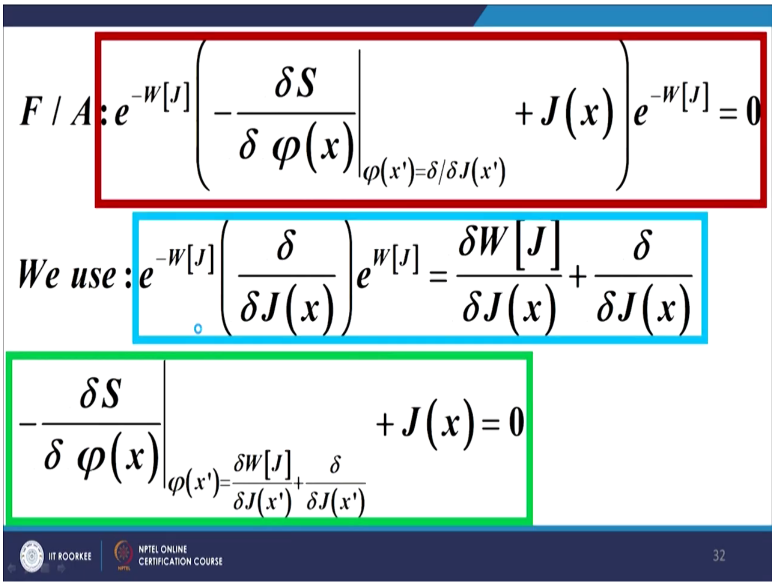

This can be simplified further by writing it in the form by writing it in the form using this expression in the blue box that we have proved just now.

Using this expression we can write this expression for the connected Green functions or the generating functional for the connected Green functions; in the form given in the green box right at the bottom of your slide.

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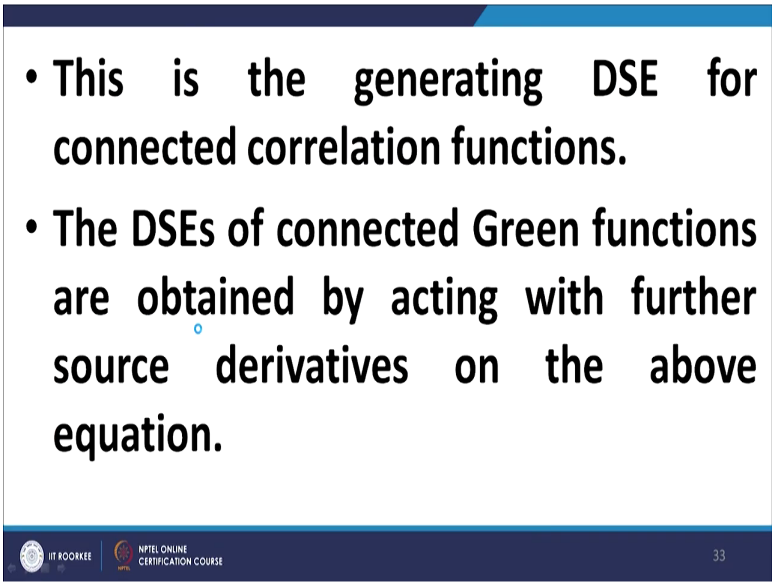

Now, the Schwinger Dyson equation for the connected Green functions can be obtained by acting with further source derivatives on the above equation.

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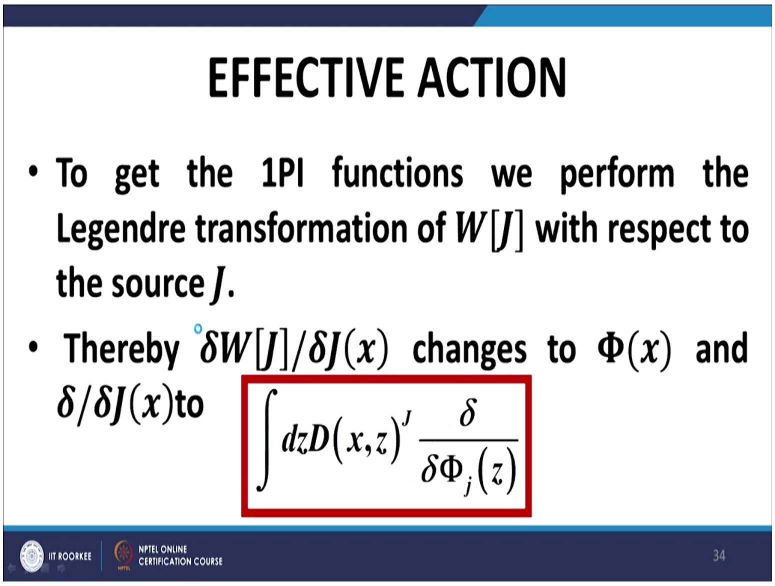

And now we talk about the effective action. The effective action as you know is the Legendre transformation of W J with respect to the source J x. And therefore, we can write in the functional derivative of W J with respect to J x as phi x. And the functional derivative with respect to J x can be written in the form that is here in the red box at the bottom of your slide.

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$$\frac{\delta}{\delta J(x)} = \int dz \frac{\delta \Phi(z)}{\delta J(x)} \frac{\delta}{\delta \Phi(z)}$$

$$= \int dz \frac{\delta}{\delta J(x)} \frac{\delta W[J]}{\delta J(z)} \frac{\delta}{\delta \Phi(z)}$$

$$= \int dz \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta}{\delta \Phi(z)} = \int dz D(x,z)^J \frac{\delta}{\delta \Phi_j(z)}.$$

So, this is established the second. The first part is quite straightforward because the effective action is the Legendre transformation of W J with respect to the source J. Therefore, the functional derivative of W J with respect to J x will return the effective the field function phi.

And the second the proof for the second relationship is given in this slide; which the functional derivative with respect to J x can be written in the form given in the red box here right at the top of the slide; which can be simplified using the fact that the field function is the functional derivative of W J with respect to J.

We use this expression for simplifying the field function capital phi of Z; writing it as del W Z upon del J Z. And, on simplification we get the expression that is in the green box right at the bottom of your slide.

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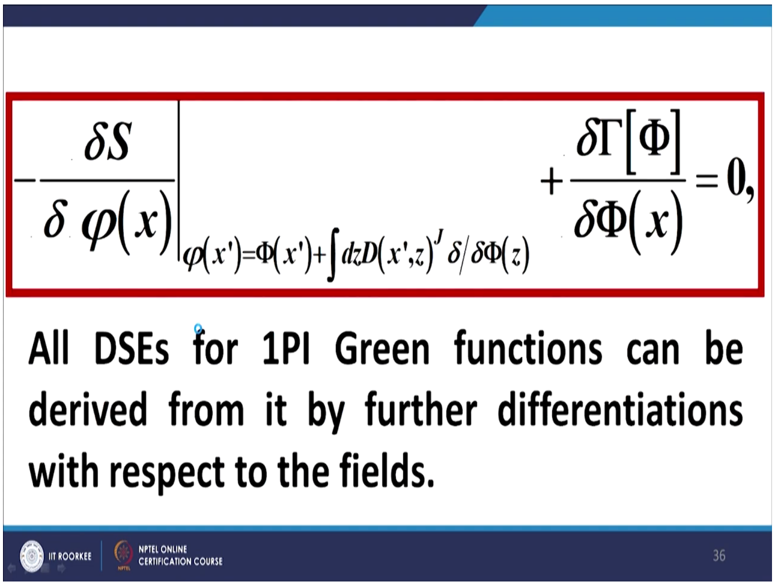

Therefore the expression for the effective field or the effective action can then now be put in the form that is shown in the red box at the top of the slide; red box at the top of the slide. Please recall that this is effective field is the generating functional for the 1 P I upon Green functions. And these 1 P I Green functions can be derived from this effective action by further differentiation.

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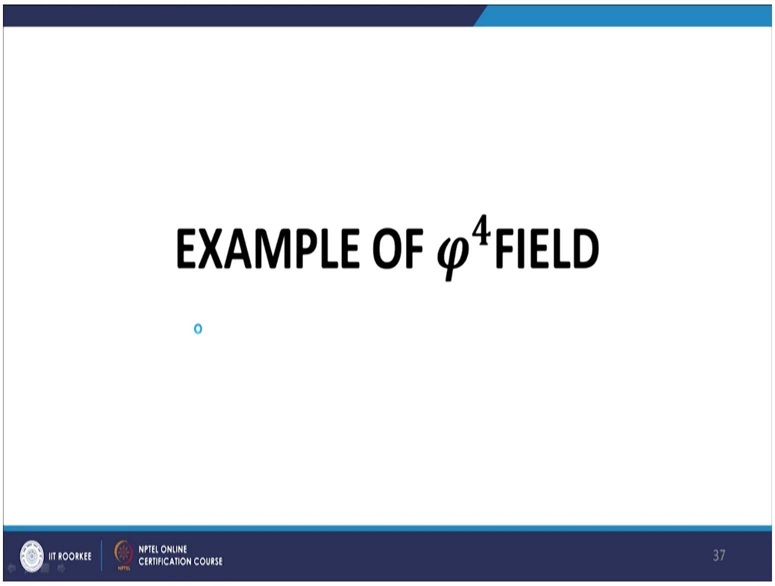

So, now we come to the example of this phi 4 field whatever we have done so far. But I will take it in the next class.

Thank you.