Path Integral Methods in Physics & Finance
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Lecture - 47
Interactive Field Theory in Minkowski Space

Welcome back, so towards the end of the last lecture; we were discussing the issue of causality in the context of the Feynman propagator.

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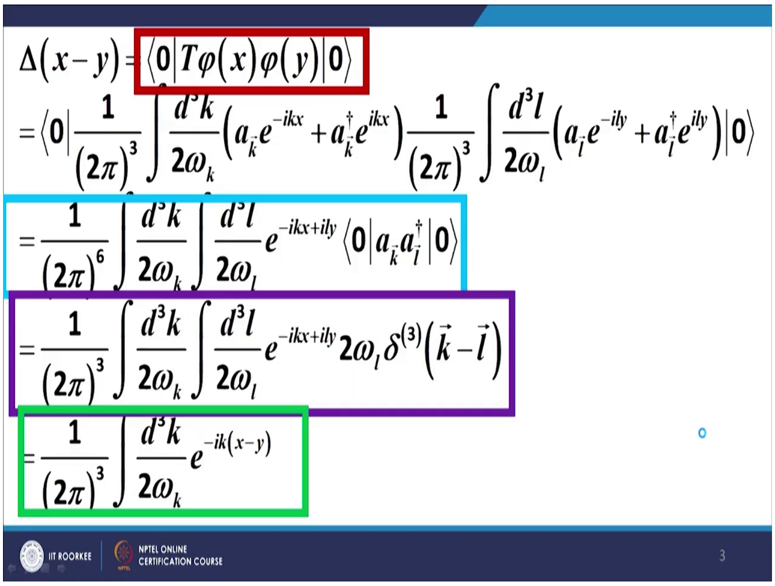

So, we started with the two point function, time ordered product and we obtained the expression for the time ordered products, starting with the expression in terms of the creation and annihilation operators and the relationship with the field variables.

We arrived, an expression for the propagator which is given in the green box right at the bottom of your slide, using the delta normalization which is; which was also explained in the previous lecture.

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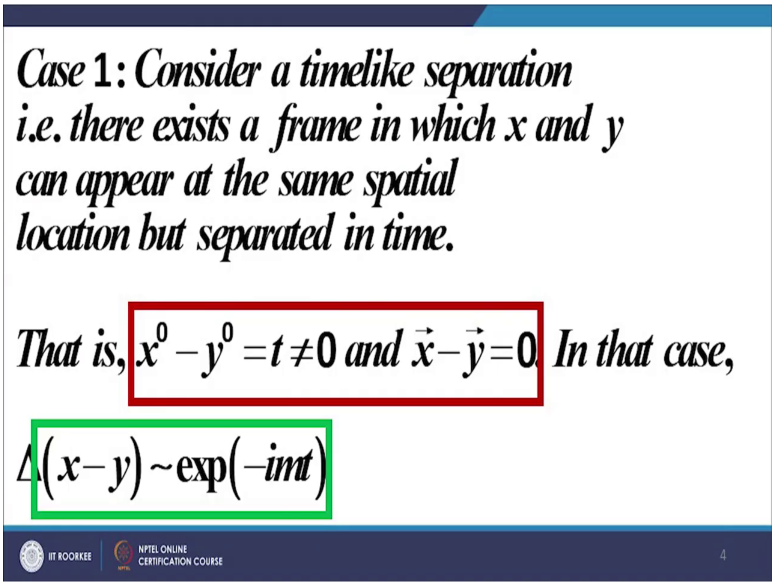

Now, then we moved on to; the; discussing the time like and space like separation of the two; points event points x and y.

We start in the first case, we discussed the time like separation of the event points x and y. And we argued that if x and y are separated in a time like manner, then there would exist a frame in which they would appear at the same spatial location and they would appear separated in time.

And under those circumstances we; after doing the contour integration, we arrived at the approximate expression for the propagator which is given in the green box right at the bottom of your slide which is importantly non zero.

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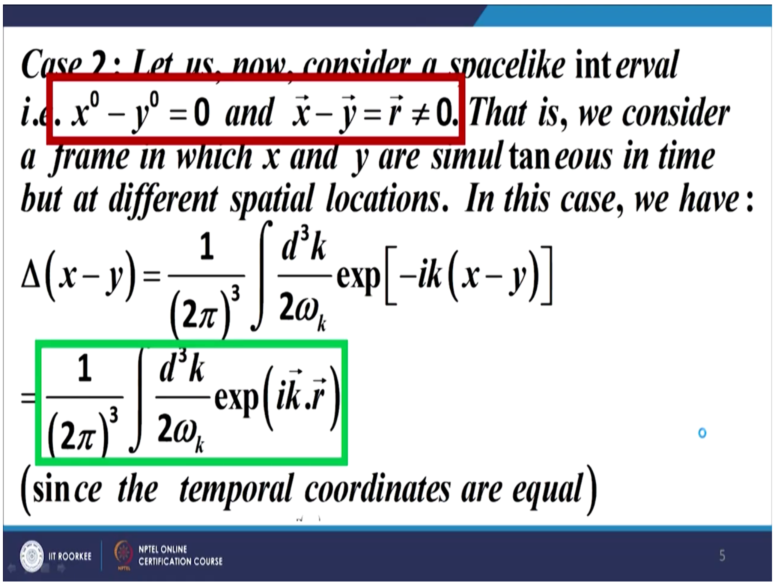

We also discussed the space like case, when the interval between x and y is space like and in such a situation; we have a scenario where the two events x and y are simultaneous in terms of time, but appear at different points, different spatial points that is say different spatial locations. And on that basis we arrive at the expression for the propagator; again which is given at the bottom of your slide in the green box and which incidentally again is non zero.

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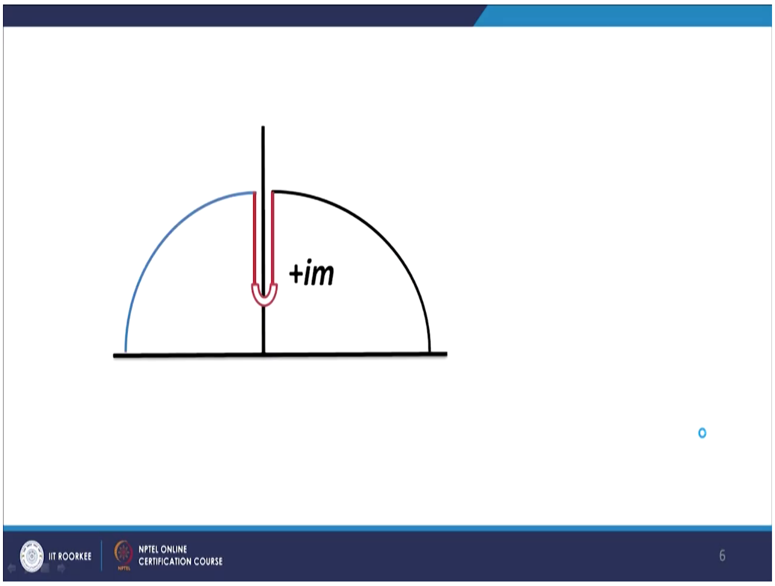

This was the counter that we used we because the integrand had poles at plus minus im.

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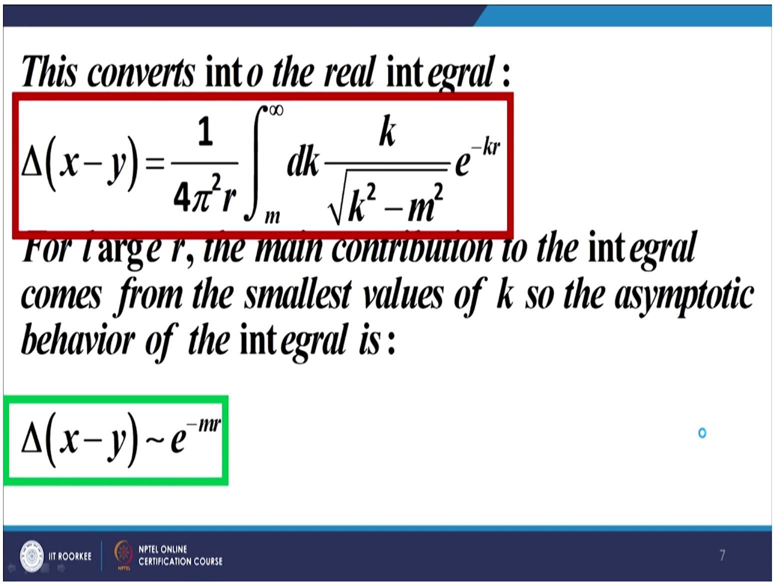

We used the contour which bypasses this poles around; moves around this poles or this branch curves and thereby, we arrived at the expression for the propagator, in the case of space like separation; as be being of the form e to the power minus r, but which importantly is non zero.

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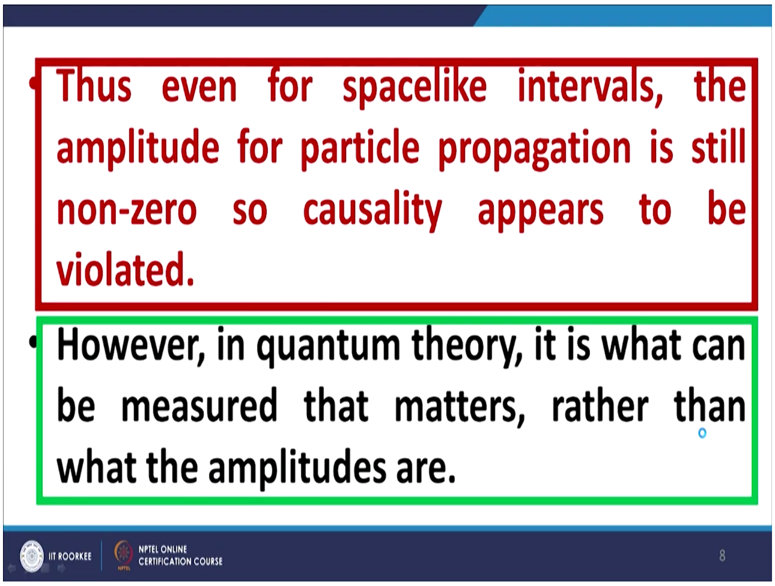

Therefore, what we conclude now is continuing from the expressions that we get for the propagator both in the case of time like and space like intervals; that in both cases the, the propagator has a non zero amplitude. And that means, that correspond stood being non zero probability of the particles being found, outside the light cone; in the context space like separation.

Now, that is apparently a violation of causality; it is clear, seems to be a violation of causality. But the important thing here is, we in the context of quantum field theory or quantum mechanics ; or quantum field theory; whatever we may be talking about; it is the measurement process that is important. It is the measurement there; it that is important, rather than the calculation of the amplitudes.

In other words, what; what is important is that although the, this transition amplitudes are non zero; we should be able to arrive at a situation or at a calculation which shows that in the case of space like separation of the two events x and y. The happening of events at one space time location will not influence the measurement of the event that occurs at the other space like location.

Let me repeat the measurement of two events; x and y are separated in space time by a space like interval. Then the measurement of the event at a point x should not be influenced by the event occurring at the point y; if the two are space like separated, that is the important point.

That is what is required for the consistency of quantum mechanics rather than the transition amplitudes being 0 per say. The transition amplitude that we have seen here in our case are non zero both in the case of time like and space like separation of the space time events x and y.

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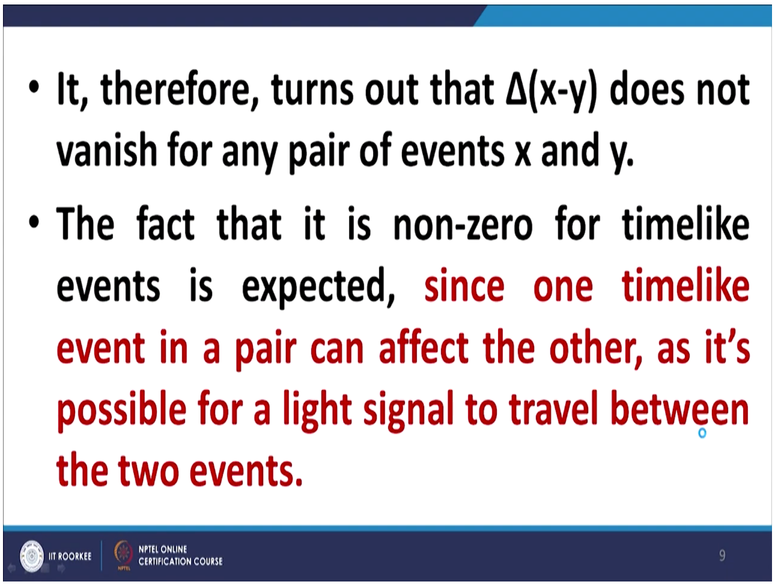

Therefore, although delta x minus y that is the propagator. does not vanish for any pair of events x and y. This does not completely violate causality, in the context of quantum mechanics because the quantum mechanics emphasizes the measurement process.

And if we can show that the measurement process of one event is not influenced by the other event, if the two are space like separated, we are still on stable ground as far as quantum mechanics is concerned.

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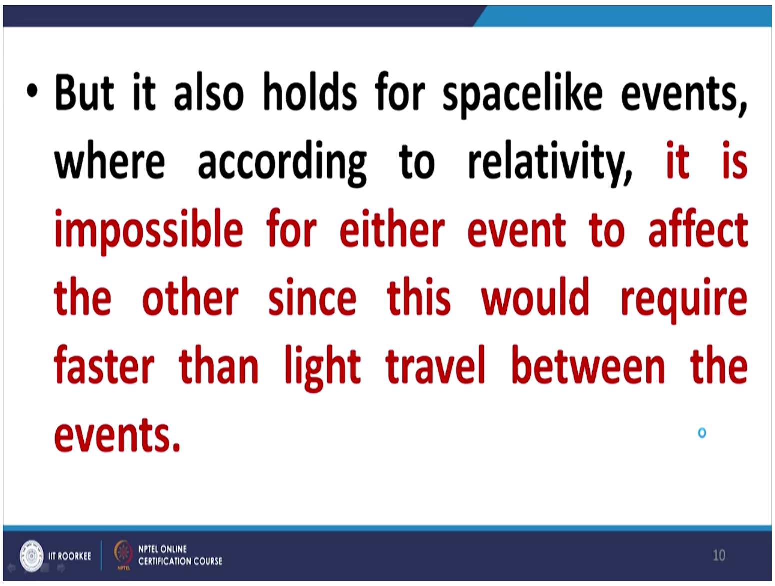

So, let us now establish that fact; in order to establish this fact, in order to establish the facts that the measurements that x and y are independent of each other. We need to establish that the commutator of the field operator at the field variables, at the point x and the point y should commute with the each other. Because, if they do not commute with each other; if they are non computing observables, then clearly the measurement at one point is influenced by the event at the other point.

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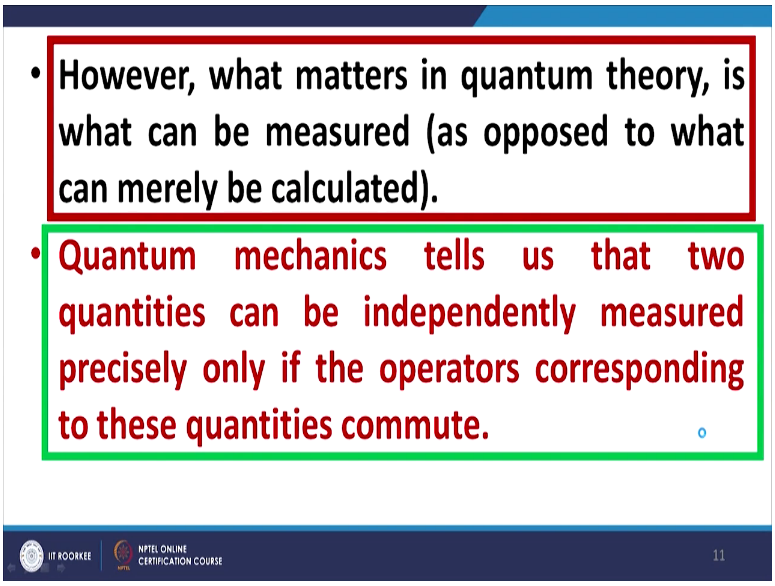

So, that is important and that is our next step; the operators corresponding to these operators commute. So, thisin the green box this is the of list of all what I have being saying. Quantum mechanics tells us that two quantities can be independently measured precisely only if the operators corresponding to those two quantities commute.

So, we need to ensure or we need to establish that the two; that the events occurring at the two points commute.

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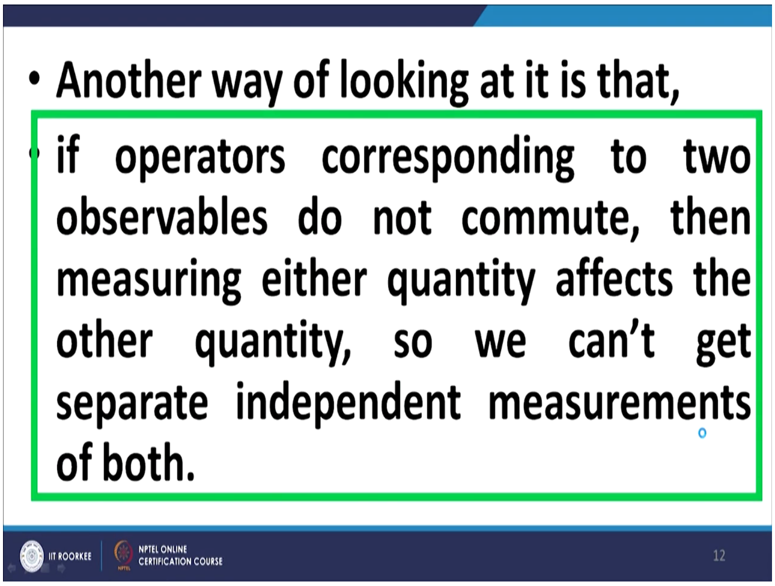

In other words, phi x and phi x commute with each other; phi x and phi y commute with each other. If operators corresponding to the two observables do not commute, then measuring either quantity affects the other quantity and then we can say that causality is being violated.

But, if the operators corresponding to the two observables do commute; then measuring either quantity is independent of each other and therefore, causality is not violated.

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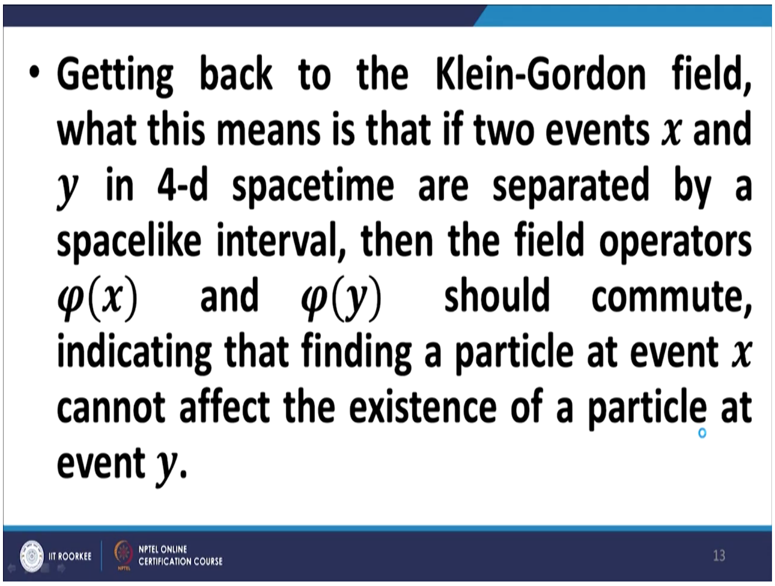

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$$\textit{From above}: \Delta(x-y) \equiv \langle 0|T\varphi(x)\varphi(y)|0\rangle$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega_k}\exp[-ik(x-y)]. \textit{ Also}$$

$$\varphi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega_k}\left(a_{\vec{k}}e^{-ikx} + a_{\vec{k}}^{\dagger}e^{ikx}\right)$$

So, let us look at what happens in our case; we have this position, we have this position that we arrived at; in from the previous work, delta x minus y. The propagator is given by the expression in the red box and the field variables in terms of the creation and annihilation operators are given by the expression in the green box.

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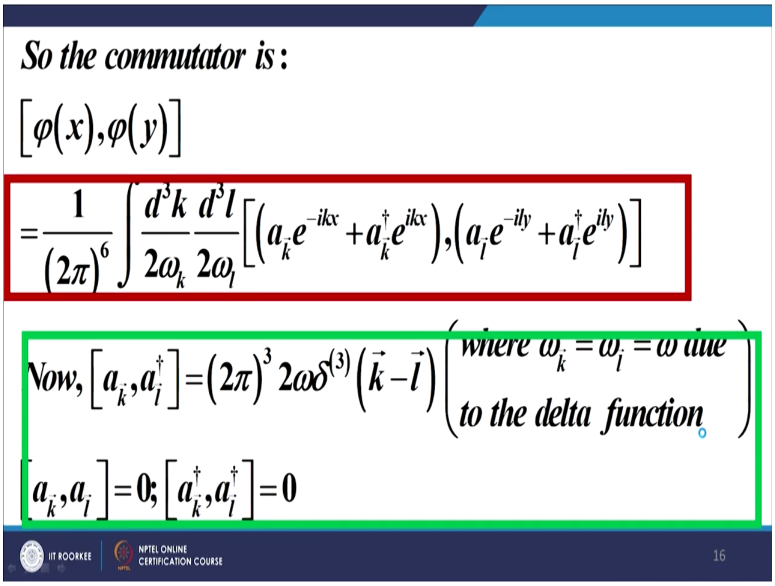

Therefore, we can write the commutator as the expression that is given in the red box and the normalization is will take the form. The commutators and the normalization between the creation and annihilation operators will take the form given in the green box at the bottom of the slide.

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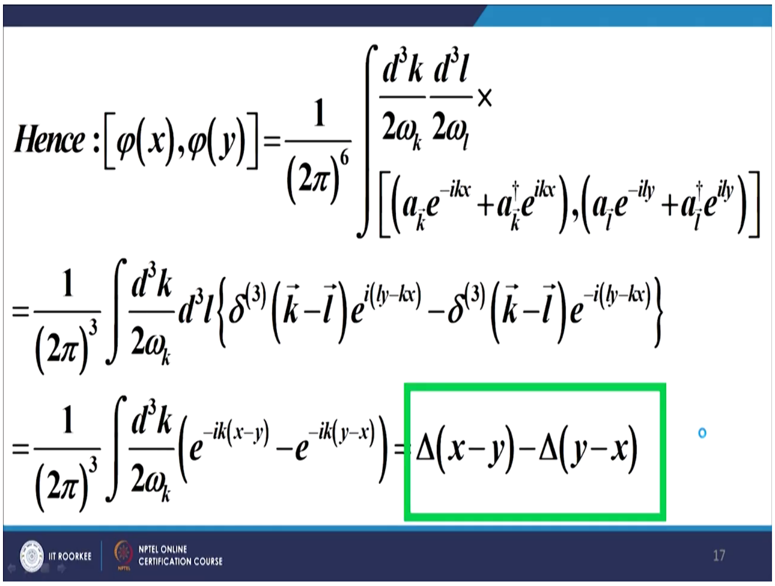

Substituting these values here in; in the expression for the commutator, we arrive at the expression which is given in the green box right at the bottom of the slide where we introduce the various commutators or we maneuver the products of the all creation and annihilation operators using these commutators.

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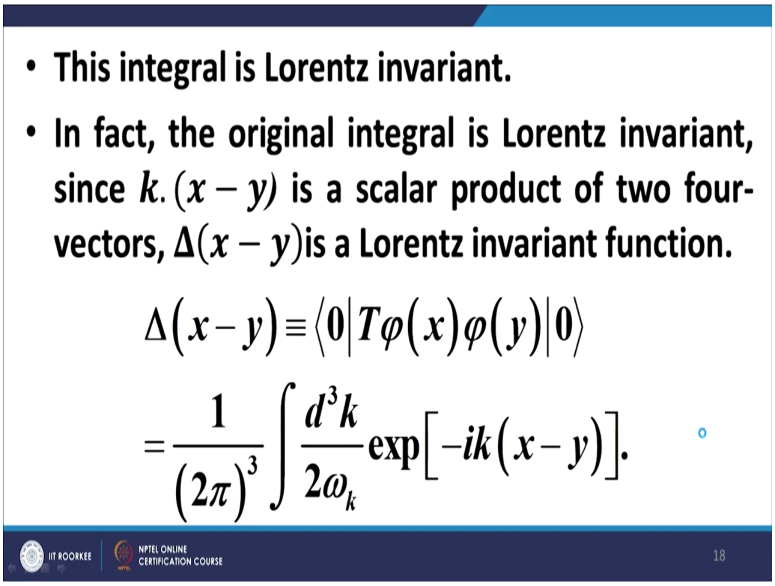

And now, the important thing is that if you look at this integral this integral is Lorentz invariant.

If you look at the left hand side the integral is Lorentz invariant clearly, why? Because the product k dot into x minus y; x minus y and k, the scalar product is Lorentz invariant. And because this scalar product is Lorentz invariant delta x minus y also becomes Lorentz invariant and therefore, the time order product also becomes Lorentz invariant.

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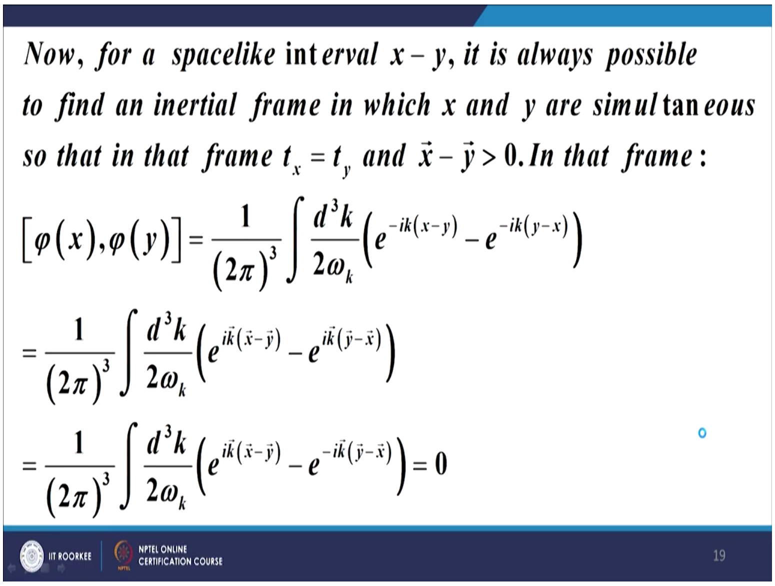

Now, now let us see what happens in the case of a space like interval. As I mentioned in the case of a space like intervals, we can always find an inertial frame in which x and y occur simultaneously in time, but they occur at different spatial locations. In other words, t x is equal to t y, but x minus y is greater than 0 in that particular frame.

If you use that expression, we arrive at the expression the right; at the bottom of your slide where the time factor will go away because t x is equal to t y. And the they are instantaneous or they are spontaneous in terms of time so that the time component appearing in the exponential goes away and we are left with the spatial components.

And when we simplify this expression; we find the result to be 0 because why the result is 0? Because if I substitute the; if I make the transformation k goes to minus k the integration variable k is overall all values of k and therefore, the substitution k equal to minus k does not change the integrand.

In other words, if in this; if the integral that is appearing at the right at the bottom of your slide, if I make the change k equal to minus k in the second term and integrate; it does not change the integral and you can see in that case the integral vanishes.

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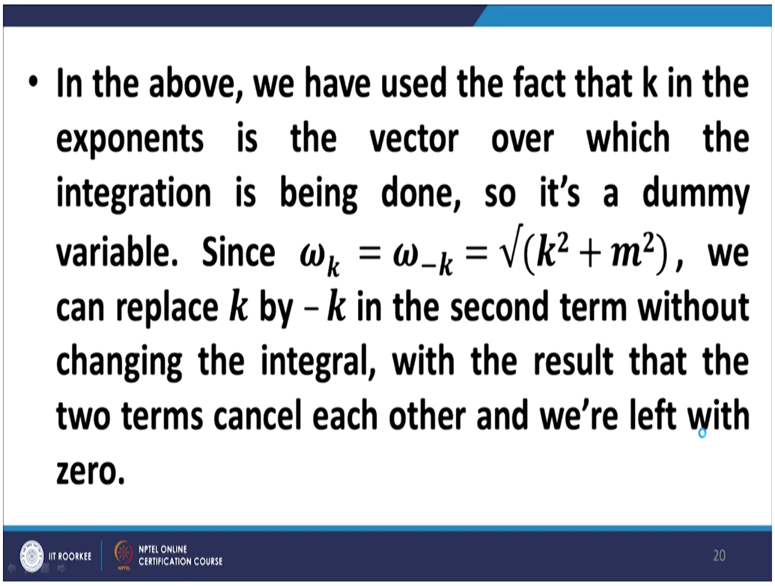

And as far as the omega k is concerned; omega k also does not change, when I substitute k equal to minus k because as is shown here; omega k is equal to under root k square plus m square. So, k transformation of k to minus k; does not change the value of omega k.

And therefore, omega k does not remain; does not change and the integration being over all values of k, it does not change by the transformation k tends to k goes to minus k and we have the same situation again the integral vanishes.

Therefore, the important thing now; what have we concluded? We have concluded that the commutator of phi x and phi y; in the case of a space like intervals, when x; x minus y is a space like interval does commute and that is what we wanted to establish.

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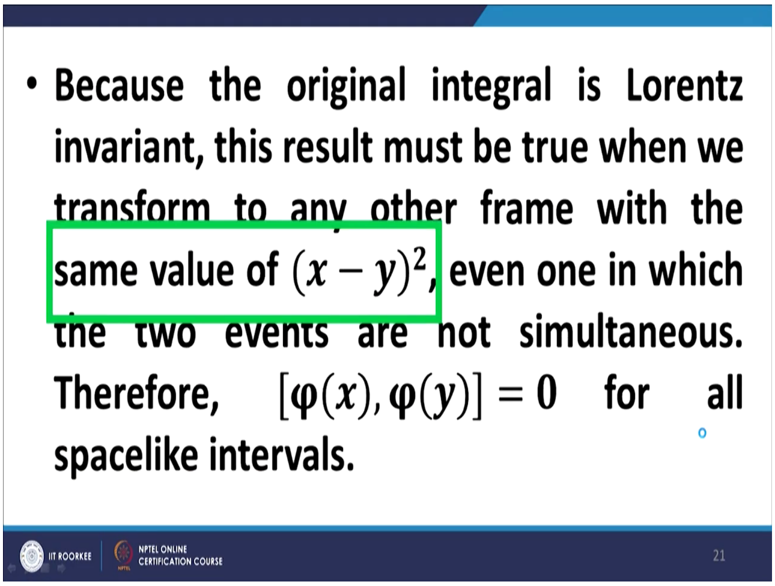

As now because this holds in a particular Lorentz frame and this is Lorentz invariant, the statement is Lorentz invariant. We can Lorentz transform to any other frame in which the time; the x and y, the spatial components are space like separated and the time and difference between the two is non zero. The time difference between the two is nonzero, but the events are still space like separated. Even in that case; even in that frame, the commutator would vanish.

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For a timelike interval $x-y$, it is always possible to find an inertial frame in which x and y occur at the same spatial location but at different times so that in that frame $\vec{x}-\vec{y}=0$

$$t_x \neq t_y \quad and \quad \left[\varphi(x),\varphi(y)\right]=\frac{1}{(2\pi)^3}\int\frac{d^3k}{2\omega_k}\left(e^{-ik(x-y)}-e^{-ik(y-x)}\right)$$

$$=\frac{1}{(2\pi)^3}\int\frac{d^3k}{2\omega_k}\left(e^{-i\omega_k\Delta t}-e^{i\omega_k\Delta t}\right)=\frac{1}{i(2\pi)^3}\int\frac{d^3k}{\omega_k}\sin(\omega_k\Delta t)\neq 0$$

So, now we look at a time like interval x minus y; in the case of a time like interval, we can find a Lorentz frame in which they occurring. The two events x and y occur at the same point in space, but they occur at different points in time. In that situation, when we simplify this expression; we get a nonzero quantity as it shown right at the bottom, right hand corner of your slide.

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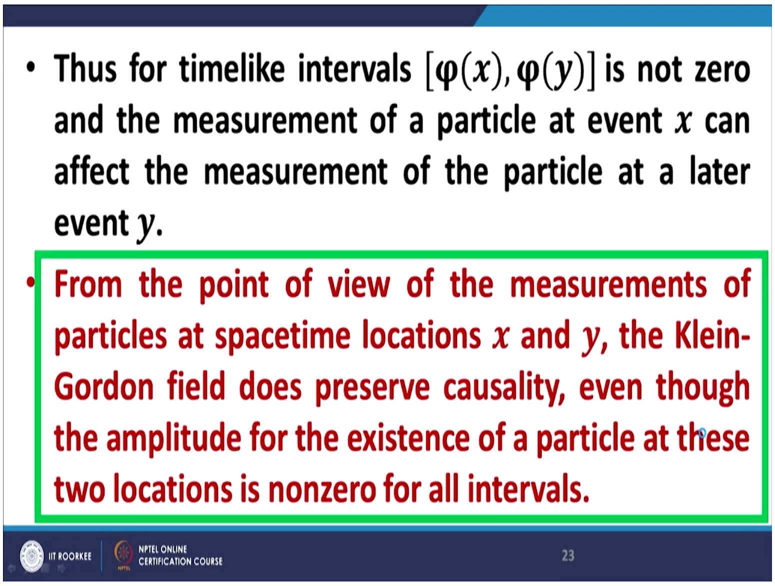

So, in this case; we find that the commutator does not vanish and thereby showing that the; the measurement of events that x will influence the measurement of event and y and vice versa.

Therefore, for time like intervals the; the commutator is nonzero and the for space like intervals that commutator is 0 and that is what the casual; causality demands. Because in the case of time like intervals the two events can be connected by; by light signals whereas, in the case of space like, space like separated variables; we cannot have.

They are supposed to lie outside the light cone and therefore, they would not be connected by signals less than the speed; less than or equal to the speed of light. And hence the commutator should have vanish which is actually true.

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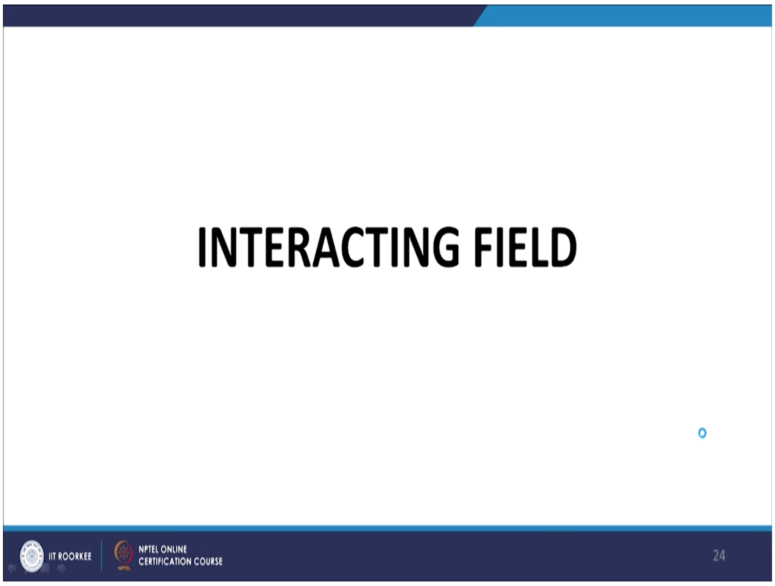

Now, we talk about the interacting field; interacting field in Minkowski space, the Klein Gordon interacting field in Minkowski space.

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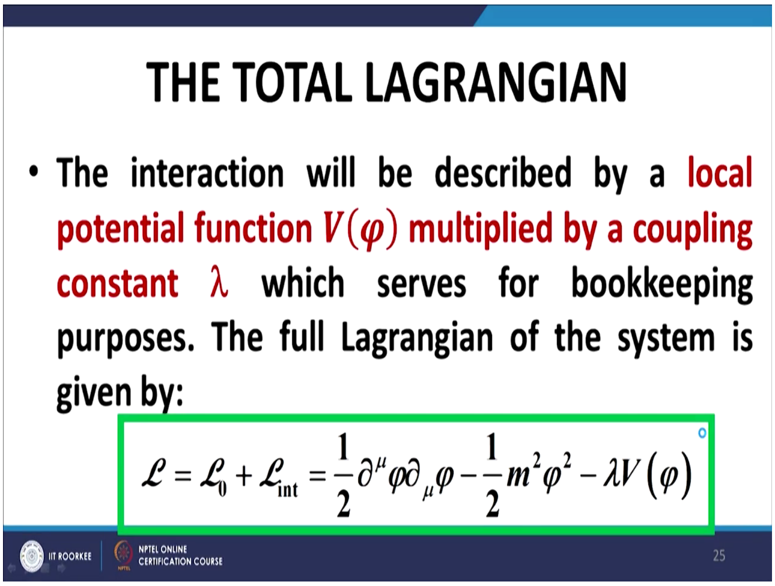

Now, the Klein Gordon the Lagrangian for the interacting field, or the Klein Gordon interacting field is given by the expression; that is in the green box here, right at the bottom of this slide.

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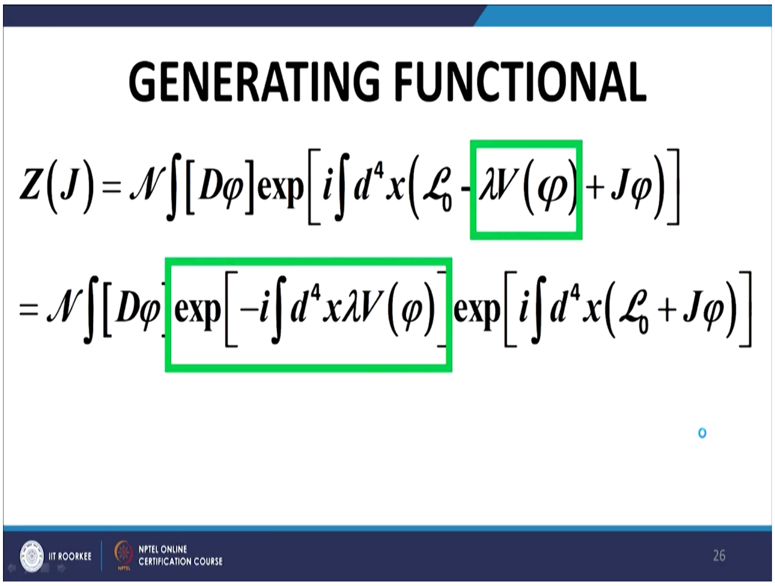

And the expression for the generating functional Z; J, you add another factor of J phi to the expression for the Lagrangian and you get the generating functional. Please note, in this case we are modelling the interaction generally as a potential term V phi with a coupling constant lambda.

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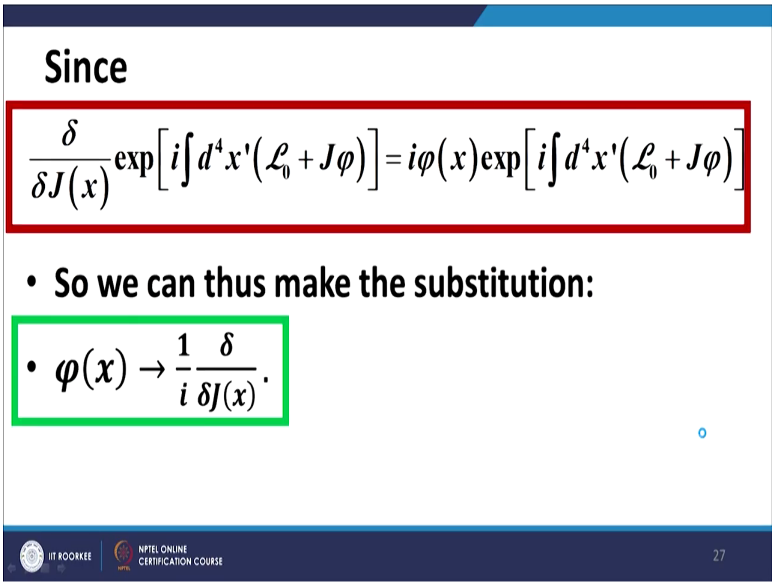

Now, because the functional derivative of this integral the; the generating functional with respect to J; x gives us i phi; x into this whole expression. As you can see, in the red box; we can substitute phi by phi x by 1 upon i by the functional derivative of J x.

So, this substitution we have done earlier in many cases; we shall be using it again.

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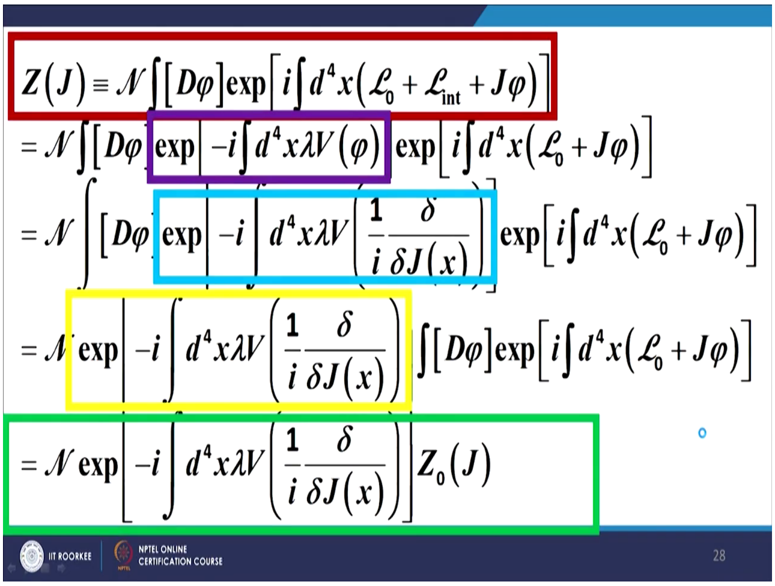

Now, this is a very important slide; let us go through it step by step. Z J is given by the normalization constant; normalization in the interacting field and this integral which we brought forward from the previous slides. Remember, the interaction term is being modelled as a potential term V of phi with a coupling constant lambda and that is what is here.

We can now because of what was mentioned in the previous slide, we can replace in V of phi, we can replace; V as a, instead of writing V as a function of phi; we can write V as a function of 1 upon i; del y, del j; as the functional derivative with respect to J; x. This operator we can substitute for the expression phi because when this operator operates on whatever is to the right hand side of this, we get phi as the outcome of the action of this operator.

We take this; the expression that is in the blue box here because it is independent of phi now. It can be taken outside the integral and that is precisely what we do; as shown in the blue box here. So, this expression we take outside the integral and whatever path integral remains, let us call it Z 0 of J, as shown in the green box at the bottom of your slide.

This is the path integral which we call Z 0 of J, the normalization is unchanged and the exponential of this term of this term which was taken outside the integral; because it was independent of phi, this expression we have taken as it is in the green box equation.

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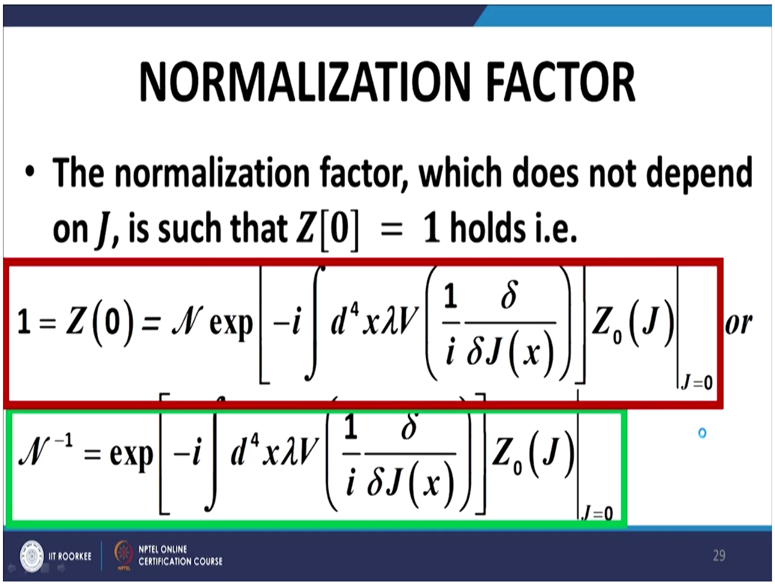

Now, the normalization factor is obtained by writing Z 0 equal to 1. We explicitly mandate that Z 0 equal to 1 be the normalization factor; that is the interaction normalization in a sense.

And that gives us the; that fixes for us the value of the normalization factor N script; N script is given by; the inverse of N script is given by the expression, that is there in the green box at the bottom of your slide; on the basis of the equation that is given in the red box equation in the red box. 1 is equal to Z 0 is equal to N exponential of this whole term. And that gives us the value of N; that fixes for us the value of N, by; we can all, we can obtain the value of N through a perturbation series.

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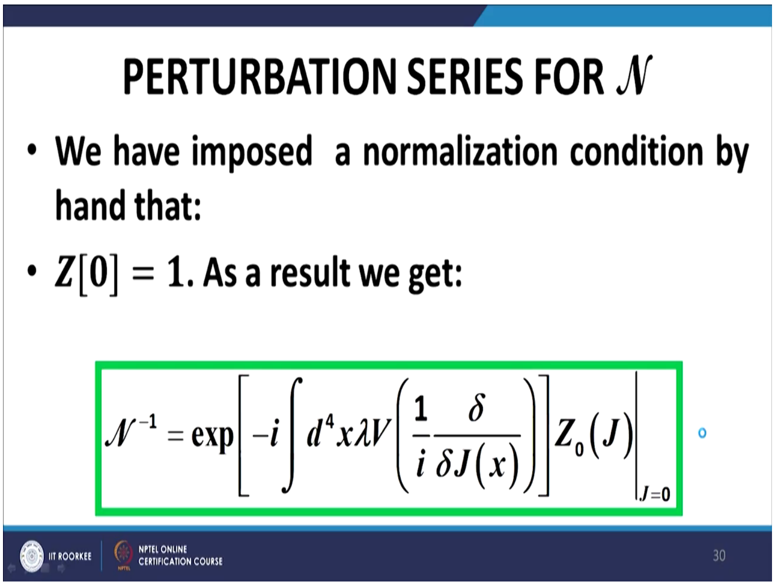

But, this is what we get for; using a perturbation series for N, N script that is the normalization, we can obtain the value of N to desired accuracy, by mandating, as I mentioned earlier Z 0 equal to 1; that is the prescribed man made mandatory normalization.

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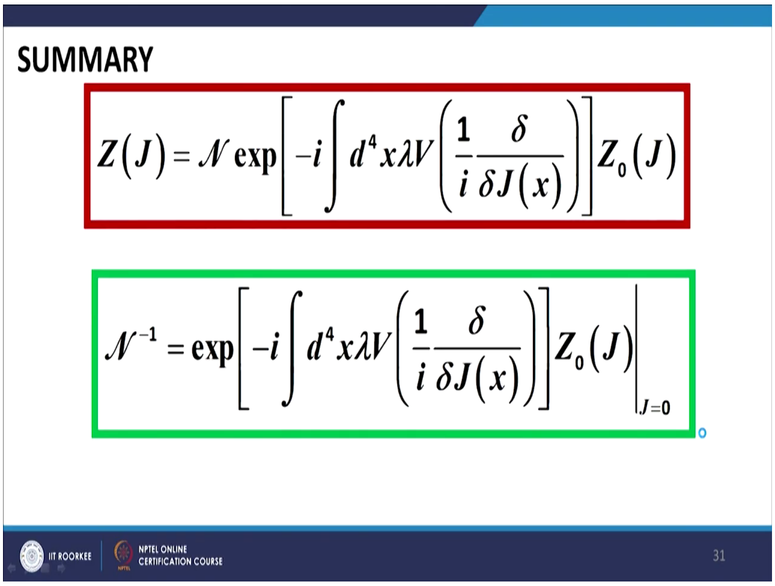

So, this is the summary of where with stand; the generating functional is given by the expression in the red box and the normalization as given by the expression in the green box here.

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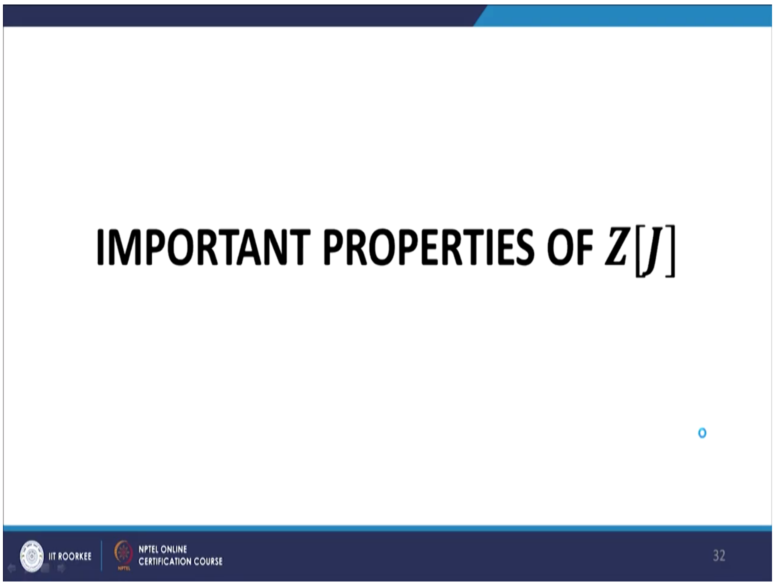

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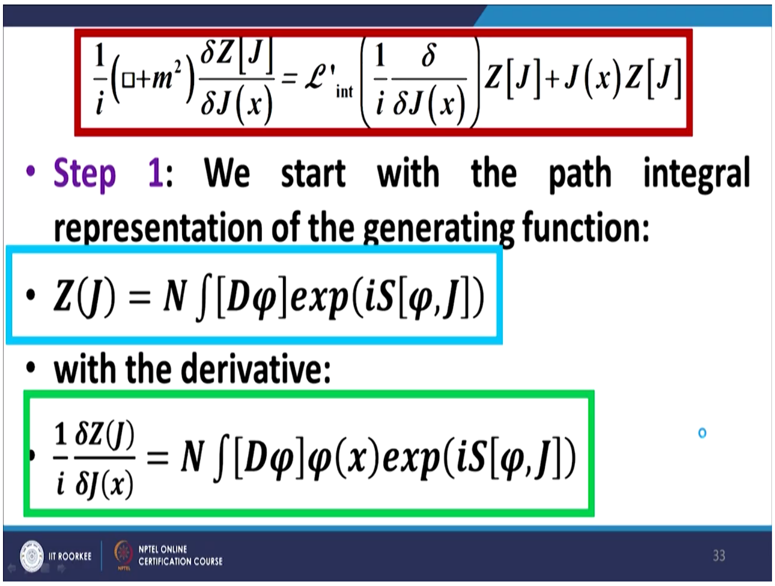

Now, let us look at the important properties of Z J; the generating function. The first property that we will be establishing is given in the red box here; we shall be establishing this. We start with the path integral representation of the generating function which we have discussed earlier. This is the; generating functional and that derivative of this expression with respect to J; x will pull down a factor of phi, as we have again and discussed just now.

If I take a derivative of the generating function respect to Jth; it pulls down a factor of phi in the path integral that is what is happened as shown in the green box.

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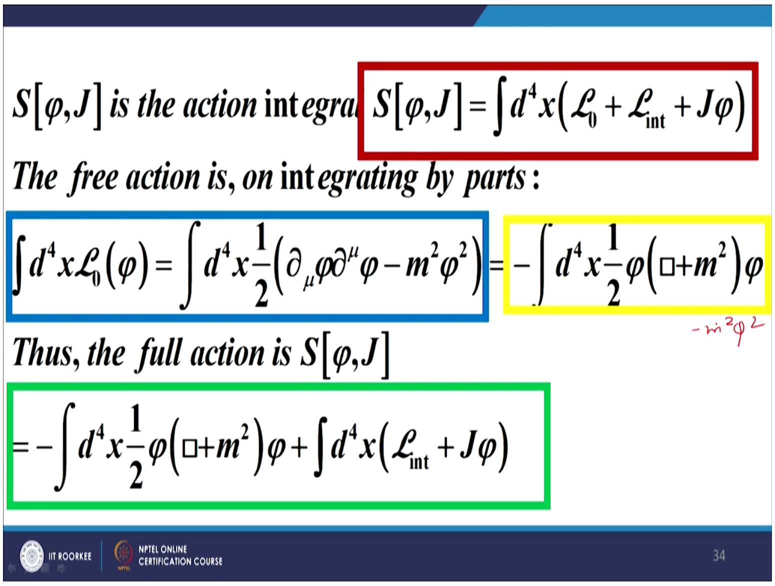

Now, for information for a recollection; for a recall, the action is given by the expression in the red box here where thea source term is included is subsumed within the action.

And the free action is given by the expression in the blue box which on an integration by parts takes the form, given in the yellow box. And the full action is therefore, given in the; as the expression given in the green box right at the bottom of the slide.

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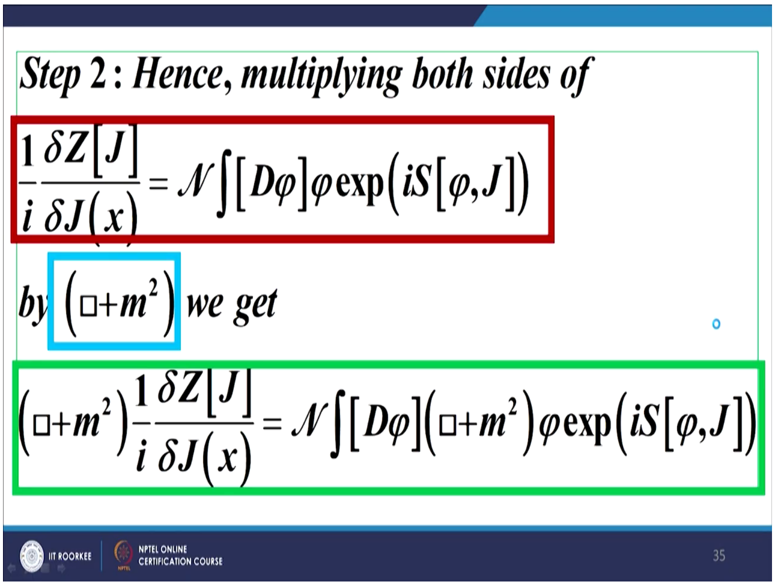

Now, multiplying both sides of this equation which is brought forward from the earlier slides by the Klein Gordon operator; what we get is this expression. And the Klein Gordon operator can be moved inside the in the path integral and we have the expression that is shown in the green box at the bottom of your slide.

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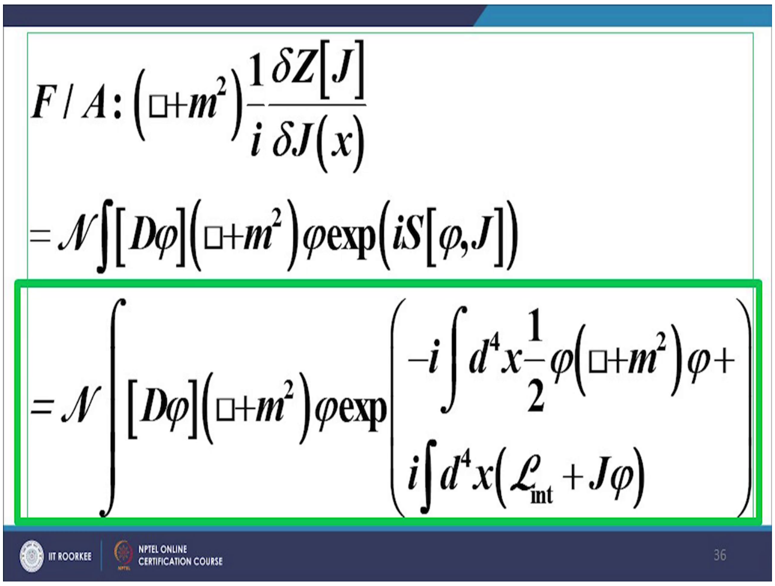

Looking at the left hand side and this is what we have from the previous slide and the box plus m squared at 1 upon i; del Z J upon and del J; x is equal to this expression, this is what we have from the previous slide.

And this expression now on the right hand side can; the action integral has been expanded to incorporate the term that involves the Klein Gordon operator and the rest of the terms.

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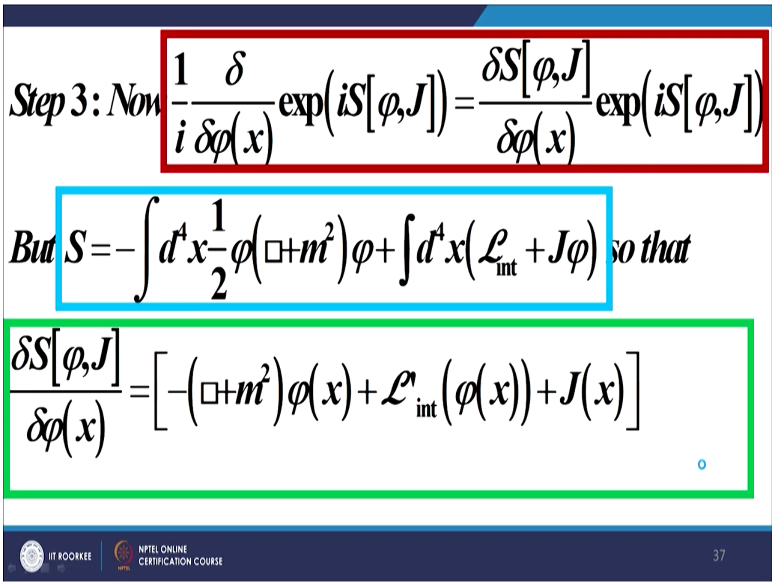

So, continuing from the previous slide; this expression now let us look at what we get from the red box here. The different; functionality differentiating exponential i S with respect to phi x; I get the expression exponential i S into del S upon del phi; that is what I get from, in the functional integration of the expression exponential i S this thing.

So, but S is equal to this whole expression and therefore, del x upon del phi x; del x upon del phi x is equal to the expression that is given in the green box. Remember, we need this expression; del x upon del phi; x for substituting in the expression on the right hand side of the equation, in the red box that is our next step.

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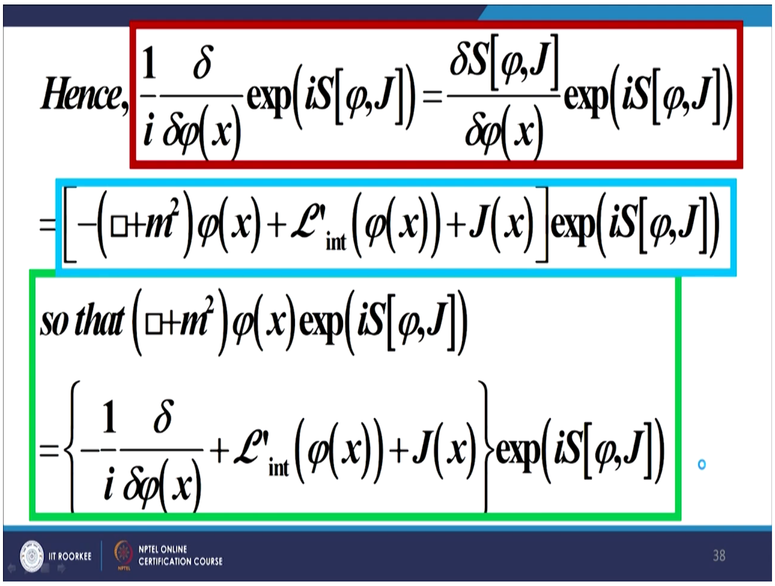

So, this is what we have from the previous slide; this is what we have obtained for del S upon del phi.

We have obtained the expression in the square bracket here and then we have the exponential i S appearing here. So, this is what we have for the expression that is given on the left hand side. You see this expression is what I want, so I take this to the left hand side from the blue box term and I take the first term on the red box to the right hand side. So, that became; that carries a minus sign and the rest of the terms in the blue box remain unchanged. So, this is the expression i get in the green box.

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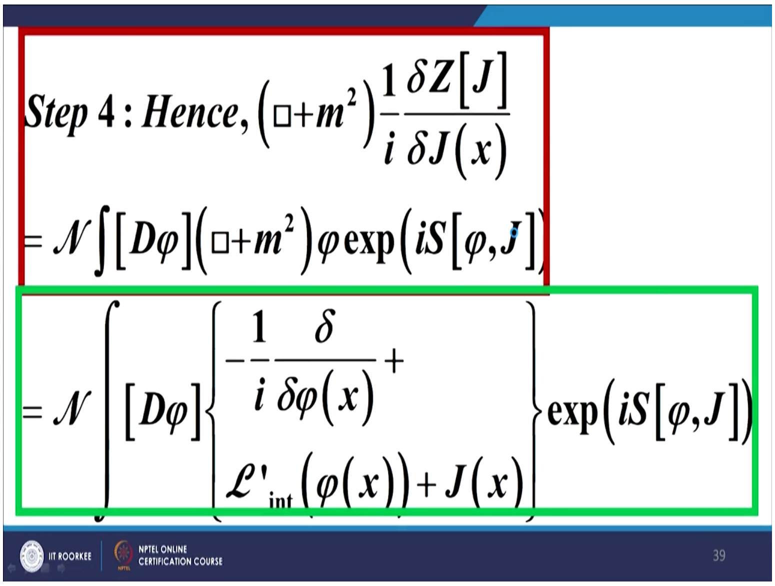

Now, we have this expression brought forward from the earlier slide and this is the expression, we got from the last slide; del plus m square 1 upon i. This whole expression and this expression del plus m square phi exponential i S ; I have got from the previous slide, as the expression within the; within the curly brackets and the exponential i S; is as it is previously.

Now, let us look at this; when you do the path integral of this is a total derivative and when it will be integrated; total derivative of exponential i S. So, when we do the integration of this term; when I do the integration of this term, I will get a surface term and under the given boundary conditions; that S vanishes in plus minus infinity, the surface term will vanish.

So, what we are left with is; the second term and the third term; L dash interaction phi x plus J; x, these are the two terms that are left. The first term; the on integration gives a 0 contribution because when converted to a surface term, it has to be; the surface term between plus infinity and minus infinity and in either case, the surface term returns a 0 value.

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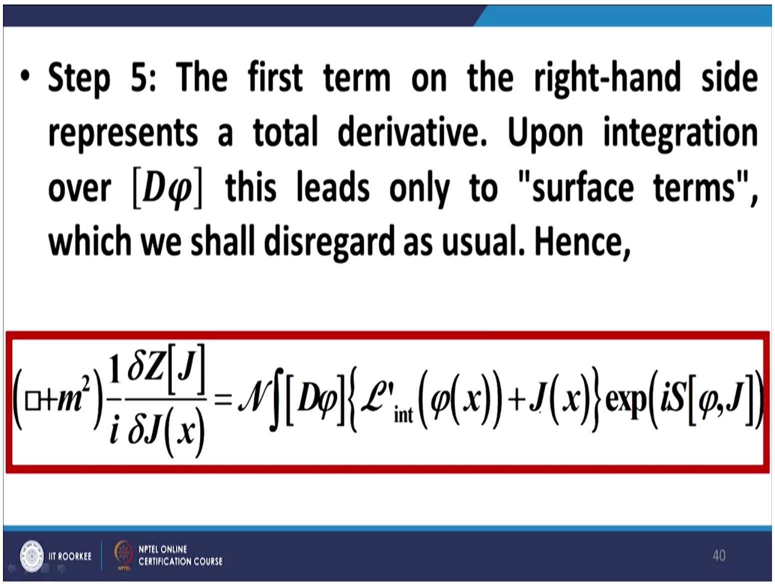

So, the net result that we get is that with the expression; that is given in the red box here.

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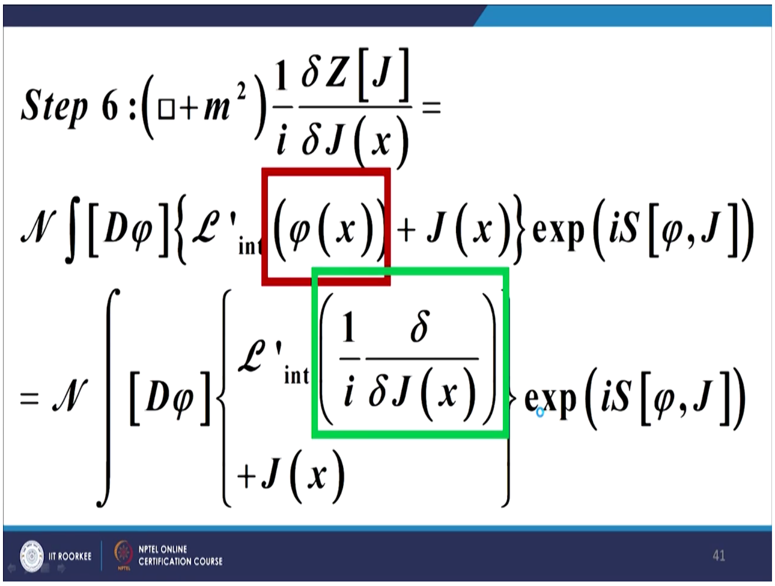

And this is now just one more step here before we finally, end up with the result. This is the expression that we had in the previous case; now phi x here can be replaced by 1 upon i and functional derivative of J; x; because the action of this expression in the green box on exponential i s x simply pull down a factor of phi.

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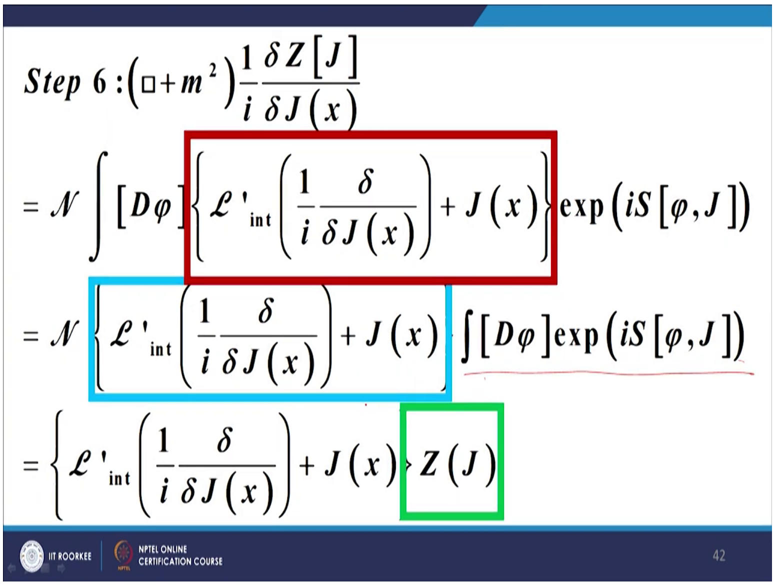

And therefore, we can replace phi x here by the expression that is given in the green box and that is; so that is; that gives us the expression in the red box here, after substituting phi x is equal to 1 upon; del of del with respect to J x of this e i S and that on simplification.

Now, this expression that is in the red box here is independent of phi; being independent of phi, being; it is also independent of the path integral measure. And therefore, it can be taken outside the integral and we take it outside the integral and what we have left is integral path; path integral exponential i S and which is; this is nothing but Z of J.

So, at the end of the day; what we have? We have L dash; 1 upon i; del of del z which is the first term here, in the curly brackets plus J x which is a second term here and multiplied by Z J; which is this particular term. And this is equal to the expression right at the top; left hand corner of the slide, box plus m square 1 upon i; del J upon J x. So, this is what we were required to establish.

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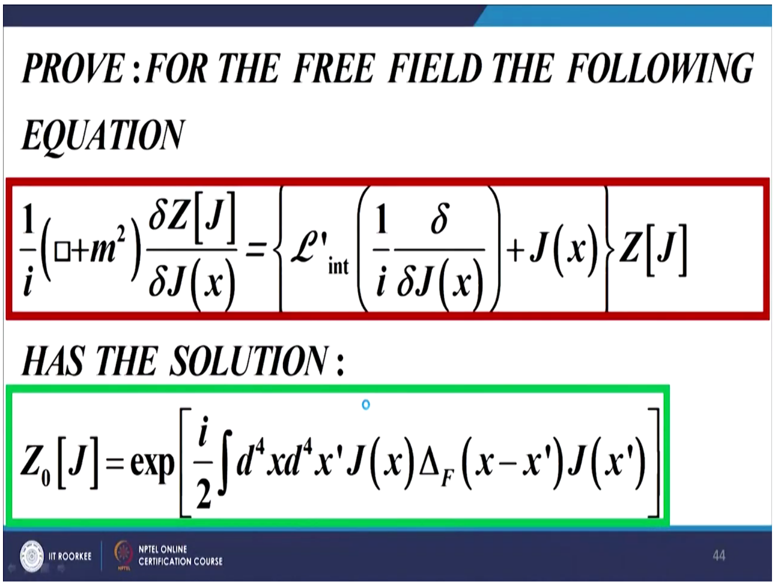

Now, for the free field the; what would be the result? Let us see; we have to prove that the solution to this equation that we have arrived at just now. In the case of the free field, is given by the expression and given in the green box right at the bottom of your slide.

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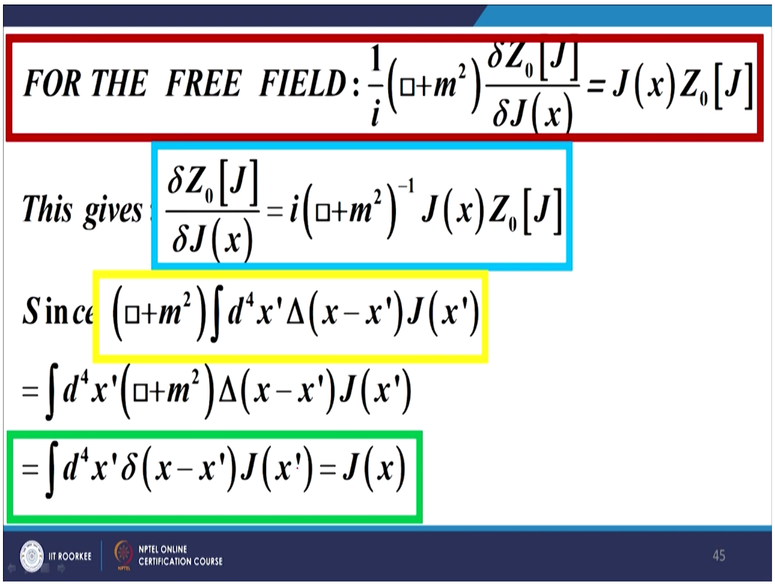

Now, for the free field; interaction term will be absent and we write Z as Z 0, Z; J as Z 0; J, we; the interaction term is absent. So, the equation that we are given takes the form of the red box at the top of your slide.

And this gives us delta of Z 0 upon delta J is; this is the inverse Klein Gordon operator, when it goes to the right hand side; it becomes the inverse Klein Gordon operator and operates on J; x, Z 0; J. In other words, we are multiplying; we are multiplying in a sense, we are multiplying left and right hand sides by the inverse of the Klein Gordon operator.

Now, the Klein Gordon operator; the as we know that the propagator is the inverse of the Klein Gordon operator. We make use of this property the propagator is the inverse of the

Klein Gordon operator. Using this property, we arrive at this relation that the Klein Gordon operator; operating on in this expression, in the yellow box gives us J x.

Therefore, on making on substituting J s; J x in terms of the propagator, what we get is the expression which is given in the; this expression is what we have from the previous slide.

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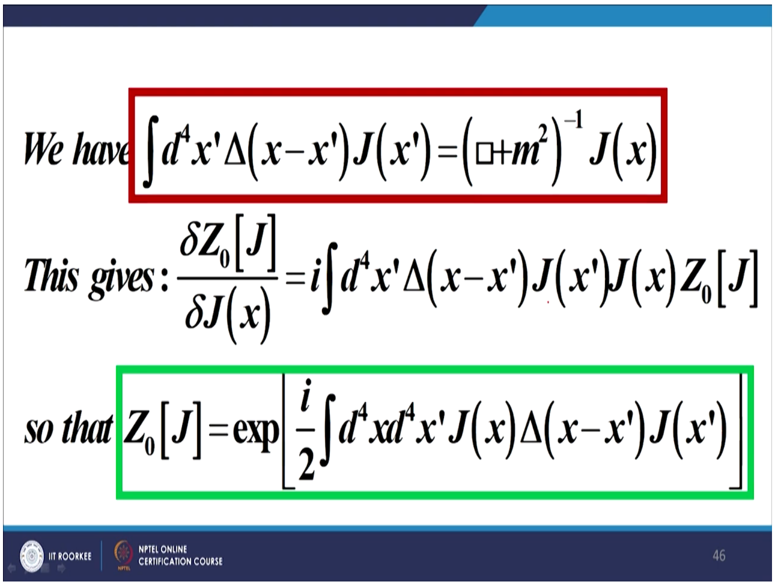

And when we substitute the expression for J x, from what we derived here from the; in the previous slide. If you substitute here for J x, we get this result which is in the center; center equation, the middle equation.

And when we integrate this middle equation by taking with respect to taking Z 0 J, on the left hand side in denominator and writing J x; on the right hand side and integrating both sides, we get the required result which is here in the green box.

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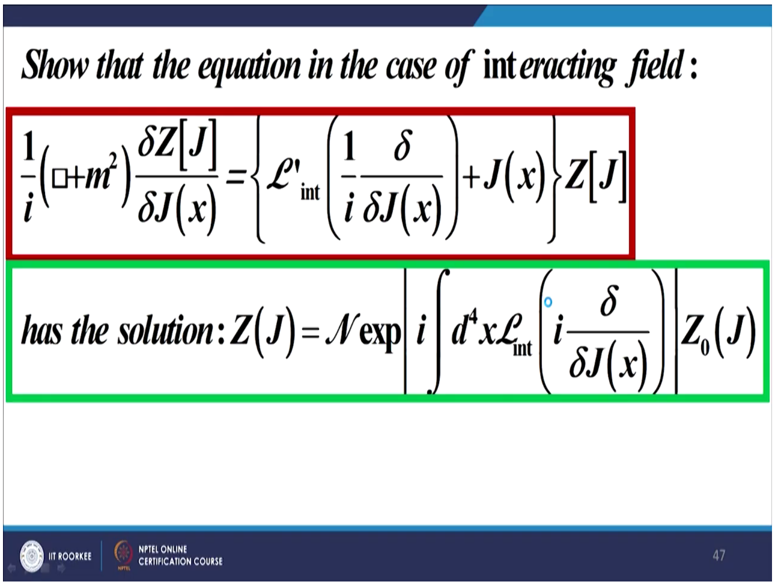

So, we will continue from here with the further properties.

Thank you.