Path Integral Methods in Physics & Finance
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Lecture - 45
Propagator in Minkowski Space

Welcome back. So, as in last the lecture we obtain the expression for the Propagator for the Klein Gordon equation. Let us quickly recap through that and then we will proceed to analyze the properties of the propagator.

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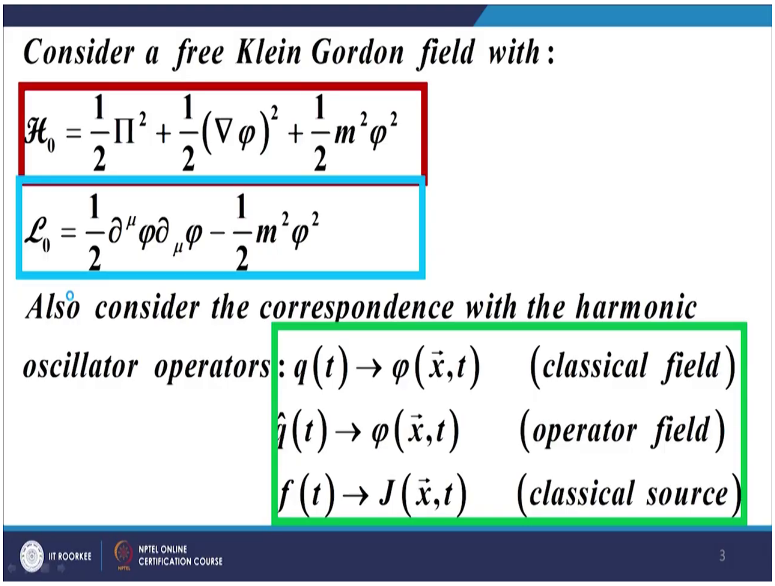

So, we started with the Klein Gordon field, the field equation, the Lagrangian for the field which is given in the blue box here on your slide and then we followed the prescription that was arrived at in the context of the harmonic oscillator. The correspondence between the harmonic oscillator operators and the field operators is given in the green box.

qt now because harmonic oscillator in the quantum version is essentially 0 plus one dimensional field theory. So, qt now becomes phi x of t depends on the space time variable underline space time variables and the other expressions also follow similarly and that is given in the green box right at the bottom of your slide right end corner.

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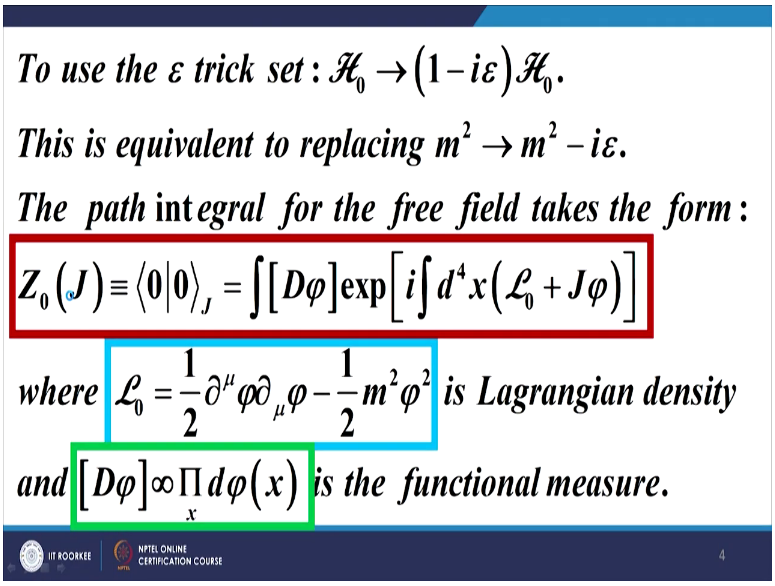

Then we introduce the i epsilon prescription by substituting m square minus i epsilon for m square to make the integrals converge. In this essentially operates as the damping factor and we wrote down the expression for the generating functional in as the expression for in the red box here, where we introduce a source term J or phi J phi rather.

The free field Lagrangian is the blue box and the integration element the path integration element now takes the integration path integration over all the field configurations. To obtain Z 0 J, we moved over from the position space to the Fourier space the wave vector space by making the Fourier transforms of the various field functions and the expression for which are given in the green box here.

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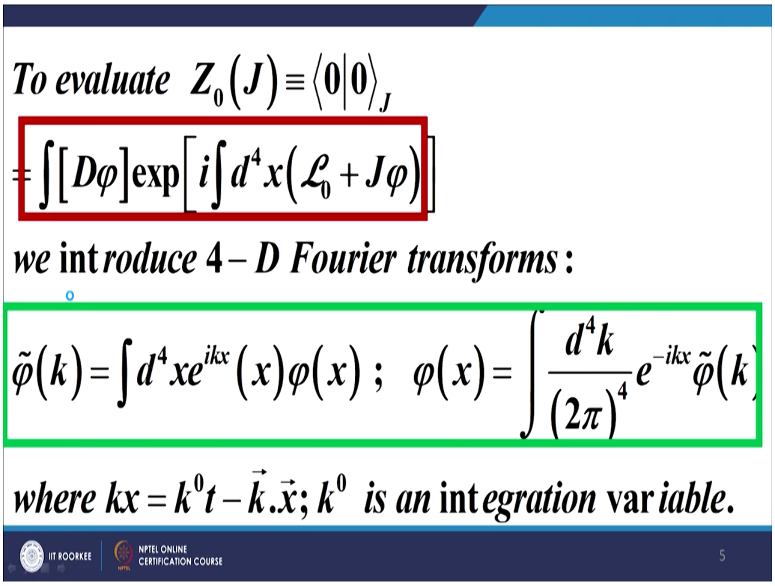

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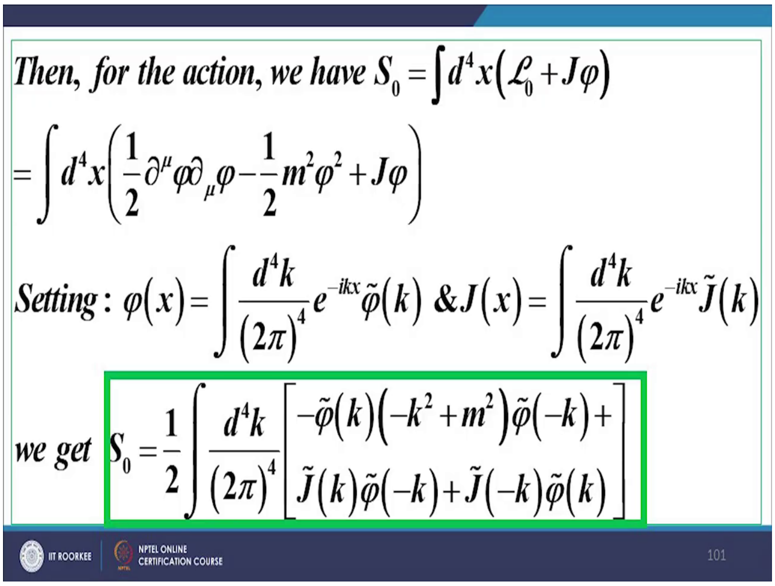

And on doing that we arrive at the expression for the action; for the action in the Fourier space as the expression given in the green box right at the bottom of your slide, where the transformations are given by the Fourier representation phi x is equal to integral over dk 2 pi to the power 4 e to the power minus x in minus ikx phi tilde of k and similarly for the J x.

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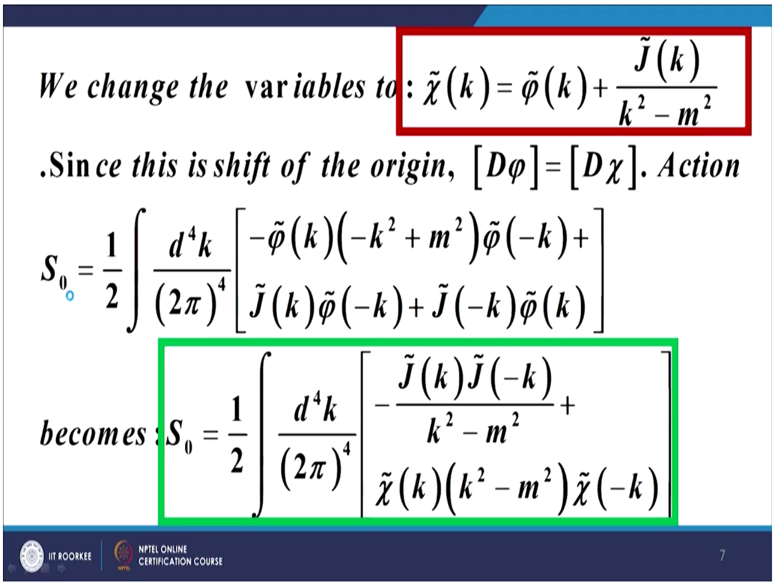

We change the introduce of change of variables chi tilde k. Now replacing phi tilde k with an additional term J tilde upon k upon k square minus m square, the entire thing is given in the red box right at the top right end corner of your slide. And, in terms of this new variable chi k, the expression for the action now takes the form given in the green box at the bottom of your slide.

The interesting part here is if you look carefully at the integrant of the action, the first term is independent of chi and the second term is independent of J. So, in a sense it operates 2 separate the separate out the terms of involving this source of the field and the field variable themselves.

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From above: $$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[-\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2} + \tilde{\chi}(k)\left(k^2-m^2\right)\tilde{\chi}(-k)\right]$$

Now $$Z_0(J) = \langle 0|0\rangle_J = \int [D\chi]\exp(iS)$$

$$= \exp\left[\frac{i}{2}\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\left[-\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2}\right]\right]\times$$

$$\times\int [D\chi]\exp\left\{\frac{i}{2}\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\left[\tilde{\chi}(k)\left(k^2-m^2\right)\tilde{\chi}(-k)\right]\right\}$$

Now, when we now move over to writing down the generating functional and we write down the generating functional in terms of the new variables that is the chi variables, it takes the form given in this integral D chi exponential i as. Recall that the path integral elemental changes does not change due to this transformation because it is essentially at shift in the origin.

And in other words, the original path integration over phi carries over to path integral over chi. So, that been the case and the because the first term in the action is or the integrand of the action independent of the integration path integration variable chi, we can take it outside the path integral. And, we can write down the first term as a free factor to the path integral in the form shown in the green box at the bottom of your slide.

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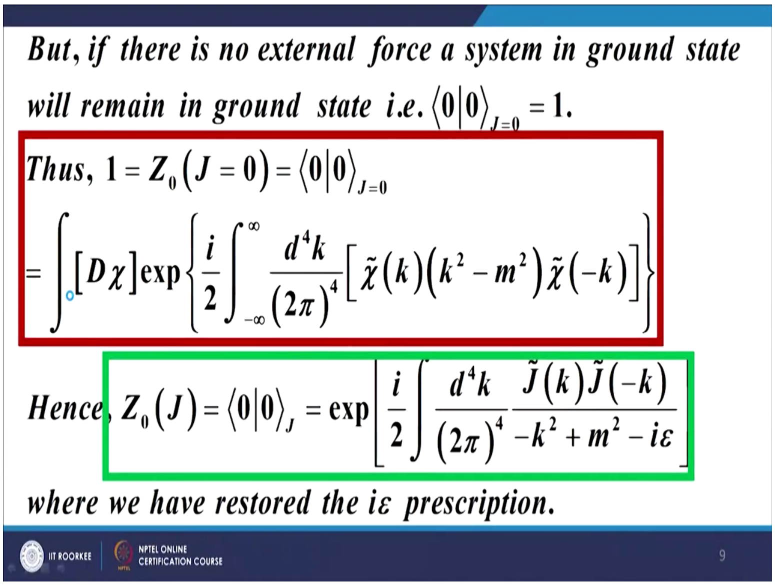

Now, when there is now comes the very important point which I emphasize it in the last class also that, if the there is no external field acting on the acting on the system then system in the ground state will continue to be in the ground state. In other words, if I work out the transition element between ground states have with J equal to 0 it must written in a value of 1. Putting that into play I have 1 is equal to Z 0 with J equal to 0 that gives us that the entire expression should be equal to 0 should be equal to 1 or 1 is equal to 1 is equal to Z 0 J is equal to 0.

So, therefore, the entire expression here in the green box must be equal to 1 in the special case when J is equal to 0; now, when J is equal to 0, if you look carefully the free factor because 1 by default because the exponent become 0. So, exponential of the exponent exponential of 0 will be 1. So, this free factor becomes 1. It therefore, follows that in the condition that J is equal to 0 the path integral must written a value of 1, but the important part is that the path is integral this is independent of the value of J.

Therefore, whatever be the value of J the path integral must written a value of 1 and that is what we make use of and therefore, we get the expression for the generating functional as the expression in the green box with the path integral returning a value of 1 for all values of J.

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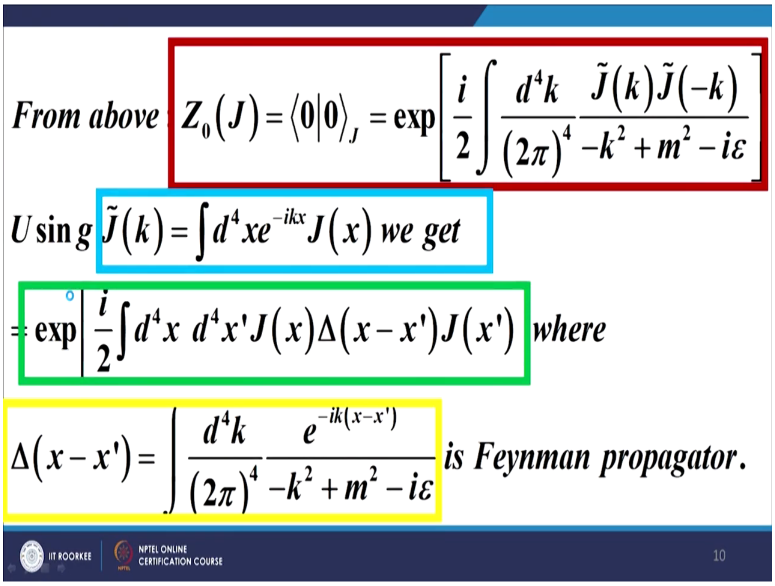

And then finally, make a Fourier transform back to the original variables and write the path integral in the form that is given in the green box; which with delta x minus x dash which is termed as the propagator and the propagator being represented by the expression given in the yellow box right at the bottom of the slide this is the Feynman propagator.

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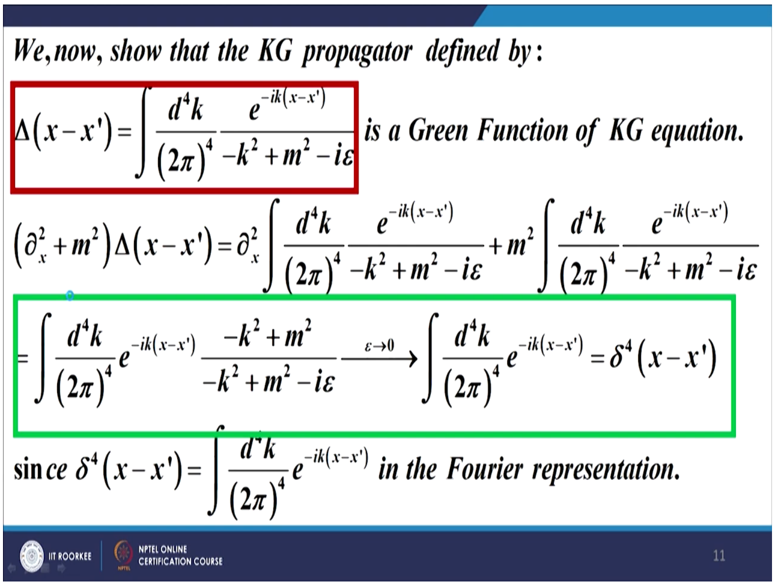

Now, it can easily be shown that the Klein Gordon propagator that you just defined earlier in the yellow box on the previous slide. This expression which have in the yellow box here satisfies the Klein Gordon equation with the delta function, delta function as the source term, in other words it is a Green function of the Klein Gordon equation. That is easily seen by writing or the by operating on the propagator with the Klein Gordon operator del square x inside the integral operating it on the e term and I get minus ik whole square that is equal to minus k square.

So, on the operation of this del x square term on the on the propagator I written a term factor of minus k square the m square till use the propagator unchange multiply that simply by m square and what I have this in the limit that epsilon tends to 0. What I have is the delta function in x minus x dash that is clearly seen by the Fourier version or the Fourier transform of the delta function, expression for the of delta function in Fourier space.

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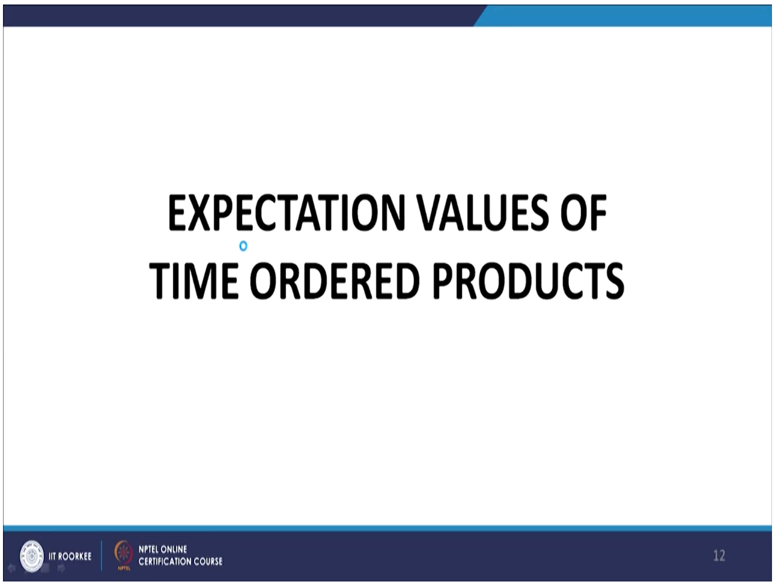

Now we come to the expectation values of the time ordered products. The time ordered products as per the formula that we arrived at earlier in the context of quantum mechanics is given by the expression that is given in the red box where the top of the slide. 1 upon i functional derivative of the generating function functional with respect to J and then and then taking the cases or setting J equal to 0. Where Z 0 J is given by the expression in the blue box that we already just now found out in the context of the Klein Gordon equation.

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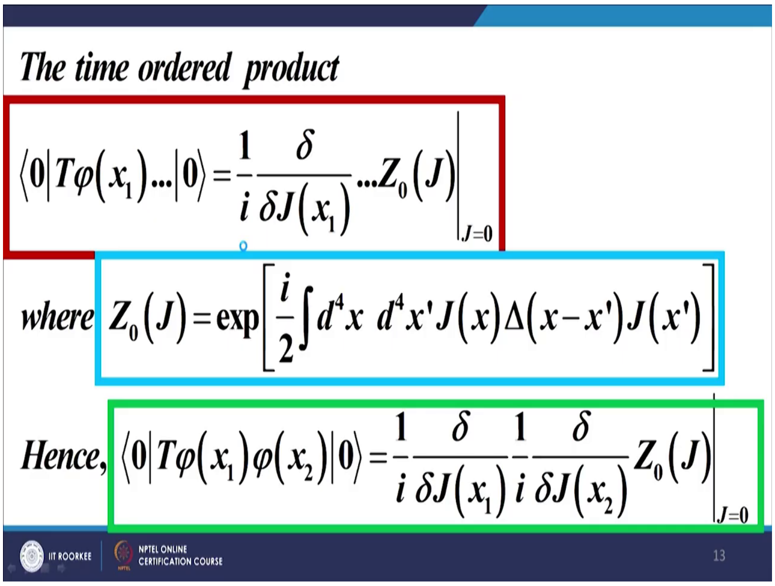

In other words, therefore, for a 2 point function we have the expression which is given in the green box are a double derivative at different points; x 1 and x 2 operating on Z 0 J and then putting J equal to 0.

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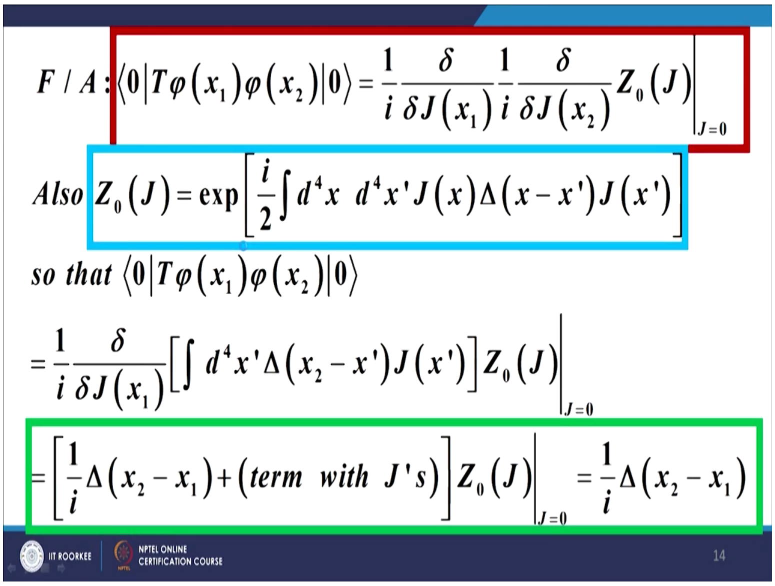

When we write down the value of, when we insert the value of Z 0 as the expression given in the blue box and then take functional derivatives of the expression of Z 0 J in the form that is given in the blue box here.

What we written is of if I take the first functional derivative I get the expression that is given in the equation just above the green box here. And please note if I put J equal to 0 in this expression, I written a 0 that is one point that we need to keep a track of. However, for getting the 2 point function we workout another functional in derivative, the second functional derivative and we get the one term that is independent of J plus term which carries the value of J and then multiplied by Z 0 J.

If I put this J equal to 0 in this second term which has a J embedded in it goes to 0 and Z 0 J at J equal to 0 gives me 1 and what I have here is 1 upon i delta x 1 minus x 2 minus x 1. That is the relationship between the time ordered products of a field operators n point functions rather there are 2 point function. In this case the 2 point function and the propagator.

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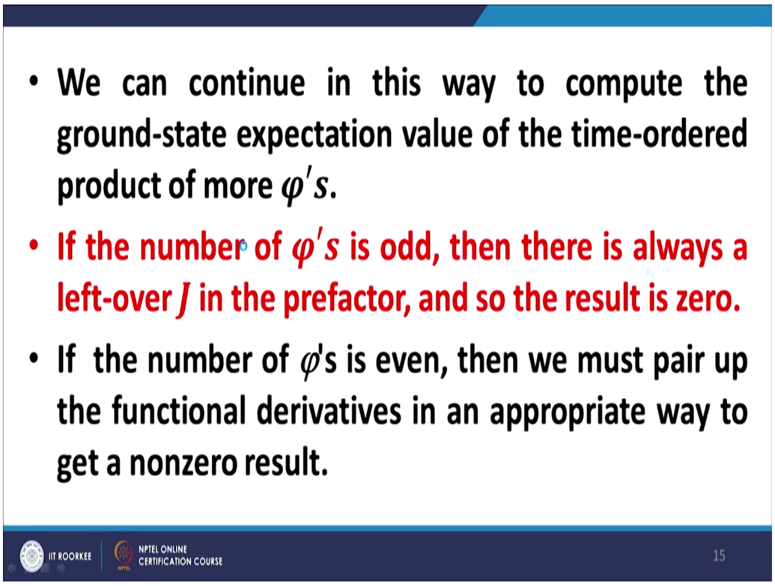

We can continue this way. We will find that for every odd point functions 2n plus 1 point functions, we get factor of J in the pre factor and when I put that J equal to 0, the entire expression vanishes.

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$$\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle$$

$$=\frac{1}{i^2}\begin{bmatrix}\Delta(x_1-x_2)\Delta(x_3-x_4)+\Delta(x_1-x_3)\Delta(x_2-x_4)\\ +\Delta(x_1-x_4)\Delta(x_2-x_3)\end{bmatrix}$$

and, in general,

$$\langle 0|T\varphi(x_1)...\varphi(x_{2n})|0\rangle=\frac{1}{i^n}\sum_{pairings}\Delta\left(x_{i_1}-x_{i_2}\right)...\Delta\left(x_{i_{2n-1}}-x_{i_{2n}}\right)$$

This is Wick's theorem.

And therefore, the expectation value in the ground state of a free field of the free Klein Gordon field gives us a value of 0, for odd point function or even point functions. Of course we can use the Wick's theorem to arrive at a formal value of the 2n point functions in the manner that we have done earlier for the 2 point functions by taking appropriate number of functional derivatives of Z 0 J. And, then substituting J equal to 0 and using then using wicks theorem to develop pairings of propagators and then summing over all those pairings.

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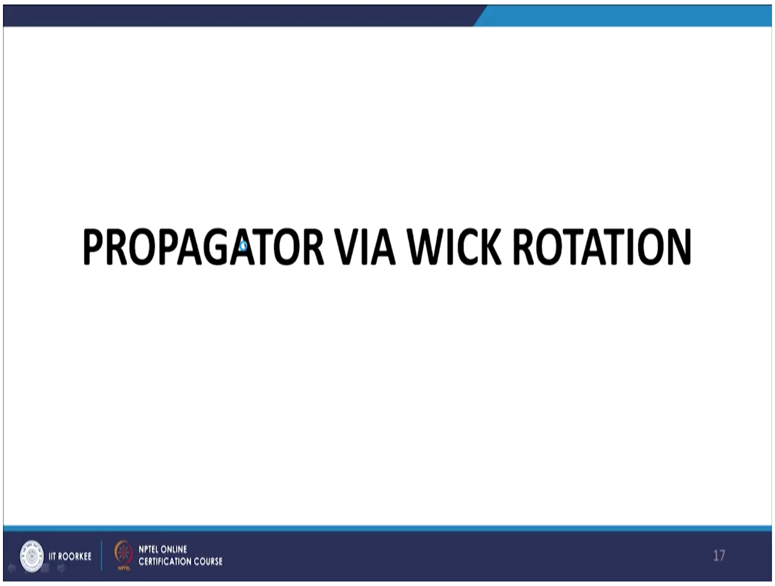

Now, we come to or we derive an expression for the same propagator using Wick rotation by way of illustrating the concept of Wick rotation. The important point that before we get into this the important point that I would like to emphasize is that in the case of the Feynmen propagator, the scheme of the poles that is envisaged or that was used by Feynmen was that the poles to the poles to the denominator here poles to the integrand here would appear.

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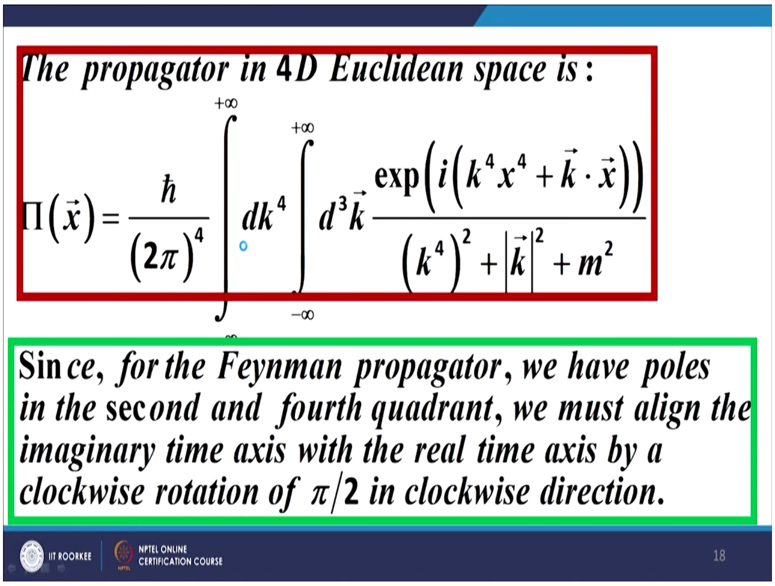

We will have 2 poles naturally. The 2 poles should appear in the second quadrant and in the fourth quadrant. That was all though there are other schemes and we will be taking about today's lecture only, but this was the scheme that was envisage by Feynmen the poles appearing in the second quadrant poles appearing in the second quadrant and the fourth quadrant.

Therefore, when I move from the Euclidean space to the Minkowski space, imaginary time in the Euclidean space to real time in the Minkowski space it entails. And if I want to avoid the poles in my integration counter, I need to integrate in the or I need to rotate by an angle of pi by 2, the imaginary axis by an angle of pi by 2 in the clockwise direction and that is precisely the what is done and that is manifest by the by the transformation x 4 goes to minus i x 0.

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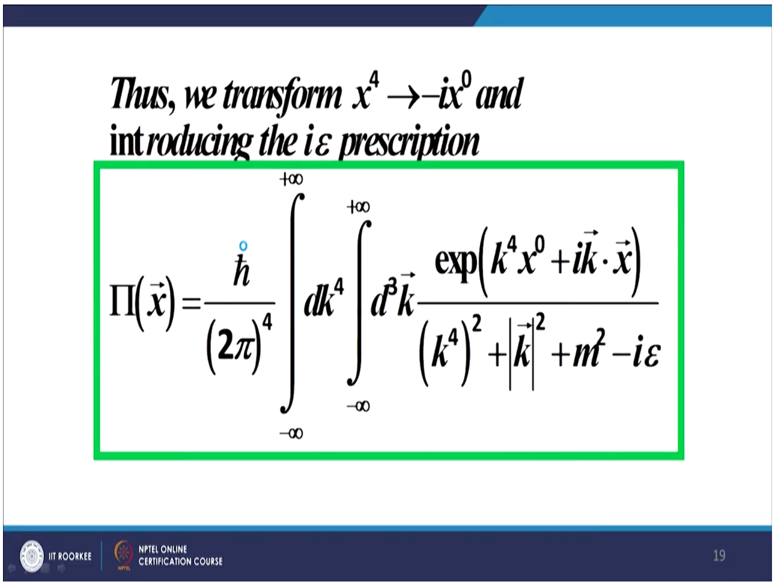

When this when I substitute this x 4 equal to minus i x 0 this is the expression that I get for the propagator starting from the Euclidean propagator. Please note that the expression here in the red box is the expression that we carried forward from the Euclidean propagator. So, for nothing has been developed the Minkowski space said about the Minkowski space and this x 4 goes to minus i x 0, now we are at transforming to the Minkowski space as for as the coordinates are concerned and this is the expression we get.

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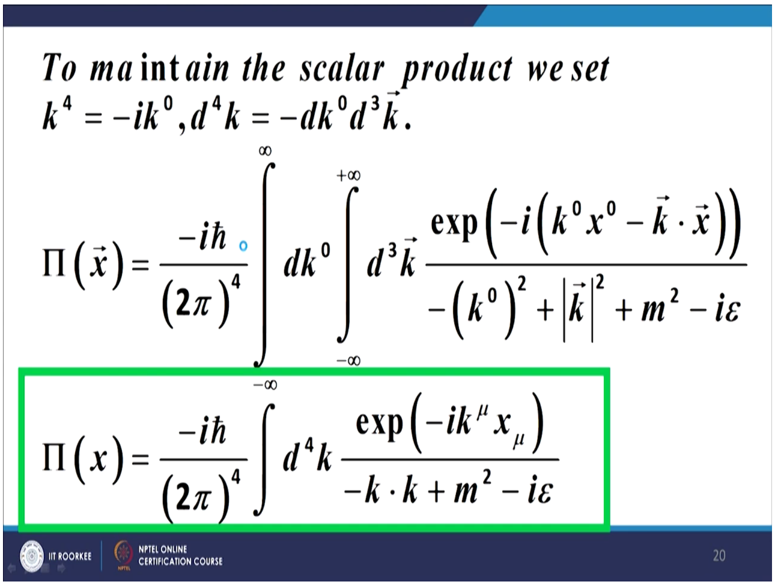

However, and to ensure that the scalar product remains invariant; we need to do a transformation of the wave vector also. The transformation for the wave vector will take the form k 4 goes to minus i k 0; in order to ensure that the that the scalar product remains unchange. And on doing the transformation, the expression that we written for the propagator in Minkowski space is the expression that is given in the green box here.

Please note in this case there is a additional minus i; here in the numerator compare to the expression that we derived using the harmonic oscillator methodology. This minus i would be absorbed when we workout the transition amplitudes and the formula for the transition amplitudes would be modified accordingly. The important point is if you are using this expression for the propagator we need to use the expression for the transition amplitudes that was derived in the context of Euclidean space and then use this formula.

Whereas, the as the expression for the propagator that was arrived at earlier and directly relates to the expression for the transition amplitude in the Minkowski space. Let me go back and explain that. Yes, here it is. In the green box, this is the expressions for the Minkowski space. So, when we are using this expression when using this expression for the transition amplitude or for the generating functional, the delta x minus x dash propagator would be expressed in terms of the expression given in the yellow box.

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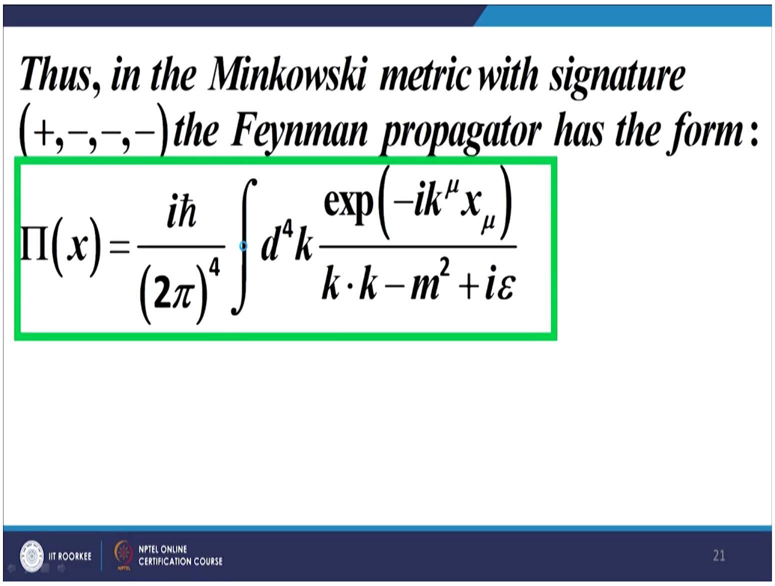

So, let us move on this is the expression for the final expression for the Minkowski space when we obtain the expression through a Wick rotation from Euclidean space.

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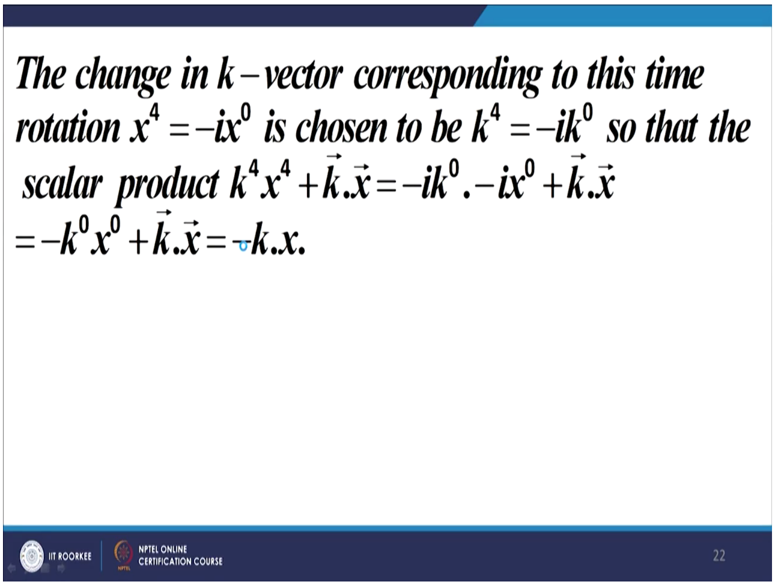

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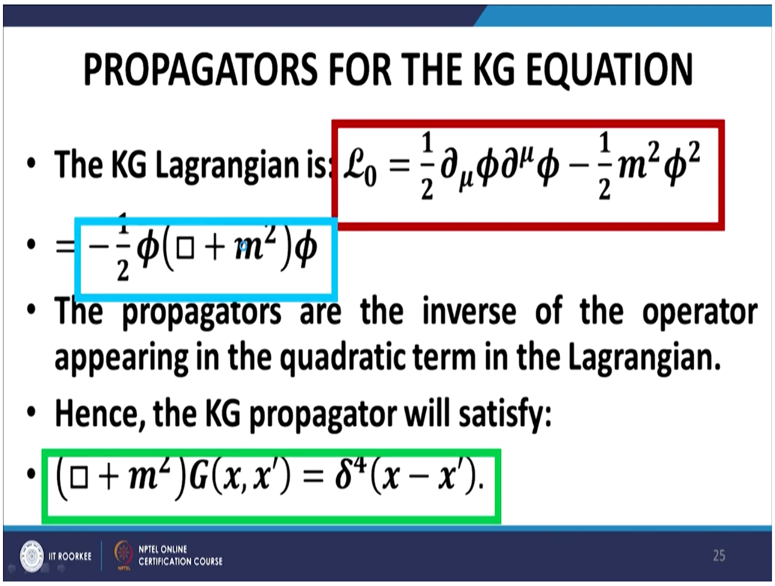

Now, we try to further simplify the expression for the propagator and for that purpose we try to carry out the time the time integral or the integral in the propagator with respect to the wave vector corresponding to the time coordinate that is our next exercise. Again Lagrangian is the Klein Gordon Lagrangian given in the red box which can through an integration by parts be expressed in the form given in the blue box here.

The propagators as we know the propagators when they operate on the on the Klein Gordon operator of for that matter any wave equation they written the delta function. So, the propagators can be regarded as the inverse of the of the quadratic term in the Lagrangian and that is precisely the what we write here in the green box here.

The Klein Gordon operator is the box plus m square operating on the propagator which we are writing as the G minus x comma x dash gives us the delta function. So, in other words we can in some sense say that the propagator is the inverse of the Klein Gordon operator in the form that it appears in the Lagrangian.

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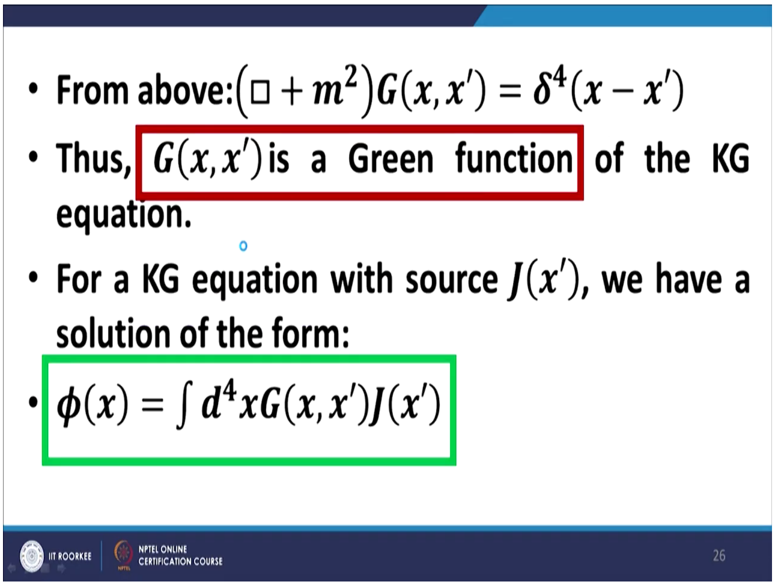

Now, this clearly shows also that the G x comma x dash is the Green function of the Klein Gordon operator as I mentioned just now. For Klein Gordon field with the source and J x dash source that x dash Klein Gordon field with the source at x dash the solution can be obtained in terms of this Green function and can be written in the form given in the green box here.

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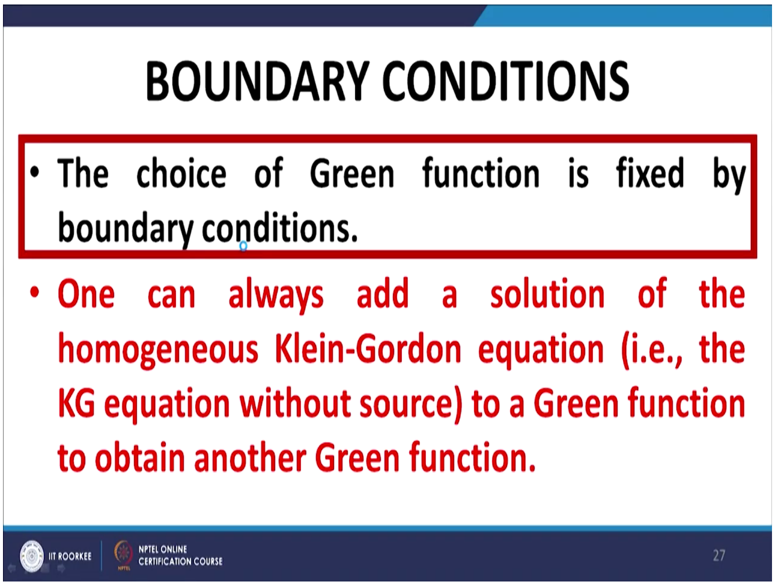

The choice of Green function will be determine by the exact boundary conditions of the problem. We can always add Green functions of the homogeneous equation to arrive at more Green functions. And therefore, the Green function in that sense are not unique and the uniqueness of the Green function we are only follow from implementing a certain set of boundary conditions.

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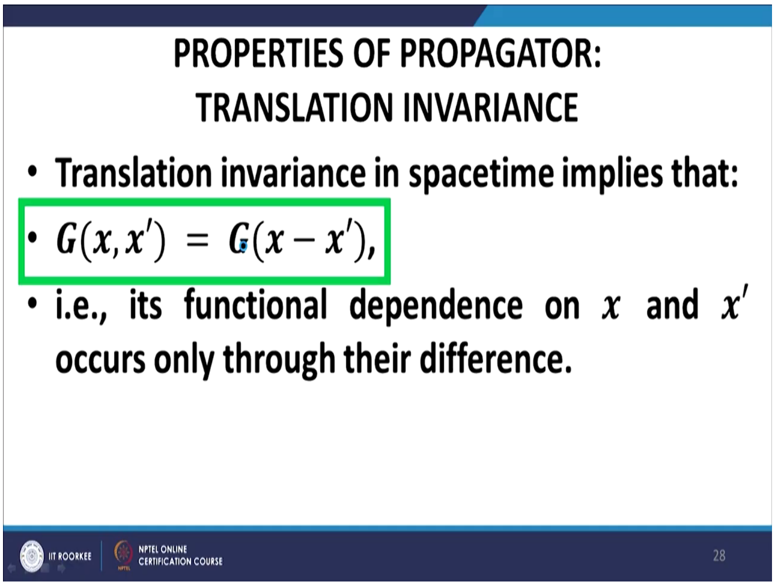

Now, the propagators are translation invariant. We discuss this in earlier context when we talked about Euclidean space. And therefore, therefore, the G x comma x dash will depend on the on the difference x minus x dash in some sense because, of the translation in variance of the propagators.

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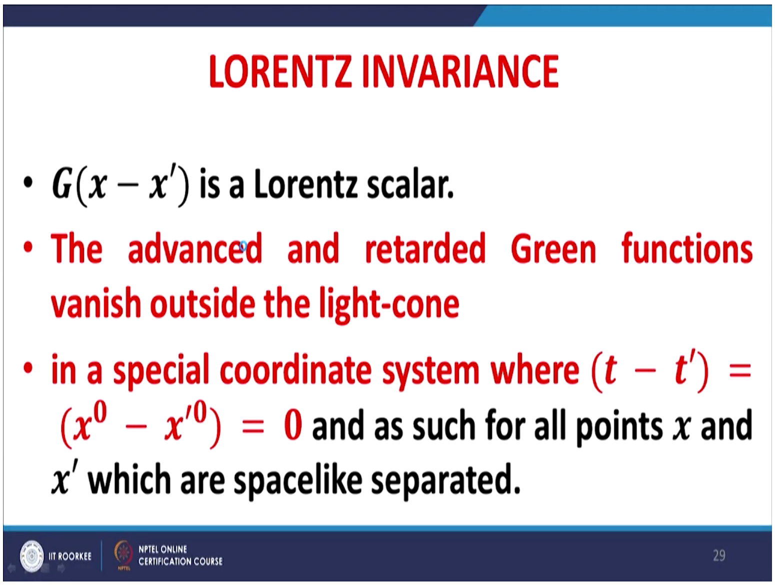

Now, the propagator is also a Lorentz scalar. The certain properties are listed on this slide then read them out. The propagator G x minus x dash is Lorentz scalar. The advanced and retarded Green functions will are defined in such way the boundary conditions of these advanced retarded Green functions are such that they vanish outside the light cone and in that sense they obey causalities in some sense, but we will come back to this point.

And in a special coordinate system, where t minus t dash is equal to x 0 minus x 0 dash equal to 0; that means, a 2 events are instantaneous in time and as such for all points x and x dash which are space like separated the advanced retarded Green functions vanish outside the light cone.

In other words, if there are 2 elements, 2 space time events and x dash and they are space like separated then the corresponding propagators vanish outside the light cone. Let me repeat. If there are 2 events at x and x dash which has space time space like separated which has spacelike separated then the propagators propagated between these 2 space time points vanishes outside the light cone corresponding to the these 2 events.

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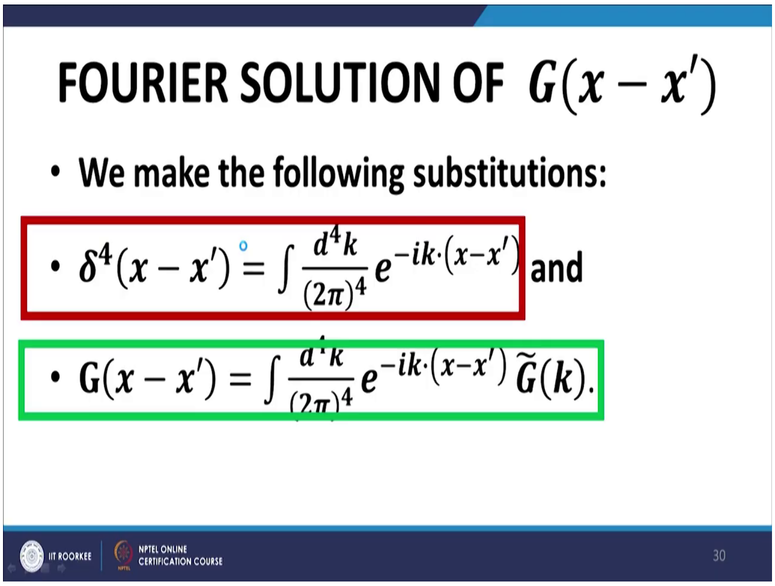

And now, we come to the solution of the solution of this particular propagator. We make the we use the you use the Fourier transforms. First solve the expression in Fourier space and then reward back to the position space. We use this 2 Fourier transform; transform for the delta function and transform for the Green functions.

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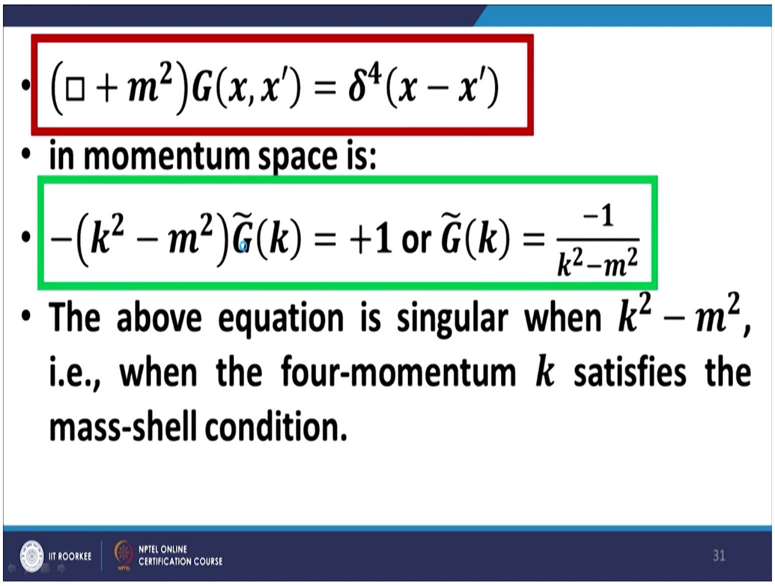

Now, from because the Green function satisfy the Klein Gordon equation, we get this in the corresponding expression in momentum space. When we workout we get the expression which is given in the green box here. G tilde k is equal to minus 1 upon k square minus m square. Please note we still not introduce on the i epsilon or you may say that i epsilon is implied here for the moment, but it will make its presence in a short while. So, this is what we have for the Fourier transform of the Green function.

G tilde k is minus 1 upon k square minus m square.

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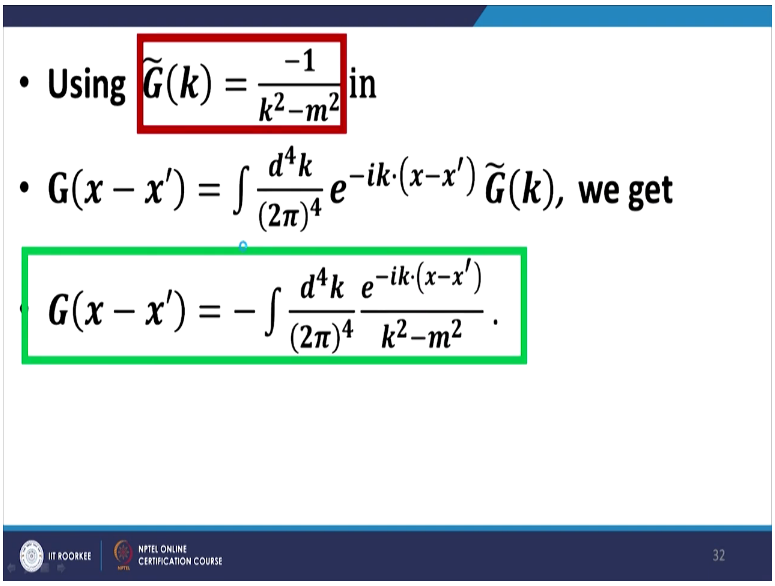

Obviously, this equation as poles at plus minus m and we now try to resolve this the issue of this pole, but before that if we reward back to position space. We get the expression for the propagator which incidentally coincides with the expression that we got earlier for through the harmonic oscillator and the expression is given in the green box here. So, different approaches yield the same expression.

This shows the internal consistency of the frame work. Now, let us evaluate the time component of this integral. In other words, we work out the integration over k 0 which is which we write as omega. Now, k square minus m square we write as, k 0 square minus k dot k, where k is the space like wave vector and k 0 is the time like wave vector or time component of the wave vector.

And k is the space component of the wave vector minus m square and we club this k square and m square term write them as omega k square.

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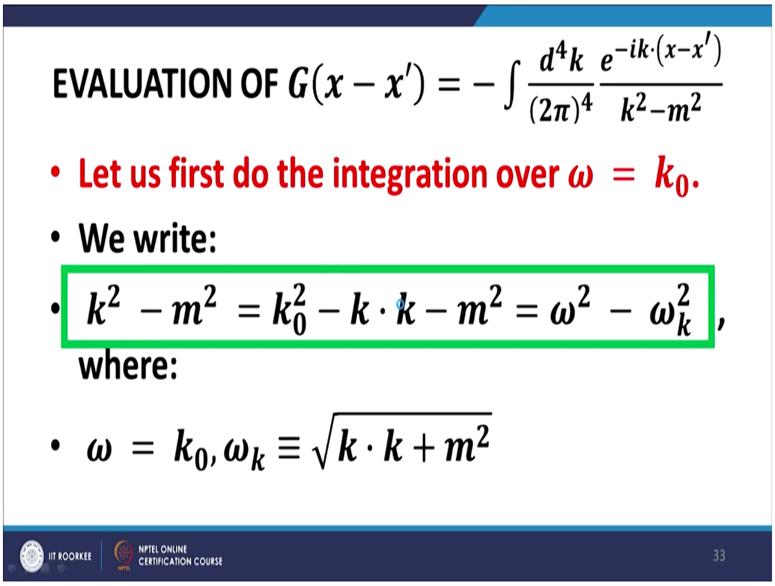

The second and the third term we club them and we write them as omega k square and k 0 square we write as omega square. So, the expression k square minus m square is now rewritten as omega square minus omega square omega k square omega square minus omega k square, I am sorry where omega square is equal to k 0 square and omega k square is equal to k dot k plus m square.

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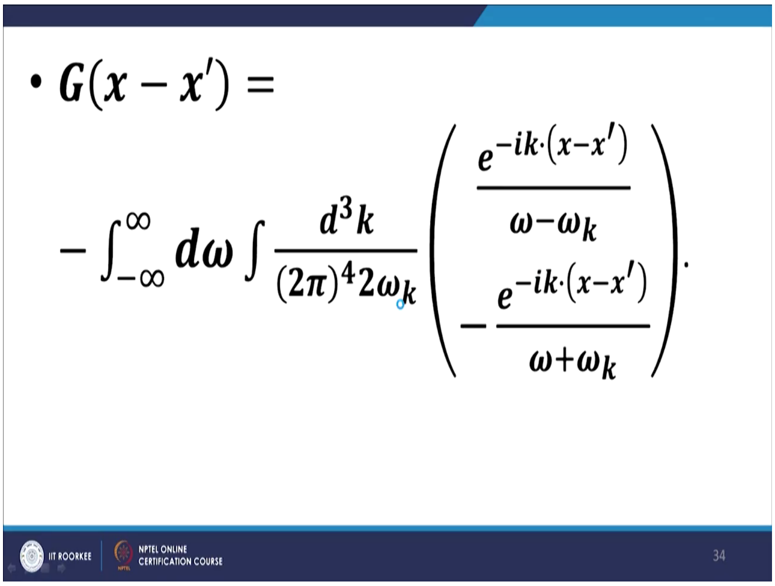

And we can write now the expression for the propagator in the form, when we factorize this expression and we do the partial fractions. We can write this expression in the form which is given in the round brackets here in this slide with of course, the factor of 1 upon 2 omega k here. So, this expression 1 upon 2 omega k with expression in the round brackets gives us the expression which we have here for the k 0 component and the integration over k 0 is written as integration over D omega.

Integration over omega and the rest is we have simply factorized omega square minus omega k square and place them as partial fractions.

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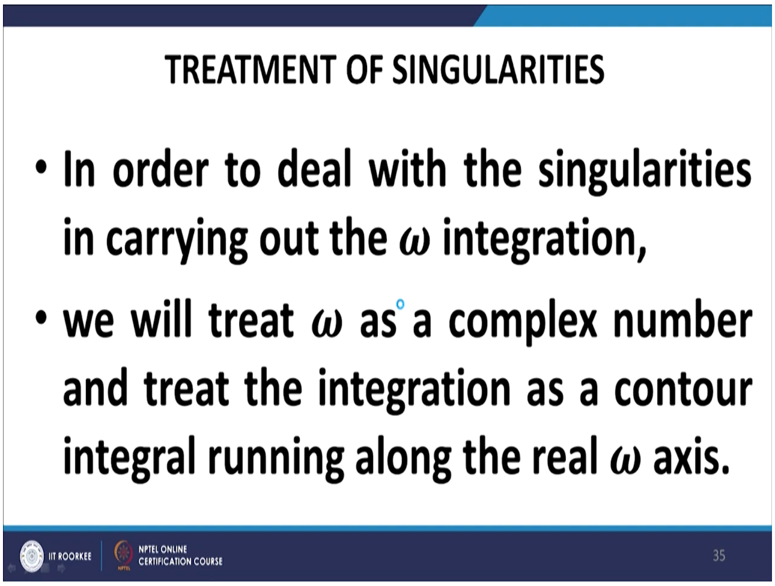

Now, clearly there are (Refer Time: 24:51) there are poles of this expression. There are 2 poles of this expression. What we will do is, we will treat omega as a complex number and treat the integration as a contour integral running along the real omega axis.

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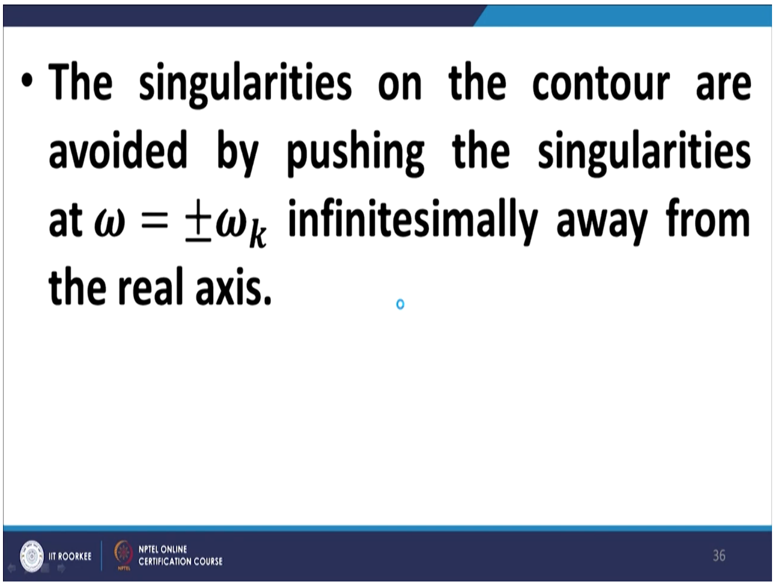

This singularities in the contour on the real axis; because the singularities are occurring on the real axis at omega equal to omega k plus minus. Of course, they are on the real axis in they are avoided during this integration by shifting them slightly either in the upper half plane or in the lower half plane.

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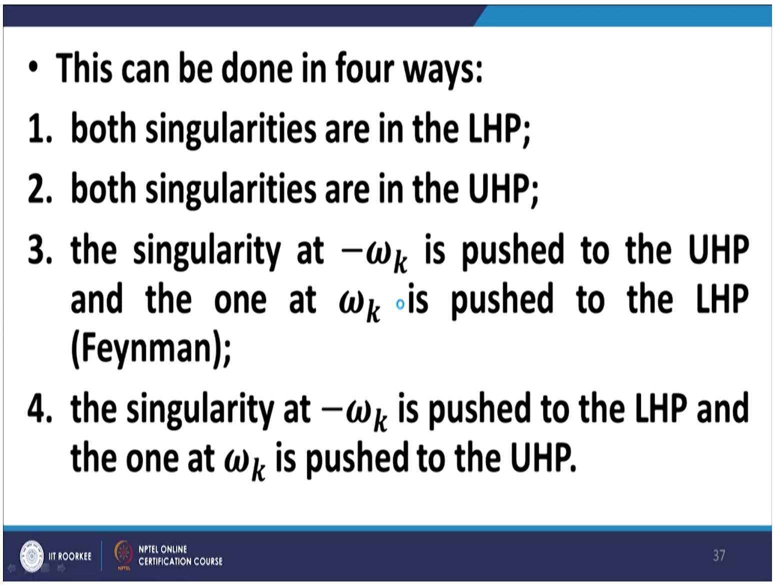

And accordingly we get 4 types of integrals depending on how they this a poles are manages to how this poles are handled, we get 4 different situations a. The first situation is when both the singularities are shifted to the lower half plane. The second is when both the singularities are shifted to the upper half plane. The third is when one singularities shifted when the first singularity at minus omega k is shifted to the upper half plane that is in the second quadrant.

And the other one in this at omega k is shifted lower half plane that is in the fourth quadrant. And the fourth is the converse of that is minus omega k is now shifted in the upper lower half plane and omega k is pushed into the upper half plane that is the converse of the one that is shown in 3.

In other words, this is having a singularities in the third quadrant and the first quadrant. And the third one is having singularities in the second quadrant and the fourth quadrant the first one has singularities in the third and fourth quadrant and the second one has singularities in the first and second quadrants.

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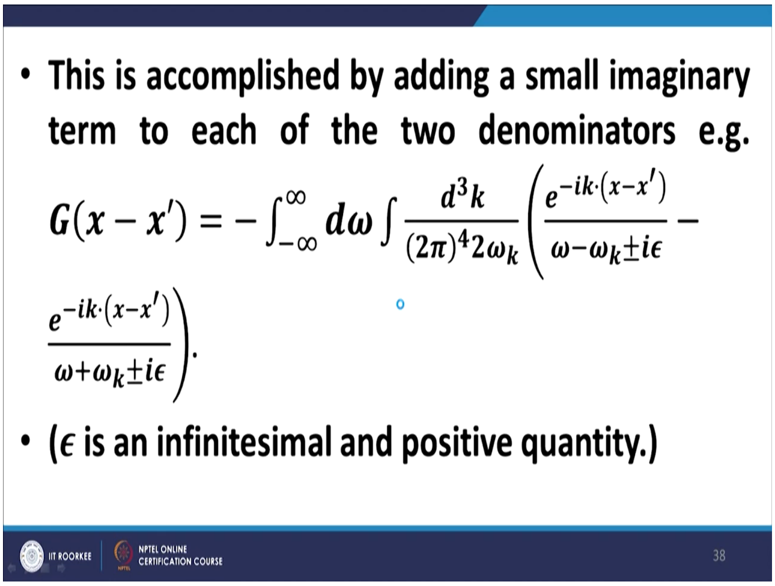

How do we do it, how do we shift this singularities away from the real axis? Well, we introduce the usual i epsilon in factor and by using the appropriate sign for this prefixing. By prefixing the appropriate sign to this i epsilon factor, we can modulate or we can manage all the singularities are to appear or how the singularities are to be treated during the process of in contour integration.

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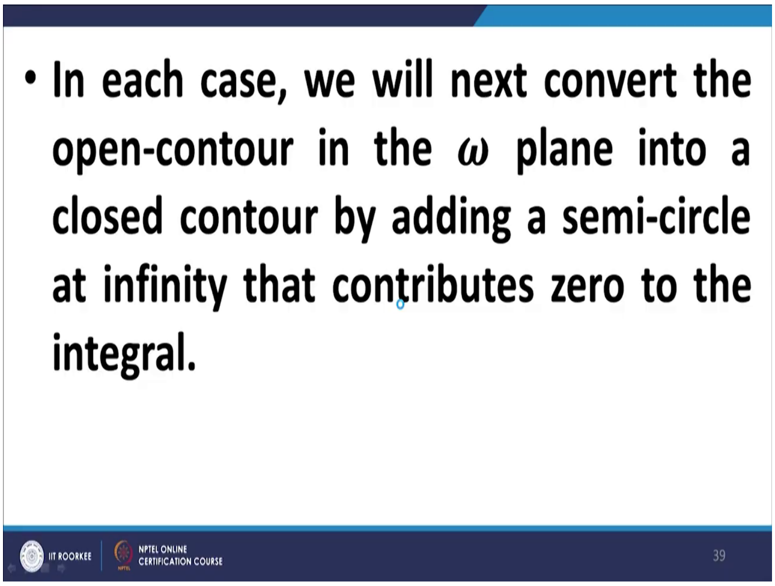

Of course, i epsilon in variably will be positive. It will be the prefix sign that will determine the location of the singularities. Then in each case we will convert the open contour that is the integration along the straight line from minus infinity to infinity.

We will converted to a closed contour by introducing semi circle either in the upper half plane or in the lower half plane as the case may decrease as the case may require case may justify and the criteria for which visual just now I had used.

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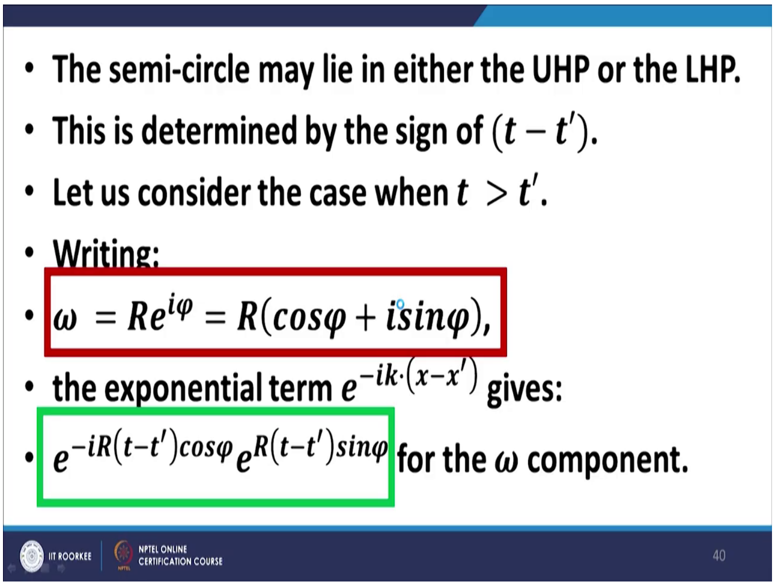

Now, suppose whether we close the contour by a semi circle at infinity in the upper half plane or in the lower half plane is determine by the sin of t minus t dash; t minus t dash determines whether they closing of the contour is to be done through the semi circle, semi circle which is latter on push to infinity in the upper half plane or in the lower half plane. Let us see the rationale behind this. Let us assume the t minus t dash is positive or t is greater than t dash.

We write omega as R e to the power i phi that is R cos phi plus a plus i sin phi. So, we can write this as R cos phi plus R i sin phi. Then the exponential in the integrand e to the power minus i k x minus x dash gives us e to the exponential minus i R minus i R t minus t dash. The t component we are confining ourselves to the t component in the exponential. Remember we are working in the matrix plus minus.

So, the t component here appears as a positive term. So, we write minus i R t minus t dash into cos phi e to the power minus i. Together with this i which is with sin phi gives us plus 1 and so, we have R t minus t dash sin phi. So, the first e is relating to the cos phi here in the red box and the second e is relating to the i sin phi component in the red box and expression we get is in the green box.

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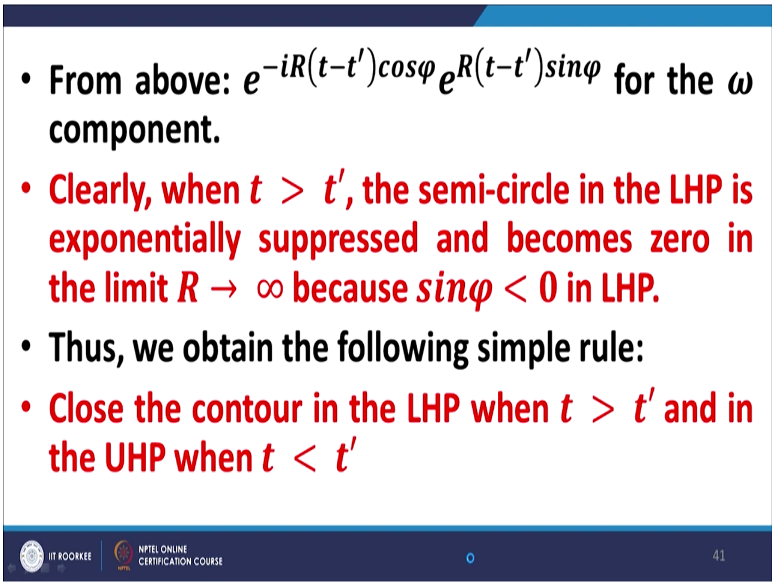

Now, when t is greater than the t dash; in other words, t minus t dash is positive, if you look at this second factor second factor what happens to this second factor as R tends to infinity? We want that the we want that the integral over the semi circle which we use for closing out the contour contributes nothing or contributes 0 to the total integral. Now, if you look at this expression R is positive t minus t dash is positive therefore, both the terms are positive.

And we want to written we want this integral to vanish in the limit that R tends to infinity sin phi needs to be negative. Therefore, sin phi needs to be negative implies that phi must lie either in the third or in the fourth quadrant. In other words, it is clear that in this situation this exponent term will suppressed only if the semi circle closing the contour lies in the lower half plane.

Therefore, we obtain the following simple rule. If t minus t dash is positive, this semicircle closing the contour must lie in the lower half plane and vice versa. We will continue from here after the break.

Thank you.