Path Integral Methods in Physics & Finance
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Lecture - 44
Field Theory in Minkowski Space

Welcome back. So, let us continue with what we were discussing.

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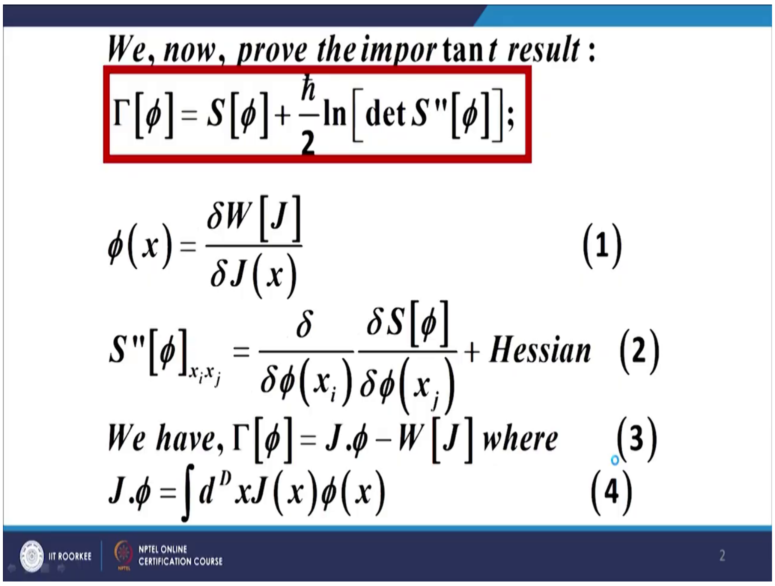

We in the earlier lectures, we discussed this very important relationship which is given in the red box at the top of your slide. We will redo it to make the exposition absolutely clear, because this is a very important relationship. And this enables us to work or to find an expression for the effective field theory or effective field corresponding to our given quantum field theory.

So, what are the relationships that we have? Let us recapitulate. The field function is given by the first derivative of W J, where W J is the generating function of the connected Feynman diagrams. The second derivative of the action will be given by this expression which is equation number 2 plus the Hessian terms.

And, we also have the effective field as given by the Lagrangian transformation of the field function J dot phi minus W J or Lagrangian transformation of W J rather given by this expression J dot phi minus W J where J dot phi is given by integral d x J x phi x. So, these are the inputs that we have equations number 1, 2, 3, 4.

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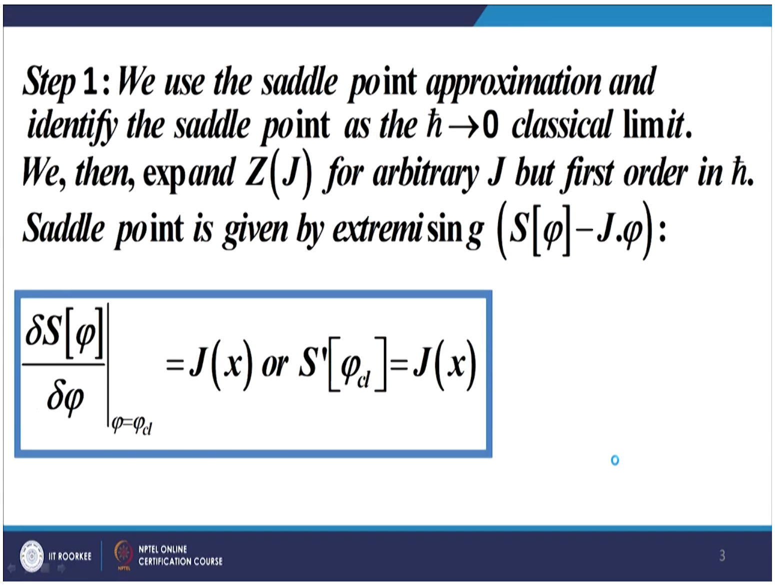

The first step is that we make use of the saddle point approximation. And we equate the saddle point approximation to the classical h tends to 0 limit, where h is h bar as the Planck's constant. We then expand Z J for arbitrary J, but first order in h bar. Now, the saddle point is given by extremising the action. So, we have for the saddle point, the functional derivative of the action is equal to J x, S dash phi classical is equal to J x. So, this is the output of step number 1.

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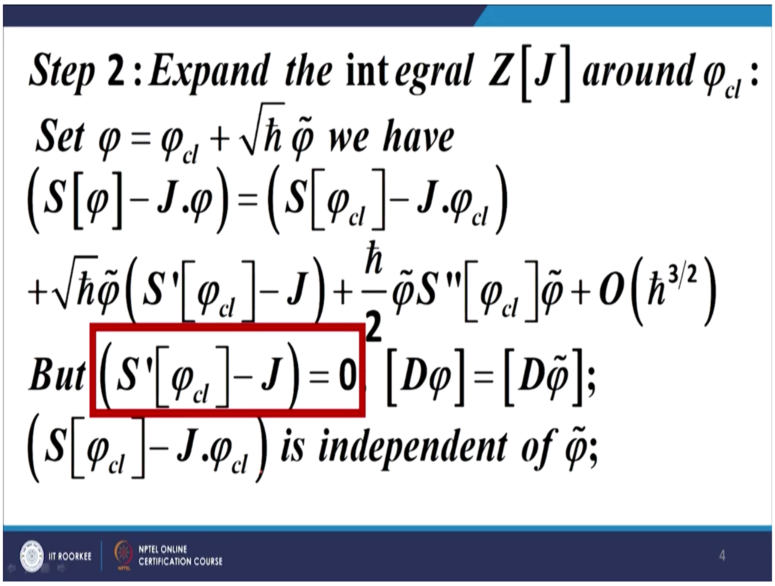

Step number 2 as I mentioned we expand Z J the generating functional around phi classical. And for this purpose we write phi has phi classical plus under root h phi tilde. And what we have is S minus J dot phi is equal to S phi classical minus J this is the 0th order term plus the first order term is the here, the second term and plus the second order term. We confine ourselves up to the second order term.

Now, the first order term, obviously, vanishes because of the saddle point condition. We also have the path integral measure with respect to phi is equal to path integral measure with respect to phi tilde. So, and we also have the this expression S phi classical minus J phi classical is independent of phi tilde, because it is given by the saddle point approximation, it is fixed.

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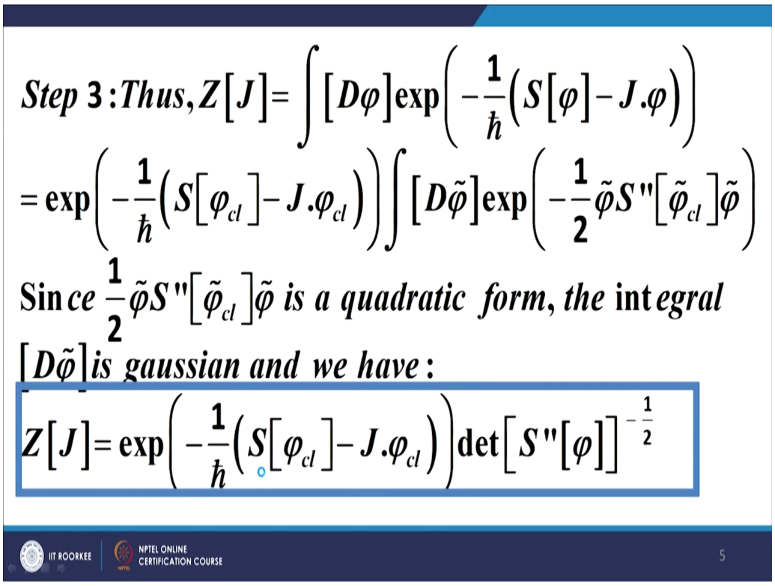

So, now we substitute these expressions in the expression for Z J, what we get is Z J is equal to we can take this first term outside the path integral because it is independent of the path integral measure D phi tilde. And therefore, we take this exponential minus 1 by h S phi classical minus J phi classical outside the integral. And inside the integral, we have this expression which is the second order term. Recall that the first order term is missing because of the saddle point requirement.

Now, if you look at the path integral, the expression inside the path integral, it is nothing but a Gaussian integral. And it is a quadratic form, and therefore, the integral is the Gaussian integral. And it can be done straight away. And as the result of which the expression for Z J is the expression that we get in the blue box right at the bottom of your slide. This is step 3.

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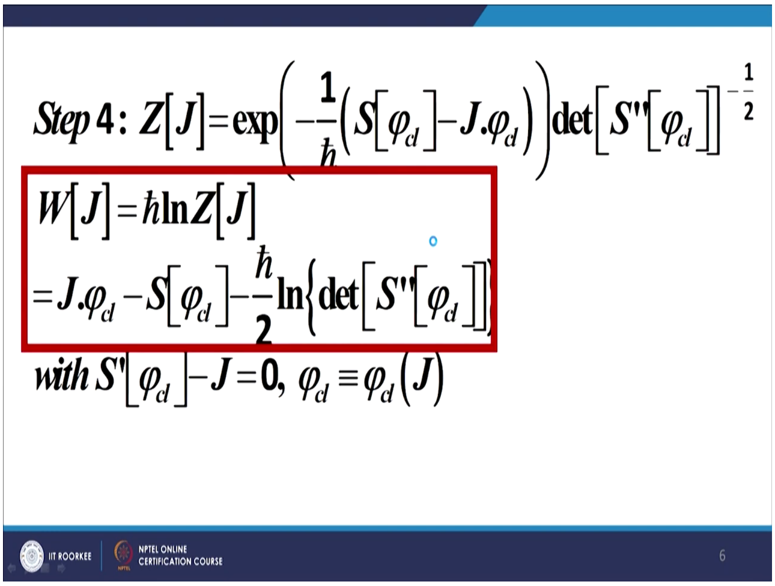

Now, we come to step 4, we start with the expression which we obtained at the end of step 3, which is given in the top equation. Now, W J is the logarithm of J Z with a pre factor of h bar that being the case we simply write out the expression for W J, it becomes J phi classical minus 1 by h minus 1 by h cancel h and h cancel out of course, h and 1 by h cancel out.

And we are left with the J phi classical minus S phi classical minus h by 2 log determinant of this the last expression. This minus 1 by 2 is here h by 2 this is part of this h by 2. Please note again we have got a S dash minus S dash phi classical minus J is equal to 0, this is important.

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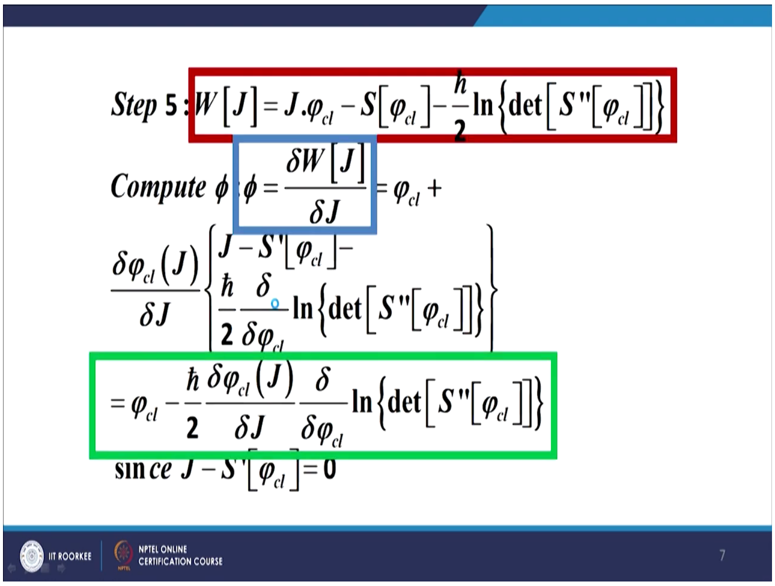

Then we come to step 5. The W J is equal to the J phi classical this we have from the previous slide. The expression in the red box we are carried forward from the previous slide. Now, we compute phi. Phi is the field function. Field function is the first functional derivative of W J with respect to J.

And that being the case, what we have is when you differentiate the expression in the when you functionally differentiate the expression in the red box with respect to J, what we get is phi classical minus, and the rest of the expressions are here with a pre factor of delta phi classical J upon delta J, and the rest of the expressions are within the brackets.

Now, of this J minus S dash phi classical is equal to 0. So, this expression simplifies to phi classical minus the rest of the expression. The plus J minus S dash phi classical is equal to 0 that part we have incorporated, and therefore, we get the expression that is there in the green box.

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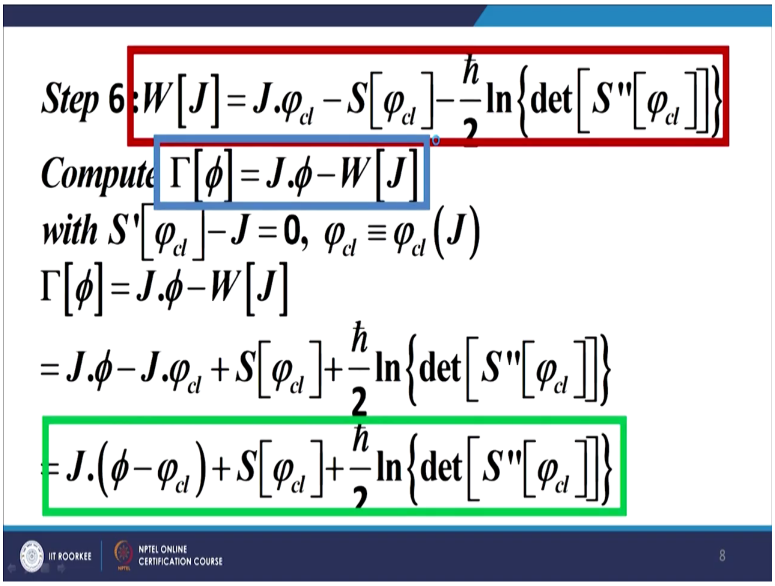

In step 6, we can we bring forward the expression that we have from the previous slide. This is what is carried forward to these this slide. And we calculate the effective field as J phi minus W J, W J is given in the red box. And therefore, we also have S dash phi classical minus J is equal to 0.

This is the saddle point condition. So, gamma phi is equal to J phi minus W J. J phi is as it is, and W J is what we have here, and we are writing it absolutely unchanged. So, if we simplify this, I get J into phi minus phi classical plus S phi classical plus this expression which is in the green box.

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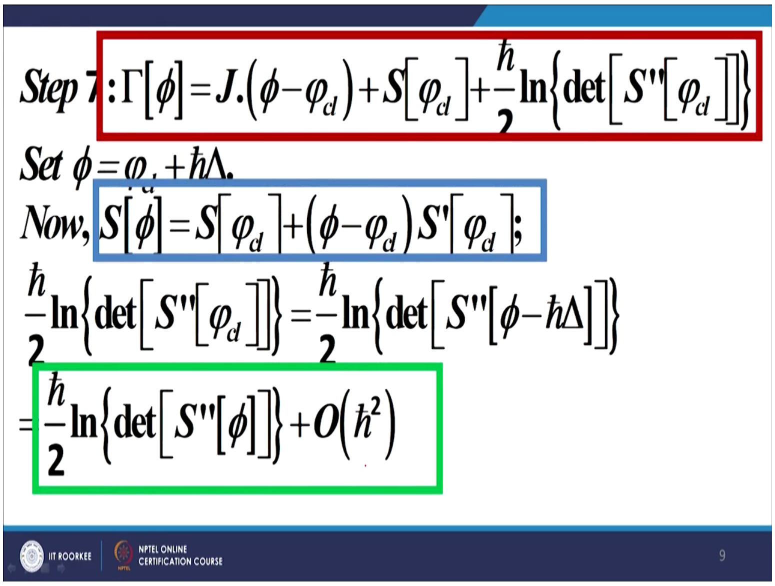

Now, this is the expression brought forward from the previous slide. The expression in the red box, it is what is brought forward in the previous slide. We now expand phi around phi classical by writing phi is equal to phi classical plus h bar and delta that gives us the expression in the blue box. And now we also note, we also note it is a very important point that the log determinant S double dash phi classical can be written as log determinant S double dash phi minus h delta with a pre factor of h.

Now, when you expand this determinant keeping in view that you have a pre factor of h bar, that means, you can retain terms and if you want to retain terms only up to order h bar that means, you can retain only terms which are of first order in phi. And you cannot retain any terms which have h bar in it in the determinant. And therefore, you write this expression as this expression becomes equal to the expression that is given in the green box at the bottom of the slide.

To reiterate please note this point phi minus h delta when I expand the determinant of this S double dash phi minus h delta phi minus h delta if when I expand this determinant the because I want to retain terms only up to h bar order I have to through away all terms which contains h bar and delta inside this expansion because this is already in h bar here, so that means I can retain only term which have phi and them.

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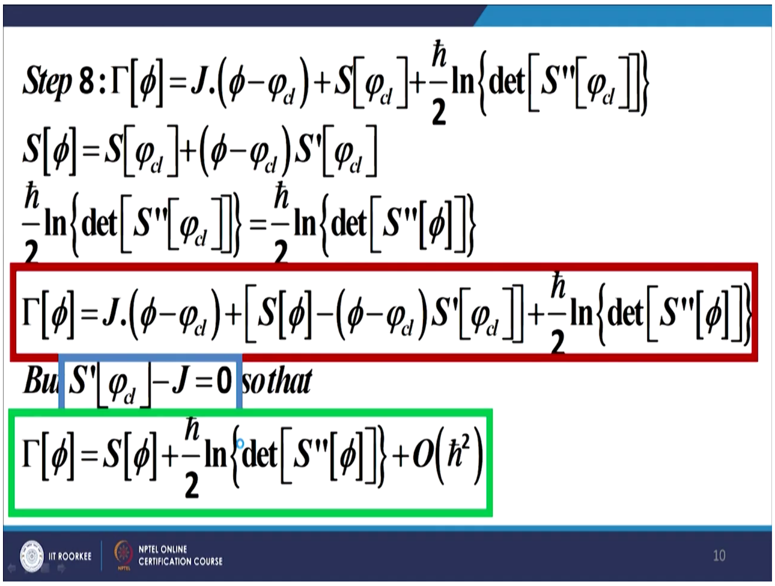

So, that being the case, we now simplify this final step is here. We have this expression from the previous slide for gamma phi. This is the expression which is there in the red box, we brought it forward to in the first equation here in this slide. And also we have S phi in terms of

S phi classical can be written as the second equation on the slide. So, and we have also establish the existence of this equality to order h bar.

So, that being the case, we make the substitutions. And when we make this substitutions we find that J minus S dash phi, it has a common factor of phi minus phi classical, but S dash phi minus J is equal to 0. So, both these terms will vanish. And what we are left with is S phi plus h bar by 2 and then this log term, so that is exactly what we wanted to establish.

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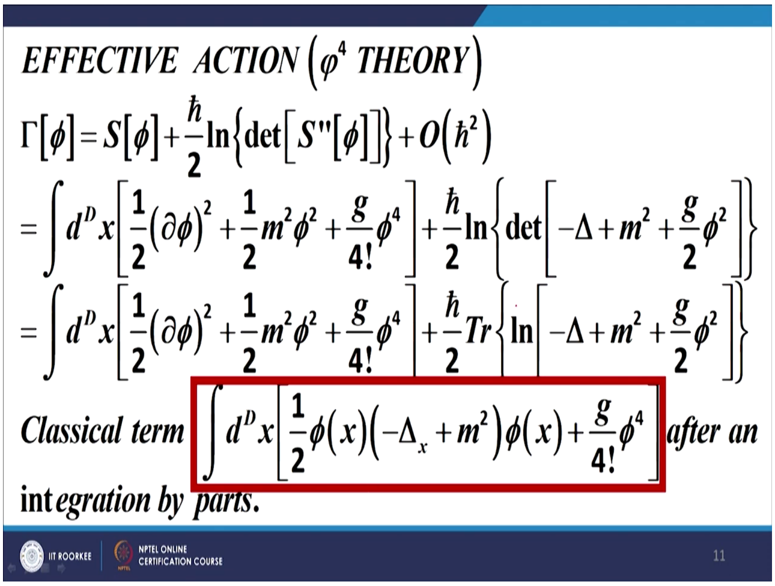

Now, on the effective action for the phi 4 theory, what happens in the case of the phi 4 theory? In the case of the phi 4, this is the equation that we have just establish for the phi 4 theory S phi is given by this expression the first integral. And the second expression is the second the second expression is the second term given in the expression for gamma phi the effective action effective field.

Now, as far as the first part is concerned, we do not bother. The first part in fact, represents the classical term this is the first part is the classical term which can be through an integration by parts, it can be put in the form which is in the red box here. If you integrate the first term by parts, you get the expression which is there in the red box. This represents the classical term. The second term represents the quantum correction or the quantum term. This is the quantum term.

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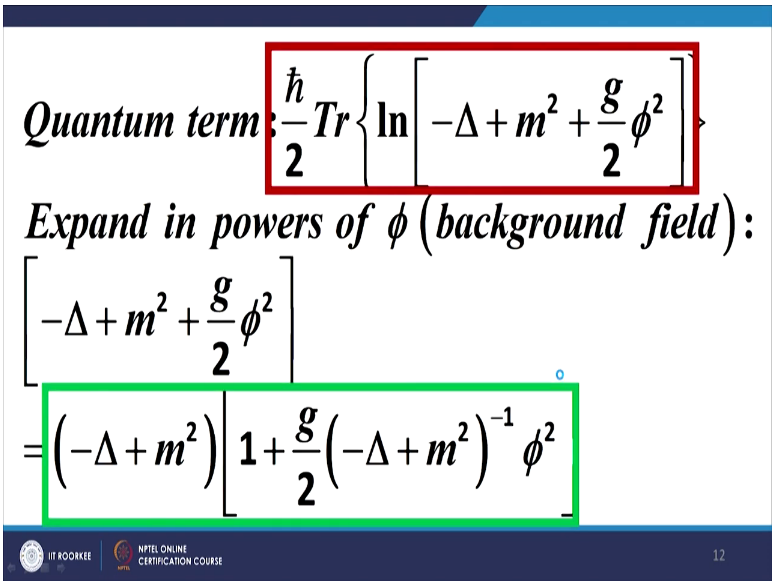

We can expand this in terms of in powers of phi by the process which is given here. We take minus and this triangle represents the Laplacian. So, minus delta plus m square and common, and we have the expression inside the square bracket in the green box. And please note this is the minus 1 here, so inverse because it the when you take this common, the inverse will come before the phi squared term here.

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$$\text{Write } \square = Tr\left\{\ln\left[-\Delta+m^2+\frac{g}{2}\phi^2\right]\right\}$$

$$= Tr\left\{\ln\left(-\Delta+m^2\right)\left[1+\frac{g}{2}\left(-\Delta+m^2\right)^{-1}\phi^2\right]\right\}$$

$$= Tr\left\{\ln\left(-\Delta+m^2\right)\right\}+Tr\left\{\ln\left[1+\frac{g}{2}\left(-\Delta+m^2\right)^{-1}\phi^2\right]\right\}$$

So, now, this expression which you have in the square box here, when you take the trace of when you first of all you take the log of this whole expression and the two terms separate out. The log of this expression is one part, and the trace of log of this expression is the other part. So, when you then you take the trace, again the trace also operates over the 2. And we have the trace of log of the first part and the trace of the log of the second part.

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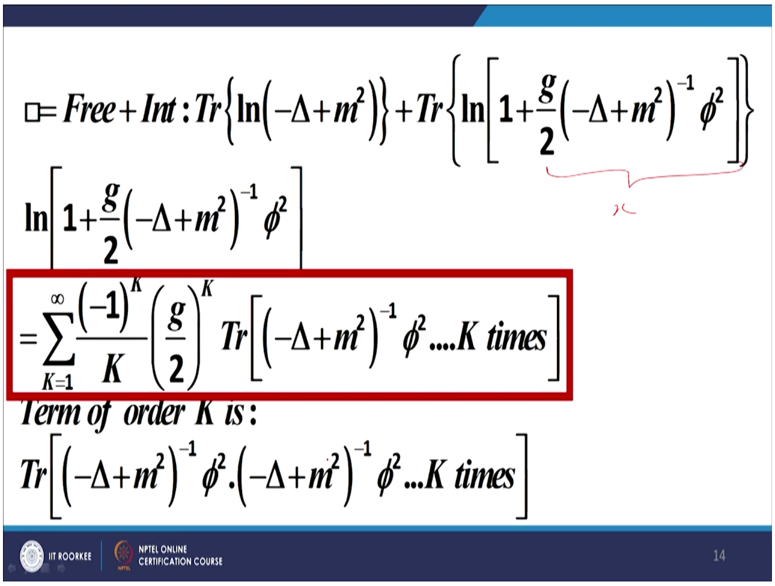

Now, the log of the second part if you look at it is of the form of 1 plus x, log of 1 plus x. So, it can be expanded as a power series in x which in our cases d by 2 x in our cases this expression g by 2 minus Laplacian plus m squared minus 1 phi square. So, that is what we have done in the red box.

This the whole expression is expand this logarithmic series has expanded as a power series in g. And the kth order term will obviously be of the form which is given in the bottom expression on this slide. So, this is all the quantum term or the effective field can be worked out, can be manipulated, can be worked out in the case of phi 4 field using the formula that we derived earlier.

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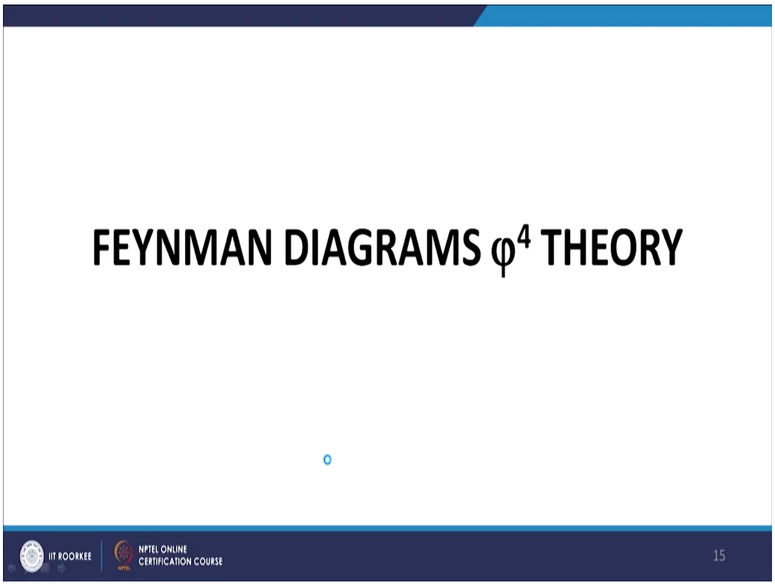

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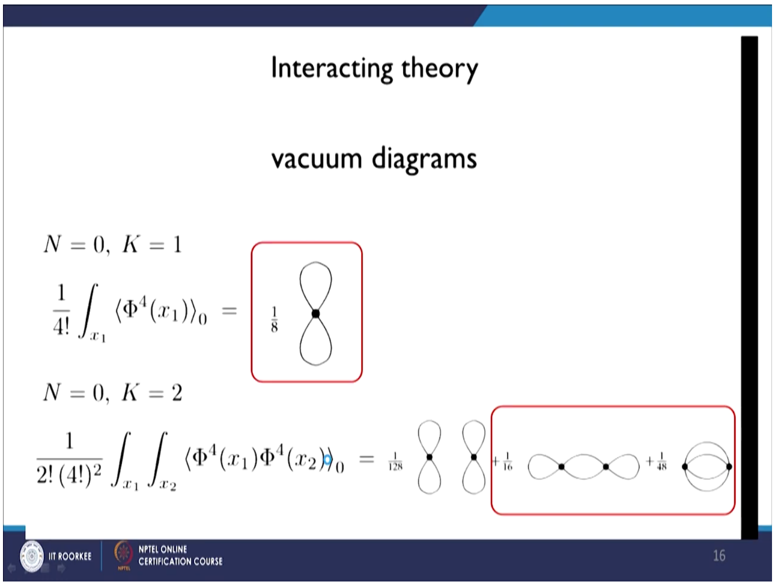

The Feynman diagrams are put them as part of the presentation for the benefit of the viewers.

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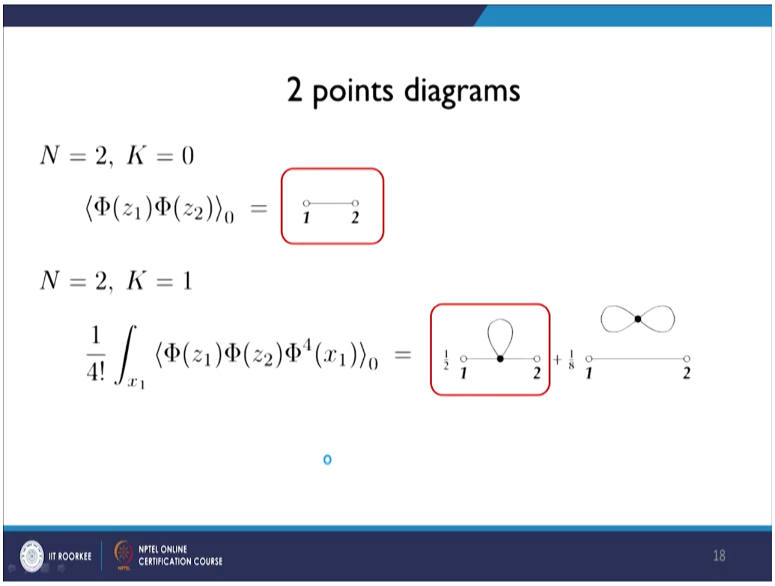

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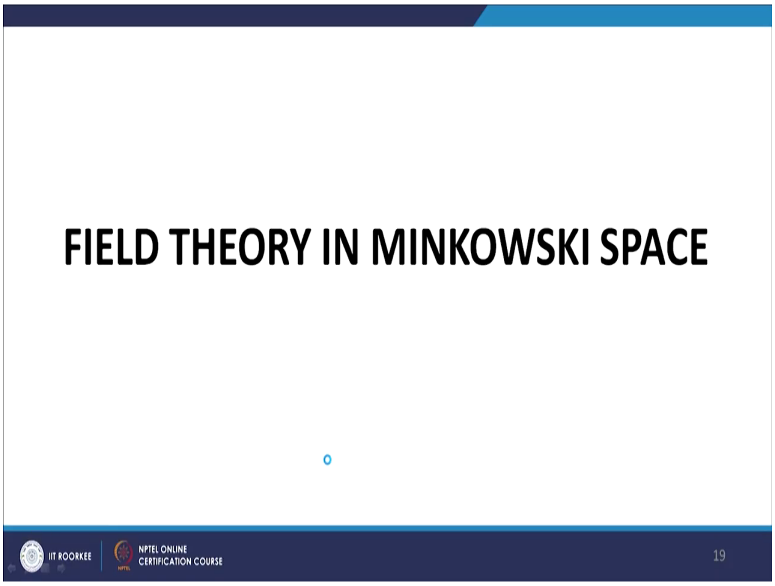

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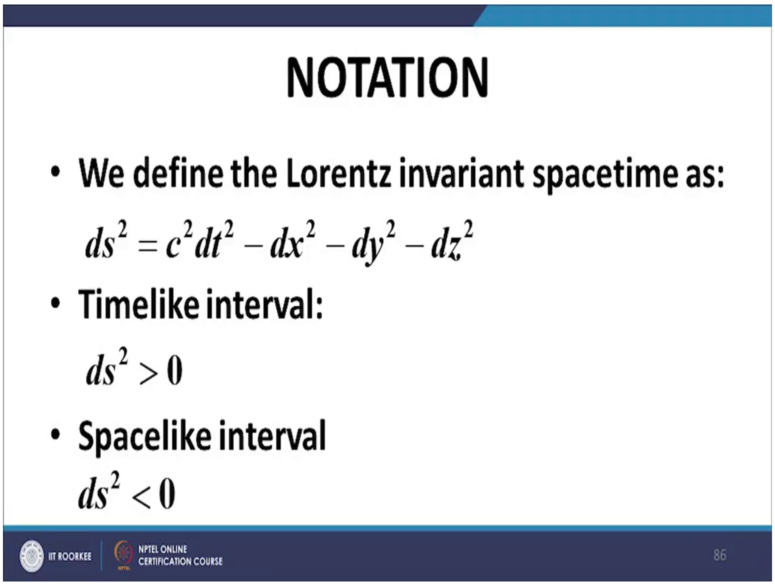

We now move to the field theory in Minkowski space. Now, the Lorentz invariant space time interval in the convention that we shall be following is given by the expression that is given on the slide c squared dt square minus bracket dx square plus dy square plus dz square that square.

Clearly the time like interval will be d s square is greater than 0, and the space like interval will be ds square is less than 0. Where the basically the time like interval is an interval where the time predominates, and the space like interval is the interval in which the space predominates.

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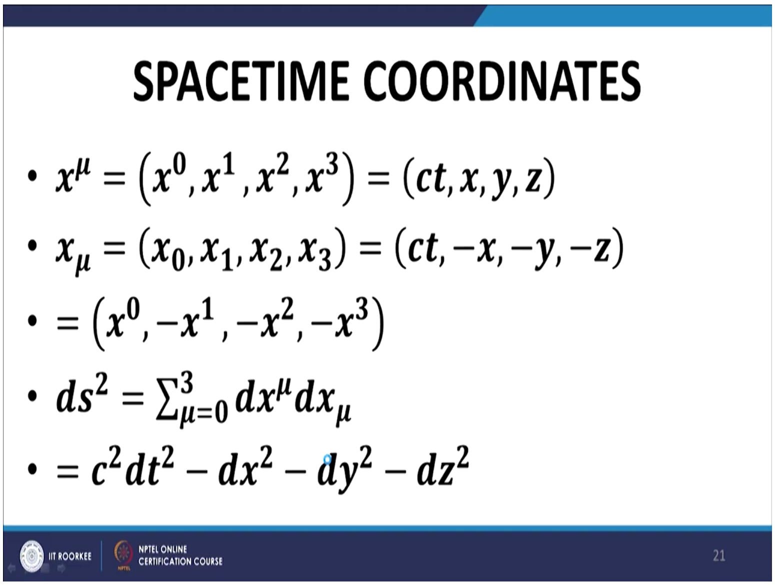

As these coordinate space time coordinates, the space time coordinates convention the contra variant coordinates will be represented in terms of upper indices as usual. And we shall use x 0 for identifying the time coordinates x 1, x 2 and x 3 for this spatial coordinates. So, x 0 is equal to ct, x 1 is equal to x, x 2 is equal to y, and x 3 is equal to z. The corresponding contra variant coordinates would be given by the lower indices, x 0 lower, x 1 lower and so on. And these would represent ct comma minus x minus y minus z.

So, clearly we have x 0 is equal to x 0 contra variant is equal to x 0 covariant, x 1 contravariant is equal to minus x 1 covariant and so on. And the interval in terms of these in this notation is given by dx mu dx mu summation over mu. Usually the summation is not explicitly mentioned when there is a repeated index was one upper and one lower, repeated index. This is the Einstein summation convention and this is what is invariably followed in all field theory and relativity. So, we shall be doing that.

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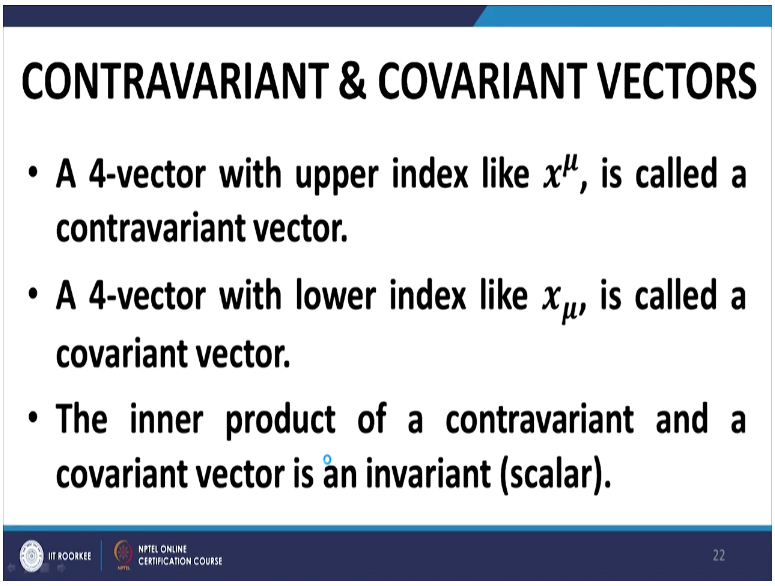

And so ds square is equal to dx mu dx mu. A 4-vector with a contravariant as I mentioned earlier a 4-vector upper index is called a contravariant vector a 4-vector with a lower index is called a covariant vector. The inner product of a contravariant, and covariant vector is a Lorentz invariant Lorentz scalar.

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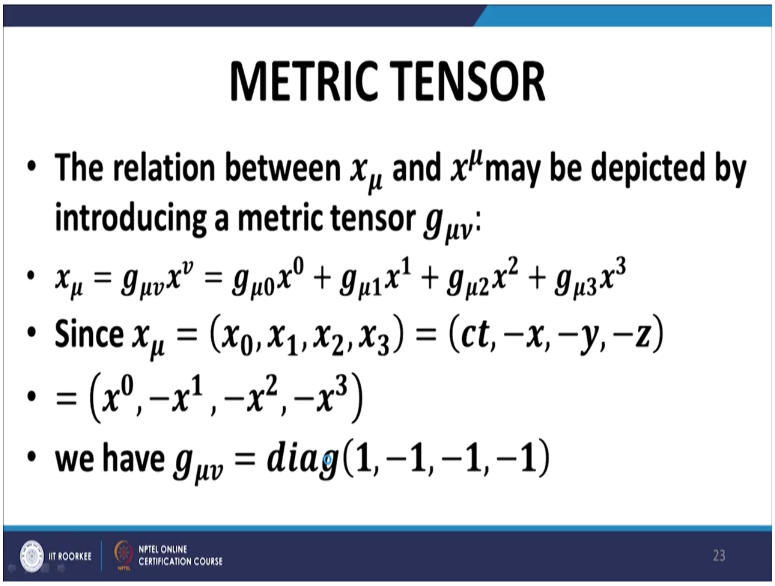

The metric tensor, we can migrate between the covariant and contravariant vectors, and covariant and contravariant coordinates by using the metric tensor. And comparisons with the between the contravariant and covariant space time coordinates enable us to fix the expression as diagonal 1 minus 1, minus 1, minus 1.

In other words, we are using a metric where the signature of the metric is minus 2. A lot of literature is there where the signature of the metric is plus 2. In other words, the metric tensor is used with the diagonal elements minus 1, plus 1, plus 1, and plus 1. The inverse matrix coincides with the original metric.

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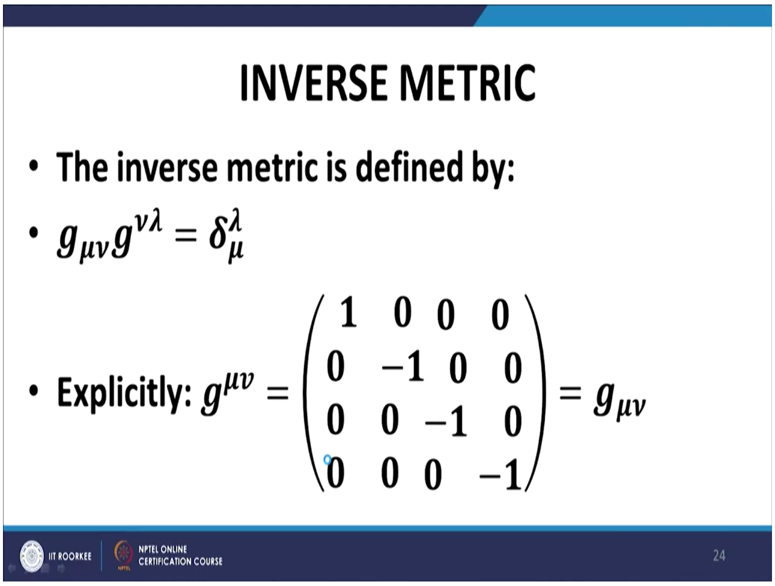

And it has the same diagonal elements 1, minus 1, minus 1, minus 1, with all other elements being 0.

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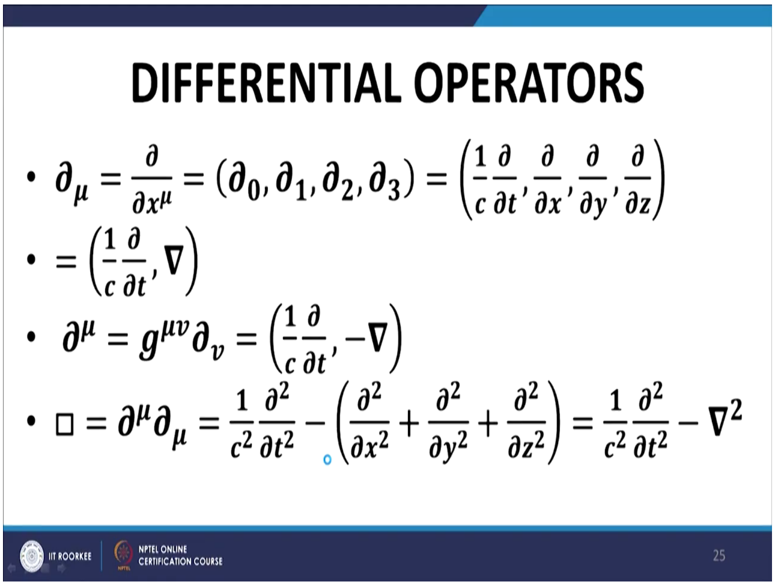

Remember this is a flat metric, it is not a curve space metric. It is a call the flat space metric or the flat metric. Please note this point. Quantum field theory, and, and this is a relevant metric in the contest of special relativity. It is only when we encounter general relativity that we need to consider metrics in curve space which of course as a much more complex form.

And in indeed, it is the function of space time parts. The differential operators in the contravariant and covariant notation are given on your slide, and the box operator is also given presented in this slide. So, we are please acquaint themselves with the notation that is here in the four first slides that are dedicated to this notation. So, make the future exposition an ambiguous.

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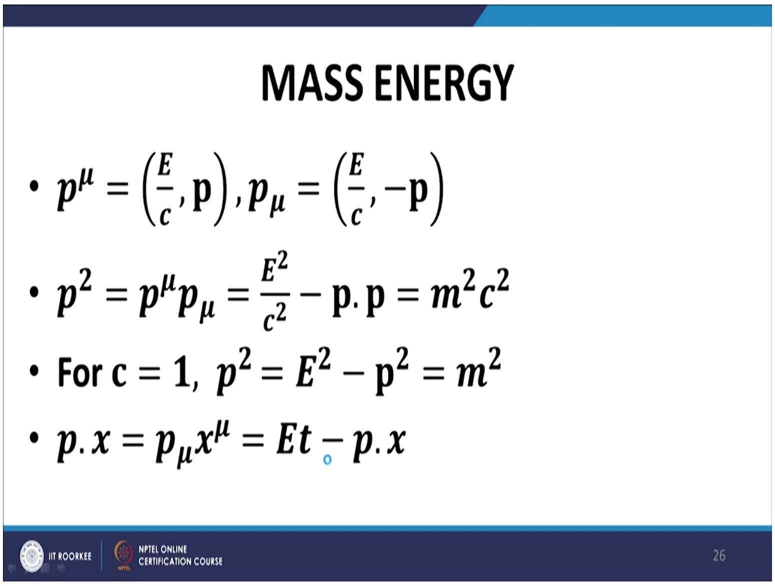

Let us continue. The mass energy relation of course, the relativistic mass can be written as a relative relativistic momentum can be written as a 4-vector with the first component being the energy component, and the spatial components being the usual Newtonian momentum.

And in terms of contravariant momentum, the space like or the spatial momentum attracts or captures the minus sign. The rest of the notation explains itself pretty much straight away, so nothing much to elucidate.

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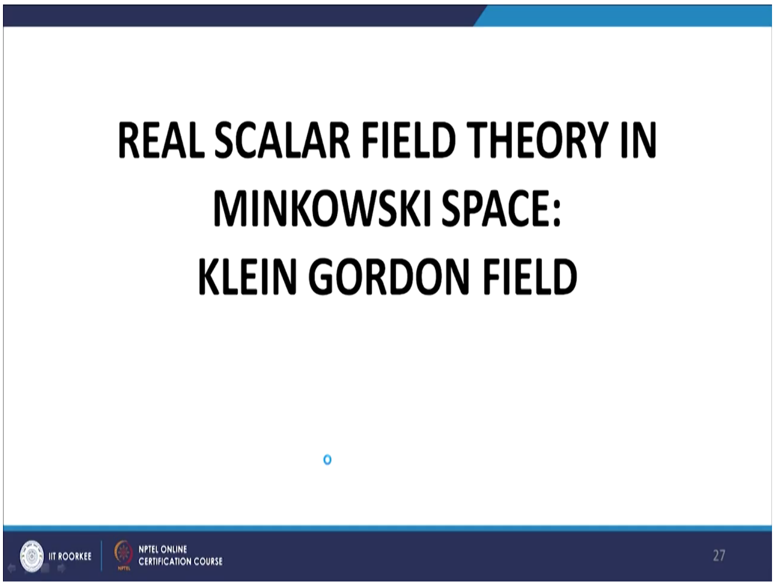

Now, we talk about real scalar field theory in Minkowski space, we talk about the Klein Gordon field.

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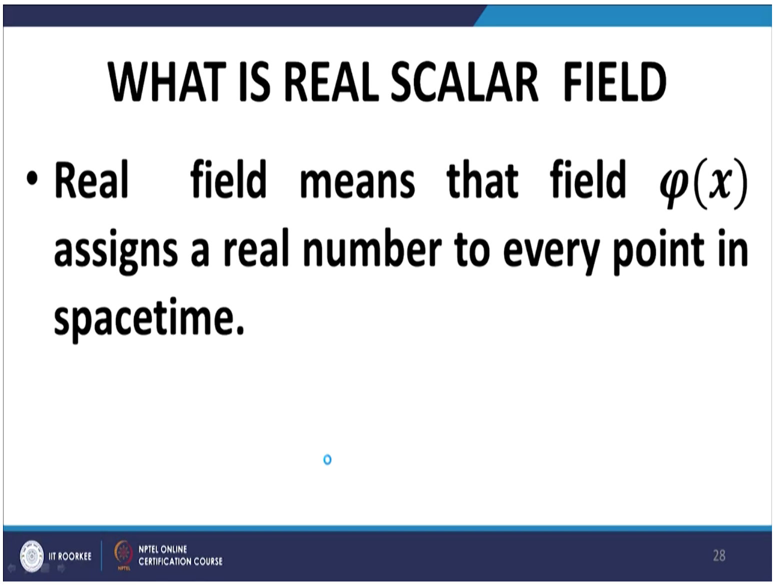

What is why the word real the word real means that the field phi x assigns a real number to every point in space time. The given field phi x assigns a real number as opposed to a complex number to every point in space time.

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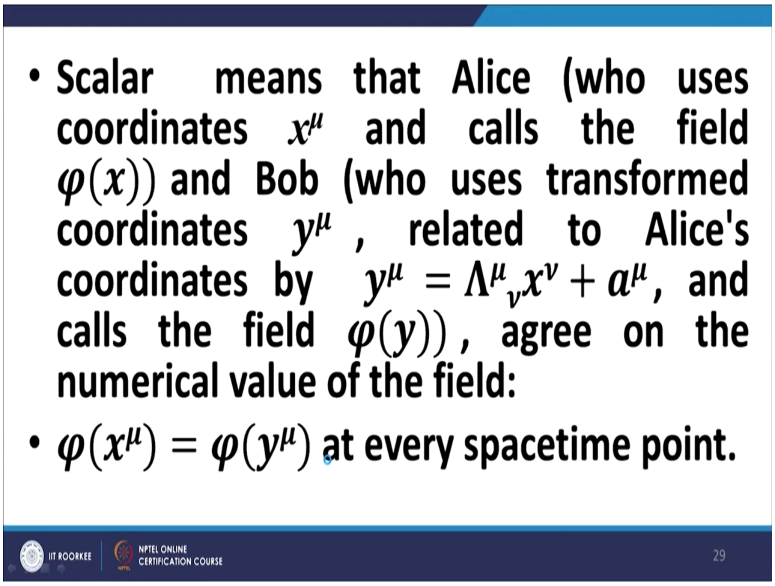

Scalar means that if they have if we have two observers Alice who has who measures coordinates in terms of xs, x mu, and who calls the field phi x and Bob who measures the space time in terms of coordinates y, and called the field phi y. And x and y are related through a Lorentz transformation. Then the both Alice and Bob agree upon the value of the field at every space time point that is what we mean by a scalar field.

Let me repeat. We have got two observers Alice and Bob. Alice is using a space time frame with coordinates x mu. Bob is using a space time frame with coordinate frame with that coordinates y mu.

The coordinates x mu and y mu are related through a Lorentz transformation, and x the Alice denotes his field by phi x, the same field Bob denotes the field by phi y. Then the both Alice and Bob agree on the value of the field that is phi x is equal to phi, phi y at every space time point, so that is the meaning of a scalar field. Therefore, the equations of motion for phi x and those for phi y will be the same.

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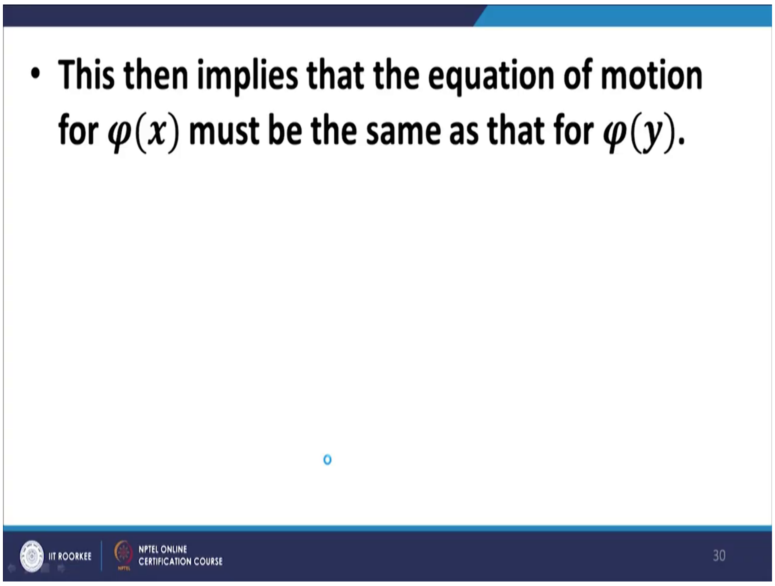

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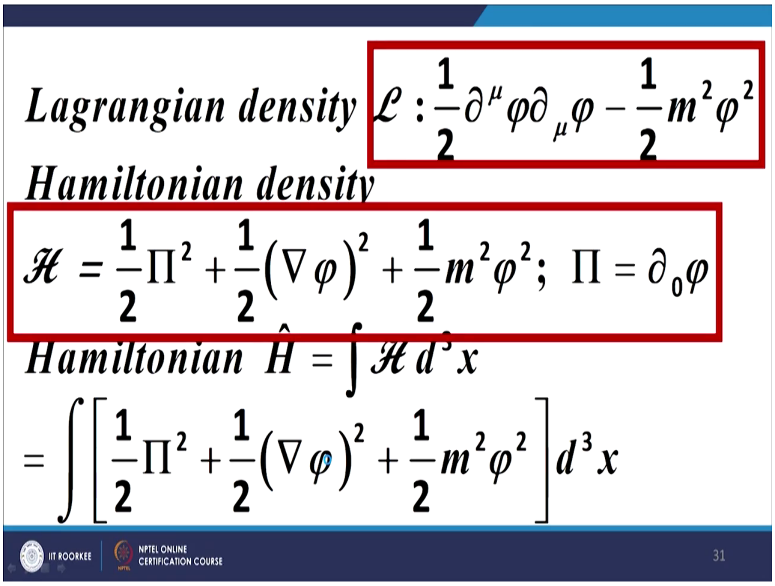

Now, the Lagrangian density for the Klein Gordon field is given in the red box at the top of the slide. The Hamiltonian density corresponding to this Lagrangian density is given in the second red box. And the Hamiltonian which is the spatial integration of the Hamiltonian density is given by the expression the third expression on this slide. The Klein Gordon path integral, we are not talking about the free field.

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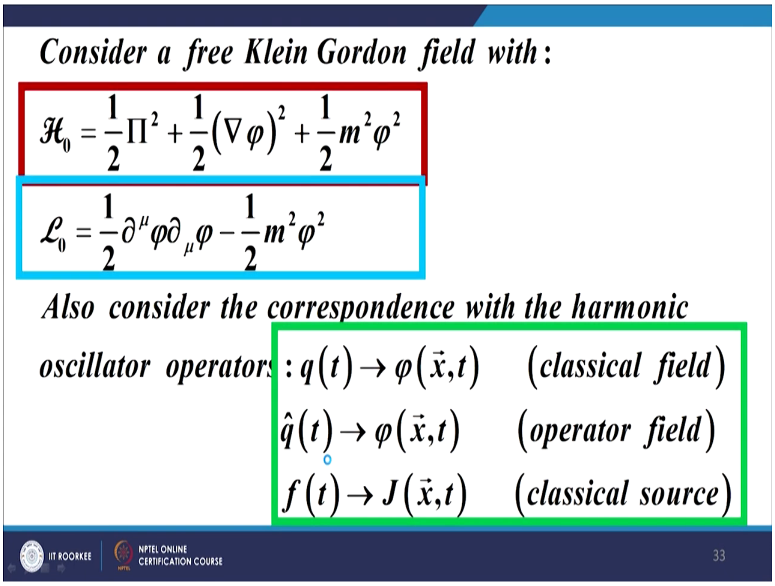

So, for the free field, we have the Lagrangian which is given in the blue box. We make the correspondence q t goes to phi x t. We make the correspondence the harmonic oscillator problem which we have done earlier. We replace the harmonic oscillators coordinates with the field operators and the field coordinates as per the scheme that is given in the green box here. q t goes to phi t, x.

Please note the field coordinates are now functions of space time. Whereas, earlier the harmonic oscillator coordinates were simply functions of time, and that is why we call the quantum mechanics, the standard quantum mechanics of 0 plus one-dimensional field theory, 0 in space, and 1 in time.

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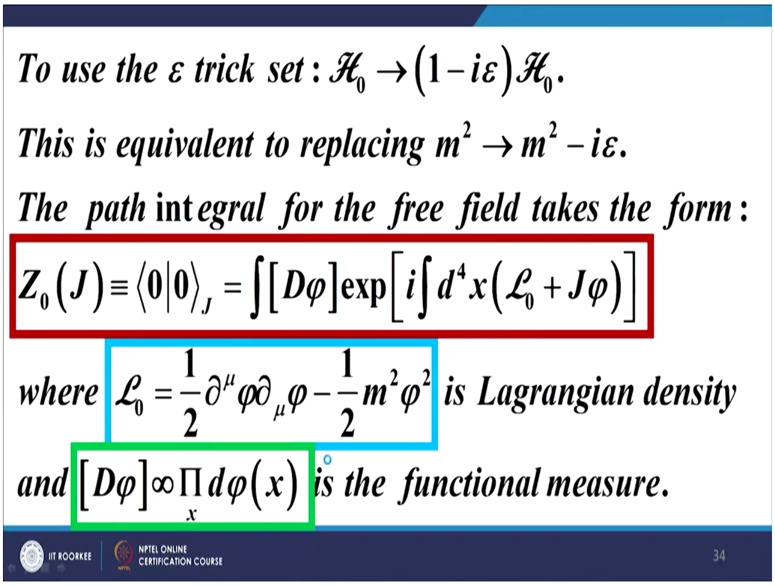

Now, introducing the epsilon trick, we set H 0 to 1 minus i epsilon H 0, which amounts to replacing the mass m square by m square minus i epsilon. The generating functional is given by this expression. Of course, GFI is added representing, the source term. And the green box represents the fact that the path integral. Now, it is an integral over all fields, field configurations.

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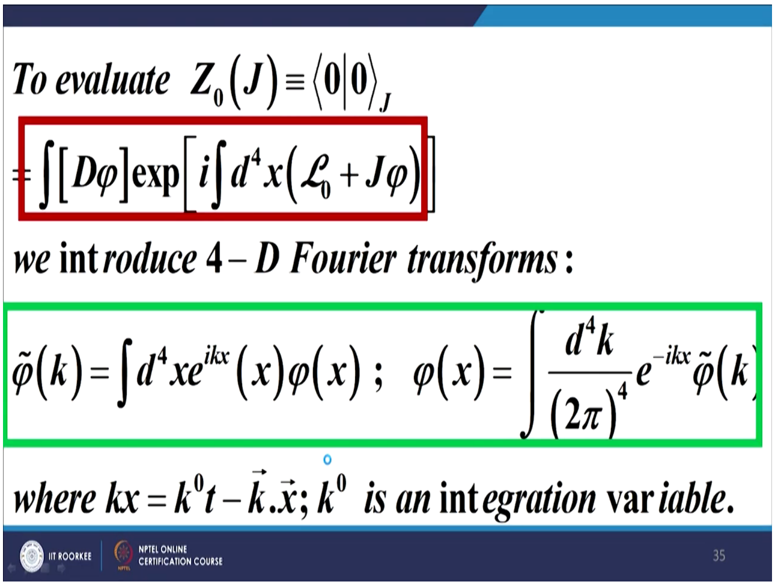

So, we have now the path integral Z J is given by the path integral that is in the red box. We move into the Fourier space. We make the Fourier transformations phi tilde k is equal to this expression, phi tilde k Fourier transformation of Fourier transform of phi x. And this is the similarly phi x is the inverse Fourier transform of phi tilde k.

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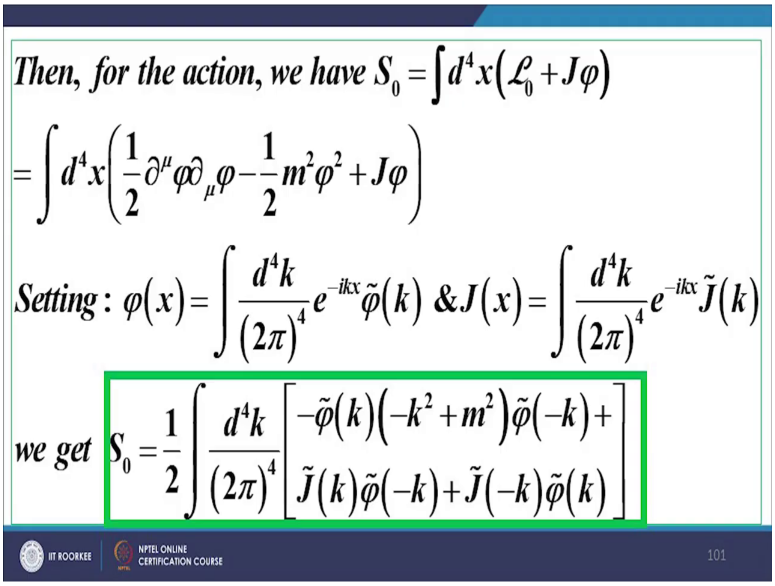

So, in other words, we now move from the positions space or the fields space to the position space to the mode space or the Fourier space. Now, what happens to the action when we introduce this Fourier transforms on the action, recall that the action is given by this expression equation 1, where we include there in the source, and substituting this Fourier transforms. Here the action takes the form given in the green box that is equation 2 at the bottom of your slide.

When we substitute the expressions for a phi x which is there in the original action that is in position space, and we substitute to the Fourier transforms we get the expression for the action. Now, this action is in Fourier space, it is represented by this form.

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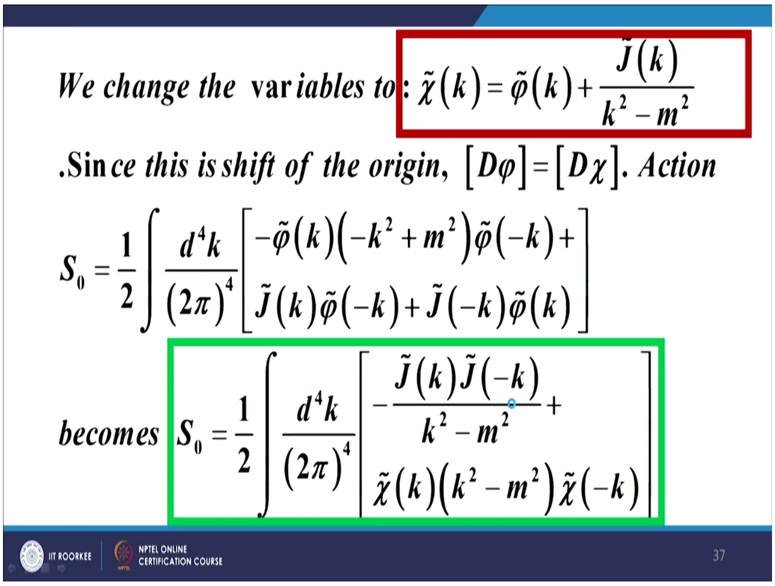

We introduce a change in variables given by the expression in the red box chi of k, chi tilde of k is equal to phi tilde of k plus J tilde of k upon k square minus m square. We introduce this change of variables it is essentially a shift of the origin.

And therefore, we have path integral measures over phi represents is the same as the path integral measure over chi. And in terms of the new in terms of the new variables chi variables, the action now takes the form which is given in the green box at the bottom of your slide right.

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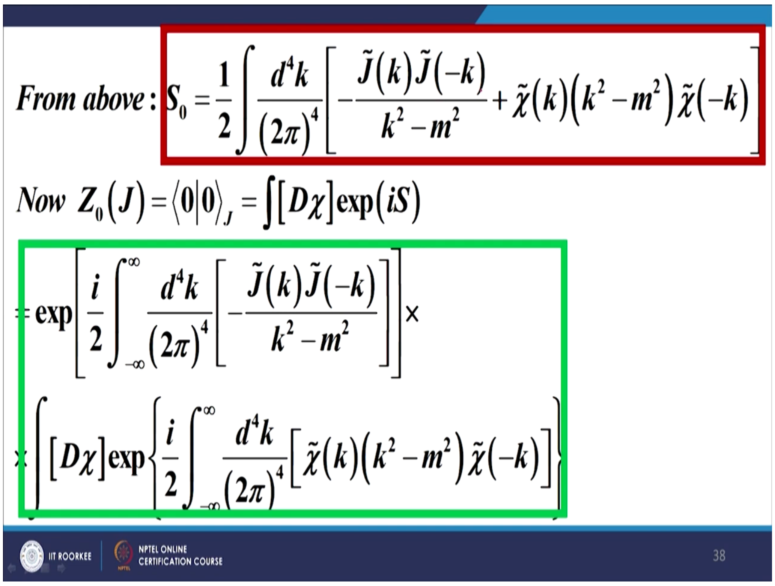

So, this is what we have from the previous slide. This is what we have from the previous slide. Now, the we can write this as the in terms of the action. Now, this is the please note this is the expression for the action, but we want the generating function. And what is the generative function it is the integral of e to the power i S. So, we integrate this expression over with the path integral that is the D of chi integral of this expression.

But the important thing is if you look at these two terms, the first term J tilde k J tilde minus k upon k square and m square this first term is independent of chi. So, because this is independent of chi, I can take this exponent exponential with this term outside the path integral, and the remaining term remains inside the path integral.

So, in other words, this expression, this action of course together with the source, this action leads to the generating functional, when I integrate over e to the power i S path integrate because this first term is independent of chi, this can be taken outside the integral.

And therefore, I have this exponential outside the integral, and the remaining exponential with depends on chi because the path integral measure is also chi, this will form the path integral. The first part in being independent of the path integral measure goes outside the integral.

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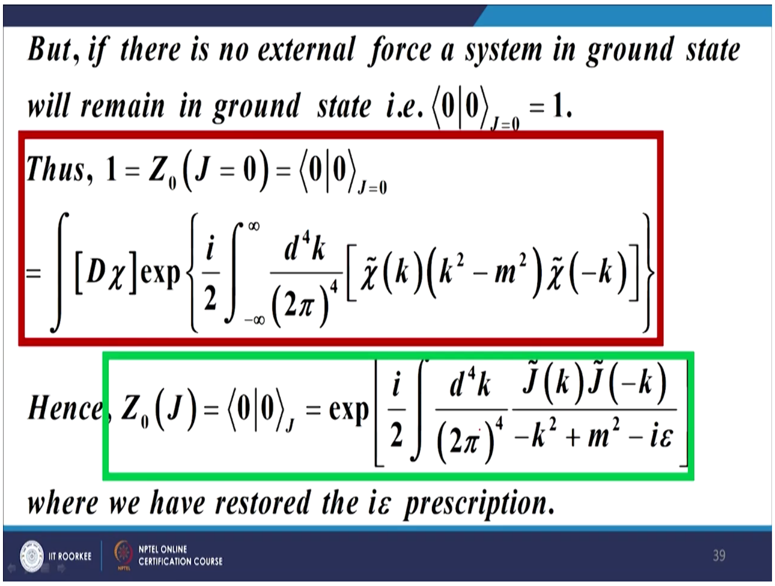

Now, comes to the important part. Now, if there is no external force, in other words, if there is no external field that is J is equal to 0. If J is equal to 0, then the field will not change, the field will remain in its crown shape, because it is the free field. It is a free field and if there is no external field or external source here, then the field remains un change the field retain remains in the ground state.

In other words, if J is equal to 0, then we must have, then we must have Z 0 is equal to 1. Z 0 at J equal to 0 must be equal to 1. Now, what is Z 0 at J equal to 0? If you note this, if you go back here, if you go back here, if you get J put J equal to 0, then this pre factor will vanish.

And whatever will remain is the path integral. And what we are saying is that if J is equal to 0, in other words, if this pre factor vanishes whatever remains must be equal to 1, so that is because why are we saying this because the field in the ground state pre field in a ground state must remain in the ground state if there is no source acting on it.

So, if there is no J, if J is equal to 0, then the free field will remain in its ground state, that means, the probability of the transition of the to from the ground state to the ground state must be 1, and the transition amplitude must also be 1, so that being the case Z 0 of J with J is equal to 0 must be 1.

And if I substitute J equal to 0, the pre factor becomes 0. The exponential of that becomes one this whole thing goes away and the whole pre factor goes away becomes equal to 1. And what we are left with is the path integral, therefore, the path integral must be equal to 1.

Hence, we have the path integral is equal to 1. And therefore, we have Z 0 of J is equal to exponential in the presence of a source in the presence of a source this the path integral becomes 1, and we are left with this Z 0 of J is equal to exponential the 1 by i by 2 and so on, this whole thing in the green bracket becomes the expression for the generating function for the pre Klein Gordon field.

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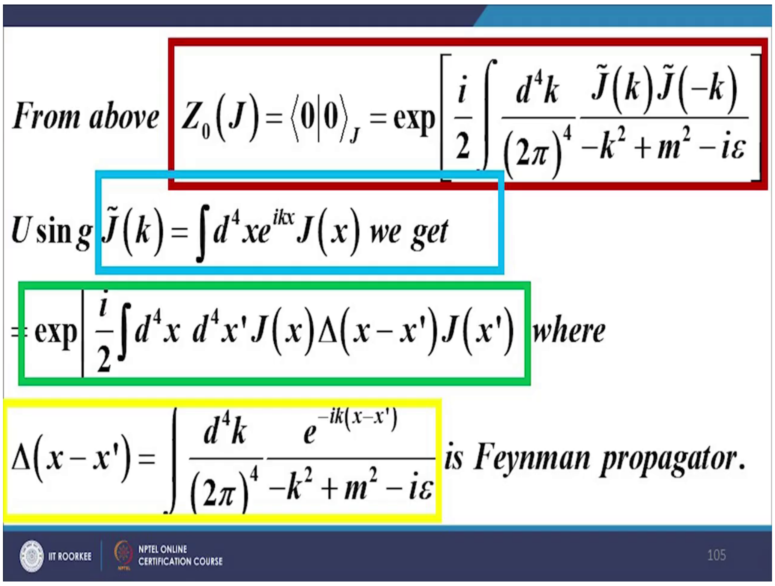

And now if we revert to the original coordinates right J k, and J k in terms of it Fourier analog and also the in terms of its Fourier analog, we get the expression for the path integral in terms of the green box that is given here, where this delta x minus x dash is called the Feynman propagators. And it is given by this expression which is there in the yellow box here right at the bottom of your slide.

So, because the in the case of a ground state, the pre factor becomes the 1, the path integral must also be 1, and therefore, the path integral and the path integral is independent of J, and therefore, the path integral will be 1 even if there is a source, even if there is a source.

In other words if there is a source the path integral does not contribute anything to the I am sorry the path integral term does not contribute anything, and the generating functional therefore, simply becomes exponential the pre factor term simply becomes the generating functional.

And finally, we shift back to our original coordinates, the position space coordinates, and we can drag the generating function in terms of the expression which is given in the green box, and which where delta x minus x is known as the Feynman propagator. And it has the expression which is given in the yellow box alright.

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We, now, show that the KG propagator defined by:

$$\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{-k^2+m^2-i\varepsilon} \text{ is a Green Function of KG equation.}$$

$$\left(\partial_x^2+m^2\right)\Delta(x-x') = \partial_x^2 \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{-k^2+m^2-i\varepsilon} + m^2 \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{-k^2+m^2-i\varepsilon}$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} \frac{-k^2+m^2}{-k^2+m^2-i\varepsilon} \xrightarrow{\varepsilon\to 0} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} = \delta^4(x-x')$$

$$\text{since } \delta^4(x-x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} \text{ in the Fourier representation.}$$

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Thank you. We will continue from here in the next class.