Path Integral Methods in Physics & Finance

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Lecture - 41
Euclidean Field Theory (1)

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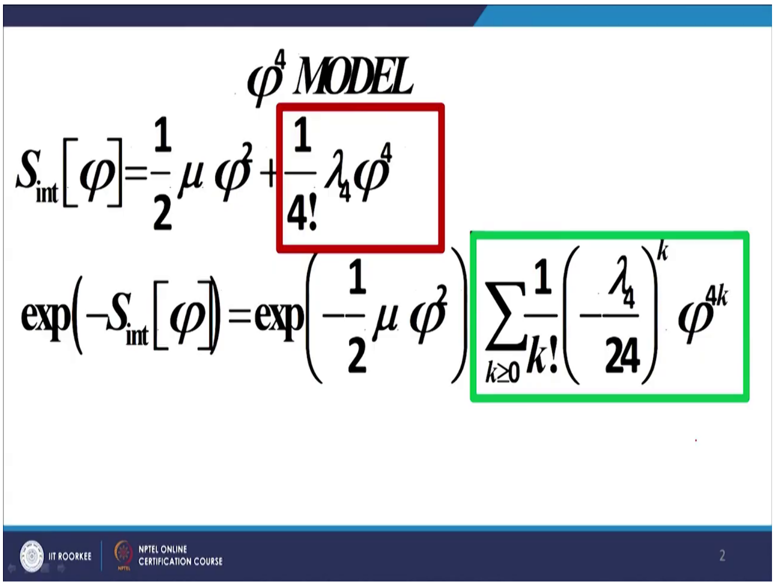

Welcome back. So, let us start. Now, today we will talk about the phi to the power 4 theory, in the context of Euclidean D-dimensional Space. The interaction, the action term then captures another interaction term involving the coupling constant. And the action can be written in the form that is given in the first equation with the with the red box representing the interaction term.

We then, as in the case of 0 dimensional theory we expand this interaction term on the premise that the coupling is small as a power series in the coupling constant.

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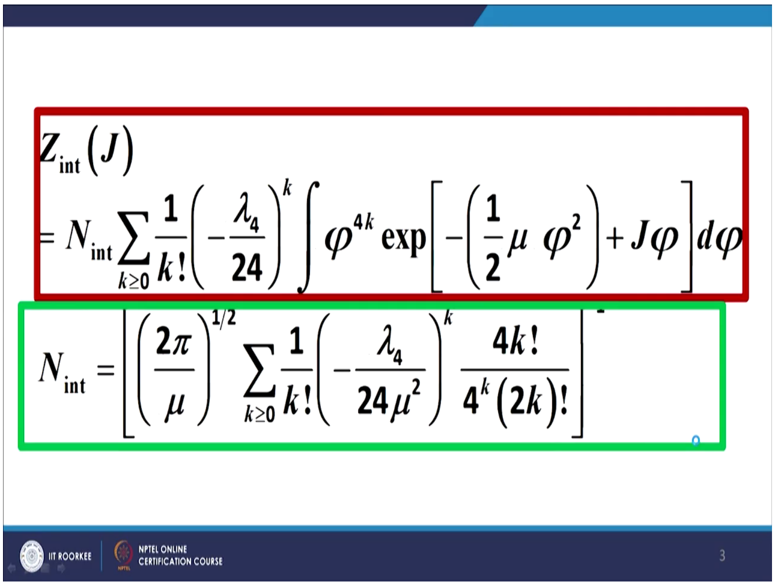

And we obtain an expression for the generating functional, and the normalization in the interaction theory as the expressions given in the red box and the and Green box, respectively. This is pretty much the situation that we encountered in the 0 dimensional field theory.

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$$G_{2n} = N_{\text{int}}\sqrt{\left(\frac{2\pi}{\mu}\right)}\frac{1}{\mu^n}\sum_{k\geq 0}\frac{1}{k!}\left(-\frac{\lambda_4}{24\mu^2}\right)^k \frac{(4k+2n)!}{2^{2k+n}(2k+n)!}$$

$$= \frac{\frac{1}{\mu^n}\sum_{k\geq 0}\frac{1}{k!}\left(-\frac{\lambda_4}{24\mu^2}\right)^k \frac{(4k+2n)!}{2^{2k+n}(2k+n)!}}{\sum_{k\geq 0}\frac{1}{k!}\left(-\frac{\lambda_4}{24\mu^2}\right)^k \frac{4k!}{4^k(2k)!}} = \frac{H_{2n}}{H_0}$$

And then we define 2 n point Green functions in terms of H2 n and 0 H2 H 0. This again follows what we did in the 0 dimensional case.

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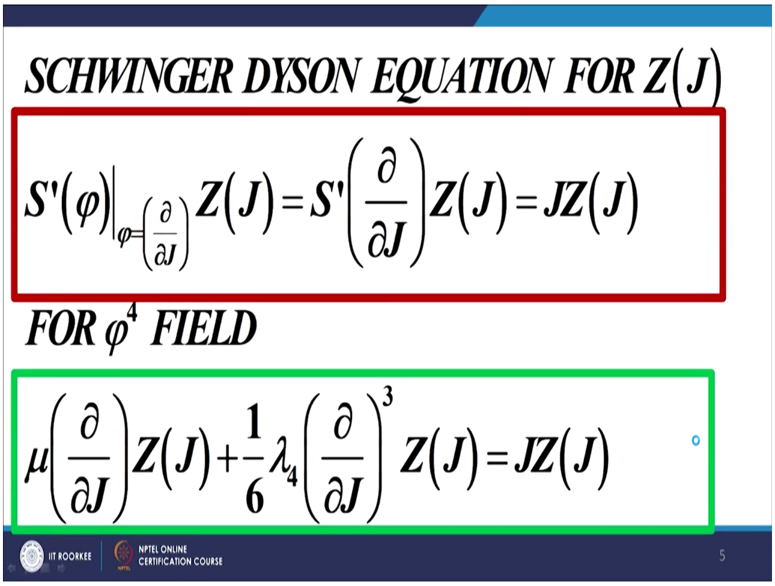

The Schwinger Dyson equation for Z J was obtained in the form given in the red box and for the phi 4 field it takes the explicit form given in the Green box at the bottom of your slide.

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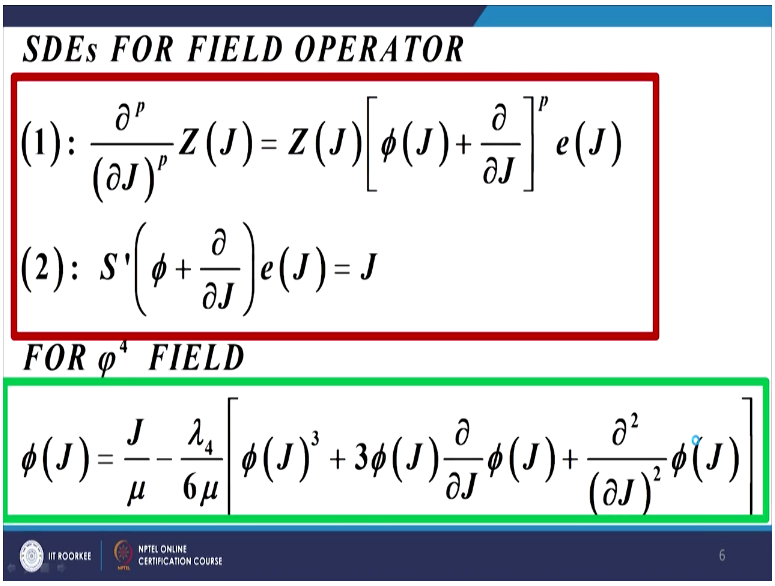

Schwinger Dyson equation for the field operators can be in either of the forms 1 and 2, and for the phi 4 field we have this expression, which is again given at in the Green box, right at the bottom of your slide.

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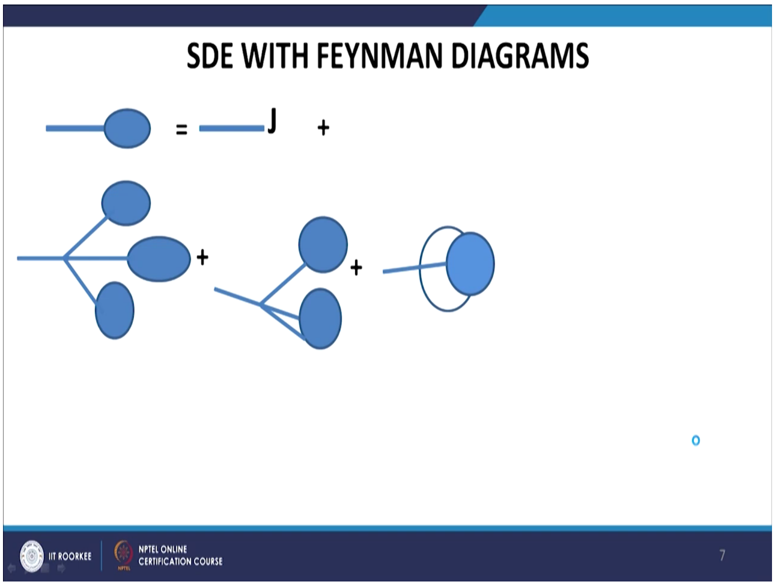

So, and the Feynman diagrams which led us to the Schwinger Dyson equation given in the Green box here for the phi 4 field in the case of 0 dimensional theory are reproduced here for the sake of continuity.

So, all the results so far that I have enumerated were delivered and discussed in the context of 0 dimensional field theory in an earlier lecture.

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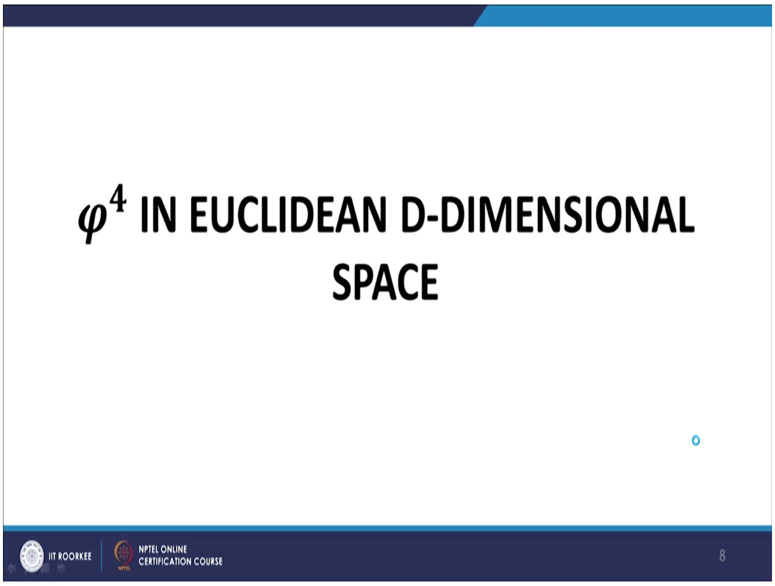

Now, we take up this the situation in the case of Euclidean D-dimensional space.

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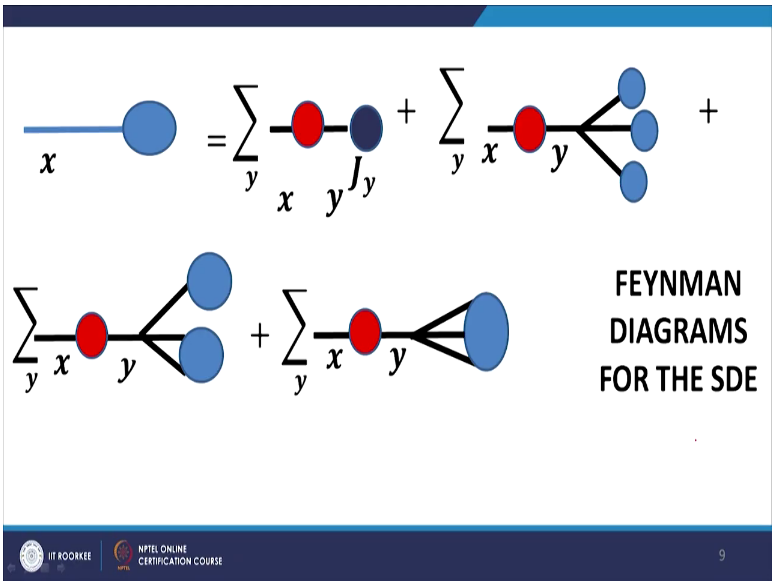

In the case of Euclidean D-dimensional space our first step obviously, is to obtain the Schwinger Dyson equation for the propagator. We consider a field function entering through an external like at the space time point x or Euclidean point x. And what are the possibilities or what are the various situations or scenarios that this particular field function can encounter on entering through an external line are depicted on the right hand side of the equality.

The first diagram represents the fact that the incoming field gets propagated to a field a new field phi at this space time point y, phi y and then it jumps into or then it encounters the source J y.

Now, because the propagator as I mentioned in our previous lecture the propagator singles out a particular y as the outgoing field, we need to sum over all possible values of y to correspond to the situation on the left hand side of your equality.

And then, this other situation that can arises that after propagating to the space time point y the fields these new field phi y encounters an interaction 4 point vertex, and each of these 4 point vertex then has the possibility of going back into the blob that is there on the left hand side.

In other words, it could further encounter the same situation on that it the new that the field entering at the point x had encountered. In other words, it could again encounter with or a propagate to another space time point and then then encounter a source there or it may encounter more 4 point vertices and so on. Each of these 3 vertices outgoing vertices or 3 outgoing lines rather I am sorry emerging from the 4 point vertex could lead to any these situations.

Then, there could be a situation that the, again the summing over the outgoing field has to be done. And then, there could be a situation where the phi x gets propagated to a new space time point y and then again it faces or it encounters a 4 point interaction. And a post interaction, after interaction the two branches emerging from the vertex again going to the same blob while one branch goes away to another blob and two and the situation where it faces either another 4 point vertex or it faces a source.

So, that is the that is the interpretation of the first diagram of the bottom line. And similarly, the first diagram and the second diagram of the bottom line can be interpreted where all the 3 outgoing lines emerging from the 4 point interaction rejoin or reunite into the same blob.

Now, we translate that to a Schwinger Dyson equation. The translation is almost absolutely parallel to what we had for the 0 dimensional case except for the fact that there is an additional summation over y. And this summation over y translates to an integral in the continuous form.

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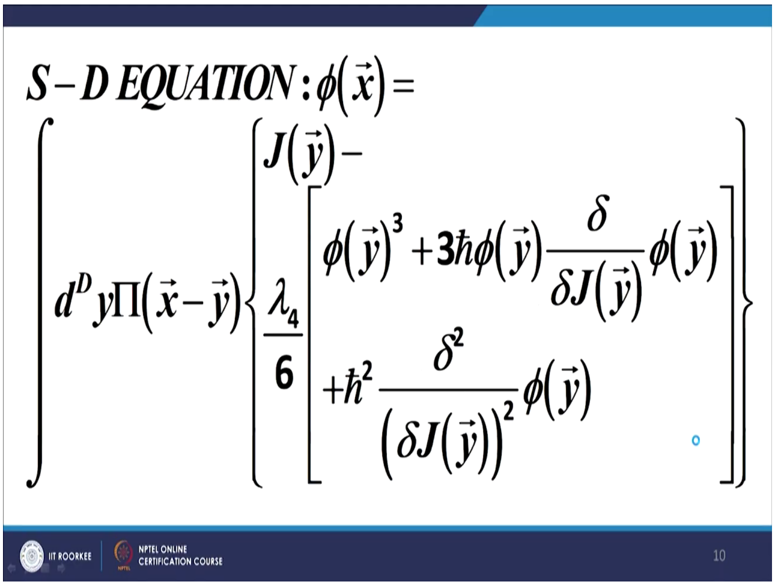

And therefore, in the Schwinger Dyson equation we have an integral additional, integral as you can see. Integral with respect to the y coordinates of the propagator and then we have this representation of each of the 4 diagrams that we discussed just now.

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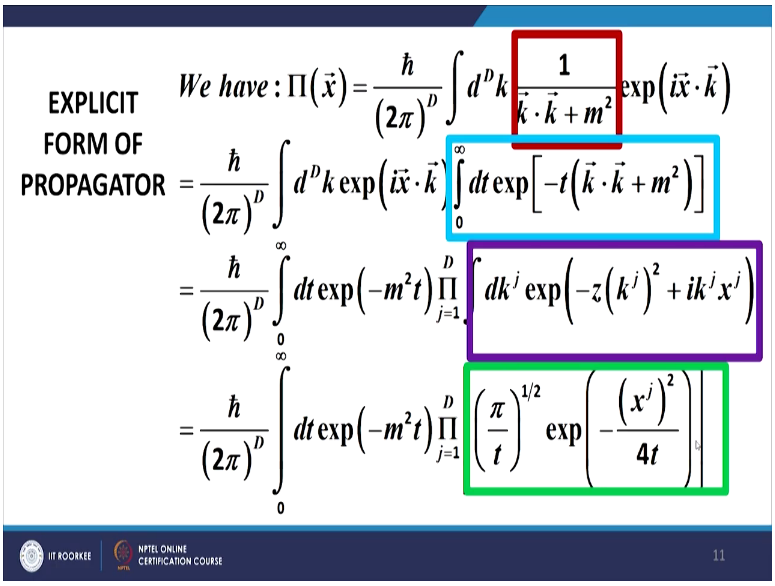

Now, we come to the explicit form of the propagator in the context of the current Euclidean space. Recall that we had worked out the propagator to the expression that is given in the first equation on your slide and the in the previous lecture. This was precisely the expression that we obtained in the continuous case of the D-dimensional Euclidean space. Now, we try to obtain an explicit expression for that a closed form expression for this to the extent possible.

Now, if you look at the expression in the red box in the top equation this expression in the red box in the top equation can be represented by an integral can be represented by an integral that is given in the blue box in the second equation on your slide. Integral between 0 to 1, 0 to infinity I am sorry, integral between 0 to infinity dt exponential minus t k dot k plus m square. If you perform this integral you get precisely 1 upon k dot k plus m square which is the expression in the red box.

So, that being the case we can replace the latter by the former that is precisely what is done and now we do the integration. We take the dt integral in to separate separately and the remaining integral now becomes clearly a Gaussian integral. And that is easily performed the expression in this black box purple box I am sorry is the Gaussian integral, and this Gaussian integral can be easily performed to obtain the expression in the Green box.

So, what we have now is the last equation with the expression in the Green box representing the solution of the Gaussian integral product of the Gaussian integrals that we have in the second last equation.

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$$\Pi(\vec{x}) = \frac{\hbar}{(2\pi)^D} \int_0^\infty dt \exp\left(-m^2 t\right) \prod_{j=1}^{D} \left[\left(\frac{\pi}{t}\right)^{1/2} \exp\left(-\frac{(x^j)^2}{4t}\right) \right]$$

$$= \frac{\hbar}{(4\pi)^{D/2}} \int_0^\infty dt \left(t^{-D/2}\right) \exp\left(-m^2 t - \frac{|\vec{x}|^2}{4t}\right)$$

$$= \frac{\hbar}{2\pi} \left(\frac{2\pi|\vec{x}|}{m}\right)^{1-D/2} K_{1-D/2}\left(m|\vec{x}|\right)$$

The product running from J to D is nothing, but the mod of x square which is what it is substituted in the second equation. And the rest of it is as it is and we get the expression in the second equation on your slide which can be written in terms of the Bessel functions a modified Bessel functions of the second kind.

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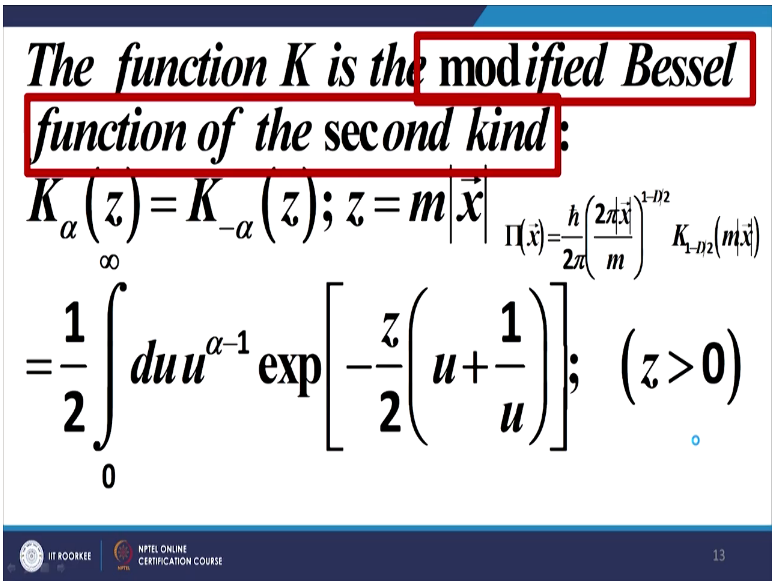

This expression here K m mod x is the modified Bessel function of the second kind. This modified Bessel function of the second kind has a representation integral representation given at the bottom of your slide here.

So, recall, just a recall z here is equal to m mod x as you can see in the previous slide. It is the modified Bessel function of the second kind of m argument is m mod x. And here the argument is z, so z is equated to m mod x and the expression for the propagator is given in the small box here at the right hand side of your slide and the integral representation of the modified Bessel function is given right at the bottom of your slide.

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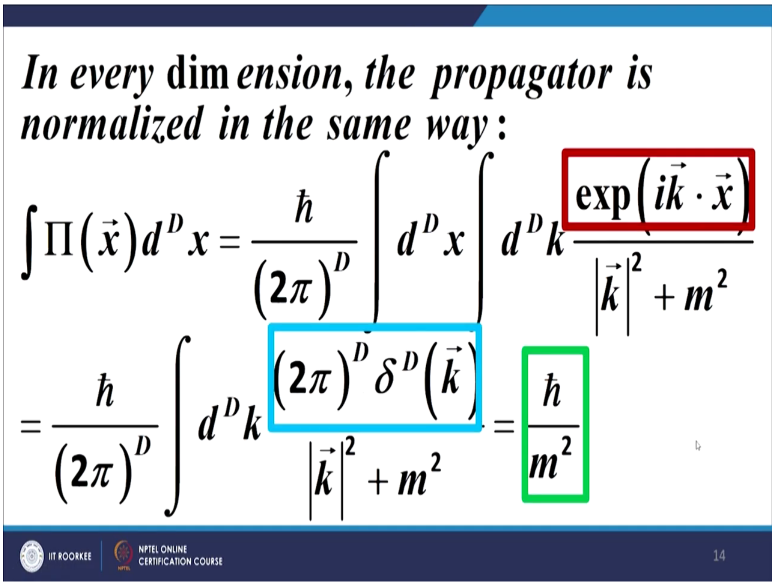

And as far as the normalization is concerned, the normalization of the propagator is straightforward integral pi of x with respect to the Euclidean volume works out to h upon 2 pi D integral d D x integral d k exponential, this is the expression for the propagator.

And if you insert there that in the delta functions that the exponential ik x becomes 2 pi to the power D delta d and then d D-dimensional delta of k. And when you do the k integration the D-dimensional delta goes and what we are left with is h upon m square because k when you do this integral the k term vanishes and the k term vanishes the delta term vanishes 2 pi, 2 pi cancel out and we are left with h upon m square. Now, that is the normalization in every dimension.

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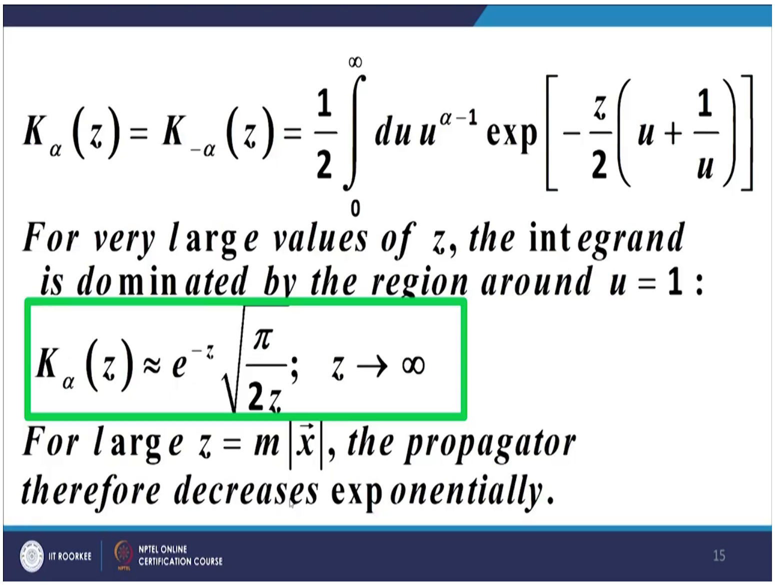

Now, we explore the behavior of the propagator in various extreme situations. For very large values of z the integral is dominated by the region around u is equal to 1. Recall that u is equal to 1 represents the stationary point or the or the saddle point. And the integral in that situation is given by the expression in the Green box.

It is clearly obvious that the as z tends to infinity the propagator decreases exponentially, falls exponentially. So, this is the large scale behavior or large distance behavior of the propagator.

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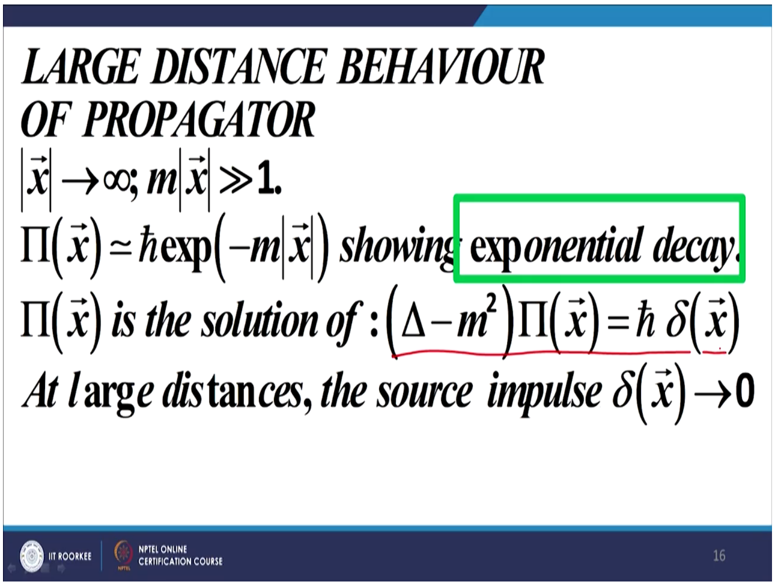

You can also obtain this expression more explicitly because pi x the propagator is a solution is a Green function of the of the Euclid, Klein Gordon equation in Euclidean space and therefore, we have this equation here which I have underlined. And in this equation, in the limit that delta x tends to 0 the sources tend to vanish and therefore, we have the expression Laplacian minus m square of pi x is equal to 0.

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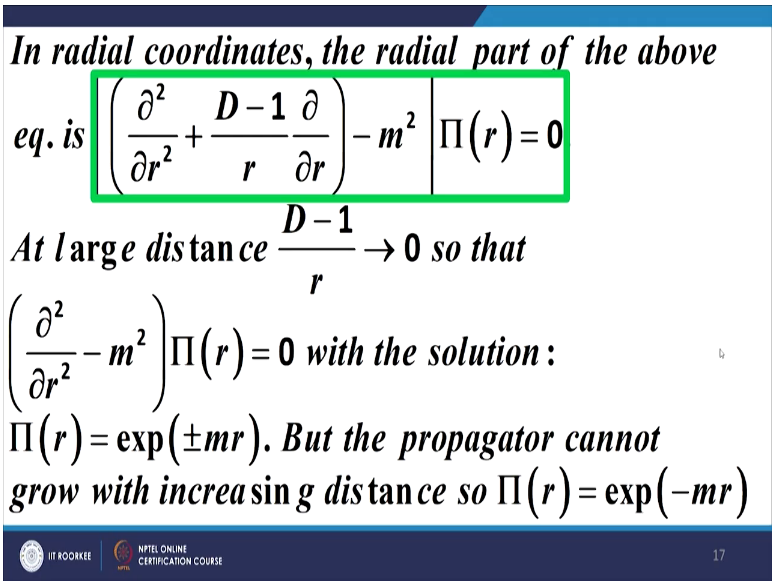

Writing out the Laplacian in terms of the explicitly in terms of spherical coordinates we get the expression that is at the given in the Green box. And further as r tends to infinity, or r increases, r becomes large the second term that is the d by dr term vanishes, and we are left with a straightforward differential equation del d square upon dr square minus m square is equal to pi r with the explicit solution pi r is equal to exponential plus minus mr.

Obviously, the propagator cannot grow with distance, therefore the plus sign is discarded and we have pi r is equal to exponential minus mr which again shows that the propagator decreases exponentially as time increases, as distance increases I am sorry.

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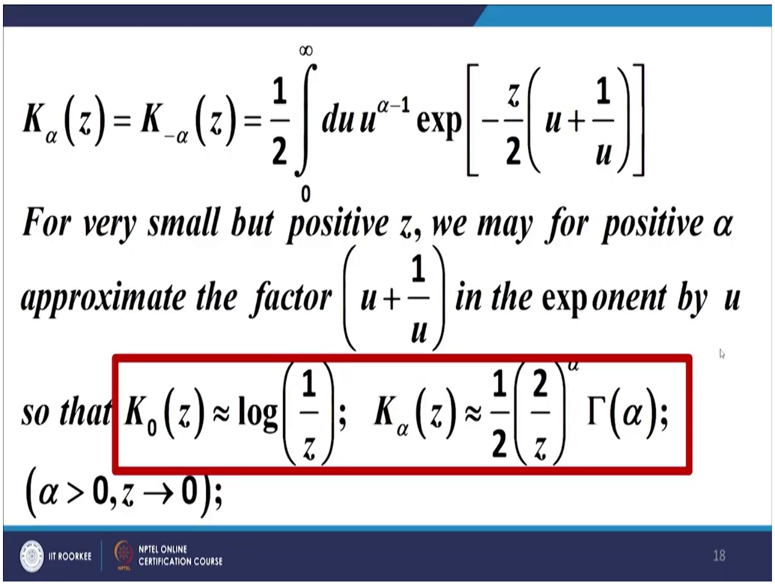

Now, for small z we may we may for the expression u plus 1 upon u we may substitute it by u and we arrive at the two expressions K 0 z is equal to log 1 upon z for that is obviously, for D equal to 2, 2-dimensional case and K alpha z for D greater than 2 is 1 upon 2 and 2 upon z to the power alpha gamma function of alpha. Now, the first one holds for D equal to 2 and the second expression for the modified Bessel function holds for D greater than 2.

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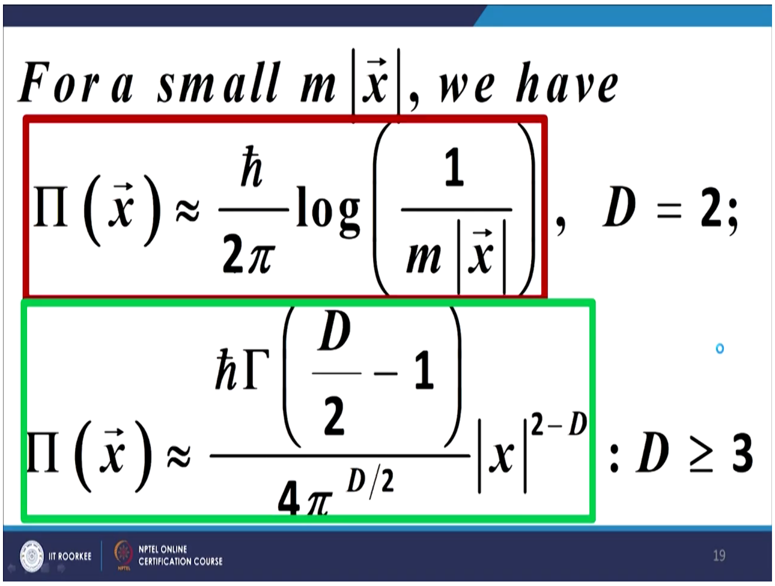

Clearly, and this leads to the expressions for the propagator given in the red box for D equal to 2 and given in the Green box for D greater than 2 that is greater than equal to 3.

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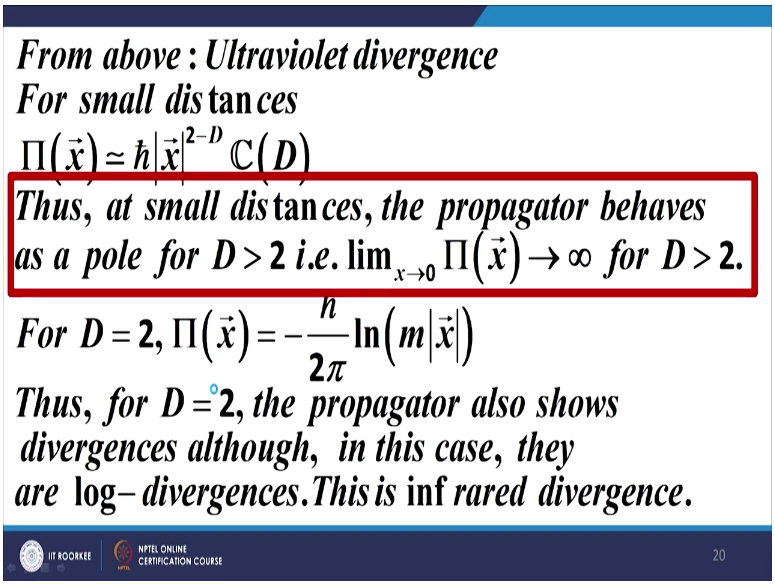

Therefore, for small distances the propagator seems to have a pole. Now, the expression for the if you look at it carefully, if you look at it carefully this was the first part the pre factor of mod x to the power 2 minus D is a quantity which depends on the dimensionality and then it has mod x to the power 2 minus D.

Clearly, as D tends to as x tends to 0, x tends to 0, where D greater than 2 the propagator approaches a pole. Therefore, in the case of small distances the propagator behaves as a pole for D greater than D, D greater than 2 I am sorry, D greater than 2 limit x tending to 0 pi x approaches infinity for D greater than 2.

What happens for D equal to 2? For D equal to 2 the propagator is given by this expression minus h bar upon 2 pi log of m mod x. Here again we encountered divergences as x tends to 0. The divergence in this case is logarithmic divergence or log divergences and this is called infra-red divergence. The previous divergence for dimensions of 3 and greater than 3 are usually called ultraviolet divergences.

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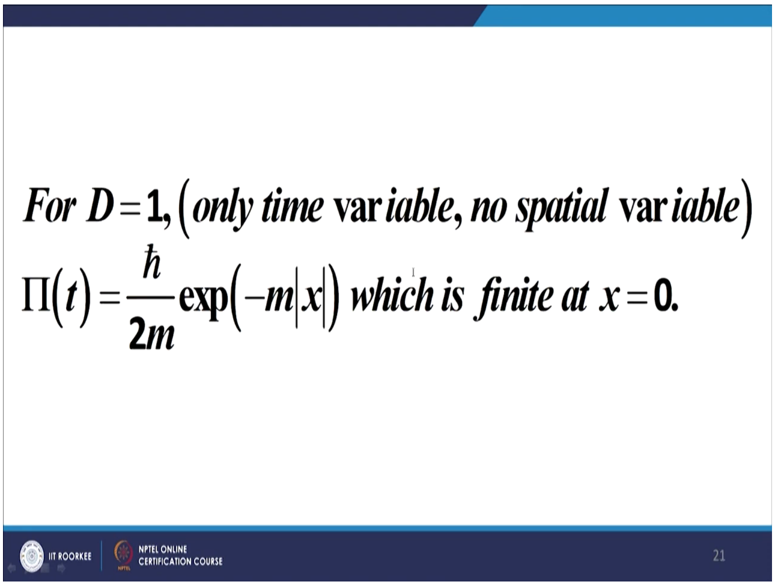

Of course, for D equal to 1 when we have only the time variable that is traditional quantum mechanics pi of t is equal to h bar upon 2 pi exponential minus m mod x which is finite at x equal to 0. So, the traditional quantum mechanics which is 0 plus 1 theory quantum field theory, it is equivalent to 0 plus 1 quantum field theory the issue of divergences in the propagator does not arise.

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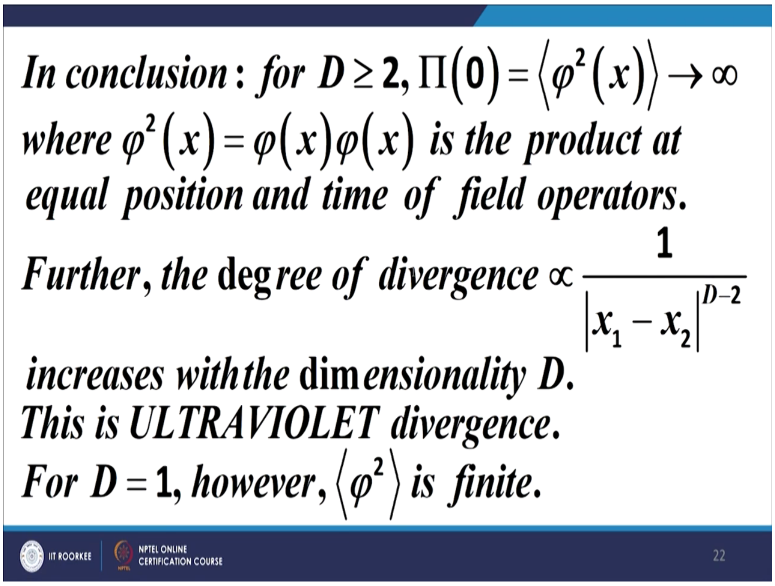

So, for to conclude here for D greater than equal to 2, pi 0 is divergent it tends to infinity, where pi 0 is the expected value of phi square x. At the same space time point, the expected value of phi square x is phi x into phi x both of them being at the same space time point the product of equal position and equal time of field operators. And this product or this expected value tends to diverge for dimension greater than equal to 2. Their degree of divergences increases with the dimensionality. This is ultraviolet dimension, ultraviolet divergence and for D equal to 2 we have infra red divergences, but the divergence nonetheless manifest themselves.

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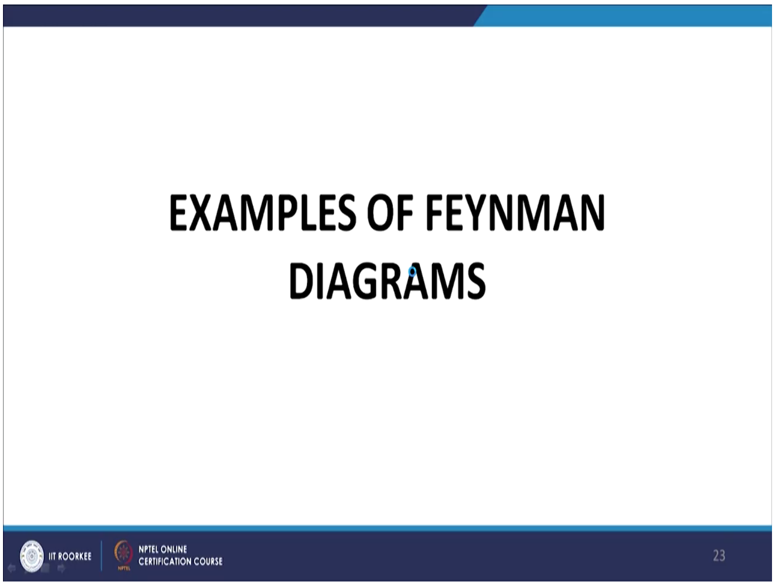

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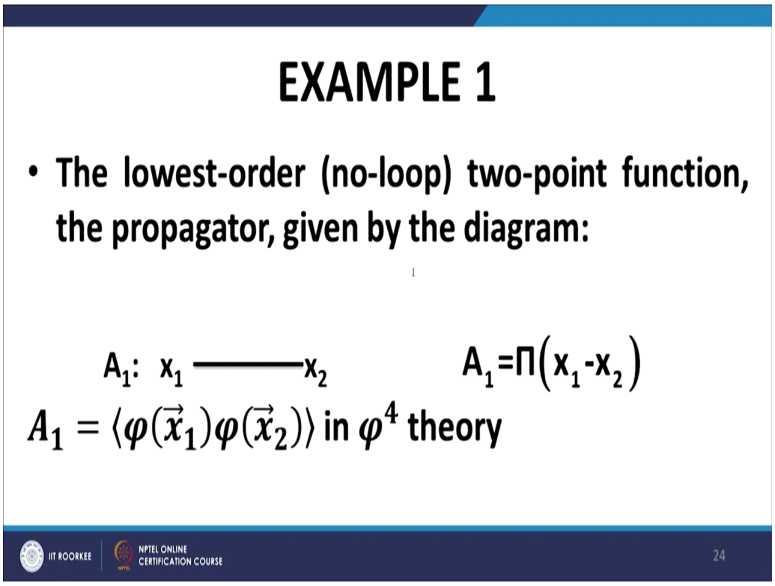

Now, we look at certain examples of Feynman diagrams in the current theory. The first one is quite simple the lowest order no loop two-point function that is going by the expected value of phi x 1, phi x 2, we call it A 1 and that is just a straight line joining x 1 and x 2 as a Feynman diagram. And that represents the propagator pi x 1 minus x 2.

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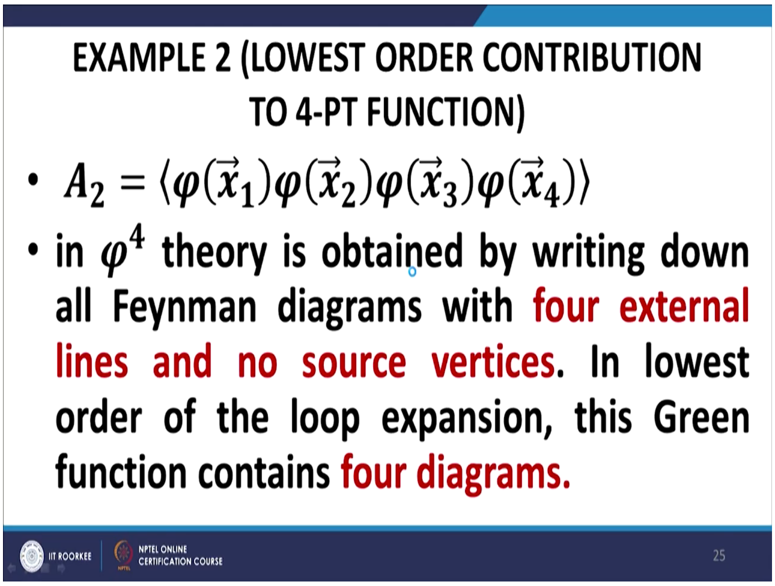

Then we look at the lowest order contribution in to a 4 point function. A 4 point function is given by the expected value of phi x 1, phi x 2, phi x 3, phi x 4. And this is obtained by writing down all Feynman diagrams with 4 external lines and no source vertices; 4 external lines and no source vertices.

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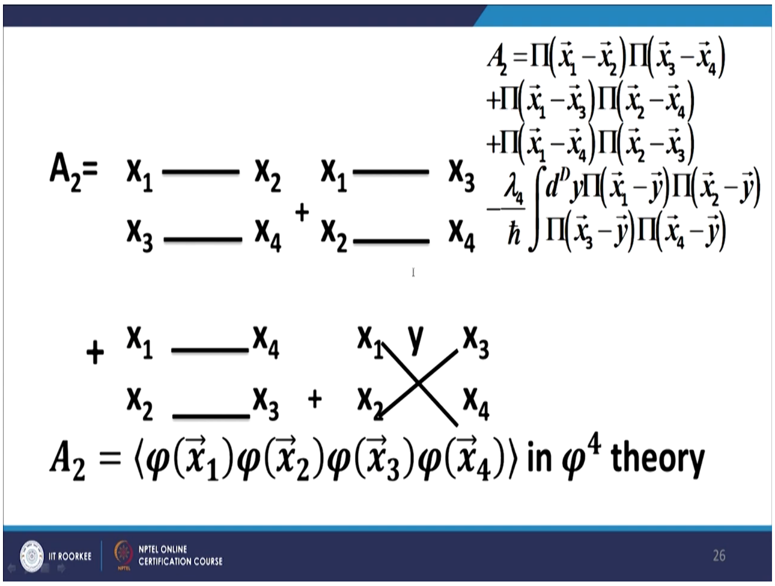

There will be for such diagrams as you can see on this slide, the first diagram, second diagram, third diagram and then we have the fourth diagram; first diagram, second diagram, third diagram and the fourth diagram.

The main diagram in the diagrams 1, 2, and 3 simply evaluate to products of the propagators. For example, the first diagram evaluates to the product of the propagators between x 1 and x 2 and propagator between x 3 and x 4. Similarly, the second diagram evaluates to the product of the propagators between x 1 and x 3 and the propagator between x 2 and x 4 and the third diagram as well.

However, in the case of the fourth diagram because we have a vertex here we need to the and the factor of a lambda 4 comes into play and this vertex evaluates to minus lambda 4 upon h. And the propagators evaluate as usual corresponding to the 4 lines you have 4 propagators.

But then, another important thing is that because y we are now talking about space time points in D dimensions we need to integrate overall value all possible values of y and that is why the y integral appears here.

So, in the fourth diagram there will be a y integral here because of the existence of the vertex at the point y, and secondly, the vertex also evaluates to minus lambda 4 upon h bar.

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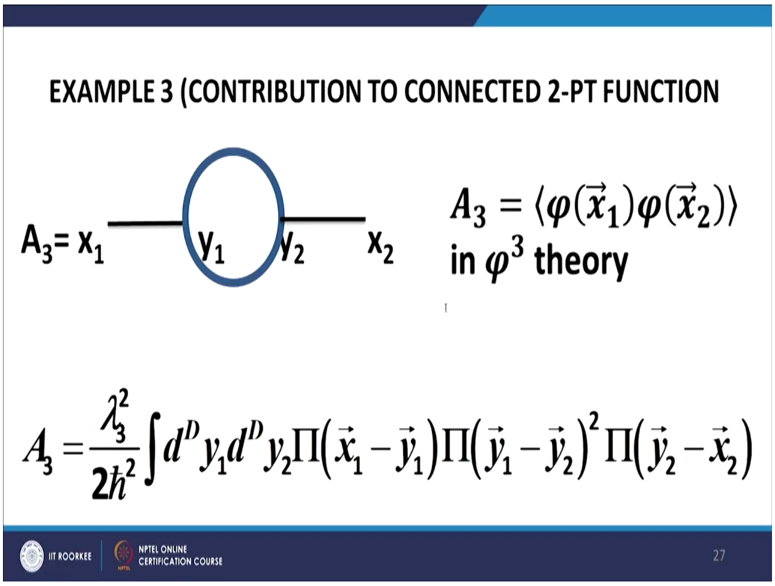

Now, we have another interesting open Feynman diagram this is a diagram in the phi 3 theory. It has two vertices as you can see, vertices at the point y 1 and vertices at the point y 2.

There are 3 point vertices, two 3 point vertices which combined together to form a loop and therefore, each of them evaluates two lambda 3 upon h bar and therefore, we have lambda 3 upon h bar square. Of course, 1 by 2 is the symmetry factor which allows for the flipping of the leaf. And we also have integration over y 1 and y 2 both, because these are two vertices in D-dimensional space time and therefore, they need to be integrated over.

And a recall, and also note that because there are two lines joining y 1 and y 2 we have the square of the propagator between y 1 and y 2 within the integrant.

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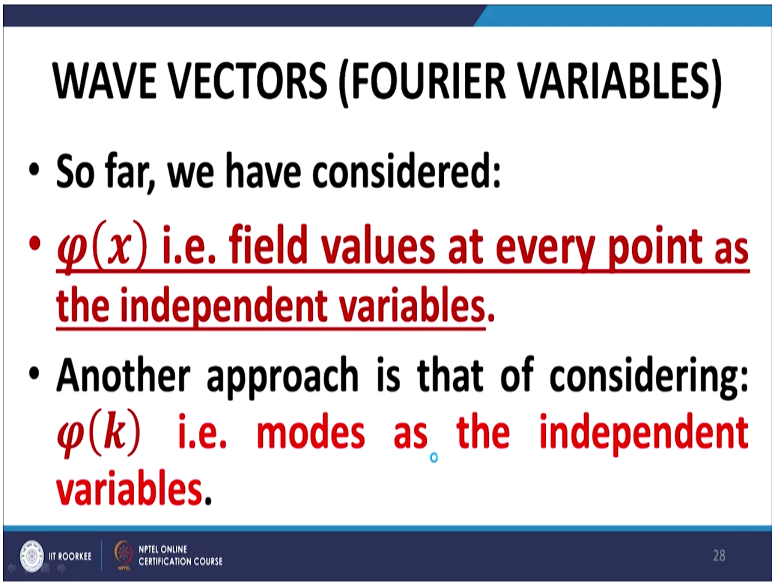

Now, we introduce a wave vectors or Fourier variables. So far you see we have considered phi x; that is wave with the independent variable having field values at every point of space time. We have considered the field values at every point of space time as the independent variables.

Now, we consider phi k as the independent variable that is the modes as the independent variable, modes, also they are also sometimes called wave vectors they are intimately related to momentum. But they are basically they are the Fourier transforms of the of the field functions in space time or and therefore, they are given by the expression given in the Green box.

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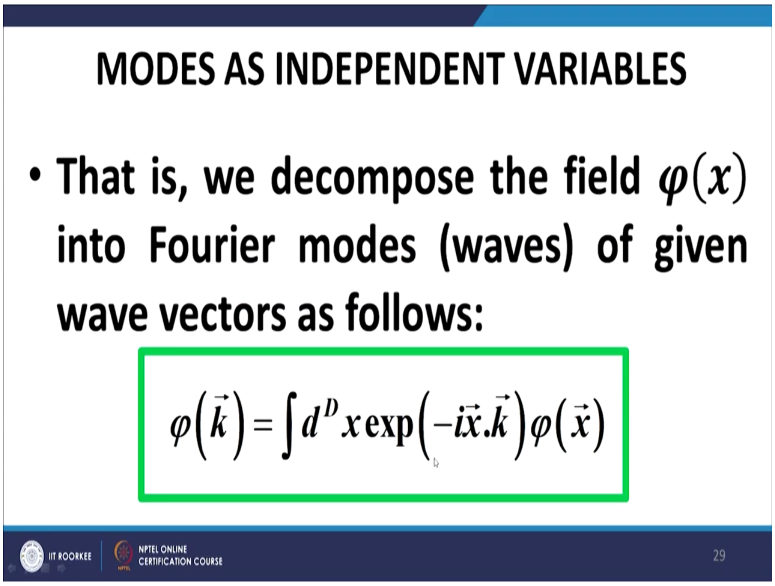

In other words, what we are simply doing is we are decomposing the field function phi x in terms of its Fourier components or Fourier modes.

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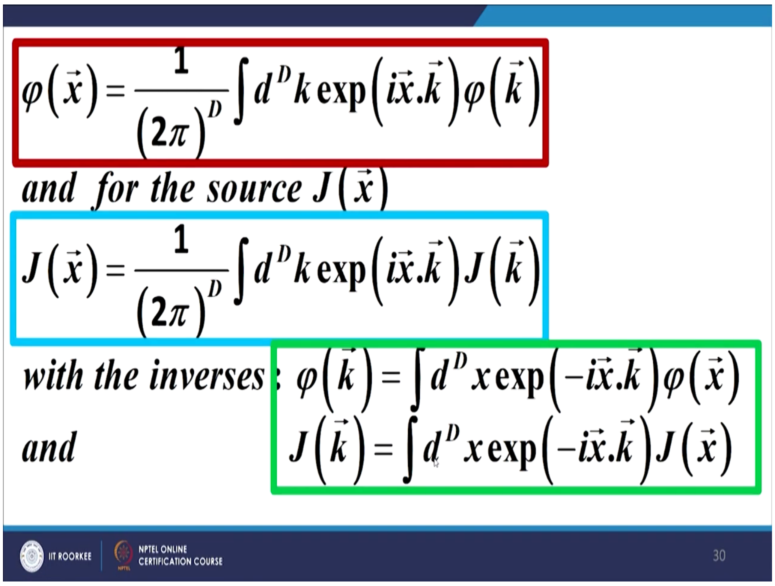

So, phi x is represented by Fourier transform and the inverses given by the expression in the Green box. Similarly, we do the same, we decompose the source J x also in terms of its Fourier modes as the expression given in the blue box with the inverse which is also given at the bottom or the second equation in the Green box.

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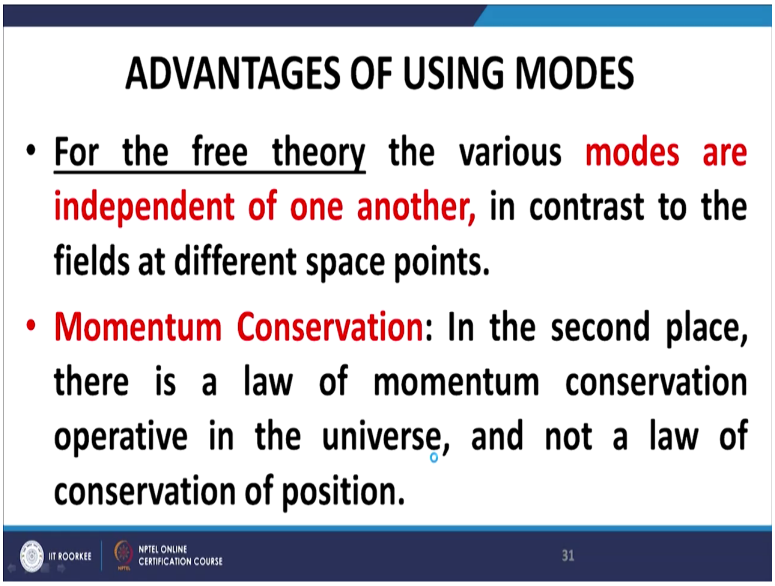

Now, there are certain advantages of using modes over the usual space time prescription. For the free theory the various modes are independent of one another, in contrast to the fields at different space time points. The fields at different space time points even in the case of free theory are not independent of each other and they are correlated. However, for the free theory the modes at different the various modes of the of the field function are independent of each other.

Then, momentum conservation is are universal law and the momentum conservation law can be directly applied in momentum space or in Fourier space.

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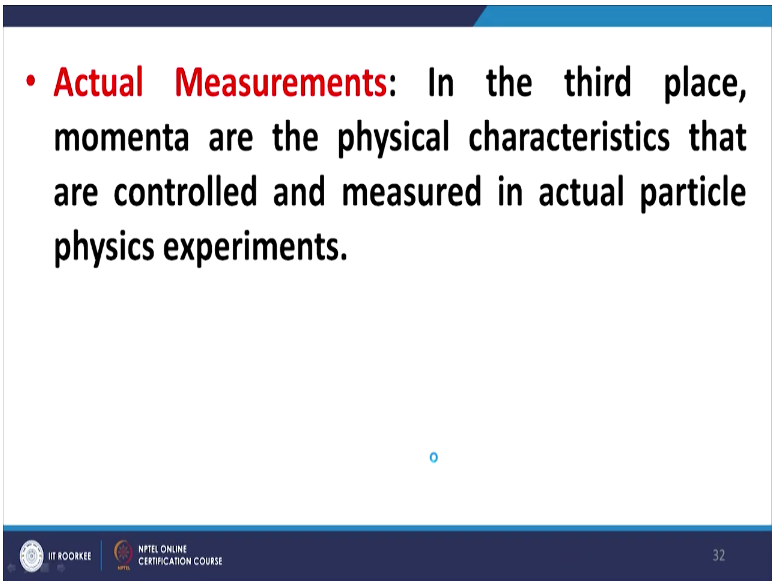

Actual measurements in experiments are usually return the momenta of interacting particles rather than this the space time points at which the interaction takes place.

So, these are 3 fundamental advantages which accrue on using the and the modes or the Fourier variables instead of the space time variables for depicting for representing the fields.

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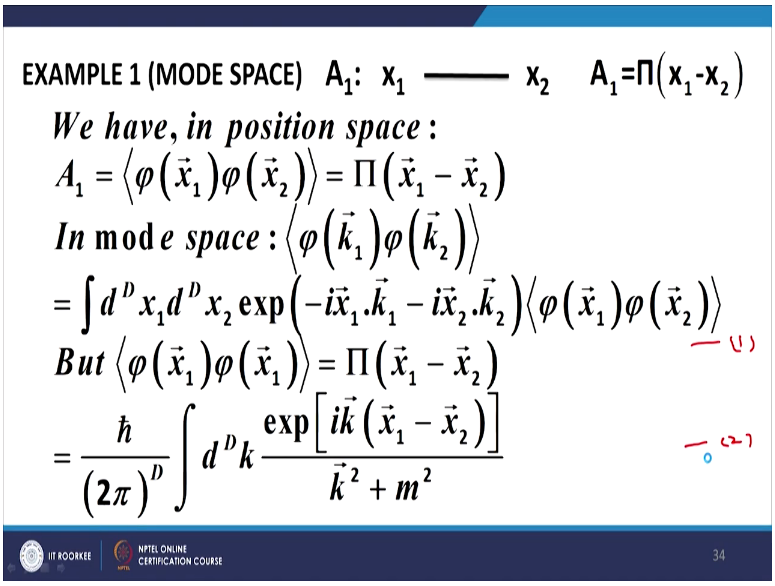

Some examples of mode space computation computations of Fourier space computations. In the position space we have for example, the two-point function is given by the expected value of phi x 1, phi x 2 this is in the position space, and this is the expression for the propagator also. In the mode space or in momentum space we will have phi k 1, phi k 2 and that will be by using the relationship between k 1 and x 1 and k 2 and x 2, the Fourier transforms of each other we get integral d over x 1, integral over x 2 exponential minus ix 1 k 1 minus ix 2 k 2, phi x 1, phi x 2. We have simply transformed x 1 and x 2 from into their Fourier transformed variables k 1 and k 2.

Now, phi x 1, phi x 2 as I mentioned earlier is nothing, but the propagator and the expression for the propagator is given at the, right at the bottom equation of your slide. So, phi x 1, phi x

2 is the expression for the propagator which is here. And we substitute this expression for the propagator phi x 1, phi x 2 into the equation let us call it equation number 1.

In other words, what we are doing is, we are substituting the expression for phi x 1, phi x 2 from equation 2 in equation 1. And what do we get? This is what we have it from, when we do the substitution what we have is we get another third integral, and this third integral comes from the expression for the propagator that is in equation 2 and the rest of it is carried as it is into this equation.

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$$\text{so that } \left\langle \varphi\left(\vec{k}_1\right)\varphi\left(\vec{k}_2\right)\right\rangle = \frac{\hbar}{(2\pi)^D}\times$$

$$\int \frac{d^D x_1 d^D x_2 d^D k \times \exp\left[\left(-i\vec{x}_1.\vec{k}_1 - i\vec{x}_2.\vec{k}_2\right) + i\vec{k}\left(\vec{x}_1 - \vec{x}_2\right)\right]}{\vec{k}^2 + m^2}$$

$$= \frac{\hbar}{(2\pi)^D}\times\int d^D k \frac{(2\pi)^{2D}\delta^{(D)}\left(\vec{k} - \vec{k}_1\right)\delta^{(D)}\left(\vec{k} + \vec{k}_2\right)}{k^2 + m^2}$$

$$= (2\pi)^D \frac{\delta^{(D)}\left(\vec{k}_1 + \vec{k}_2\right)}{\vec{k}_1^2 + m^2}$$

Now, if you look at it carefully we get certain delta functions minus ix we are looking at the exponents of at the exponential minus ix 1 k 1 you have here and ik x 1 you have here. So, this these two expressions combined together and give us a delta function over k minus k 1.

Similarly, the second delta function is k plus k 2. So, we have two delta functions here and we have and we have an integral over k. When we do the integral over k the delta function goes away and we replaced k everywhere with k 1 when we integrate over the delta first delta function that is k minus k 1. This gives us k equal to k 1 and we replace everywhere k equal to k 1 and we get this expression.

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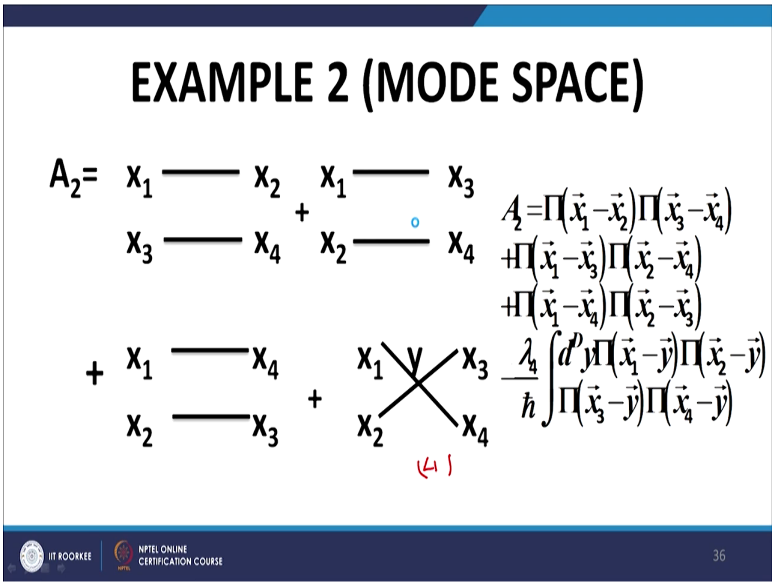

Now, we look at the second example, that is the example that is of A 2 that is the 4 point function, the lowest mode. A 2 is given by the expression that you have on the slide. We shall first analyze the last term the fourth diagram that involves a vertex, 4 point vertex. We shall first analyze this expression and then we will incorporate the rest of the evaluation which is elementary and we get full expression for the two 4 point Green function.

Let us look at the fourth diagram that is we are evaluating this diagram the forth diagram and let me mark it as 4, right. Now, we look at the 4 point function and we look at the lowest order contribution in the 4 point function.

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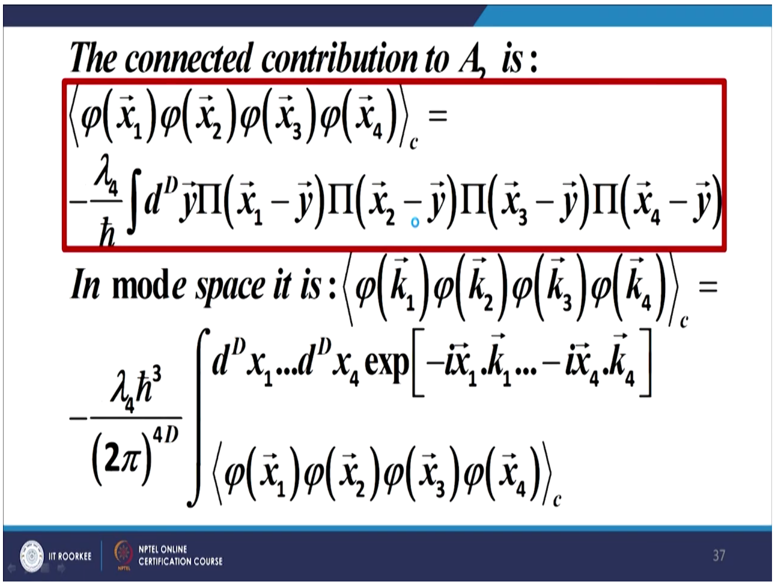

The expression in position space is given at the right hand side of your slide we look at the expression for the same thing in momentum space in Fourier space. So, this is the expression.

We first examine the fourth diagram that is diagram number 4 given here, and we look at the outcome of the fourth diagram, representation of the fourth diagram in momentum space in Fourier space. So, the 4 point function is given in terms of the expectation values in position space of phi x 1, phi x 2, phi x 3, phi x 4 which is given in the in terms of the propagator this expression is shown in the red box at the top of your slide.

Transforming it into Fourier variables we have the expression which is given at the bottom of your slide, the expected value phi k 1, phi k 2, phi k 3, phi k 4 connected. This is the 4 point function with representation in the Fourier space.

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$$\left\langle \varphi\left(\vec{k}_1\right)\varphi\left(\vec{k}_2\right)\varphi\left(\vec{k}_3\right)\varphi\left(\vec{k}_4\right)\right\rangle_c = -\frac{\lambda_4 \hbar^3}{(2\pi)^{4D}} \int d^D x_1 \ldots d^D x_4 \exp\left[-i\vec{x}_1.\vec{k}_1 \ldots - i\vec{x}_4.\vec{k}_4\right] \left\langle \varphi\left(\vec{x}_1\right)\varphi\left(\vec{x}_2\right)\varphi\left(\vec{x}_3\right)\varphi\left(\vec{x}_4\right)\right\rangle_c$$

$$But \left\langle \varphi\left(\vec{x}_1\right)\varphi\left(\vec{x}_2\right)\varphi\left(\vec{x}_3\right)\varphi\left(\vec{x}_4\right)\right\rangle_c = -\frac{\lambda_4}{\hbar}\int d^D \vec{y}\, \Pi\left(\vec{x}_1-\vec{y}\right)\Pi\left(\vec{x}_2-\vec{y}\right)\Pi\left(\vec{x}_3-\vec{y}\right)\Pi\left(\vec{x}_4-\vec{y}\right)$$

$$\left\langle \varphi\left(\vec{k}_1\right)\varphi\left(\vec{k}_2\right)\varphi\left(\vec{k}_3\right)\varphi\left(\vec{k}_4\right)\right\rangle_c = -\frac{\lambda_4 \hbar^3}{(2\pi)^{4D}} \int d^D x_1 \ldots d^D x_4 d^D \vec{y} \exp\left[-i\vec{x}_1.\vec{k}_1 \ldots - i\vec{x}_4.\vec{k}_4\right] \Pi\left(\vec{x}_1-\vec{y}\right)\Pi\left(\vec{x}_2-\vec{y}\right)\Pi\left(\vec{x}_3-\vec{y}\right)\Pi\left(\vec{x}_4-\vec{y}\right)$$

Now, this is what we have from the previous equation. The first equation is what we have from the previous equation, but the expected value of phi x 1, phi x 2, phi x 3, phi x 4 in terms of propagator we have already seen, in the previous slide it is there. Here it is the equation in the red box. We use this expression and we write the 4 point function in momentum space in terms of the propagators of a position space.

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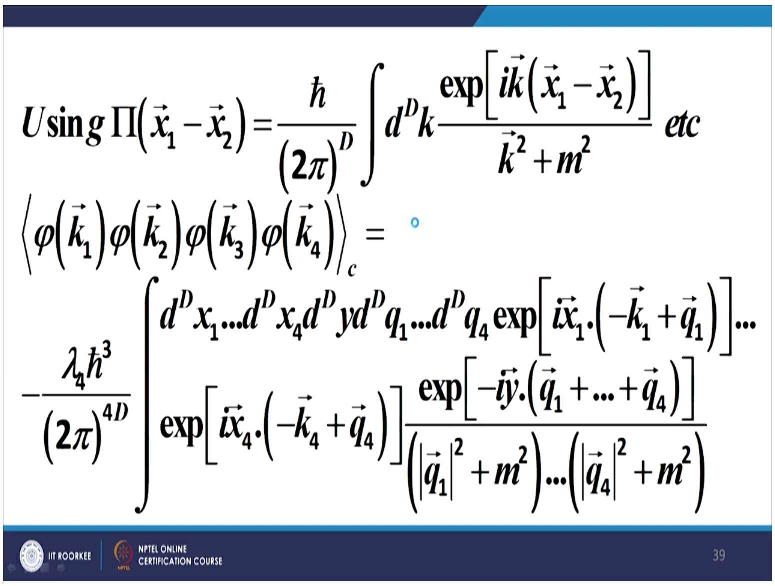

Thereafter, we use these explicit expressions for the propagators we substitute each of these propagators with the explicit expression given at the top of the top most equation on this slide. And we get the expression in the second equation on the slide.

This equation obviously, can be transformed to using the methodology in the earlier case. We have a several number of delta functions emerging from the coefficients of the exponentials. And when we integrate over these delta functions, we in other words we do the d q 1, d q 2, d q 3 and d q 4 integrations the expression that we end up with is the expression given in the Green box on the slide, right.

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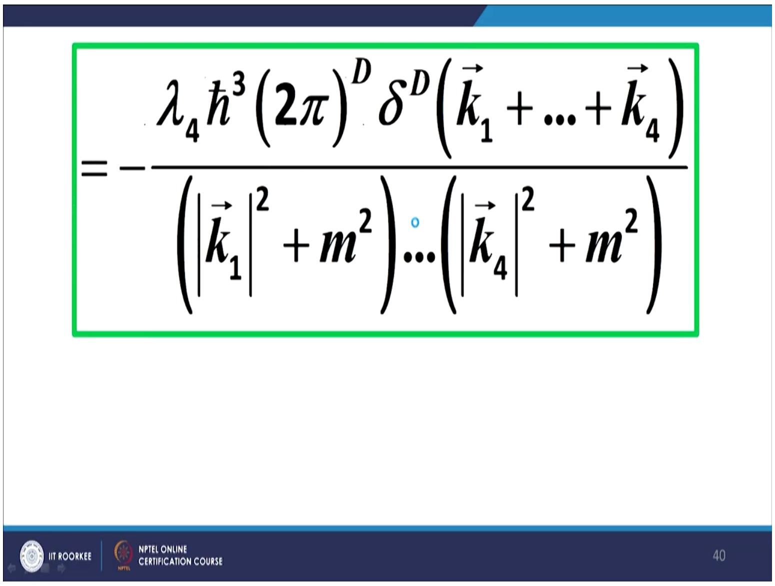

We will continue from here after the break.

Thank you.