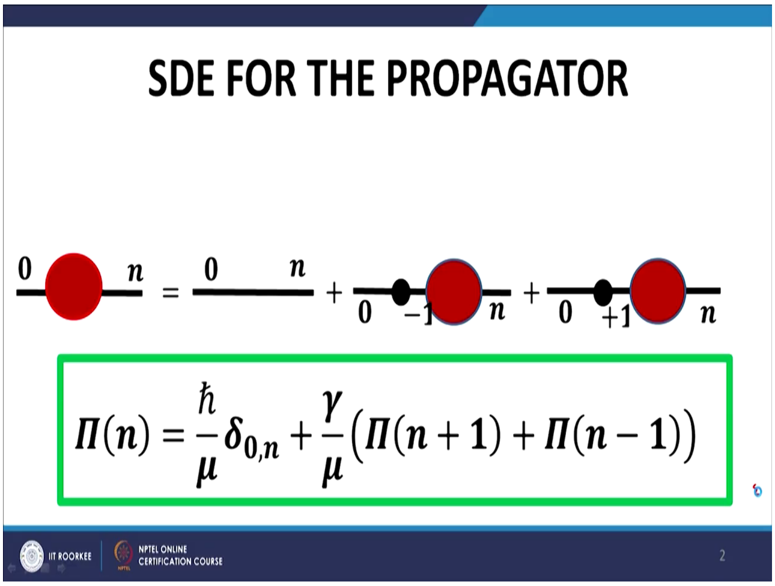
Path Integral Methods in Physics & Finance
Prof. J. P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee

Lecture – 40
Field Theory in 1-D (2)

(Refer Slide Time: 00:33)



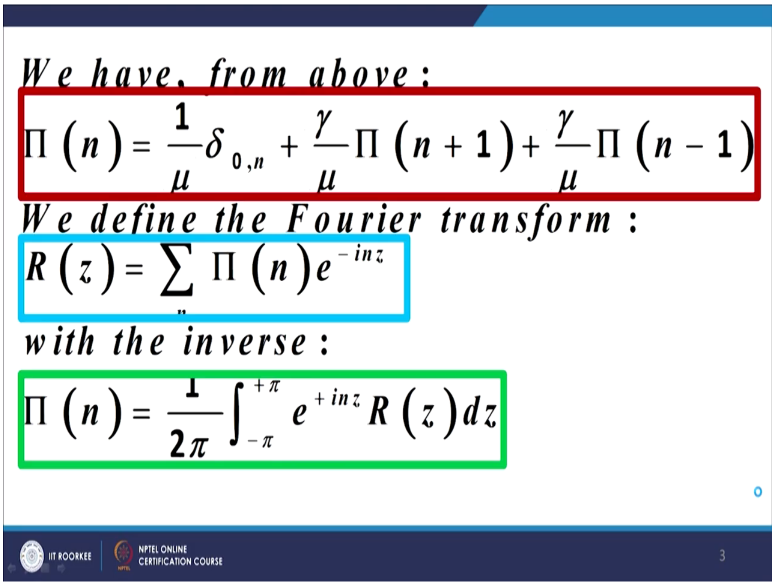
Welcome back. So, before the break I were discussing this equation which is there on the in the green box on your slide. This represents the recursive relation for the propagator and we shall now proceed to solve this equation to arrive at an explicit expression for the propagator.

So, let us do that. Just to recap the first diagram represents a field phi 0 entering the system and a field phi n emerging from the system. This can happen in the first way when it if it does not encounter any vertices then without interaction and phi 0 leaves at phi 0 this will happen even phi n when n is nothing, but 0. So, that is represented by the first term.

In the second case, we the field encounters an interaction with the neighbour's and it gets converted to phi minus 1 and, which then gets converted by a propagated to phi n which is represented by the first term pi n plus 1 because we are moving from minus 1 to n. So, it will be n plus 1 and in the last case it will be n pi n minus 1. Furthermore, the blob evaluates to gamma the black blob or dot evaluates to gamma.

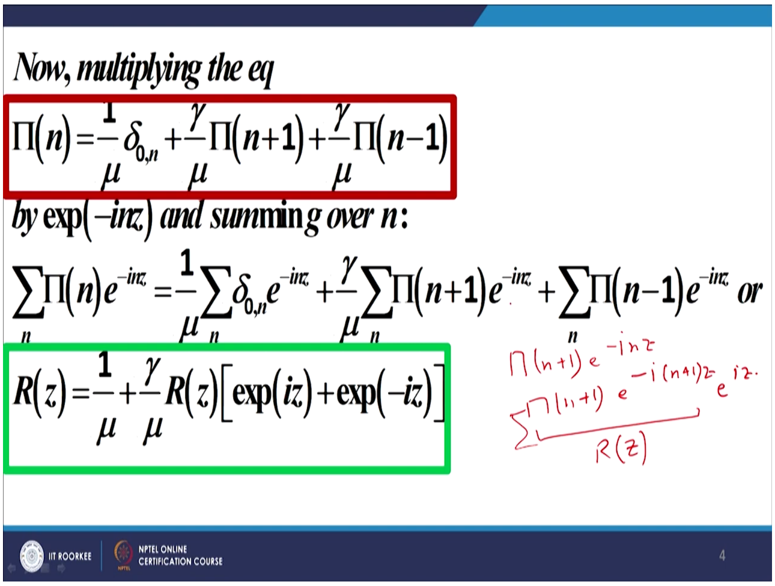
The entering line evaluates to 1 upon mu and the red blob with the outgoing line evaluates to pi n plus 1 being the incoming line being minus 1 and the outgoing line being plus 1. So, that is how the second term is evaluated. The third term is evaluated similarly.

(Refer Slide Time: 02:14)



So, we have this recursive relation with us at the beginning at the red box at the top of your slide. We write down its Fourier transform and discrete Fourier transform which is given in the blue box. Introduce the Fourier variable z and we write down it Fourier transform R z a summation pi n exponential minus inz and the inverse is obtained. Inverse gives us the propagator pi n is equal to 1 upon 2 pi integral minus is pi to plus pi minus pi to plus pi e to the power plus inz R z d z.

(Refer Slide Time: 02:57)



We multiply this recurs this equation in the red box which we have with us from the Feynman diagram. We multiply it by e to the power minus inz and sum over all values of n. When we sum over all values of n, what we get is on the right hand side 1 upon mu summation delta 0 and e to the power minus inz. This will pick out n equal to 0 and which will give us 1 upon mu which is the first term in the bottom red bottom green box of your slide for R z.

The second term when you evaluate that it is nothing, but if you look at it carefully it is exponential iz into the propagator pi n plus 1 e minus i n plus 1 into z which is nothing, but R z again. Similar manipulation happens here. Let me do one here. We have what do we have? We have pi n plus 1 e minus inz.

I can write this as pi n plus 1 e minus i n plus 1 z e iz and this expression the first expression is nothing but R when you sum it over because you are summing over n from n you summing over n and up to infinity. So, whether you use n or n plus 1 makes no difference. So, this will become R z and this will become e to the power iz. So, this is R z e to the power iz. Similarly, the last term is manoeuvred.

(Refer Slide Time: 04:57)

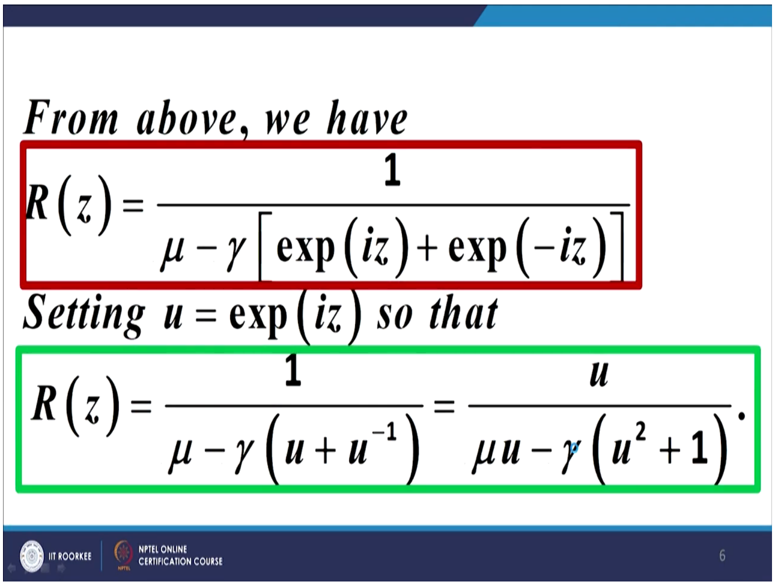
$$F/A: \boxed{R(z)=\frac{1}{\mu}+\frac{\gamma}{\mu}R(z)\left[\exp(iz)+\exp(-iz)\right]}\ or$$

$$R(z)\left\{1-\frac{\gamma}{\mu}\left[\exp(iz)+\exp(-iz)\right]\right\}=\frac{1}{\mu}\ or$$

$$\boxed{R(z)=\frac{1}{\mu-\gamma\left[\exp(iz)+\exp(-iz)\right]}}$$

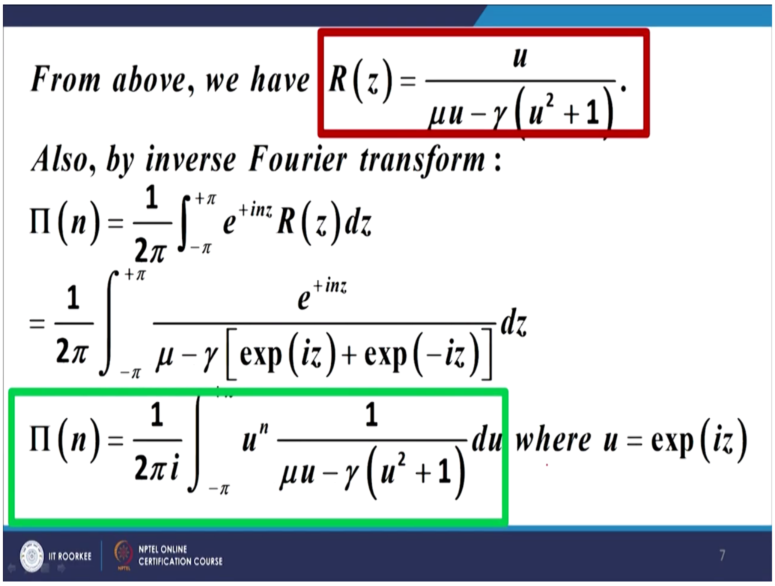
Now, if you simplify this expression we get the expression for R z as the expression given at the bottom green box of your slide.

(Refer Slide Time: 05:07)



Let us substitute u equal to exponential iz. When you substitute u is equal to exponential iz; you get the expression that is again in the bottom green box of your slide simple algebraic substitution nothing else.

(Refer Slide Time: 05:21)



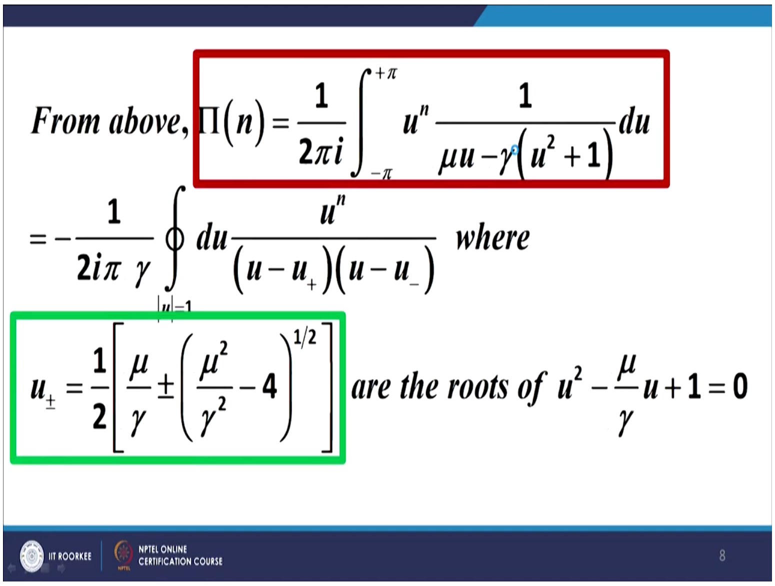
And now, this is what we have from the previous slide. This is the expression that we have in the red box has been brought forward from the previous slide and by an inverse fourier transform as we had at the beginning.

The pi n the propagator was given by 1 upon 2 pi minus pi to plus pi e to the power inz R z dz. We substitute the value of R z from the red box here and we get the expression that is here in the second last equation of your slide 1 upon 2 pi minus pi 1 1 upon 2 pi minus pi to plus pi e to the power inz divided by mu minus gamma exponential iz plus exponential minus iz.

This is by substituting the value of R z in this expression and when I substitute R z in terms of the expression that is given in your red box at the at the top of your slide. What I get is the expression at the bottom of your slide where of course, I have used as mentioned earlier I have used u is equal to exponential iz. You can see it here u is equal to exponential iz. We got this expression and using that expression again we get the expression here for the propagator.

When I substitute in the inverse Fourier transform expression which represents the propagator as given by this expression when a substitute u is equal to exponential iz; I get this expression which is the expression in the green box at the bottom of your slide.

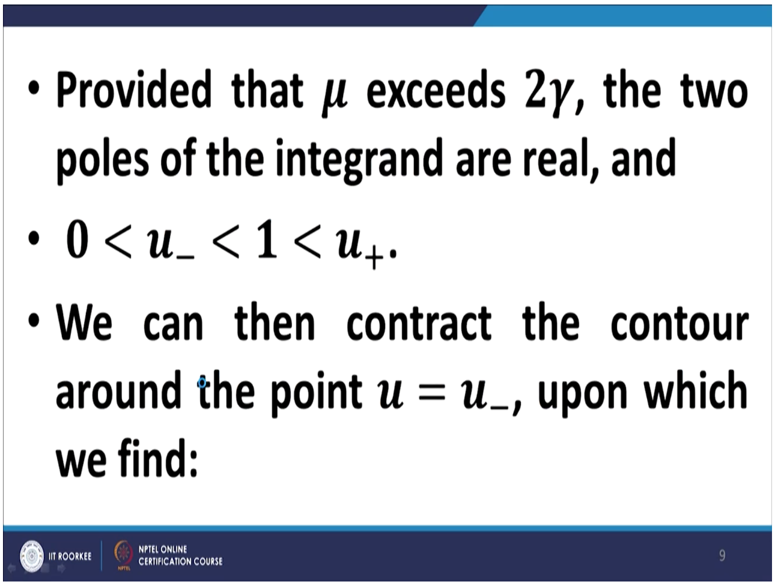
(Refer Slide Time: 07:14)



Now, this expression from the top of your slide in the red box is what we obtained from the previous slide. We factorize the new denominator. We factorize the denominator and we find that it has poles at u plus and u minus where u plus minus are given by solving the quadratic equation.

And, we find their values as the expressions given in the green box at the bottom of your slide which are the solutions of the quadratic equation given here u square minus mu upon gamma u plus 1 is equal to 0 which is the cast of the express denominator recast form.

(Refer Slide Time: 07:57)



Now, if mu exceeds 2 gamma, then the 2 poles of the integrand are real. You can see here if mu exceeds gamma then mu x is 2 gamma then this expression turns out to be real. And if this expression is real we both the poles are real and secondly 0 will be less than u minus less than 1 less than u plus. We can contract our contour around u minus.

(Refer Slide Time: 08:27)

From above,

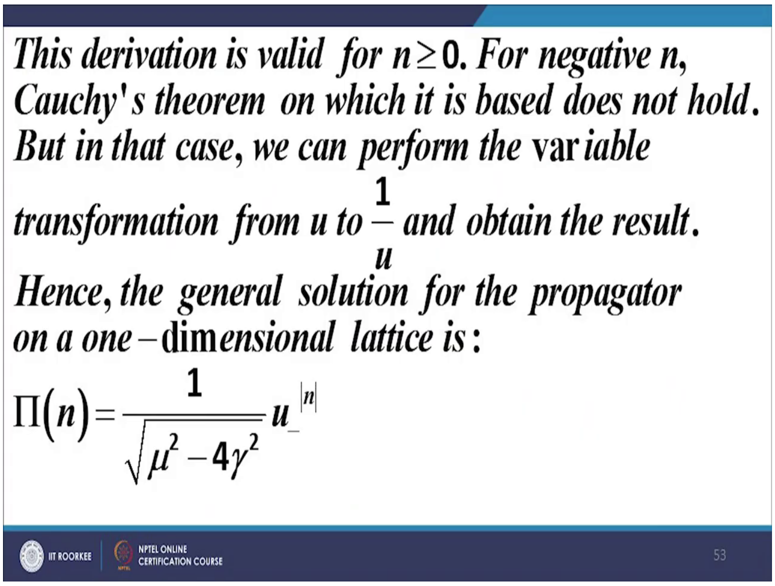
$$\boxed{\Pi(n) = -\frac{1}{2i\pi\,\gamma}\oint_{|u|=1} du\,\frac{u^n}{(u-u_+)(u-u_-)}}$$

$$or\ \ \Pi(n) = \frac{u_-^n}{\gamma\,(u_+ - u_-)},\ \ n \geq 0$$

$$\boxed{\Pi(n) = \frac{u_-^n}{\gamma\sqrt{\left(\frac{\mu^2}{\gamma^2} - 4\right)}} = \frac{u_-^n}{\sqrt{(\mu^2 - 4\gamma^2)}}}\ \ n \geq 0$$

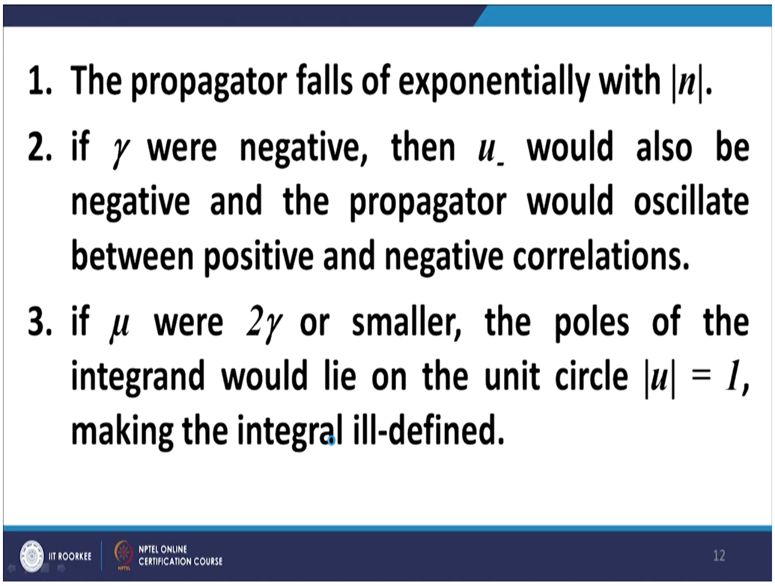
And on doing that we can the expression for the integral the contour integral can be obtained as the expression here in the bottom green box of your slide which simplifies to this u minus to the power n upon under root mu square minus 4 gamma square n greater than equal to 0. So, this is what we have for the discrete case for the one dimensional discrete case for the propagator.

(Refer Slide Time: 09:03)

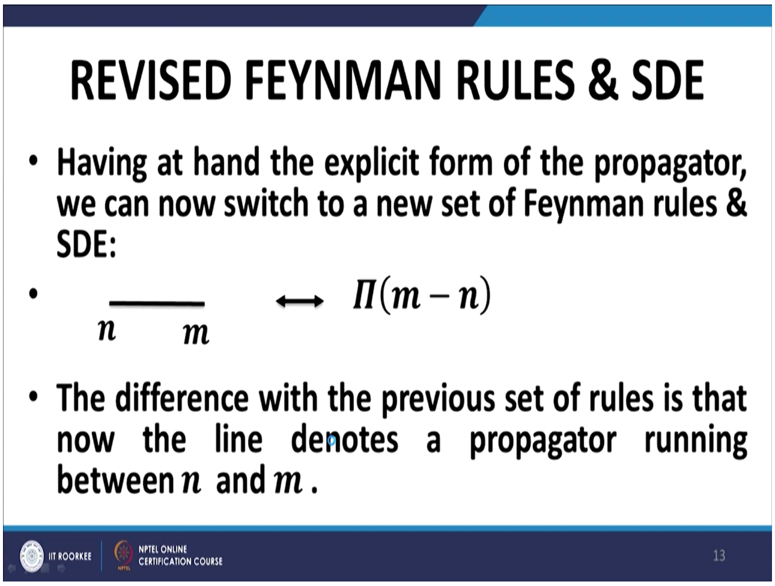


This is the expression for the propagator in the case of a one dimensional discrete case. It is it may be noted that this expression is true for any greater than 0. But by transforming u to 1 upon u we can also work out the expression for the say we are can also arrive at the same expression for the case when n is less than 0.

(Refer Slide Time: 09:23)

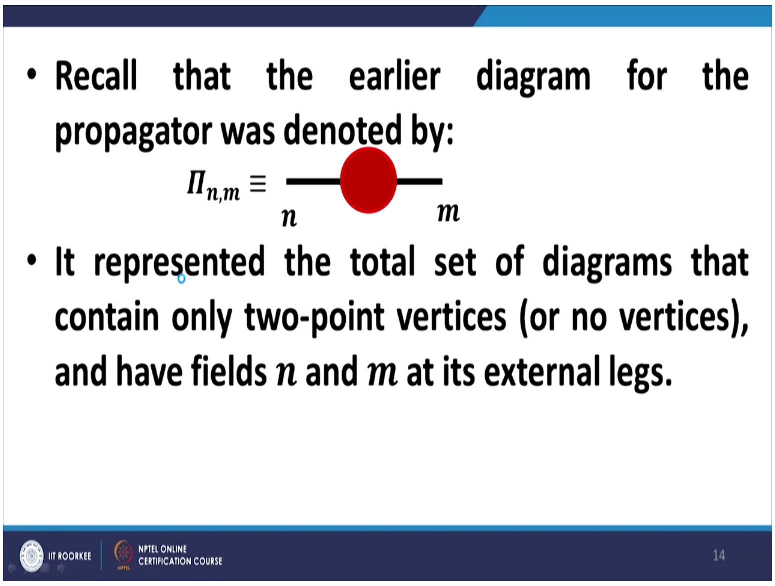


(Refer Slide Time: 09:28)



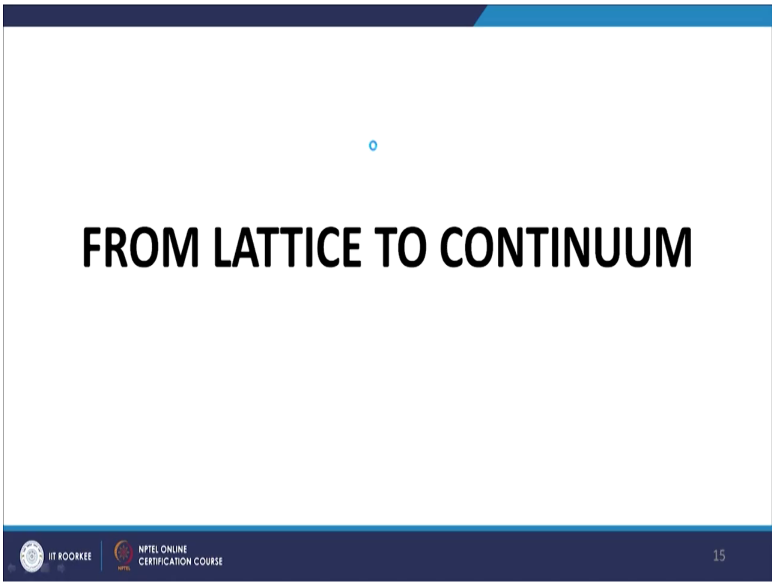
Now, we shall explore the properties of the propagator, but before we explore the properties of the propagator we go back to the Feynman rules. And because now we have the explicit form of the propagator with us we can introduce a new set of simplified Feynman rules. Instead of having blobs as the as the propagator we simply use a straight line for the propagator. A straight line between connecting states phi n and phi m represented by n and m represent the propagator pi m minus n or n minus m as I mentioned because of parity invariance it does not matter.

(Refer Slide Time: 10:06)

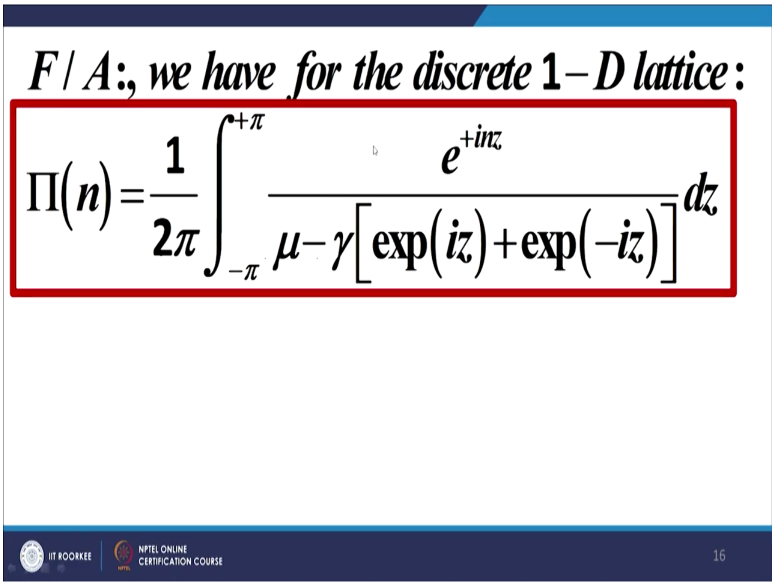


So, instead of this red blob that we had been using earlier we shall now be using the straight line for the propagator. And please note this represents the total set of diagrams that may contain 0 or 1 or more 2 point functions that that have the property that we have the incoming field as phi n and the outgoing field as phi n. These are the constraints incoming field phi n outgoing field phi m and within that we may have in this blob we may have 0 interaction vertices or 1 or more as many two point interaction vertices.

(Refer Slide Time: 10:53)

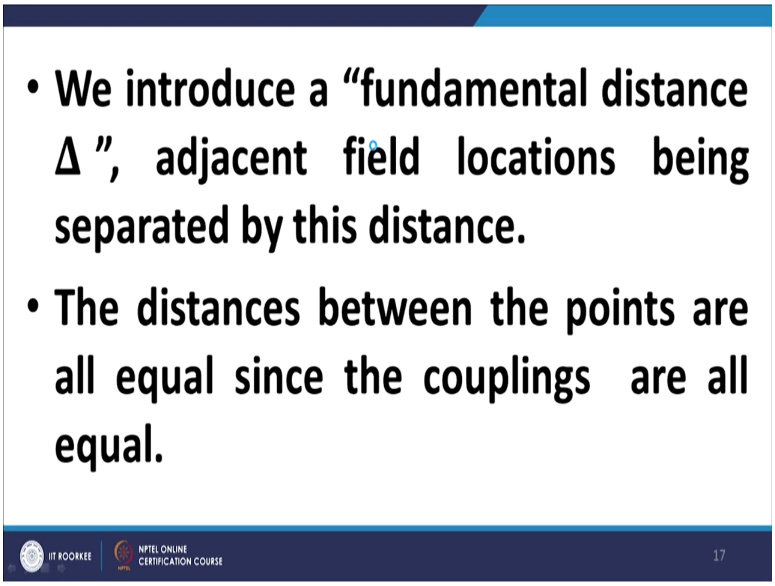


(Refer Slide Time: 10:54)



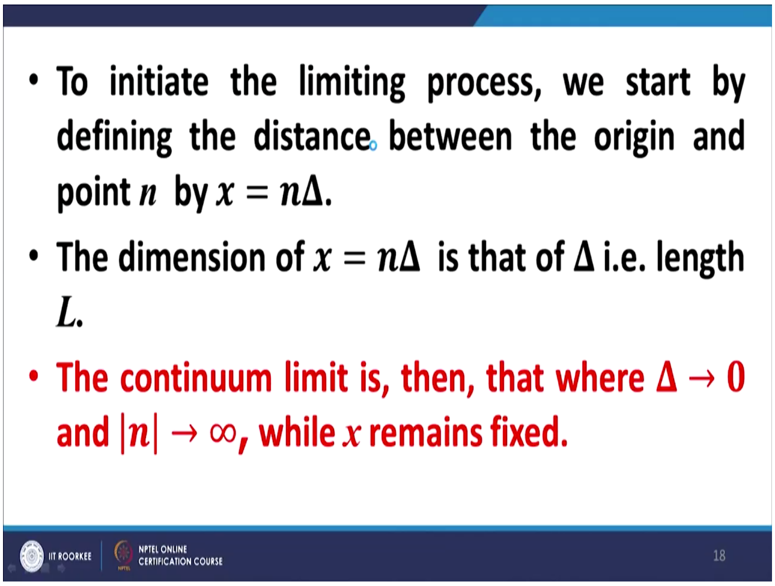
Now, we move from the lattice structure to the continuum. We have from the lattice structure we have this expression for the propagator. We shall now develop we shall now take a limit of this expression in such a form that we get a sensible result. Let us see how we go about it.

(Refer Slide Time: 11:13)



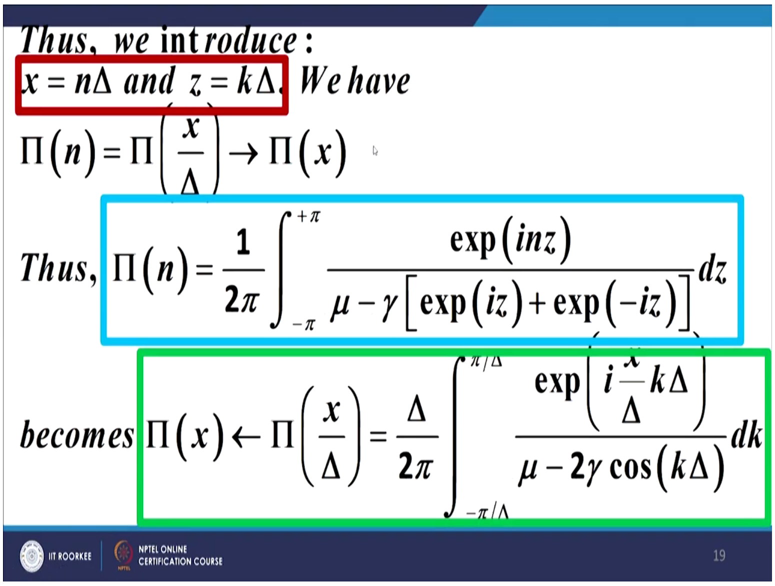
First thing as I mentioned the distance between two neighbouring lattice points we define as delta and of course, we assume the delta distance to be equal across the lattice. In other words any two points are separated by the same delta.

(Refer Slide Time: 11:36)



Now, to initiate the limiting process we define the distance between the origin and any point n by x is equal to n delta. The dimensions of x are obviously of delta which is length. Now, the continuum limit will take of this distance by taking delta tending to 0 and n tending to infinity provided x tends to remain fixed or x remains unchanged. So, x remains unchanged delta tends to 0 n tends to infinity this gives us the continuum limit.

(Refer Slide Time: 12:11)

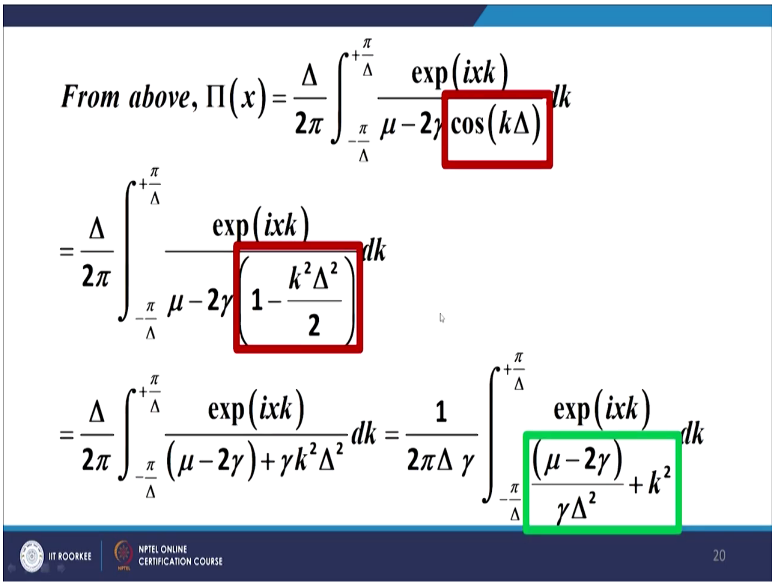


So, we write x is equal to n delta we write z is equal to k delta and for the expression of the propagator what do we have pi n and pi n now becomes pi x upon delta which we represent by pi x it is now a function of x.

So, we call it pi x. So, pi x this expression with which was therefore, pi n now becomes pi x is equal to delta upon 2 pi where that this delta come from this delta comes from this change of variable from dz to dk because z is equal to k delta. So, dz is equal to delta dk. This delta is taken outside the integral.

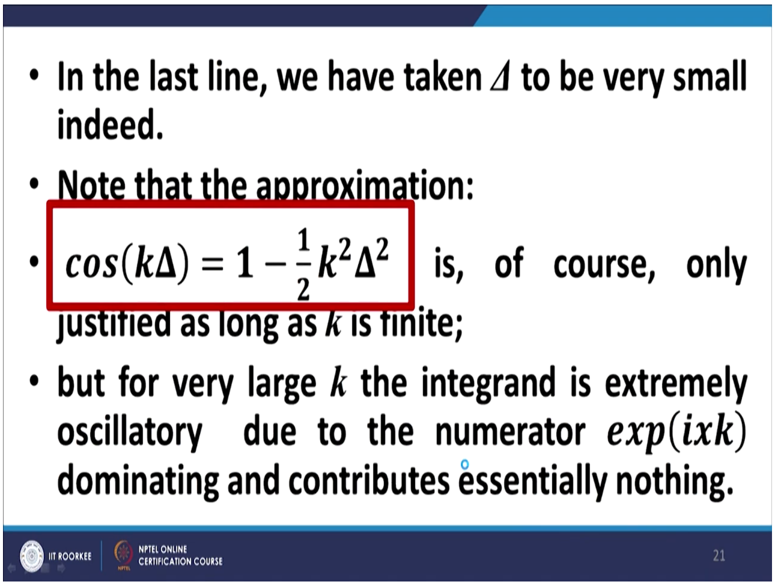
So, this is this delta i n is equal to x upon delta and z is equal to k delta. So, delta and delta cancel out we get i x k in the numerator exponential i x k in the numerator. In the denominator what do we get? Denominator it is exponential iz iz becomes i k delta and exponential minus iz that becomes exponential minus i k delta; so, that is nothing, but cos with a factor of 2 that becomes cos k delta. In other words, this e to the power i theta plus e to the power minus i theta is 2 cos theta that is what we have used here.

(Refer Slide Time: 13:46)



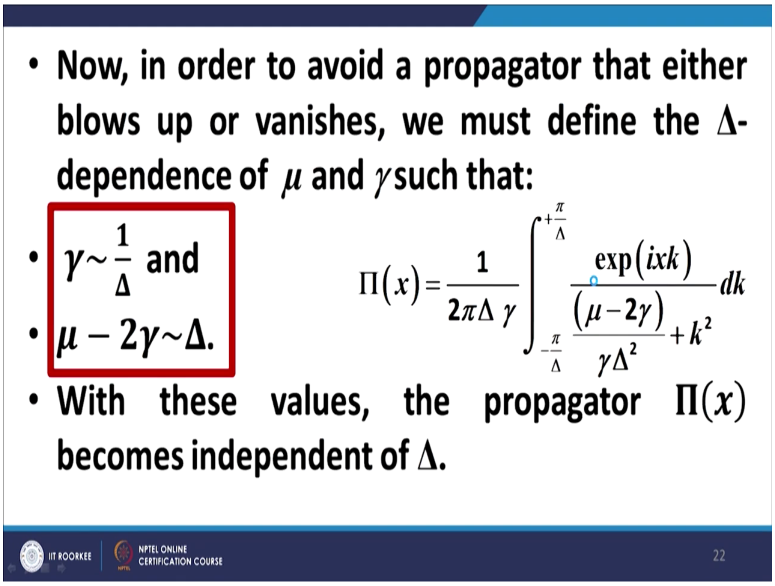
Now, we have expanded cos theta as a series in theta cos theta is equal to what 1 minus theta square upon 2 that is precisely whatever we have done expanded to the first significant term and we have the rest of it is as same. So, on simplification what we have is the expression in the right hand green box the denominator becomes the expression that we have in the right hand green boxes your slide.

(Refer Slide Time: 14:32)



Now, the important things arise. Firstly, because delta has to be very small and indeed delta is small because we are taking the limit delta tending to 0. And in fact if we have very large k if we have very large k, then the exponential i k x expression becomes extremely oscillatory and dominates the denominator and the integral becomes oscillatory and it contributes nothing.

(Refer Slide Time: 14:55)

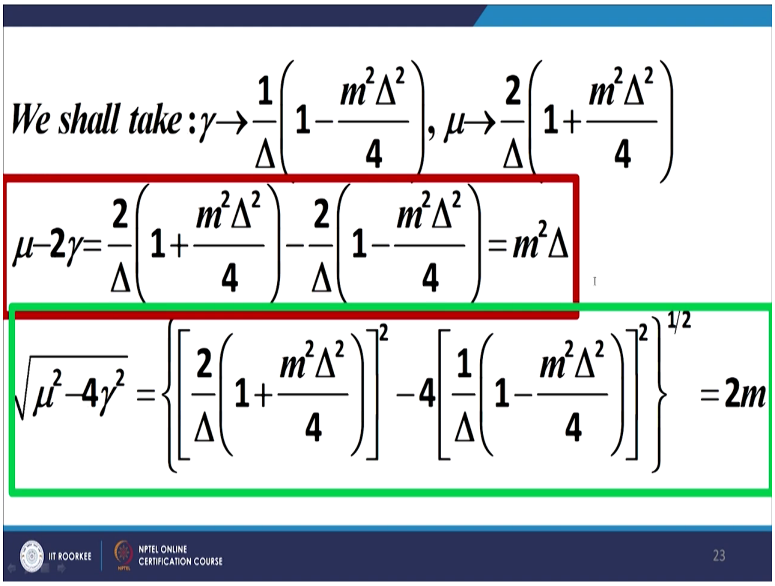


Integrand becomes extremely oscillatory in the limit that k tends to infinity k becomes very large and therefore, it will not contribute significantly to the integrand. So, we have k as finite and we have delta tending to 0 that enables us to expand cos theta as 1 minus theta square upon 2 now we want a sensible limit. So, we want a propagator which neither blows up nor vanishes.

So, let us see how we can do it the expression for the propagator is given in this expression. If we have delta is of the order of 1 upon gamma or either way around gamma is of the order of 1 upon delta then this expression cancels out literally the prefactor to the integral gives me 1 upon 2 pi.

So, that is one part. Then what do we have here? Gamma delta squared. Gamma delta square you have gamma delta is almost 1. So, we have a delta here now in order that this expression does not blow up gamma mu minus 2 gamma must also be of the order of delta. So, these are two important inferences that we draw if you want to have a sensible limit.

(Refer Slide Time: 16:18)



So, what we do is we take gamma as 1 upon delta into 1 minus m square delta square upon 4. And we take mu as 2 upon delta upon 1 plus m square delta square upon 4 mu minus 2 gamma give becomes m square delta which is of the order of delta as we as we wanted mu minus 2 delta is of the order of delta which is what we are getting here. And gamma is of the order of 1 upon delta which is very clear what do we get for the propagator denominator under root mu square minus 4 gamma square turns out to be 2m.

(Refer Slide Time: 17:03)

The propagator is $\Pi(x) = \dfrac{\Delta}{2\pi}\displaystyle\int_{-\frac{\pi}{\Delta}}^{+\frac{\pi}{\Delta}} \dfrac{\exp(ixk)}{(\mu - 2\gamma) + \gamma k^2 \Delta^2} dk$

$$= \frac{\Delta}{2\pi}\int \frac{\exp(ixk)}{\left(m^2\Delta\right) + \frac{1}{\Delta}\left(1 - \frac{m^2\Delta^2}{4}\right)k^2\Delta^2} dk$$

$$= \frac{1}{2\pi}\int \frac{\exp(ixk)}{k^2 + m^2} dk$$

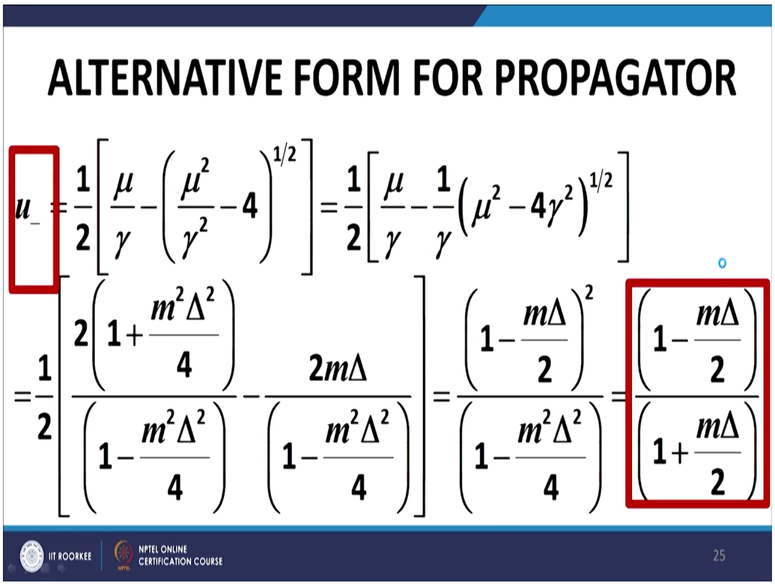
NPTEL ONLINE CERTIFICATION COURSE

IIT ROORKEE

24

So, the propagator now becomes if you have simplify everything we have the propagator as 1 upon 2 pi exponential i x k upon k square plus m square d k. So, this is the propagator in the continual one dimensional space continuous line or a continuous lattice in one dimensional we have this expression for the propagator.

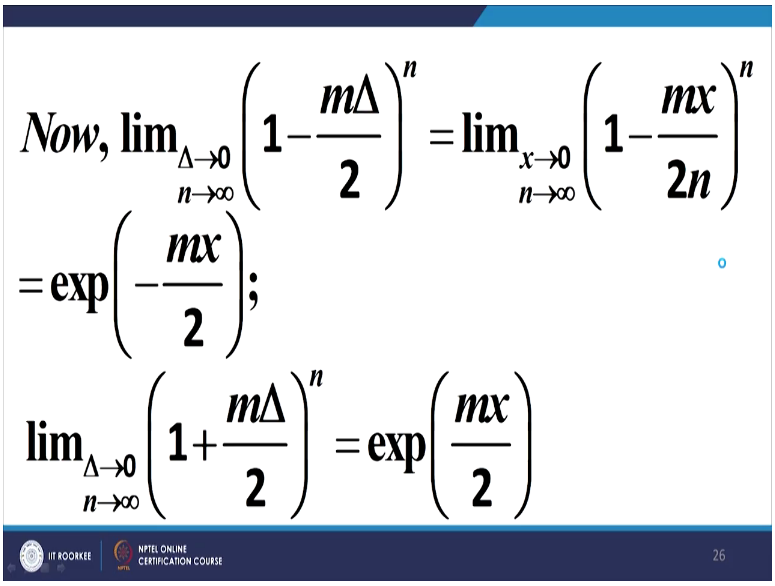
(Refer Slide Time: 17:34)



You can also arrive at another expression for the propagator um. Recall that u minus was defined as this this expression from the previous slides. It was yes it was quite a way back so. But, it was defined as per this expression when we talked about being the counter integration of the proper expression for the propagator.

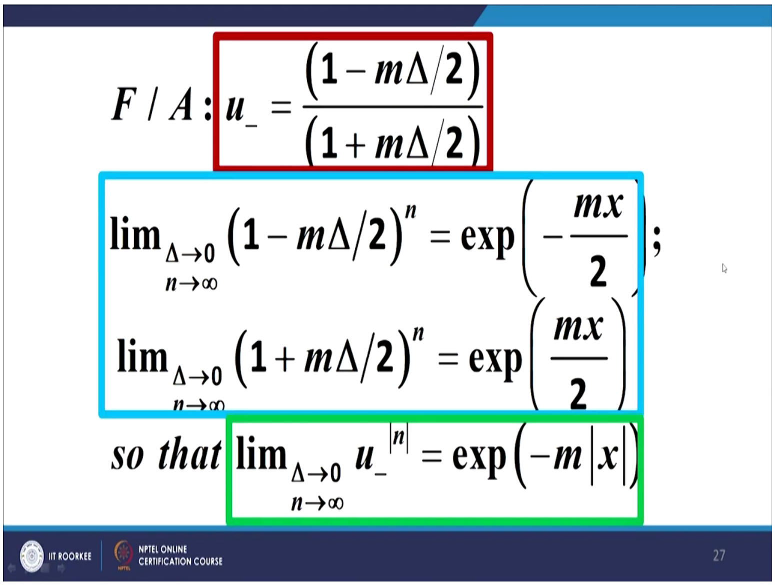
So, that from there we have u minus is equal to this. We substitute this in terms of the new parameters the revised expression and we get u minus is equal to 1 upon m delta upon 2 1 minus m delta upon 2 divided by 1 plus m delta upon 2.

(Refer Slide Time: 18:27)



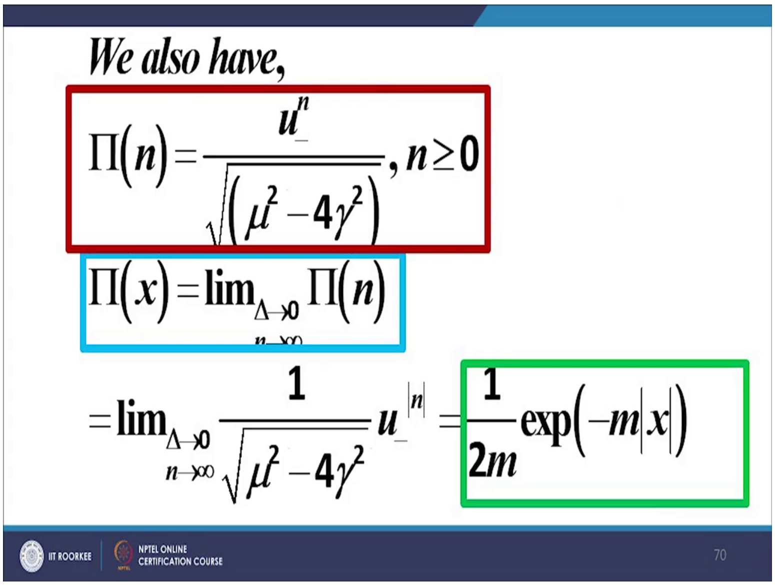
But, we want u minus to the power n. So, let us see what we get. When we take the limit delta tending to 0 n tending to infinity that is the continuum limit 1 minus m delta upon 2. The numerator here 1 minus m delta upon 2 to the power n we get e to the power exponential minus m x upon 2. Similarly, for the expression 1 plus m delta upon 2 which was the denominator here. This denominator to the power n because we are evaluating u minus to the power n we get exponential m x upon 2.

(Refer Slide Time: 19:06)



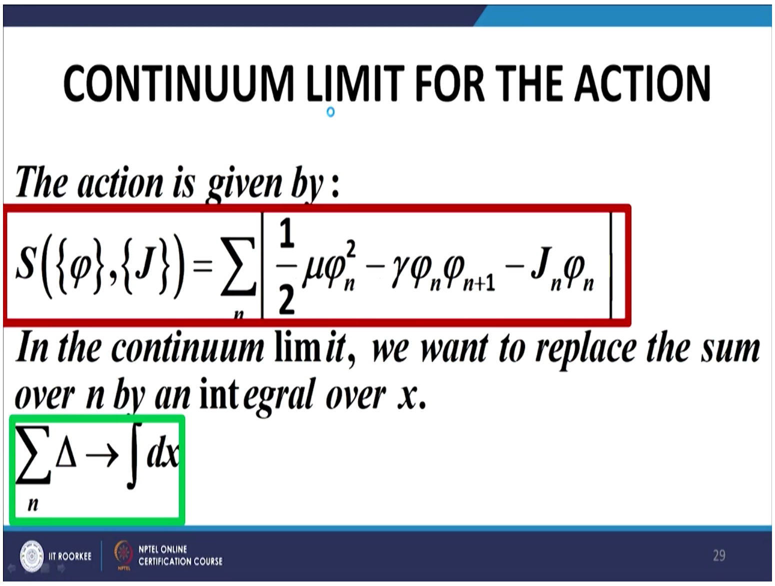
And because you have the mode of n we have this is equal to exponential minus m mod of x of course, in the continuum limit delta tending to 0 n tending to infinity.

(Refer Slide Time: 19:21)



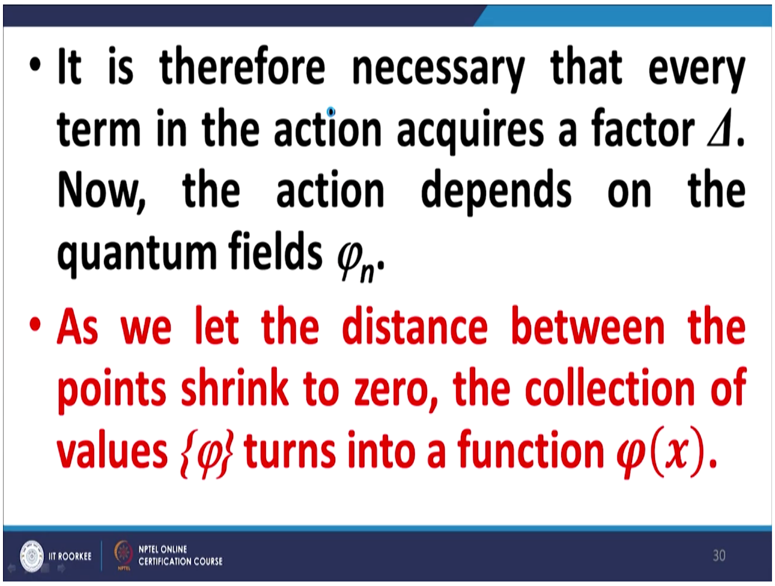
Hence, the propagator becomes pi x is equal to 1 upon 2 m exponential minus m upon mod x. This this is the denominator under root mu square minus 4 m square is nothing but 1 upon 2 m and the u minus to the power mod n is nothing but exponential minus m mod x right.

(Refer Slide Time: 19:53)

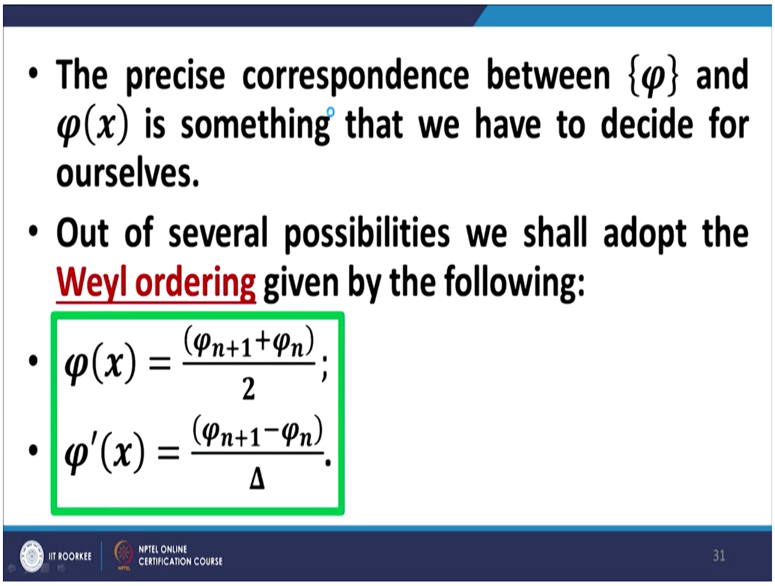


Now, we can explore continuum limit for the action is quite straightforward in fact and let us do it. The action form of the action we have if we subsume the sources within the action and the action is given by this expression in the red box. Please note we have included the sources within the action. In the continuum limit, what we have to do is we have to shift the summation by integrals.

(Refer Slide Time: 20:18)



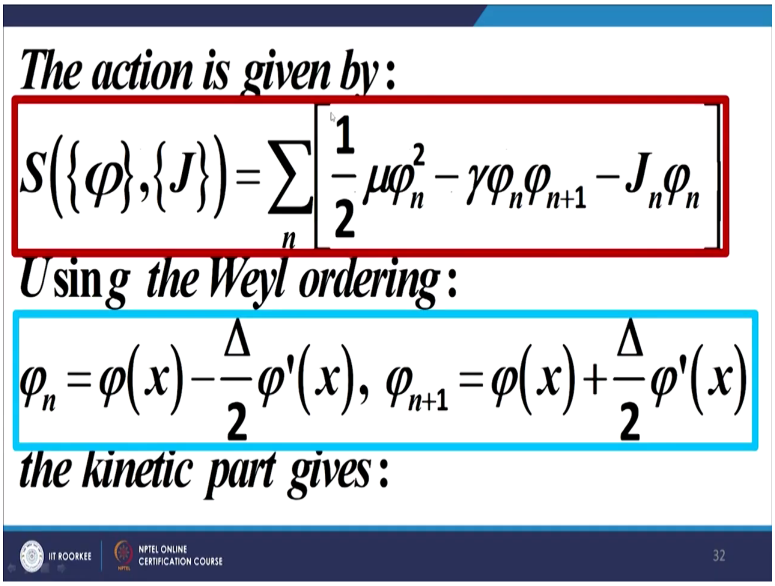
(Refer Slide Time: 20:21)



And therefore, how do we do it? We do it by using the Weyl ordering. Weyl ordering enables us to substitute continuum based variables for the field variables, for their discrete field variables using the prescription that is given in the green box particularly the first one.

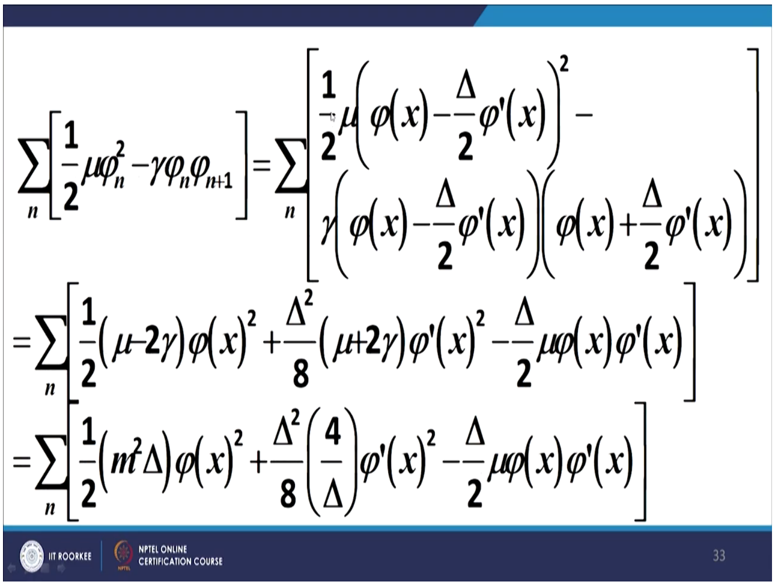
Phi x is equal to phi n plus 1 plus phi n upon 2 in other words in a sense the field variable in the continuum framework is located midway between the 2 discrete points. So, and the other one is of course, phi n plus 1 minus phi n upon delta. Delta is the interstitial spacing in the discrete lattice.

(Refer Slide Time: 21:04)



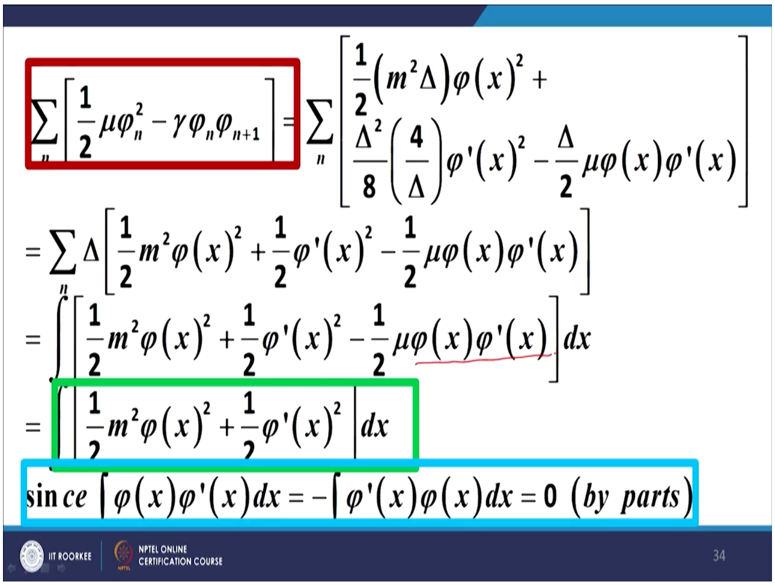
So, the action is given by when we put in these expressions for the phi n. And phi n plus 1 in terms of the corresponding continuous phi variables or field variables continuous field variables we have the expression which is given in the blue box.

(Refer Slide Time: 21:23)



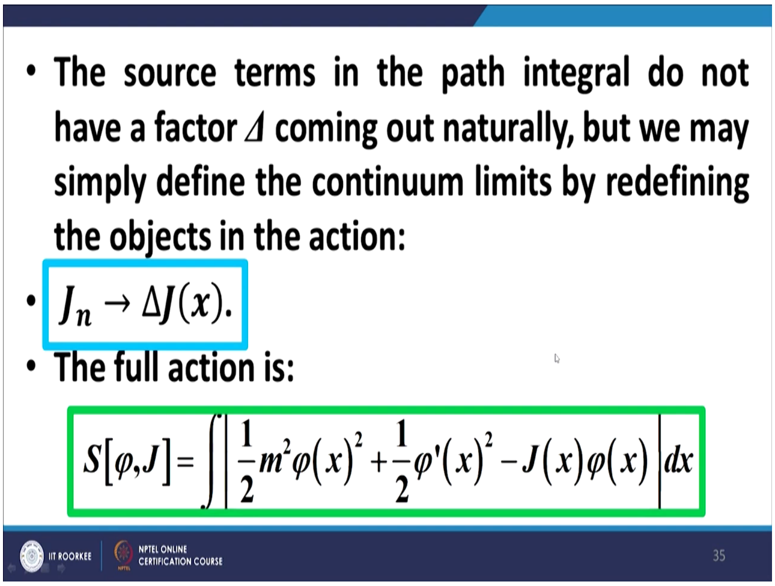
And when we substitute this into the expression for the action what we get is on simplification simple algebraic manipulations no technical literature involved really.

(Refer Slide Time: 21:28)



What we have is the action is turns out to be 1 upon 2 m square in m square phi x square plus 1 upon 2 phi dash x square d x. Of course, integrated over a x and where we have used this result which is in the blue box as far as this expression is concerned. This expression evaluates to 0 when we do an integration by parts because phi x integral of phi x phi dash x dx if you integrate by parts gives you this is equal to integral of phi dash x phi x dx minus with a minus sign that implies that the integral vanishes.

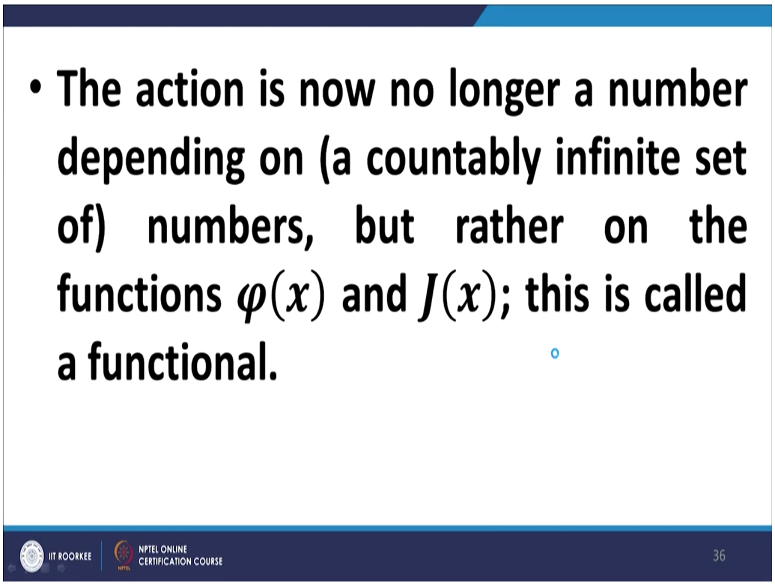
(Refer Slide Time: 22:27)



Now, the source terms also need to be redefined. We do we have worked out the action without the source terms. If you want to include the source terms in the action and have an have a continuous version of the source terms we need to redefine the source term by introducing therein a factor of delta so that it can be captured in the integral, but writing it in the expression given in the blue box.

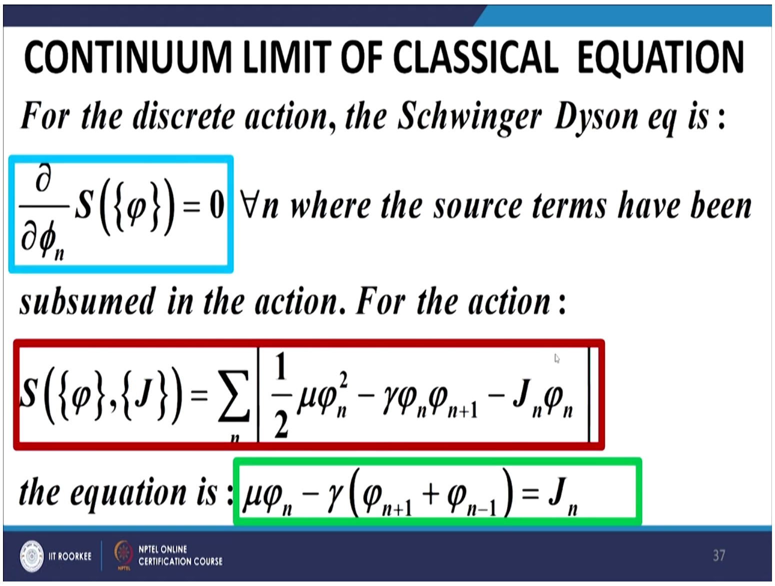
And, with that with this expression we can now incorporate the source also in the continuous action and we have the expression for the action including sources as the expression given in the green box.

(Refer Slide Time: 23:08)



The important thing is that the action is now no longer a number. It is a functional. It is a function of functions and therefore, it is a functional.

(Refer Slide Time: 23:18)



Now, as far as the continuum limit of the classical equation is concerned again it is quite straightforward. If, what is the classical equation? The classical equation is S dash of phi is equal to 0 and when we work it out in the discrete framework we get the expression that is in the green box right at the bottom of your slide. This is the expression in the case of the discrete framework discrete one dimensional lattice.

(Refer Slide Time: 23:52)

The S – D eq is: $\boxed{\mu\varphi_n - \gamma\left(\varphi_{n+1} + \varphi_{n-1}\right) = J_n}$

Making the Weyl and other substitutions:

$$\varphi_n = \varphi(x) - \frac{\Delta}{2}\varphi'(x),\ \varphi_{n+1} = \varphi(x) + \frac{\Delta}{2}\varphi'(x); \gamma = \frac{1}{\Delta}\left(1 - \frac{\Delta^2 m^2}{4}\right)$$

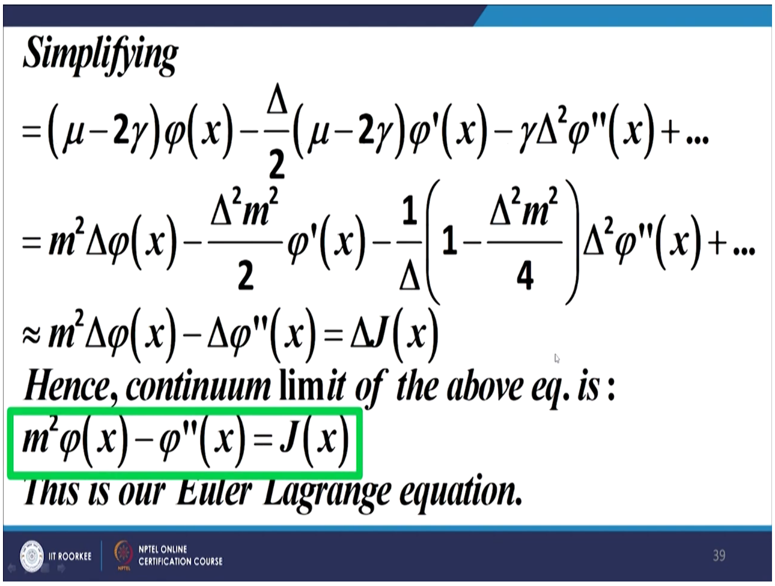
$$J_n \to \Delta J_n;\ \mu - 2\gamma = m^2\Delta; \mu + 2\gamma = \frac{4}{\Delta}$$

$$= \mu\left[\varphi(x) - \frac{\Delta}{2}\varphi'(x)\right] - \gamma\left\{\begin{bmatrix}\varphi(x) + \frac{\Delta}{2}\varphi'(x)\end{bmatrix} + \begin{bmatrix}\varphi(x) - \frac{3\Delta}{2}\varphi'(x) + \Delta^2\varphi''(x)\end{bmatrix}\right\} + \ldots$$

IIT ROORKEE | NPTEL ONLINE CERTIFICATION COURSE 38

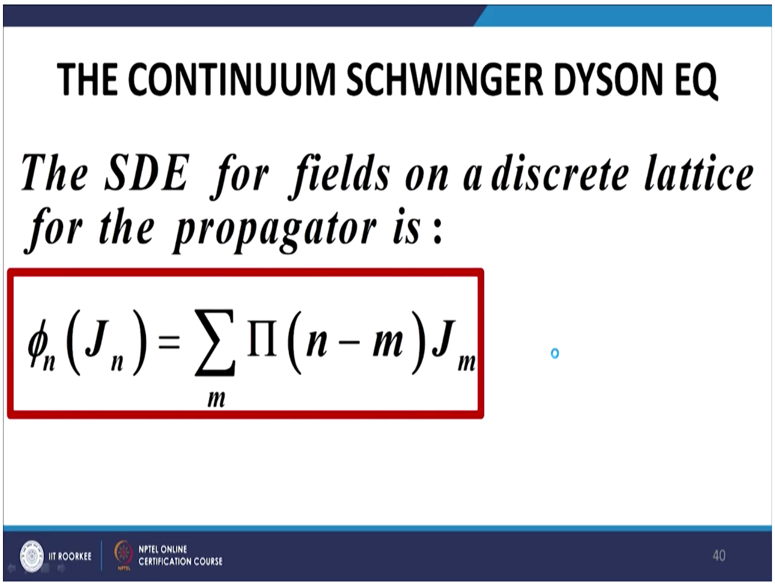
Now, we worked equivalent expression using the Weyl ordering using the Weyl substitutions and other substitutions.

(Refer Slide Time: 24:05)



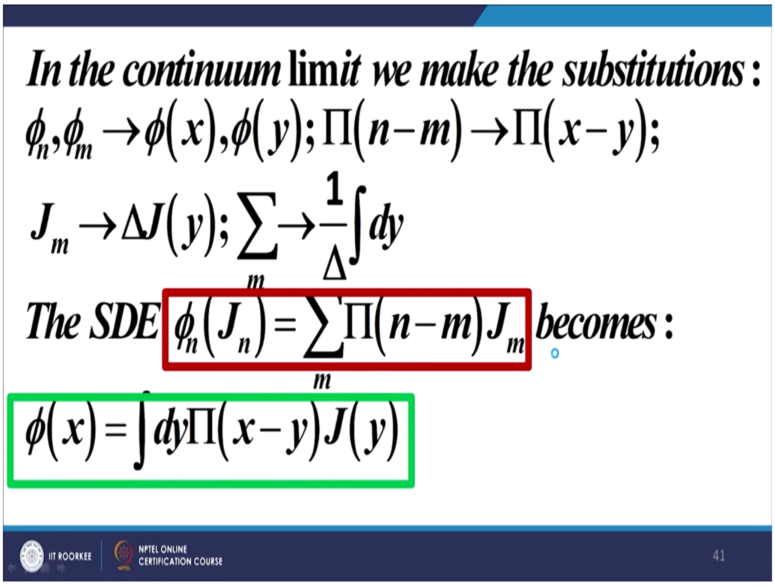
We when we make these substitutions what we get at the end of the day is that m square minus phi double dash square is equal to J x which is nothing but the Euler Lagrange equation.

(Refer Slide Time: 24:14)



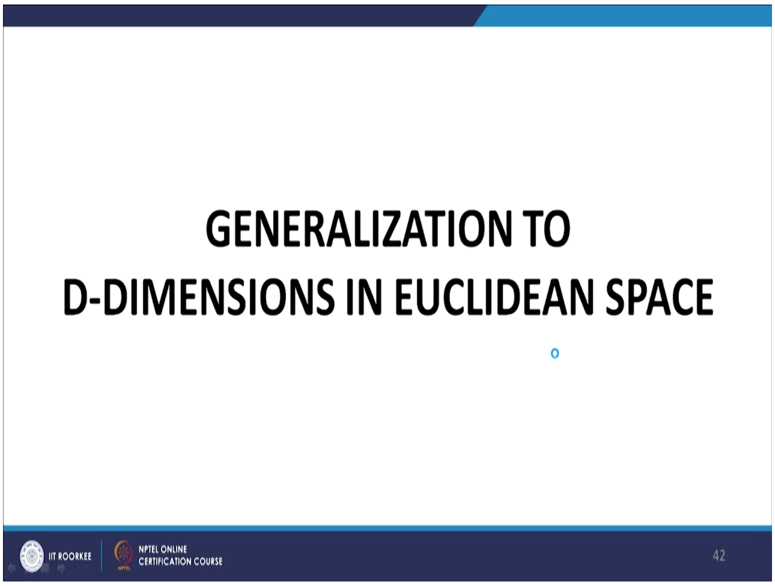
And the Continuum Schwinger Dyson equation can be easily read off. It is quite straight forward; from the Discrete Schwinger Dyson equation for the propagator or field function in terms of the propagator. We have already discussed it in detail.

(Refer Slide Time: 24:29)



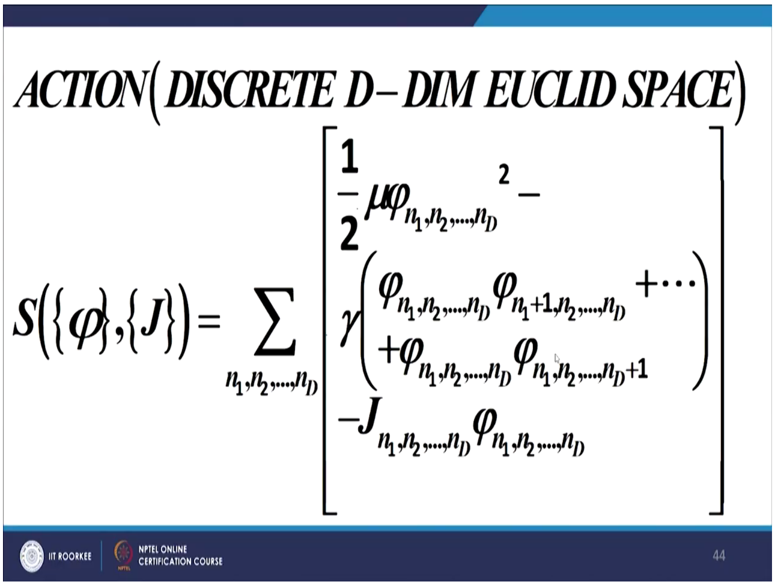
It becomes phi x is equal to integral d u. The summation is replaced by the integral the discrete version of the propagator is replaced by the continuous version. The discrete version of the sources is also replaced by the continuous version and we have phi of x is equal to integral dy pi x minus y J y.

(Refer Slide Time: 24:55)



Now, we can now we will literally generalized we have literally moved from the 0 dimensional framework to a one dimensional framework. These results can now be extended to d dimensional framework, but in an Euclidean space. We are still not entered into Minkowski space. I would like to first generalize these results which is rather trivial in a sense, but nevertheless we should know the results before we enter into Minkowski space. So, generalization of these results in D dimension in Euclidean space.

(Refer Slide Time: 25:40)

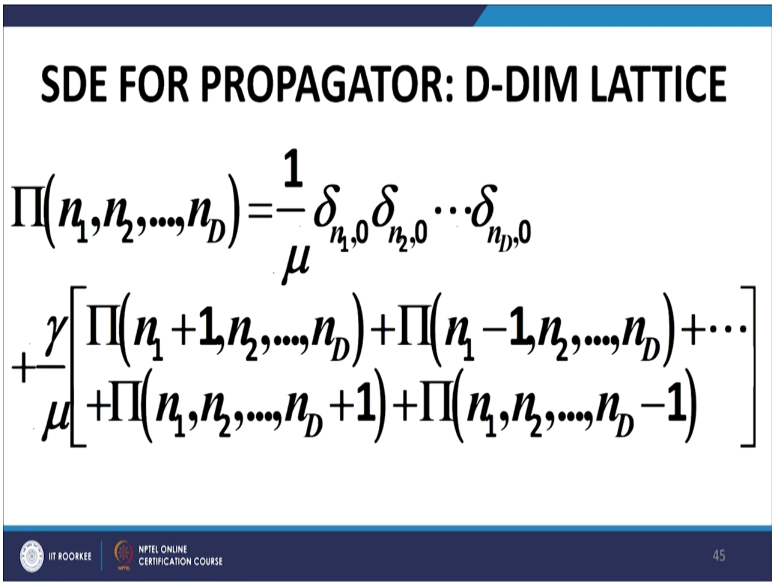


The most important thing is the action you see. In the one dimensional case, east field had 2 nearest neighbours with which it could interact. Now, if you have d dimensional space obviously, the neighbours that as a particular field has increases. For example, if you have a 2 2 dimensional space the number of neighbours becomes 4 and so on.

So, the important thing is when you have a d dimensional space and you still retain the fundamental property of the field of the field variables being that it they interact with only the nearest neighbours the action will take the form which is given in this slide.

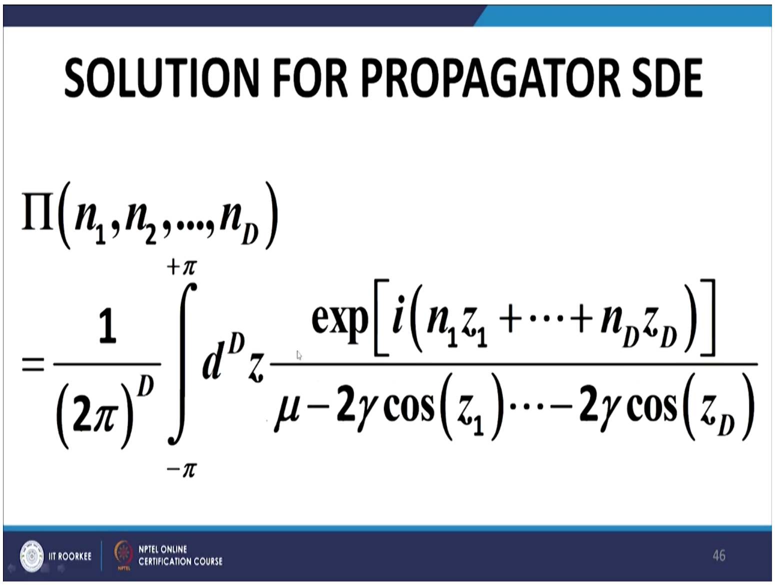
Please note here the please note the gamma based interaction terms phi n 1 is interacting with phi n 1 plus 1 phi n d is interaction with phi with n d phi n d plus 1; so, each of these indices carries a neighbour each of these indices carries a neighbour. And therefore, obviously as many indices are there the neighbours also multiplies accordingly and the interactions also increase accordingly.

(Refer Slide Time: 27:10)

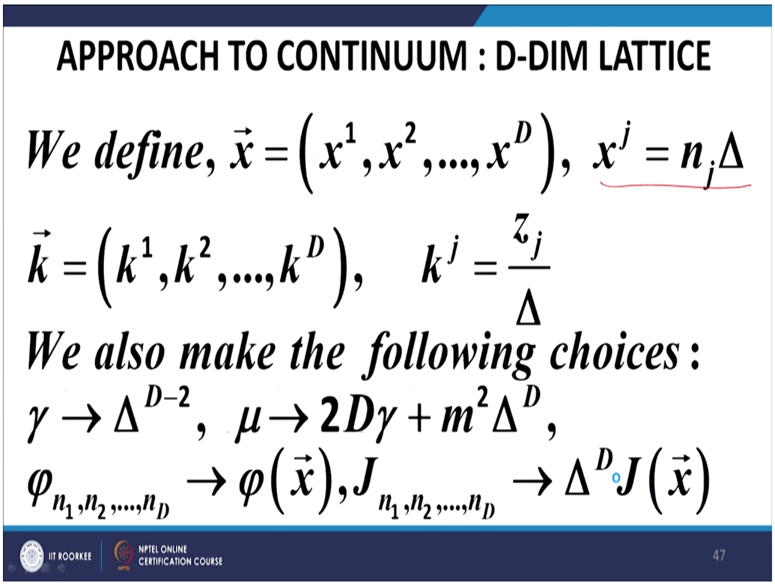


The propagator again we have a similar equation to what we have in the one dimensional case of course, the number of terms increases, each corresponding to an interaction, n 1 interaction n 2 interaction and so on.

(Refer Slide Time: 27:28)



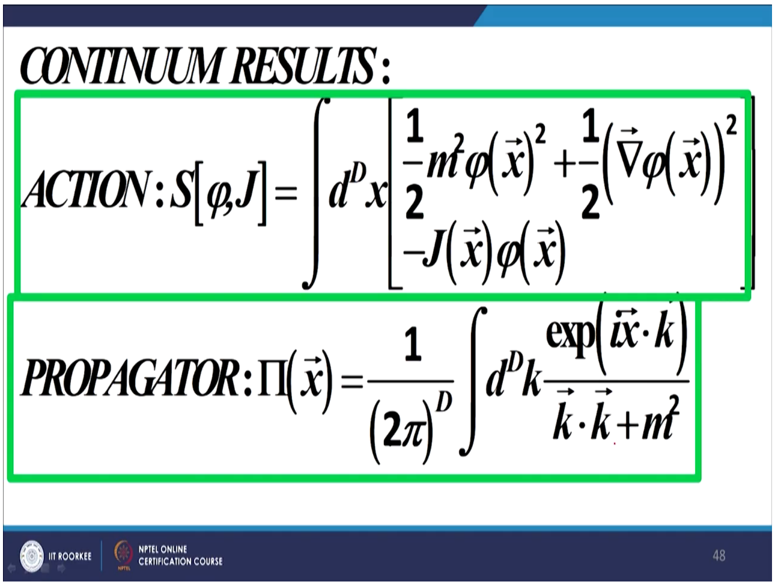
(Refer Slide Time: 27:36)



And the solution of the propagator is more or less parallel is absolutely is similar. The and we now simply replace instead of making replacements by numbers by numbers we have in a sense to replace the number by vectors and the with various components of those vectors being replaced in the form given by this expression x J is equal to n J delta.

Similarly, the k expressions k number is replaced by the k vector and each component of the k vector k j is replaced by z replaces z j upon delta and we also make this set of choices. And on the basis of this set of choices and we define the field discrete field variables as continuous. We substitute the discrete field variables in terms of the continuous field variables, substitute the discrete sources in terms of the continuous sources, integrate over D dimensions.

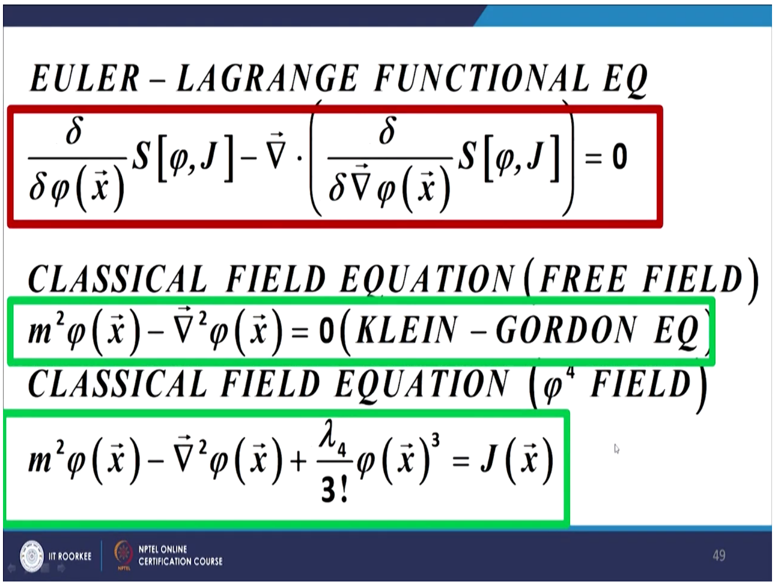
(Refer Slide Time: 28:45)



After doing all that the action becomes the expression that is given in the green box. The upper green box and the propagator becomes the expression that is given in the bottom green box. Just recall that earlier we had x and now we have earlier we had scalar x now we have a vector x.

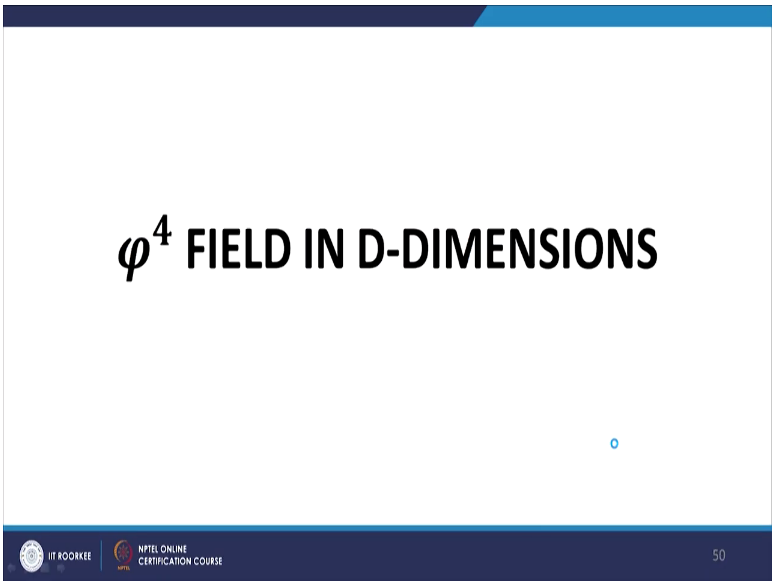
And similarly for k we have a vector k with the components being replaced in this form which are given on your slide. So, each component was replacement is required and on that basis the k square term the scalar k square term is now replaced by the scalar product of vectors k dot k and we have this expression for the propagator.

(Refer Slide Time: 29:32)



The Euler Lagrange equations and the classical field equations also become more or less the same or at least retain the similar form as they were there in the case of the one dimensional situation.

(Refer Slide Time: 29:49)



From here we will continue in the next class phi 4 field in D dimensions.

Thank you.