Path Integral Methods in Physics & Finance
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Lecture – 39
Field Theory in 1-D (1)

Welcome back. So, in the last lecture we concluded our discussion of the Field Theory in zero-dimensional space time. Today, I propose to extend the framework to one-dimensional lattice, move onto the continuum version of the one-dimensional framework. And then, we will graduate on to the Minkowski space in the next few lectures.

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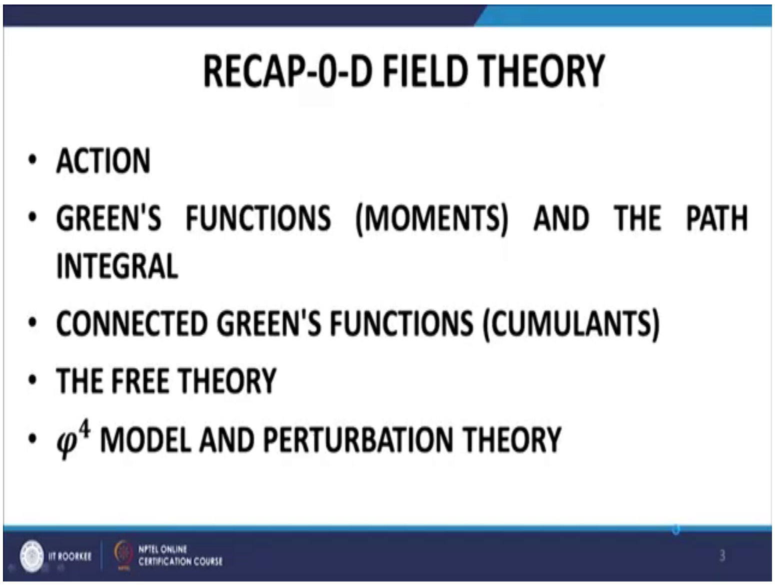

But, before we get into the nitty-gritty, a brief recap on where we stand at the moment. As far as zero-dimensional field theory was concerned we started with the action, we defined the

Green's functions and the connected Green's functions for the theory. And we evaluated the relevant Green functions for the free theory.

We then introduced, in the interactions through the phi 4 model by adding a term to the action, and developed this as a perturbation expansion, perturbation theory in powers of the coupling parameter coupling constant.

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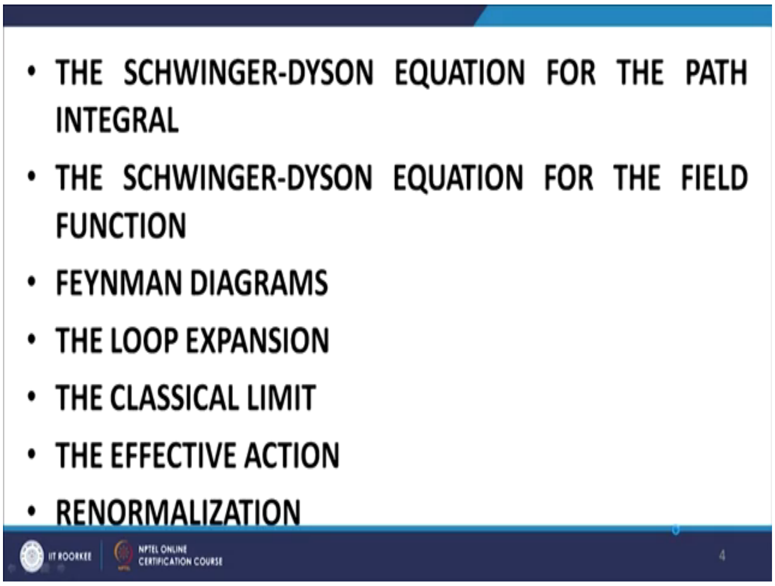

And, we also arrived at the Schwinger-Dyson Equation for the path integral we defined the generating functional for the theory and we arrived at the Schwinger-Dyson equation or the evolution equation for the path integral or the generating functional.

We also arrived at the Schwinger-Dyson equation for the field functions. Then, we moved on to the concept of Feynman diagrams, which perform which constitute a wonderful recipe for a diagrammatic representation of various issues in relation to quantum field theory; they form the cornerstone of literally all field theoretic calculations. So, we introduced it Feynman diagrams at the most elementary level possible starting with the Green function for the free theory, represented by Feynman diagrams pairs of straight lines. We then moved onto the issue of loop expansion.

So, the perturbative expansions, in the context of the classified on the basis of the number of loops involved in the Feynman diagrams by introducing the parameter h bar; we identified h bar with the conventional h bar of quantum mechanics the Planck's constant.

And then, took up the issue of the classical limit of the path integral we arrived at the classical limit through the saddle point approximation. And we observed that, the behaviour of the various paths in pathways in relation to the classical trajectory which is only one of the paths.

Then we defined the effective action as the action which whose classical solution corresponds to the full quantum solution of the given action. We then, discussed renormalization in a lot of detail and that is where we concluded the last lecture.

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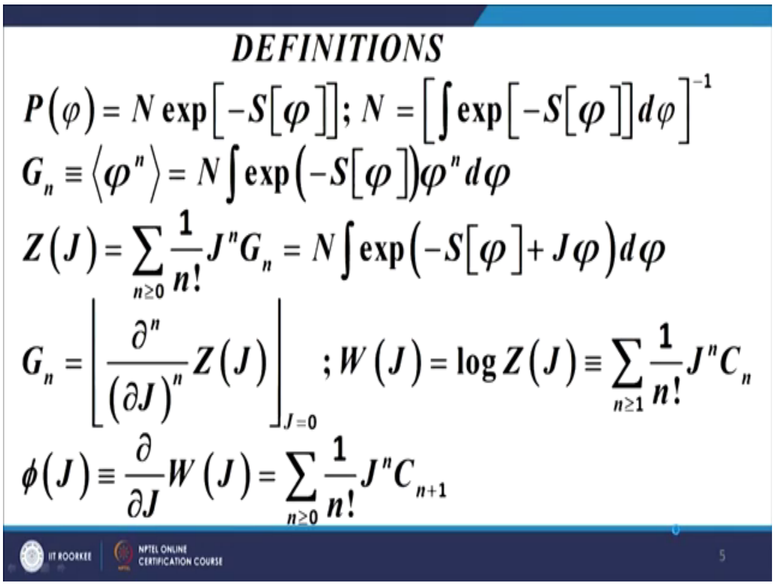

Just the gist of whatever has been done so far is presented in the next few slides. This first slide covers the definitions.

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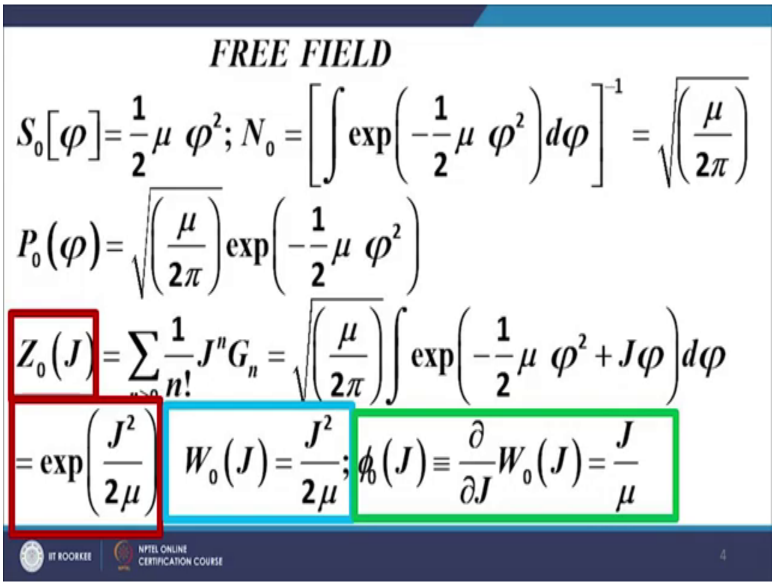

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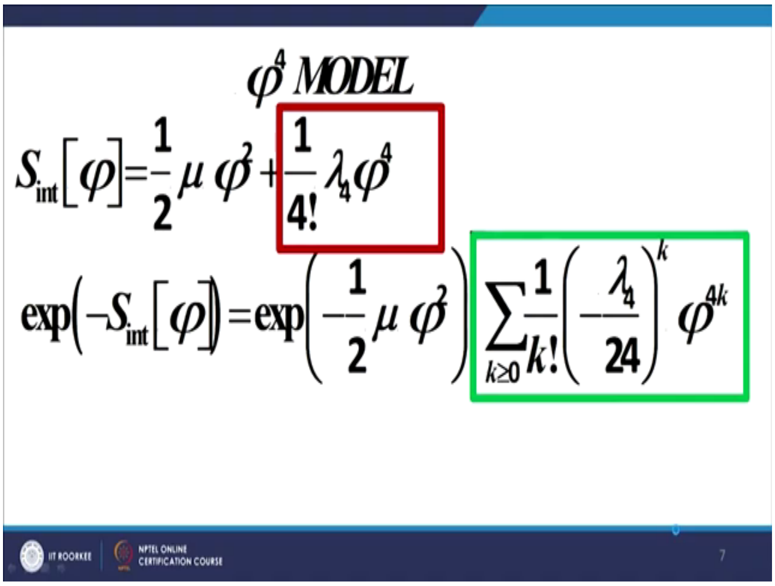

The free field parameters are given in this slide. Then the phi 4 model is introduced through the interaction term given in the red box and the perturbative expansion in terms of the coupling constant given in the green box.

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$$Z_{\text{int}}(J) = N_{\text{int}} \sum_{k \geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24} \right)^k \int \varphi^{4k} \exp\left[-\left(\frac{1}{2} \mu \, \varphi^2 \right) + J\varphi \right] d\varphi$$

$$N_{\text{int}} = \left[\left(\frac{2\pi}{\mu} \right)^{1/2} \sum_{k \geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24\mu^2} \right)^k \frac{4k!}{4^k (2k)!} \right]^{-}$$

We develop the partition function or the normalization of the interaction model as a perturbative expansion, and we work out the Green functions.

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$$G_{2n} = N_{\text{int}} \sqrt{\left(\frac{2\pi}{\mu}\right)} \frac{1}{\mu^n} \sum_{k \geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24\mu^2}\right)^k \frac{(4k+2n)!}{2^{2k+n}(2k+n)!}$$

$$= \frac{\frac{1}{\mu^n} \sum_{k \geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24\mu^2}\right)^k \frac{(4k+2n)!}{2^{2k+n}(2k+n)!}}{\sum_{k \geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24\mu^2}\right)^k \frac{4k!}{4^k (2k)!}} = \frac{H_{2n}}{H_0}$$

We 2n point Green functions and represent them in terms of the normalization, 1 upon H 0 and the parameter or the quantity H 2n.

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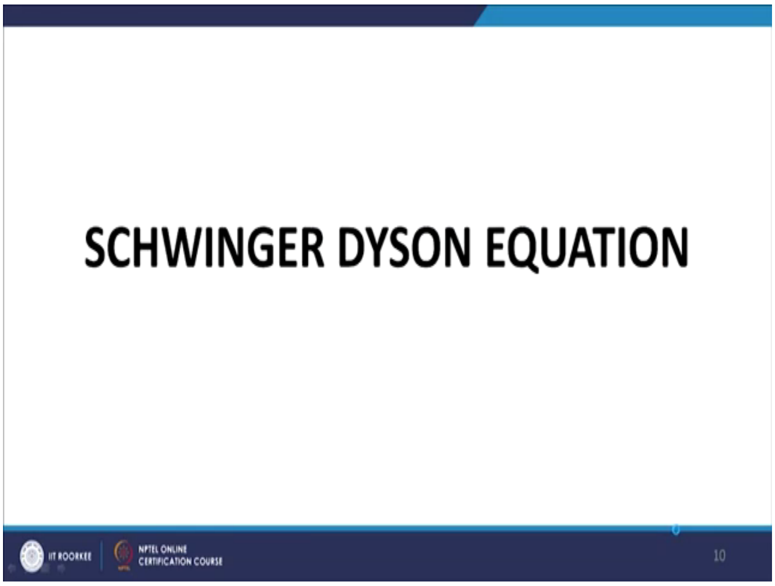

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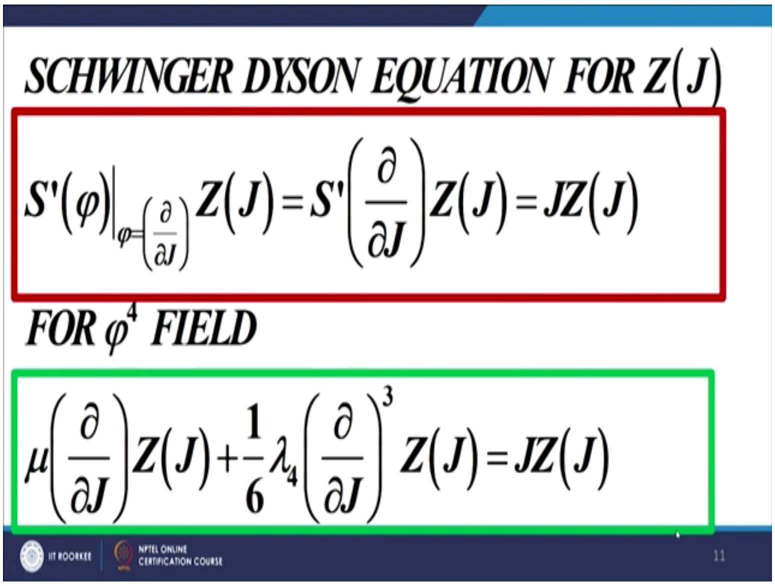

We the Schwinger-Dyson Equations for the generating functional given in the red box, and which for the phi 4 field that we have been discussing throughout takes the form given in the green box here.

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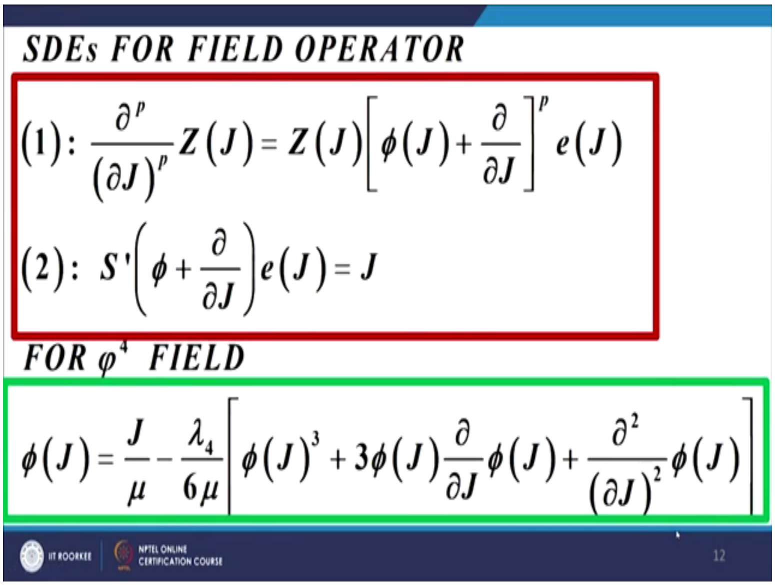

The field operator obviously, Schwinger-Dyson equation, that is given in the red boxes, it is given in 2 forms both of them hold for the phi 4 theory it takes the form given in the green box.

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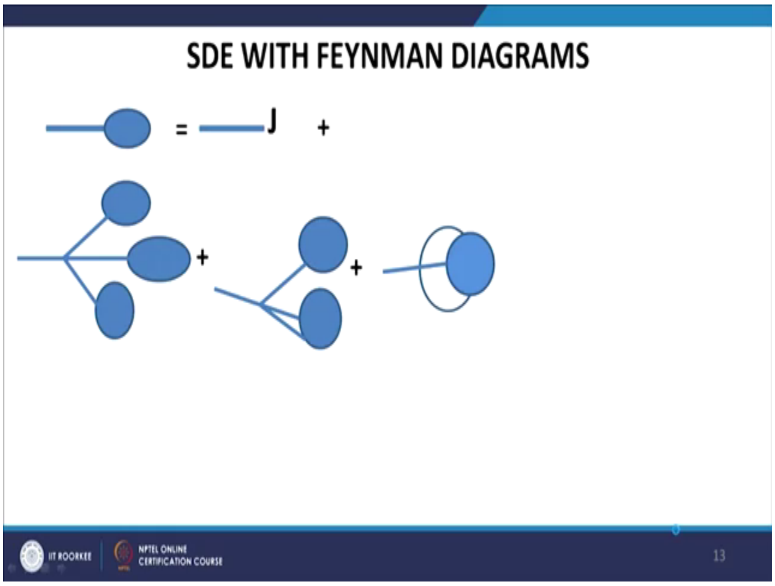

The Schwinger-Dyson equation can be arrived at by reference to the Feynman diagrams which are given on this slide.

The first diagram represents no interaction. The second diagrams represents an interaction with a 4 point vertex with each vertex branching of forming another blob or ending up at another blob. The second situation is where, 2 of the; 2 of the branches of the vertex reunite into a single blob. And the third situation is where all the 3 external lines from the vertex reunite to form into the same blob.

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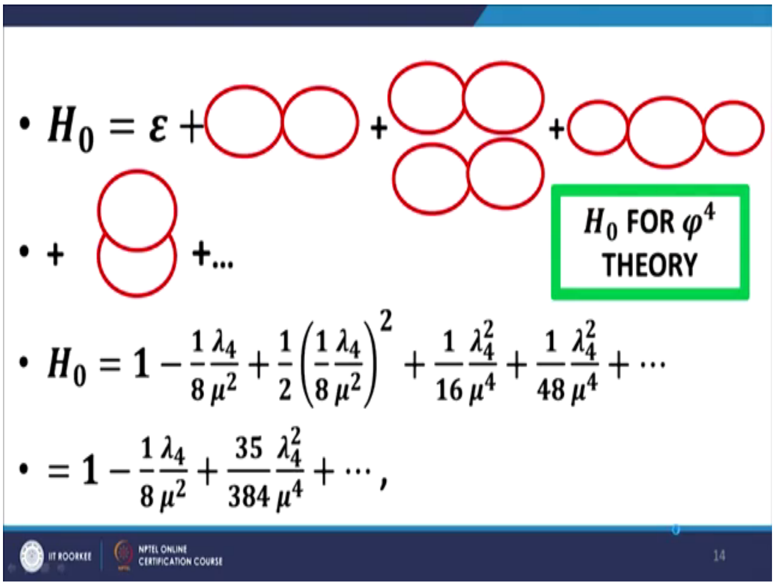

And, these 4 terms correspond respectively, to the Schwinger-Dyson equation that is given in the green box. The free field propagator is given in or the free field term is given in the first term. And the term the other 3 figures, the other 3 figures which involve a vertex in each case interaction with at the 4-point vertex in each case are given by the, given by the terms in the square brackets.

The first term phi J cube, the second term involving phi J with the first derivative because, 2 branches emanating from the 4-point vertex reunite to a single blob, and in the third case the 3 branches emanating from the vertex all reunite to a common blob which is the last term on the green box given here.

The issue of vacuum bubbles was also discussed. These are the series of vacuum bubbles they are infinite series and they would perform a part of every expansion or every perturbative expansion and they are excluded by dividing the entire perturbative expansion by this expression. In other words, when we write down the Feynman diagrams for a particular Green function or particular expansion we exclude the vacuum bubbles from the right hand side.

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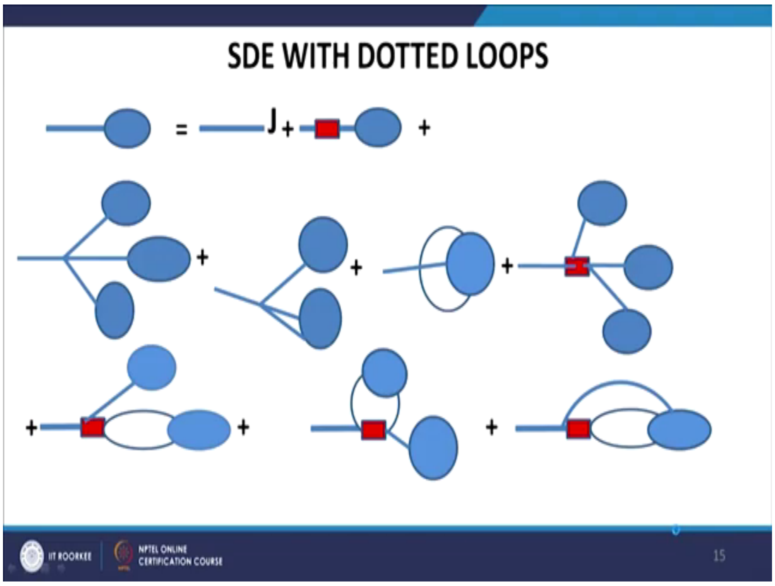

Then, we discussed the issue of loop divergences we represented them as dotted loops and, Feynman diagrams corresponding to the phi 4 theory with dotted loops takes the form which is given in your figure.

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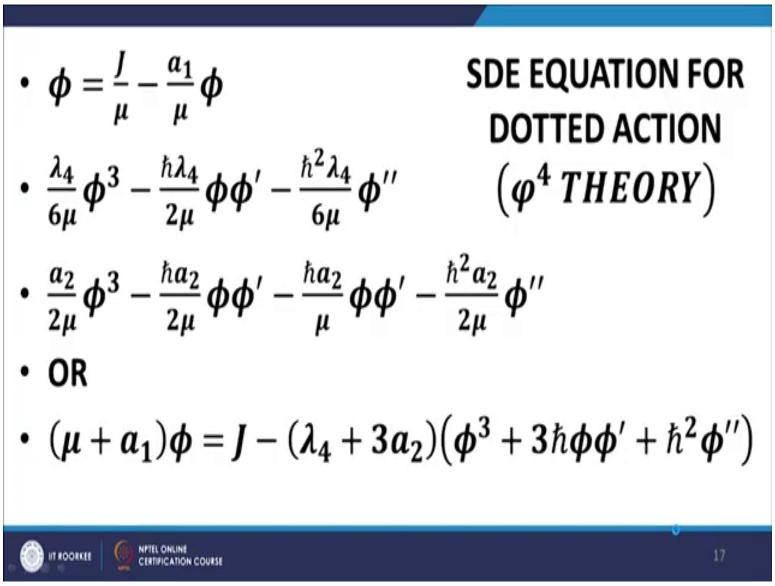

The Schwinger-Dyson equation, when we factor in the dotted diagrams, the loop divergences takes the form which is given in this figure. The interesting part is as we as I discuss towards the end of the last lecture, that the bare parameters of the action combined together or with a 1 mu combines with a 1 and lambda 4 combines in 3 a 2 in the Schwinger-Dyson diagram.

Which implies that whenever, an experimental measurement is made, the measured quantity will be mu plus a 1 and lambda 4 plus 3 a 2 rather than mu and lambda and that enables us to do a renormalization in the event that a 1 and a 2 move towards infinity or tend to infinity by redefining mu and lambda 4 also as infinities. So, that their sum appears in the theory as a finite quantity and the expression that we measured through experiments is a finite quantity.

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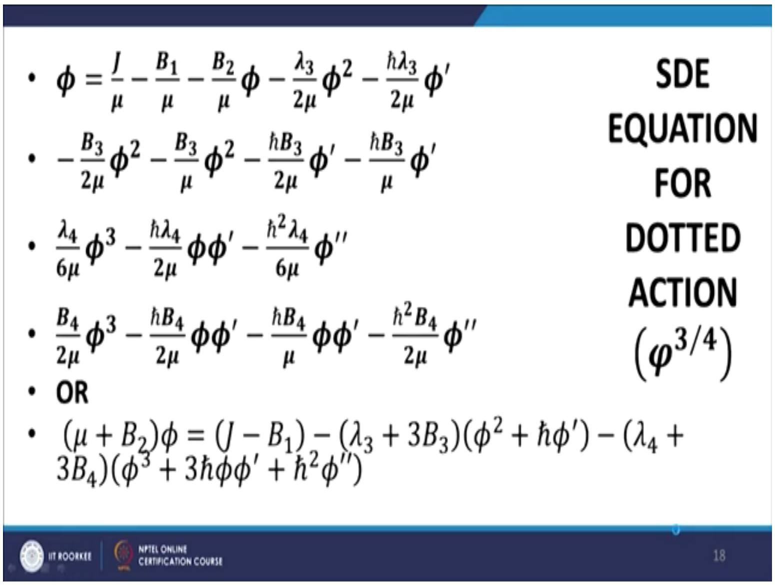

And therefore, the theory can be expressed in terms of finite quantities.

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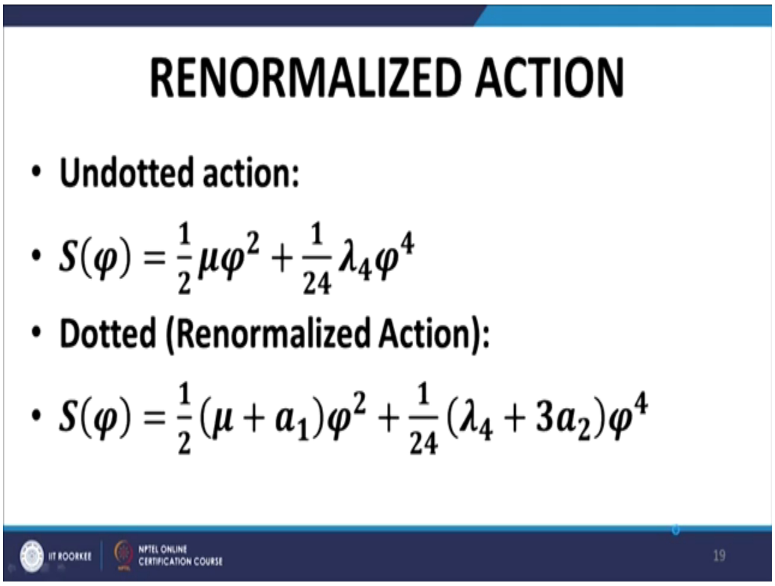

The as I mentioned just now, the original action was the bare action and the bare parameters was mu and lambda 4 given in the first equation on your slide. However, the dotted or the renormalized action takes the form mu plus a 1 as I mentioned mu is invariably replaced by mu plus a 1, lambda 4 is invariably replaced by lambda 4 plus a 2.

And, these are the quantises, that will be experimentally measured that are experimentally measured and therefore, if a 1 and a 2 tend to infinity we can redefine mu and lambda 4 also in similar manner so, that these expressions continue to be finite, and we get finite expressions for the Green functions.

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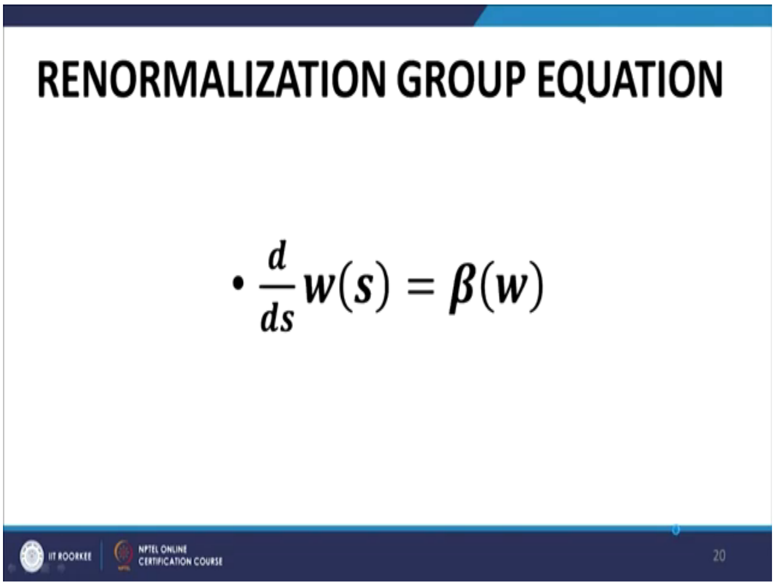

We also arrived derived the renormalization group equation for the variation in scale variation energy of the renormalized parameters of the action and for this 0-dimensional field theory.

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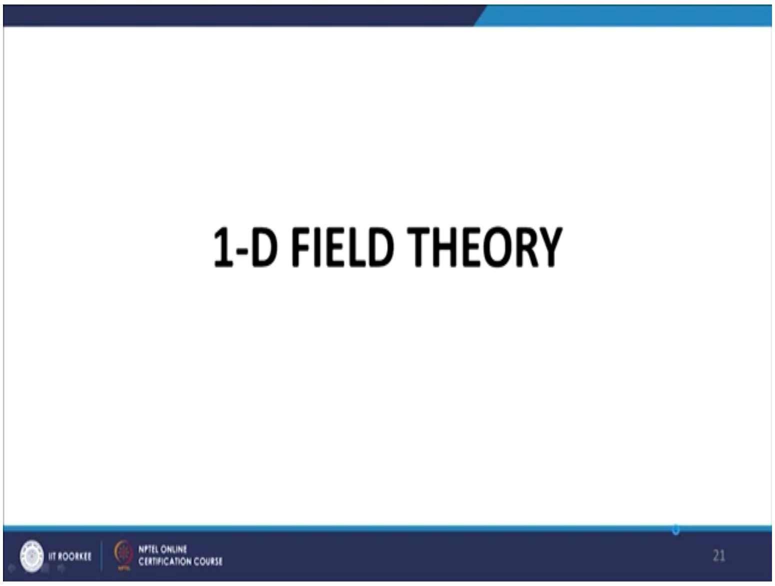

So, that is what we have done. So, far as far as zero-dimensional field theory is concerned.

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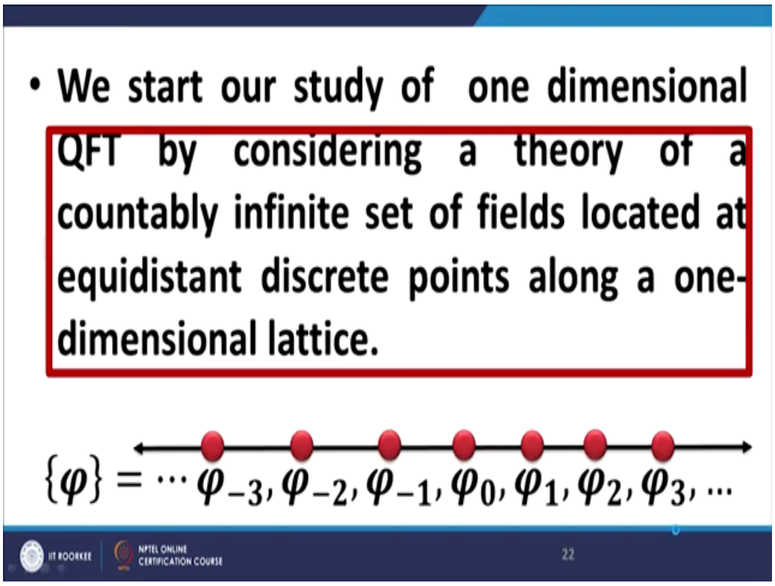

We now, start one-dimensional field theory start the one- dimensional field theory by considering a lattice of one-dimensional lattice of equidistant points, discrete equidistant points the distance between each of them as we shall see later will represent by delta.

And, we shall go from this discrete model where each in the lattice is a discrete lattice and we shall move towards continuum by reducing the distance between the various points in the limit, that delta tends to 0; delta is the inter spatial distance.

And, we at each part on the lattice we identify a field with the respective suffixes phi 0, phi 1, phi 2, phi 3 and similarly, to maintain uniformity to or reflective symmetry we have phi minus 1, phi minus 2 and phi minus 3 and so on. We take the origin as the centre of the lattice.

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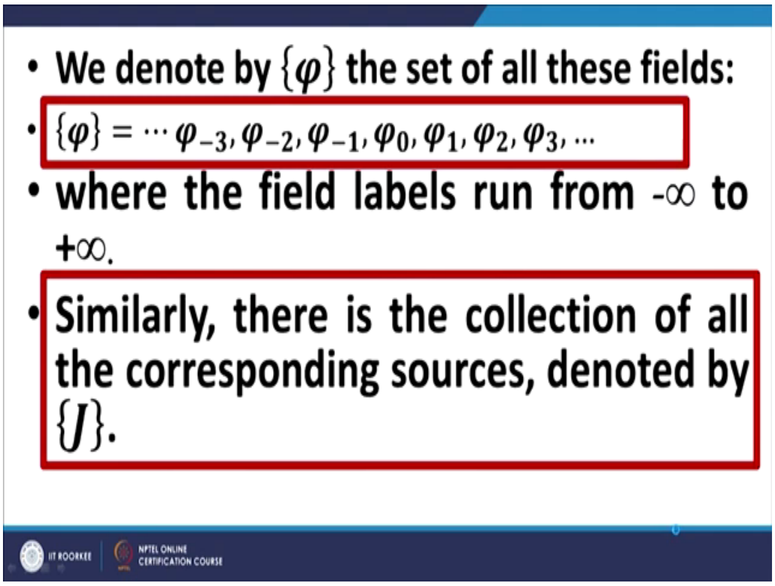

Collectively the fields are denoted by phi, the set phi and the field labels as I mentioned run from minus infinity to plus infinity. And similarly, we have a collection of all sources and the sources are also collectively represented by J.

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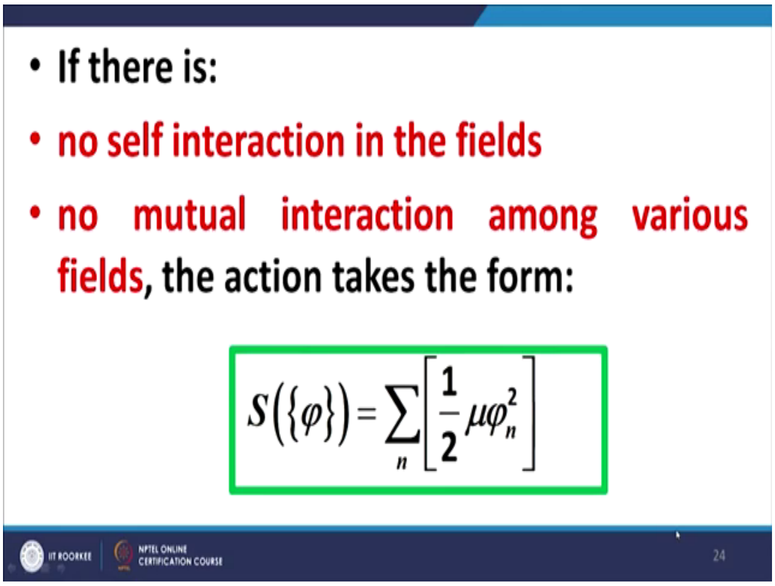

For the moment, let us start with the simplest case if there is no self-interaction and there is no interaction between the neighbouring or any interaction between distinct fields between different fields then the action takes the form given in the green box here.

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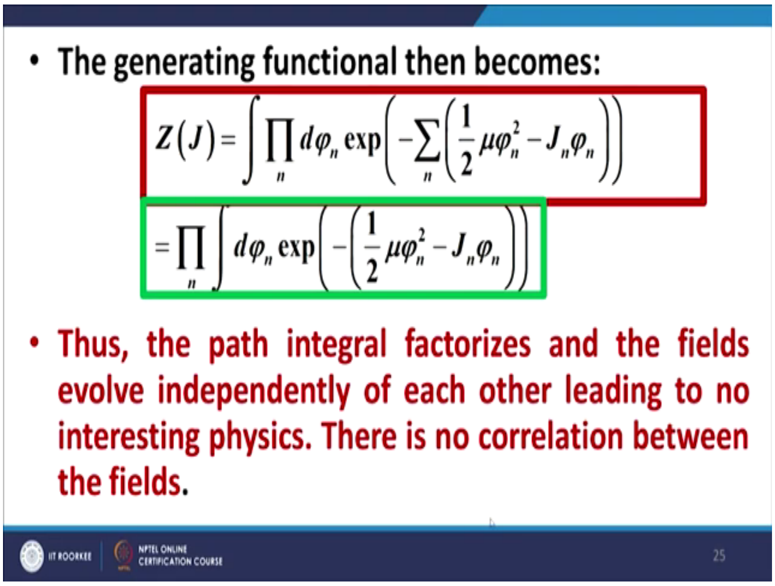

And, the generating functional can be simplified to take the form given in the green box here. Straight away it factorizes the generating functional factorizes itself, and each field in a sense evolves independently of all others there is no correlation and therefore, there is no interesting physics involved in this particular case.

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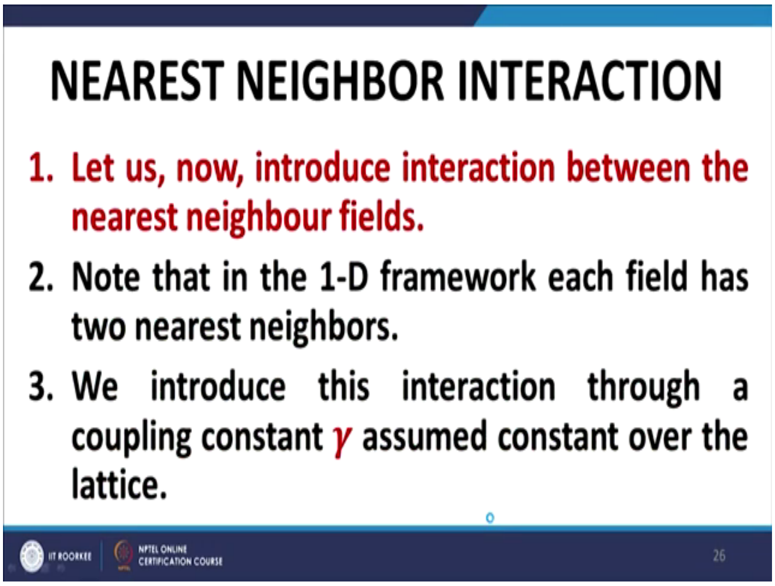

So, what we do now is we introduce some kind of interaction having studied the phi 4 interaction already, we keep it in abeyance for the moment we introduce interaction between nearest neighbors. In other words every field on the lattice interacts with one neighbor towards left, and the other neighbor towards right.

Recall, that in this framework each in the 1-dimensional lattice framework each field is surrounded by two neighbors, one to the left and one to the right. And we assume that the field interacts with only these two neighbors that is the neighbor to the left and the neighbor to the right.

And the coupling or the interaction is reflected by a coupling constant that is gamma and that is constant throughout the lattice. In other words that, justifies the assumption that the lattice spacing is uniform.

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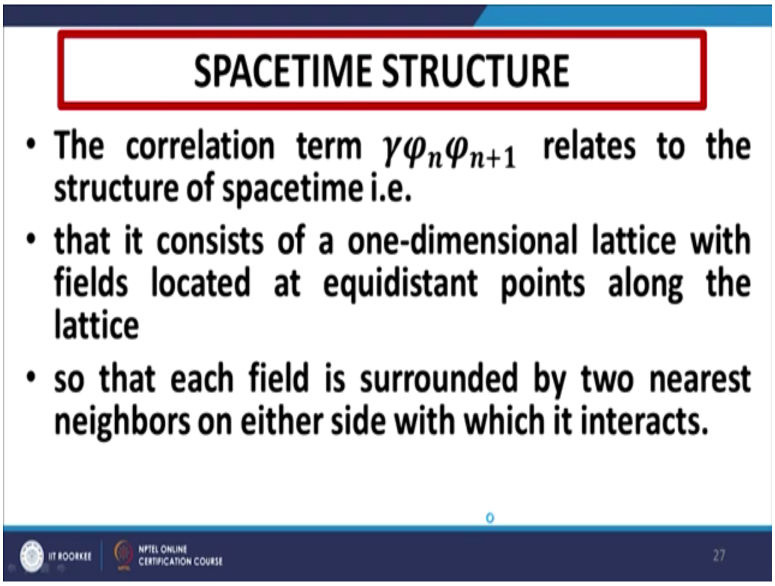

So, this correlation term will take the form gamma phi n phi n plus 1. And the very fact that it takes this form related intimately to the structure of the lattice being a 1-dimensional lattice, and therefore, two neighbors one to the left and one to the right the field interacts with these two neighbors, which is captured by this particular term which as you will see goes into the action for the field theory present field theory.

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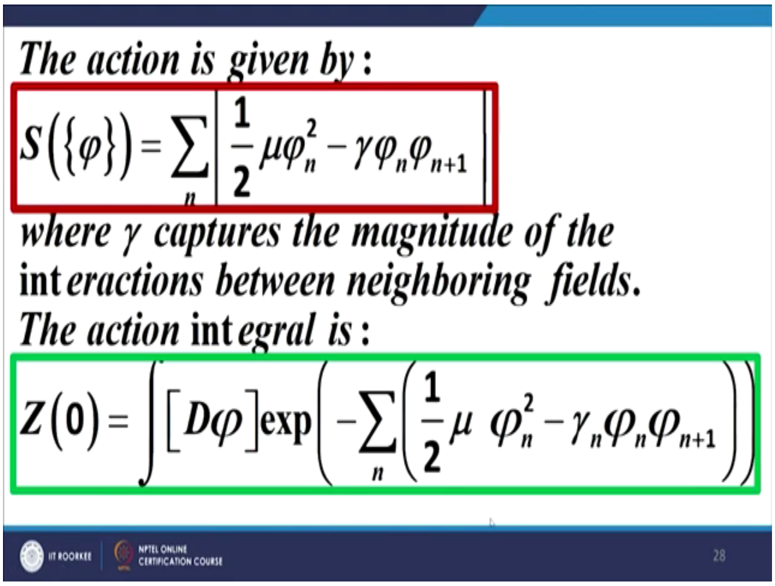

So, the action is given by as I mentioned they action by incorporate the mutual interaction term in addition to the free action 1 by 2 mu phi n square; the red box expression minus gamma phi n phi n plus 1. The second term is obviously, the term that captures the mutual interaction between neighbouring fields.

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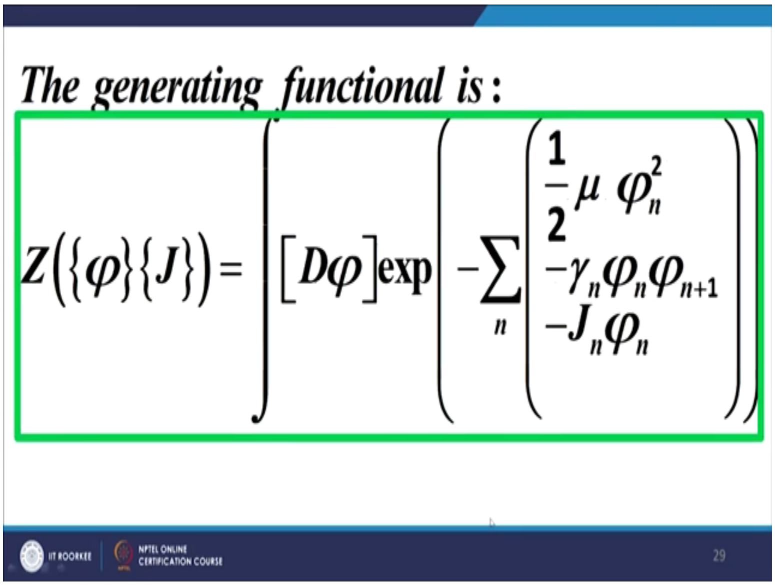

And, the action integral takes this form. The generating functional and the generating function, for the moment is still the function it takes this form we introduce sources each source corresponding to each field independent one to one field source relationship and that is reflected in the generating function given in the green box on this slide.

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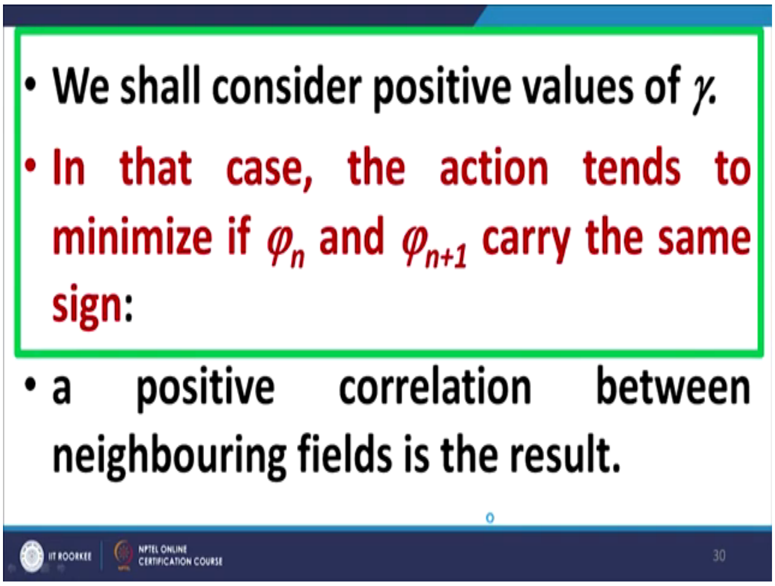

We consider positive values of gamma, positive values of gamma implies that the action between phi n and phi n plus 1 minimizes if they carry the same sign. In other words if phi n and phi n plus 1 carry the same sign and gamma is positive the action will minimize. So, there is a positive correlation between the neighbouring fields in view of gamma being taken as positive.

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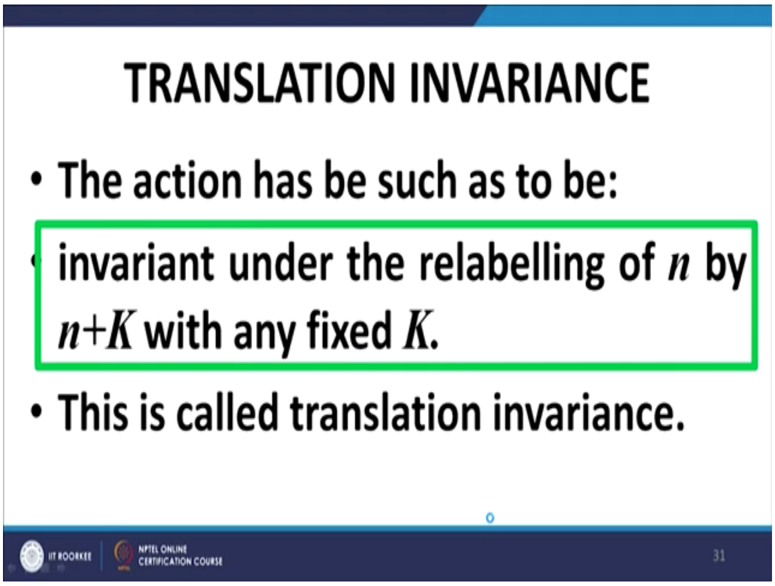

Then, we have translation invariance, in other words we want that if the entire set of fields the entire structure of fields is shifted by a constant or the indices are shifted indices of each fields are shifted by a constant k, the structure of the physical theory underlying physical theory does not change.

That means if we re label; n by n plus K, capital K for any fixed K each of these; each of these labels are renamed by as 1 say phi 0 is named as phi k, phi 1 is named as phi 1 plus k and so on. The field theory does not change.

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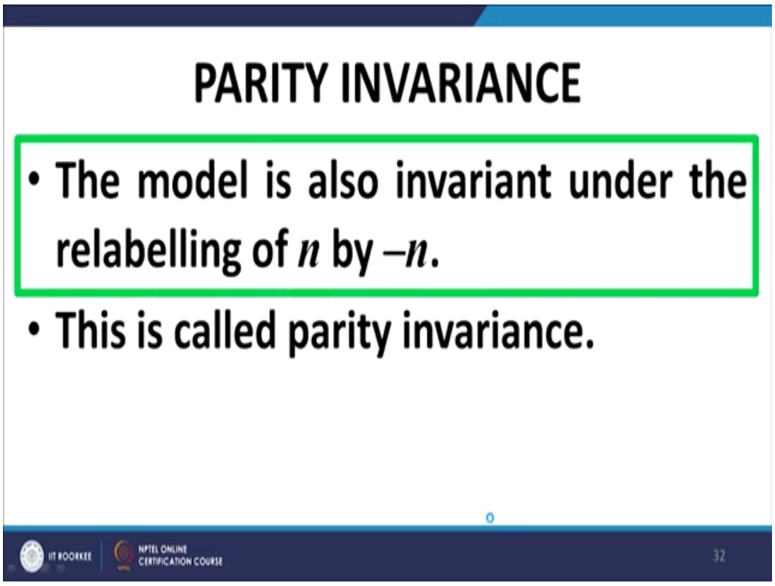

Then, we need also involve we also have parity invariance, in other words if we change phi, if we interchange n and minus n, the theory should or the theory will not change.

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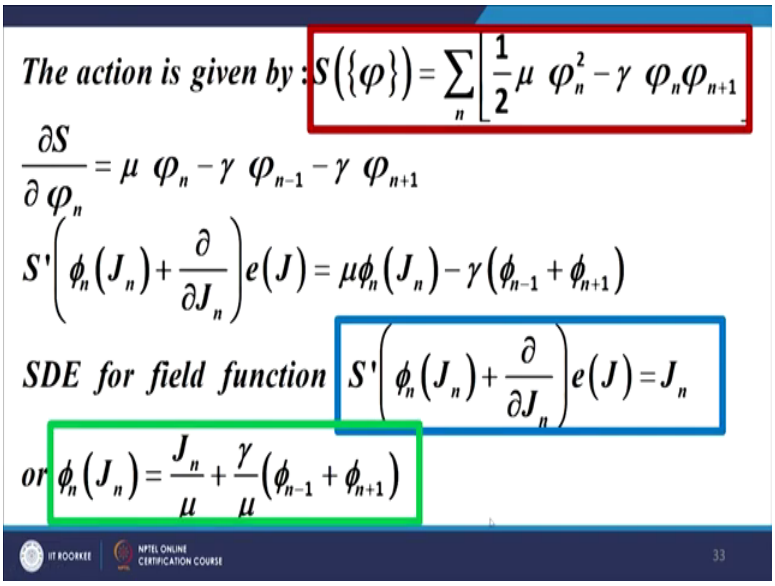

So, the action as I mentioned earlier is given by the expression in the red box above. The first derivative of the action with respect to the field variables is given by in this expression mu phi n minus gamma phi n minus 1 minus gamma phi n plus 1.

Please note, there will be 2 interaction terms: one corresponding to each of the neighbors in left neighbor and the right neighbor. The n minus 1th neighbor and the n plus 1th neighbour. In other words phi n interacts with the phi n plus 1th neighbor and the phi n minus 1th neighbor.

So, Schwinger-Dyson equation therefore, takes the form which is given, this Schwinger-Dyson equation for the field function is given by S dash of phi plus del by del J of the unit function which on simplification gives us the expression right at the bottom green box of the of your slide. This is the Schwinger-Dyson equation for the field function of the current model.

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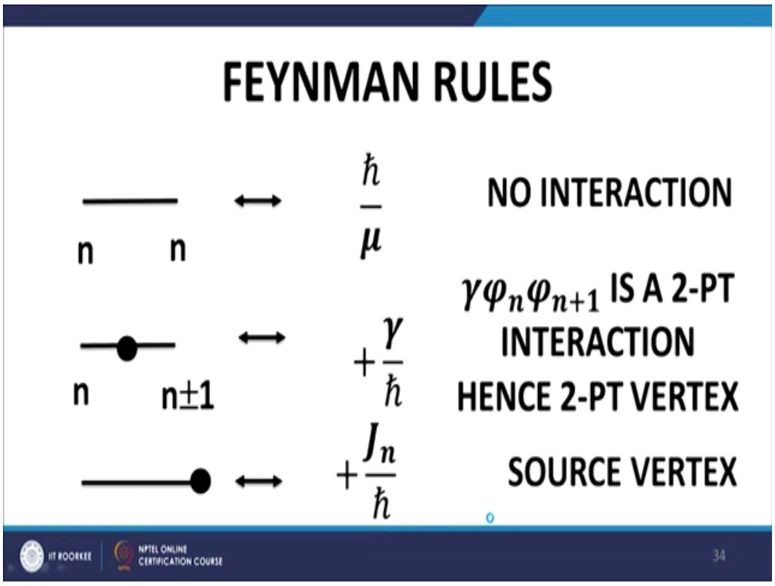

The corresponding Feynman rules, are simple. The first rule represents no interaction. The second rule represents interaction at one-point; the two-point vertices because; that means, the field is interacting with its nearest neighbor therefore, with the Feynman rule involves the field n, the incoming field n and the outgoing field either n plus 1 or n minus 1. This is a two- point vertex because, the interaction involves the incoming field and this nearest neighbor.

And, the interaction term is given by gamma phi n phi n plus 1. So, the vertex captures the two-point interaction between the fields, the field n and the field n plus minus 1; field at n plus minus 1, field at a point n and the field at point n plus minus 1. And then, the rule for the source is given right at the bottom of your slide. The line plus the blob at the right hand is the source vertex.

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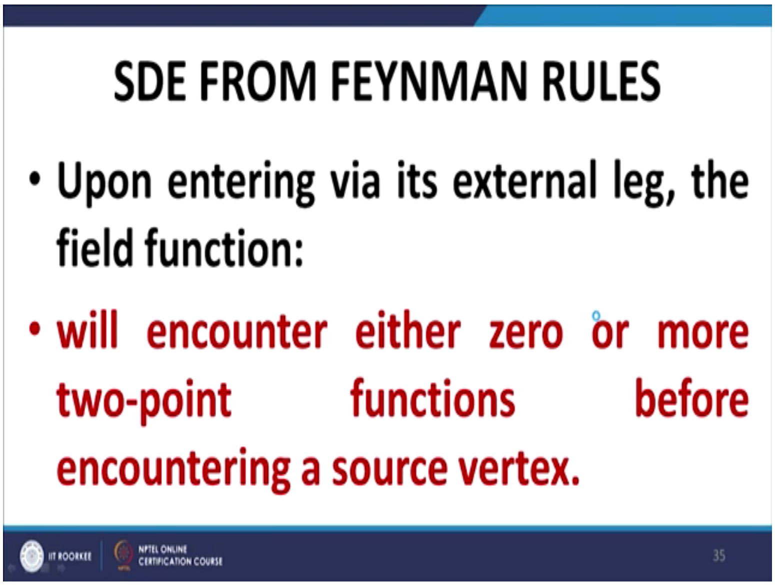

So, Schwinger-Dyson equation can be derived from the Feynman rules, upon entering the external leg, the field function will either encounter no two-point vertex or one or more two-point vertices and the situation is reflected in the Schwinger.

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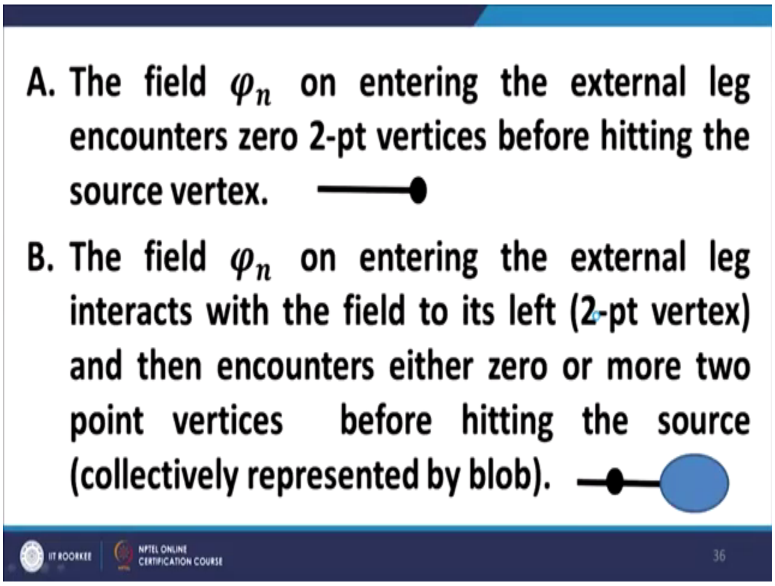

If the field, phi n encounters no 2-point vertices zero 2-point vertices, and hits the source you get a diagram of the form that is given in in point a. If the field encounters one vertex, then the diagram then the interaction is given in the diagram the blob the black blob and thereafter, which may again encounter either a zero vertex or another 2-point vertex and so on. Which is the entire thing is captured by the big blue blob.

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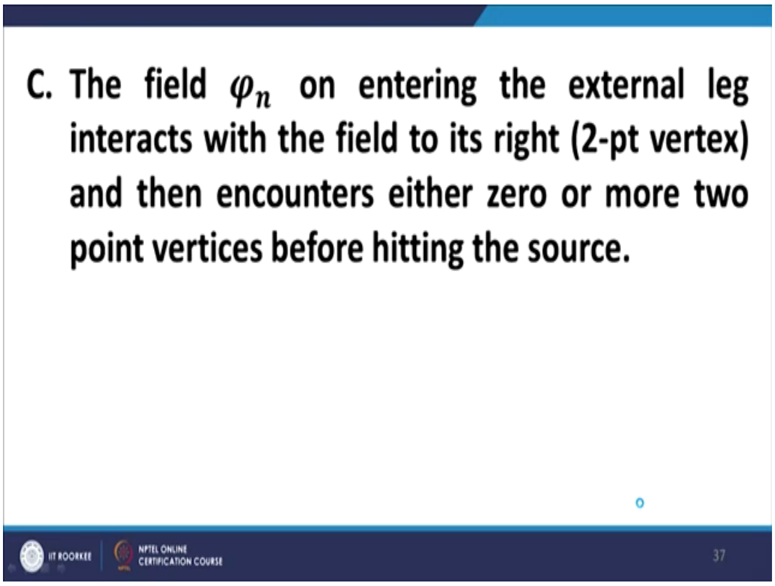

And similarly, in the interaction with the field to the right will be represented by a similar figure.

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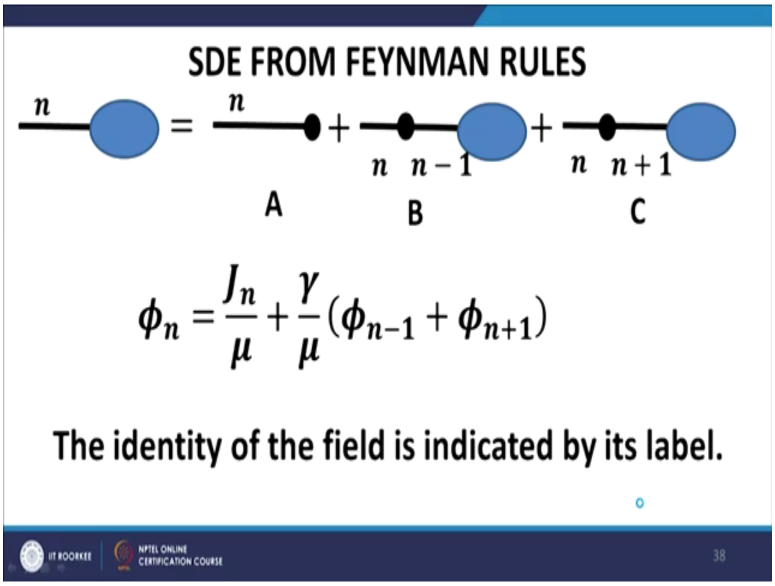

So, in totality what we have is if a field enters the field function enters through an external leg either it encounters 0 vertex, and it directly hits the source which is given in the first diagram, or it encounters one-two point vertex which changes it into either n minus 1 or n plus 1 and then, it again may encounter a 0, may encounter a no vertex or encounter one or more two point vertices that entire story of encountering no vertex or two-point one or more two-point vertices is captured by the blue blob.

Which is simply, a reflection of the blue blob that you have on the left hand side. What are we representing on the left hand side, we are representing an entering field phi n and which may encounter either no vertex or it may encounter one or more two-point vertices the entire thing is captured by the blue blob.

Here, the same thing is happening it encounters one vertex, which with which it interacts two-point interaction and then again it ends up with a situation where the field n minus 1 either encounters a 0 vertex, as I encounters a 0 vertices or encounters one or more two-point vertices which is again captured by the two-point by the blue blob.

Similarly, in the third case the field n interacts with its neighbor to get the field the field n plus 1, and again the field n plus 1 may either encounter no vertex or it may encounter one or more two-point vertices, the entire situation of the field n plus 1 cap and encountering zero or more than one or more two- point vertices is captured by the blue blob. So, this is a kind of a recursive relationship.

And, what we get is phi n is equal to this particular first case, which is which evaluates this dot evaluates to 1 to J n. And the straight line evaluates to 1 upon mu. So, it gives us J n upon mu, here the interaction what else evaluates to gamma, the entering line evaluates to 1 upon mu and then, we have this blue blob which evaluates to phi n minus phi n minus 1 and similarly, the third figure also evaluates to gamma upon mu phi n plus 1. So, we have this recursive relation which is the Schwinger-Dyson equation for the given field phi n.

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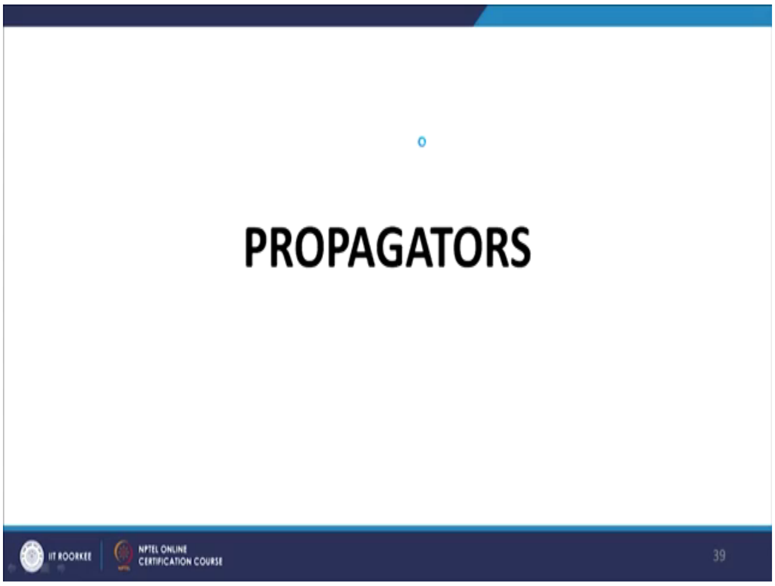

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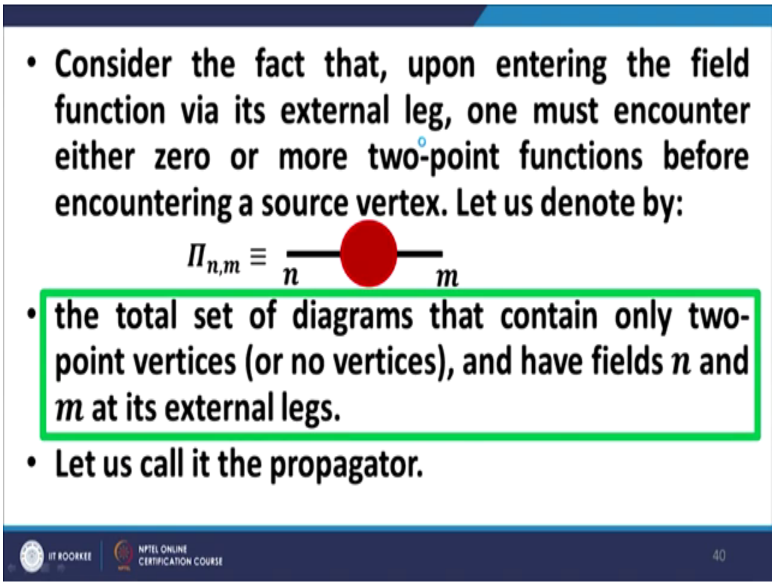

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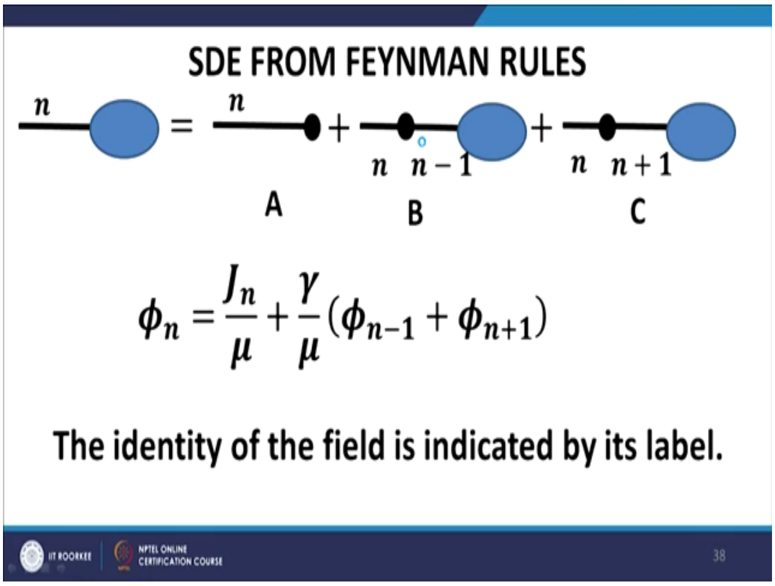

Now, we introduce the concept of propagators. Here, just let me going back a bit, here what we were talking about is the field function, entering through one of the external legs, and emerging in any of the possible states. Any possible states that is given by the blue blob.

Here, we have a more specific expression capital pi n comma n represents if nth field entering and the interaction and emerging as a field m. So, here in the previous case we were we had the emerging field could be any field.

Here, in the case of a propagator we identified, the emerging field the outgoing field the outgoing leg, as leg m. So, we have an inter incoming leg incoming external leg, phi n that is the incoming field and the outgoing field is phi m. So, that is represented by the propagator pi n comma m. So, this is the difference, this is very important that the as far as the incoming field is concerned we have a single field here, we have a single field in the previous case.

However, in the outgoing case we did not restrict ourselves to any single field in the previous case here, we are restricting ourselves to a particularly identified field that is field m. So, this red blob represents the total number of diagrams that contain only two-point vertices or no vertices, no vertices or two-point vertices and that has field's n and m at the external legs.

So, the field's n and the field m are now identified. If you compare this with the previous case here, we the incoming field was identified. But, the outgoing field was not identified, the outgoing fields could have been any field and this blob consisted of all two-point diagrams zero or more then one or more two-point diagrams all these were captured in this blue blob.

Zero or more two-point vertices I am sorry, not diagrams zero or more two-point vertices.

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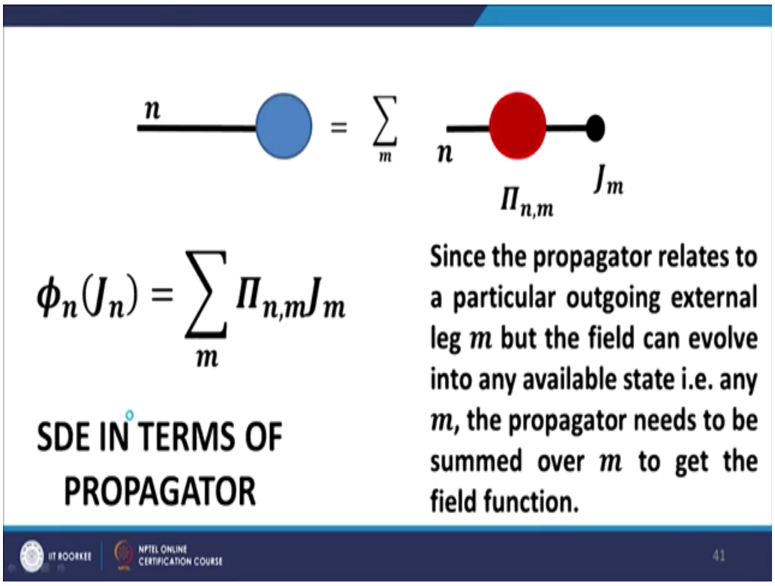

So, that is the difference. So, in other words if we have to express the field function in terms of the propagator, we have the field n; now this field n after in the blue blob can emerges any one of the out possible outcoming states of the system.

Where there whereas, in the propagator we have the field n and it could emerge as a particular outgoing state m and therefore, the relation between the two diagrams is that the right hand side diagram must be summed over all the possible values of out outgoing states m to arrive at the diagram given on the left hand side.

In other words, what we get is phi n is equal to summation over m summation over the possible outgoing fields pi n m that is the propagator J m, J m is the source corresponding to the field m, J m is the source corresponding to the field phi m. So, this is the Schwinger-Dyson equation in terms of the propagator for the simple theory that we have.

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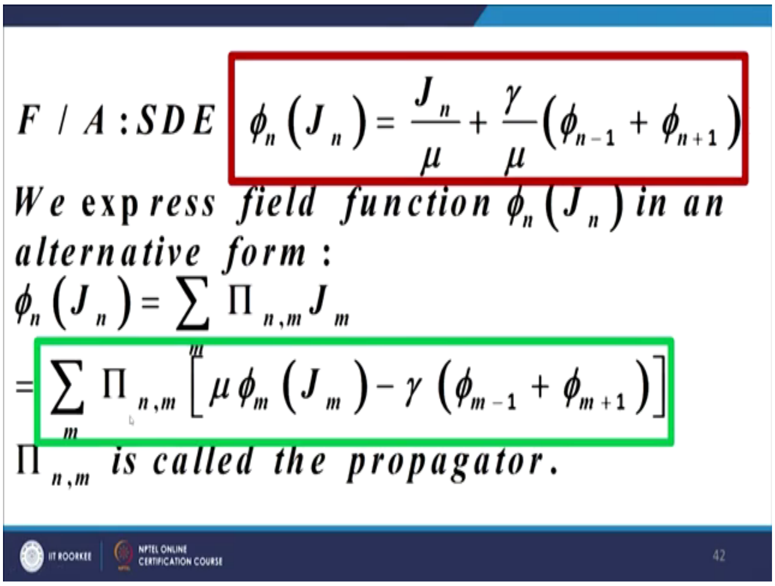

This is we can express this as a recursive relation, we have the Schwinger-Dyson equation for phi n therefore, we can change indices and we can write the Schwinger-Dyson equation for phi m in the this form, simply substituting n equal to m and we have the expression for phi m; we substitute that value of phi m in this expression we substitute the value of J m from this expression for phi m and we arrive at the expression in the green box.

Let me repeat, we have this Schwinger-Dyson equation for phi n given in the red box. From here, we can obtain a change indices to m, and we can write the expression for phi m from where we can obtain the expression for J m and we can substitute J m in the expression for the field n phi n using the propagator expression and we arrive at the expression recursive relation given in the green box at the bottom of your slide.

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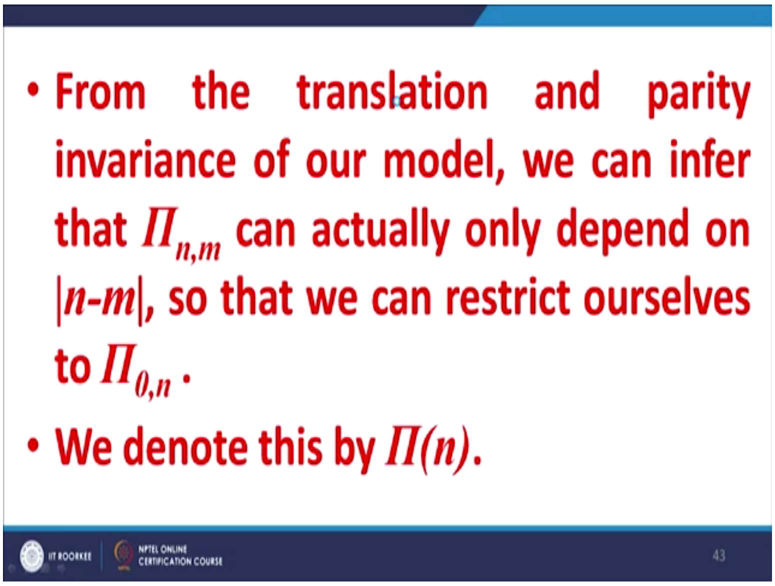

Now, because of the translation and parity invariances we can make a simplification in the representation of the propagator pi n comma m, because of the translation invariance, it the propagator must depend only on the difference n minus m, and secondly, because of parity invariance it can the propagated from n to m; would be the same as the propagator from m to n therefore, we can write it as pi n m k is equivalent to writing pi n mod of n minus m.

We can further simplify it by taking by using one of them as 0. As, the origin because it is translation invariant. So, the origin can be shifted without modifying the propagator and therefore, we can write pi n m as pi 0 n also. And which we represent by pi bracket n.

So, to repeat because of translation invariance and parity invariance operating in tandem, we can write pi n m as pi 0 n because risk first of all because of translation invariance, you can change shift the origin and because of parity invariants 0 to n or n to 0 makes no difference. So, you can identify it either way and you can write it as pi n.

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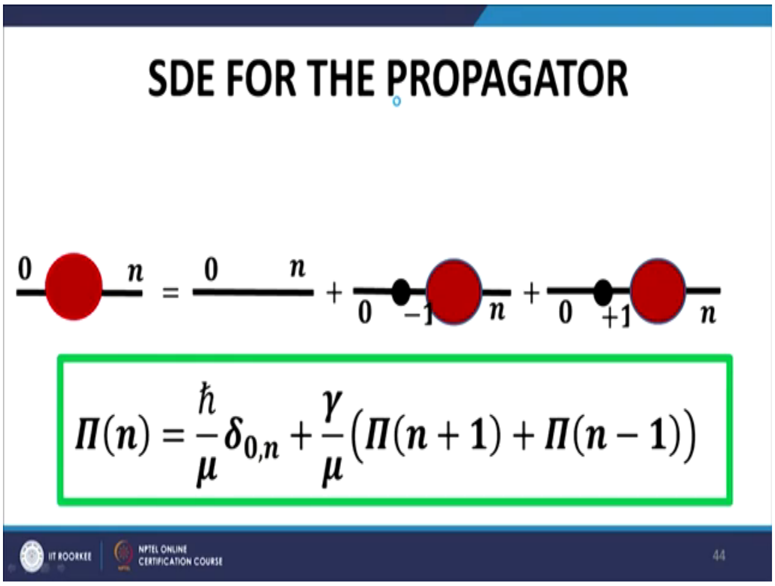

Now, Schwinger-Dyson equation for the propagator, let us talk about the propagator pi n is supposed to represent pi 0 comma n. In other words what is pi 0 comma n? Pi 0 entering the propagator the field phi 0 entering and the field phi n emerging from the external leg, that is what the propagator phi 0 comma n is supposed to represent, that is the left hand side here.

Now, it can so happen that n is nothing, but the 0th field, in other words the particle does not interact at all and comes out without interaction which is represented by the straight line, but which will happen only if n is equal to 0.

So, that is represented by delta 0 n of course, the straight line will evaluate to 1 upon mu then, the particle could interact have a could face a 2 point vertex, a 2 point interaction and therefore, it can either convert which can either convert it to plus 1 or it can convert to minus 1 and after conversion to plus 1 or minus 1 the particle will again face a situation where, because it has to emerge at the field n.

Emerge at the field n it has to face a situation where, the it faces a propagator which transforms it from the state minus 1 to the state n. In other words let me repeat if phi 0 enters the system, interacts at the two-point vertex giving phi minus 1 and phi minus 1 then, is transformed by the propagator to phi n. Because, we want the external leg outgoing external leg to be phi n.

And, the third case can be phi 0 encounters the two-point vertex gets converted to phi 1, and again it interacts with the propagator because it has to be converted to phi n. So, making use of these 3 these are the 3 possible diagrams, if you evaluate the second first diagram I have discussed.

If you evaluate the second diagram the black blob will evaluate to gamma, the entering line will evaluate to 1 upon mu and because we are transforming or the propagator is transforming from minus 1 to n, it gives me pi n plus 1, and similarly the last diagram evaluates to pi gamma upon mu pi n minus 1. And the equation in the green box therefore, is the recursive relation for the propagator. After the break we will proceed to solve this equation.

Thank you.