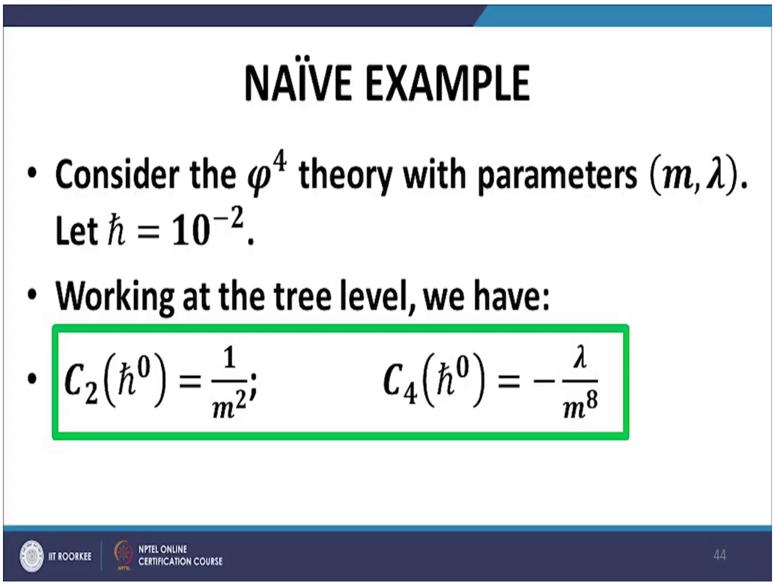
Path Integral Methods in Physics & Finance
Prof. J. P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee

Lecture - 38
Renormalization in O-D

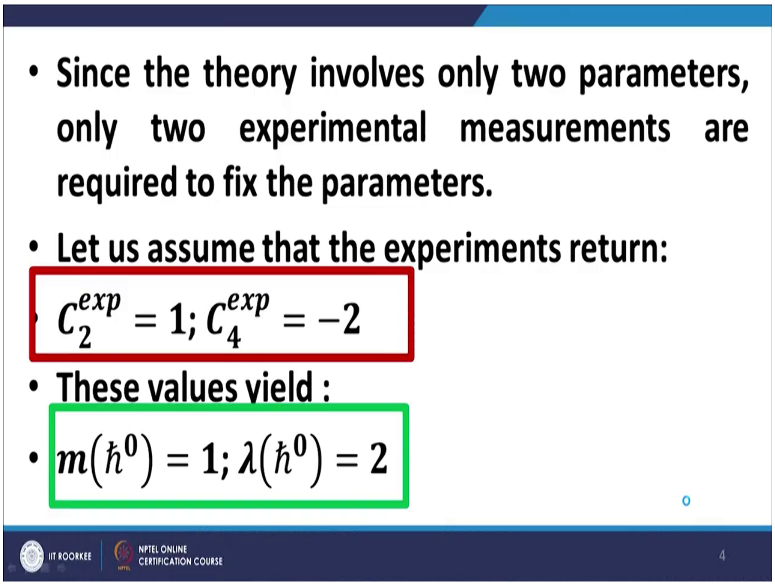
Welcome back. So, we are talking about Renormalization. Let us quickly recap the example.

(Refer Slide Time: 00:31)



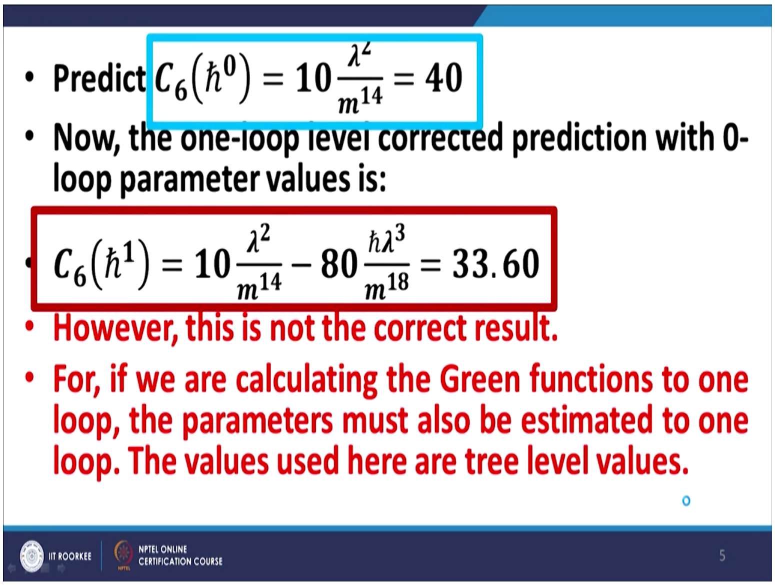
We are talking about in phi 4 theory. The two parameters are m and lambda, h bar is taken as 10 to the power minus 2.

(Refer Slide Time: 00:39)



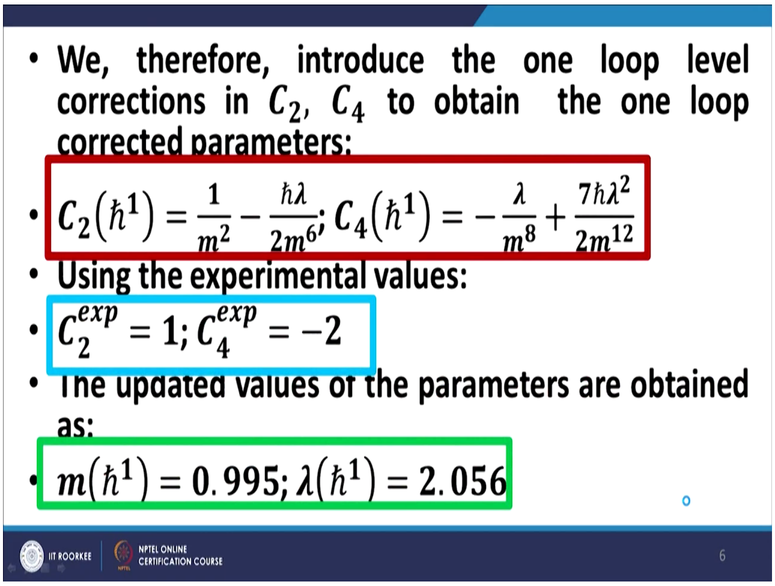
The expression for the 2 point and the 4 point functions at tree level and even in the green box. And we worked out the experimental values of these two Green functions C 2 and C 4, and we found them to be 1 and minus 2. These values at the tree level, if we if we remain at the tree level give gave us the value of m and lambda as 1 and 2, respectively.

(Refer Slide Time: 01:05)



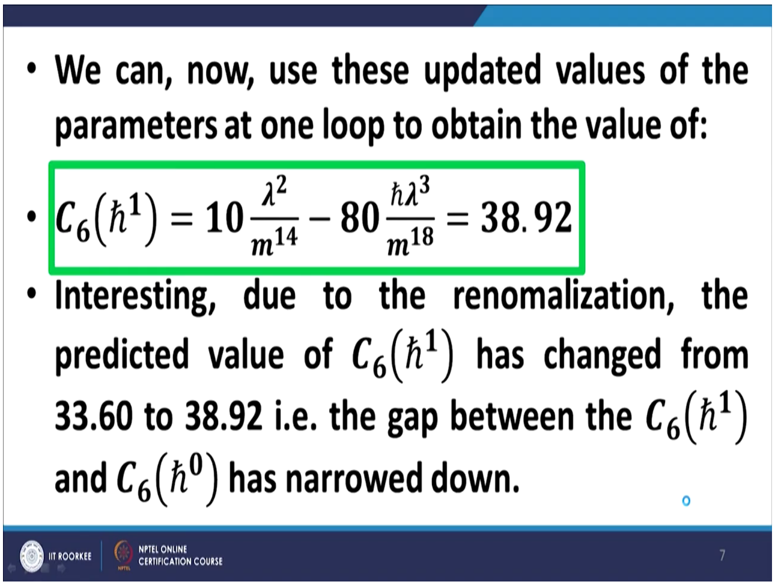
On that basis, we worked out the value of C 6 that the 6 point green function and we found it to be 40. We thereafter increased the level of perturbation from 0-loop level that is the tree level to the one loop level for C 6. And we found that if we use the existing values of a m and lambda that we have obtained earlier, that we obtained earlier m equal to 1 lambda equal to 2, we get the value of C 6 at the one loop level as 33.60. That is what we obtained.

(Refer Slide Time: 01:47)



Then, we went backwards. We worked out the formula for C 2 and C 4 also at one loop level. And we have introduced corrections for the one loop for the first loop level and we found on that basis we using C 2 and C 4 formula at the one loop level refund m h 1 to be 0.995 and lambda at the one loop level as 2.056. If we use these new values for working of the C 6 at the one loop level we find it to be 38.92.

(Refer Slide Time: 02:17)

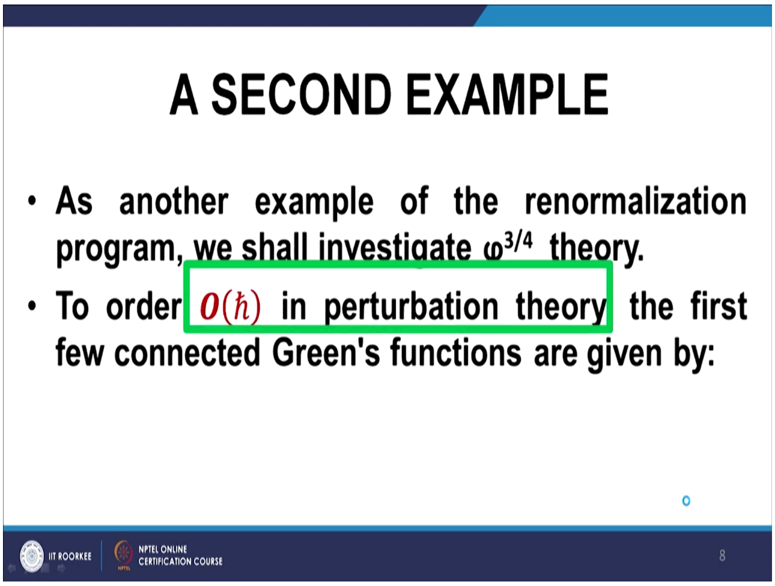


Now, the important things are that firstly, this process shows how renormalization should be carried out. If you are going to work out C 6 at one loop level you must use the bare parameters in the action, also worked out at one loop level. In other words, the green function that have been used for working out the action parameter must also have a consistent or an equivalent level of accuracy, level of perturbation to the Green functions that are being predicted. So, that is what renormalization mandates.

The second interesting part is that when we use the action parameters at the tree level we got the value of C 6 at one loop level as 33.60, and when we use the expression for the action parameters at one loop level and worked out C 6 also at one loop level we got 38.92.

Now, obviously, it is very clear from very obvious from this point that the difference between the between the C 6 values without renormalization is much much more than the difference between the values at successive levels of perturbation worked out at after renormalization. So, that is important. Renormalization tends to make the values converge rather than diverge, that is a very important aspect that and that was formalized in the process of working out the theory of renormalization.

(Refer Slide Time: 03:25)



Let us take another example, slightly more involved example. We again you in this case we use the phi 3 by 4 theory.

(Refer Slide Time: 04:38)

$$C_1 = \hbar\left(-\frac{\lambda_3}{2\mu^3}\right) + O\left(\hbar^2\right)$$

$$C_2 = \hbar\left(\frac{1}{\mu}\right) + \hbar^2\left(-\frac{\lambda_4}{2\mu^3} + \frac{\lambda_3^2}{\mu^4}\right) + O\left(\hbar^3\right)$$

$$C_3 = \hbar^2\left(-\frac{\lambda_3}{\mu^3}\right) + \hbar^3\left(-\frac{4\lambda_3^2}{\mu^6} + \frac{7\lambda_3\lambda_4}{\mu^5}\right) + O\left(\hbar^4\right)$$

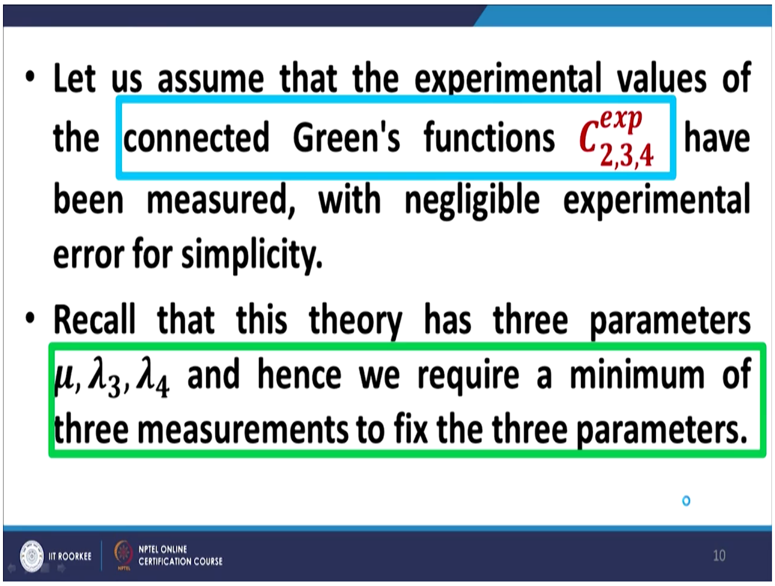
$$C_4 = \hbar^3\left(-\frac{\lambda_4}{\mu^4} + \frac{3\lambda_3^2}{\mu^5}\right) + \hbar^4\left(\frac{24\lambda_3^4}{\mu^8} + \frac{7\lambda_4^2\lambda_4}{2\mu^6} - \frac{59\lambda_3^2\lambda_4}{2\mu^7}\right) + O\left(\hbar^5\right)$$

$$C_5 = \hbar^4\left(\frac{10\lambda_3\lambda_4}{\mu^6} - \frac{15\lambda_3^3}{\mu^7}\right) + \hbar^5\left(\frac{605\lambda_4\lambda_3^3}{2\mu^9} - \frac{192\lambda_3^5}{\mu^{10}} - \frac{80\lambda_4^2\lambda_3}{\mu^8}\right) + O\left(\hbar^6\right)$$

IIT ROORKEE | NPTEL ONLINE CERTIFICATION COURSE | 9

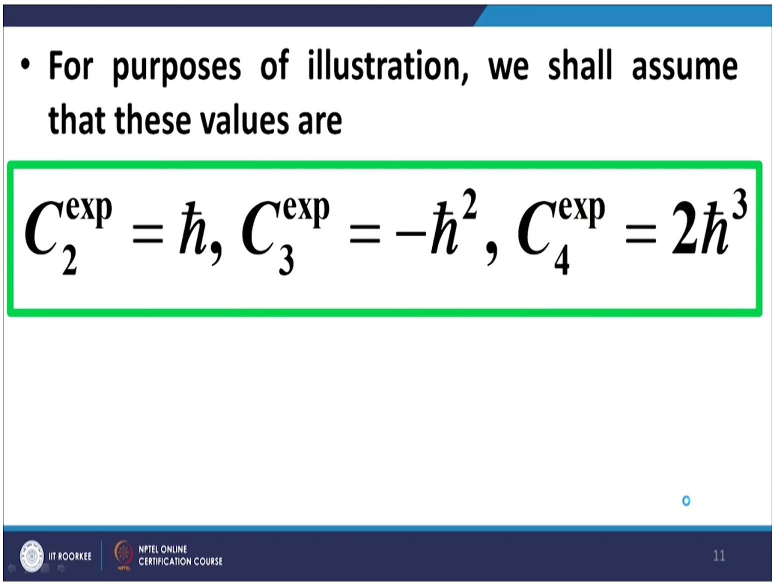
And, the values of the various green function connected Green functions to first order in perturbation are given in your slide C 1, C 2, C 3, C 4 and C 5.

(Refer Slide Time: 04:51)



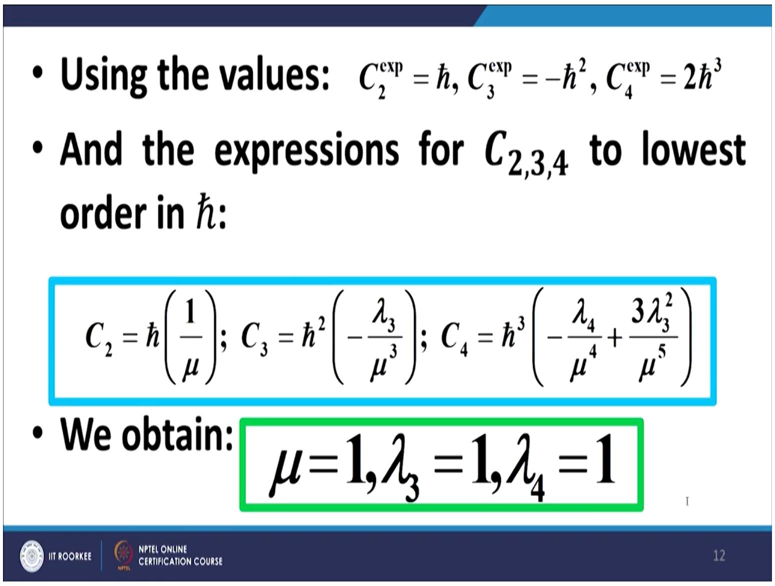
Now, we assume that the experimental results, we experiments carried out scattered, scattering experiments are carried out and connected Green functions C 2, 3 and 4 experimental were measured and obtained. And these experimental values were used for calculating mu lambda 3 and lambda 4, lambda 4 at the first level of perturbation, at the one loop level of perturbation.

(Refer Slide Time: 05:17)



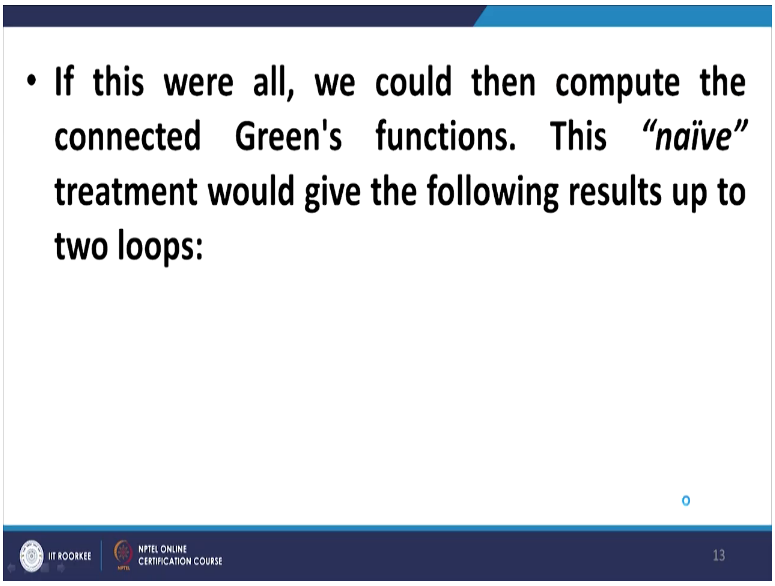
And let us assume that we get the experimental values as C 2 as h bar, C 3 as minus h square and C 4 as 2 h cube.

(Refer Slide Time: 05:33)



Now, using these experimental values and using the expressions for C 2, C 3 and C 4 at lowest level in h bar what we get is mu is equal to 1, lambda 3 is equal to 1, lambda 4 is equal to 1. This is what we obtain from the equations that are there in the blue box and we get the results that are there in your green box at the bottom of the slide.

(Refer Slide Time: 06:00)



Now, what we are do is if we use these results; if we use these results and we work out the values of C 1, C 2, and so, at two level two loop level what we get is the results that are shown in your slide, in this slide.

(Refer Slide Time: 06:02)

$$C_1^{naive} = -\frac{1}{2}\hbar + \frac{1}{24}\hbar^2 + O\left(\hbar^3\right);\ C_2^{naive} = \hbar + \frac{1}{2}\hbar^2 - \frac{3}{4}\hbar^3 + O\left(\hbar^4\right)$$

$$C_3^{naive} = -\hbar^2 - \frac{1}{2}\hbar^3 - \frac{131}{24}\hbar^4 + O\left(\hbar^5\right)$$

$$C_4^{naive} = 2\hbar^3 - 2\hbar^4 - \frac{147}{4}\hbar^5 + O\left(\hbar^6\right)$$

$$C_5^{naive} = -5\hbar^4 + \frac{61}{2}\hbar^5 + \frac{5665}{24}\hbar^6 + O\left(\hbar^7\right)$$

$$C_6^{naive} = 10\hbar^5 - 295\hbar^6 - \frac{5105}{4}\hbar^7 + O\left(\hbar^8\right)$$

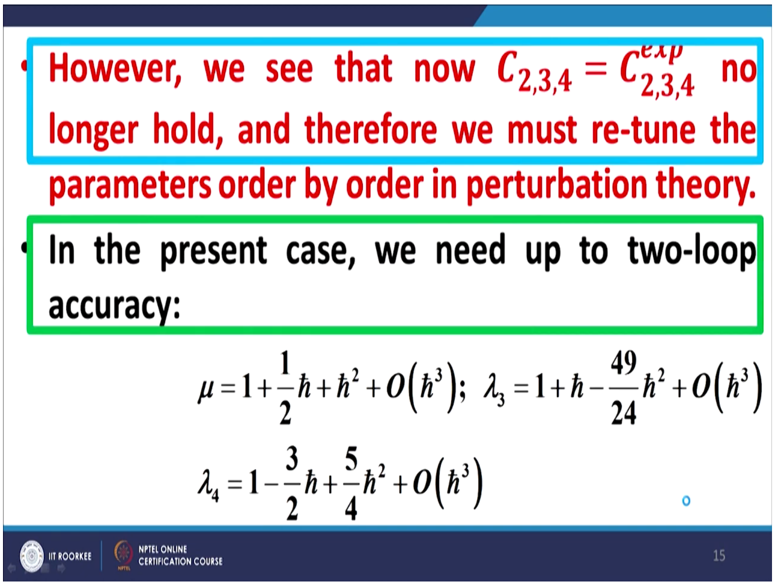
$$C_7^{naive} = 35\hbar^6 - \frac{5195}{2}\hbar^7 - \frac{47075}{24}\hbar^8 + O\left(\hbar^9\right)$$

IIT ROORKEE | NPTEL ONLINE CERTIFICATION COURSE 14

In other words, using the one loop value, these are one loop values the values given at the in the green box or one loop values, one loop perturbation values and if we use this values and we work out the connected functions at two loop levels we get these results.

Now, it is very clear from these results that the values of C 2, C 3 and C 4 that we that is given in this slide do not coincide with the value that were observed experimentally. You can clearly see here C 2, C 3 and C 4 here on this slide, and C 2, C 3 and 4 here at the top of the slide they do not coincide.

(Refer Slide Time: 07:10)

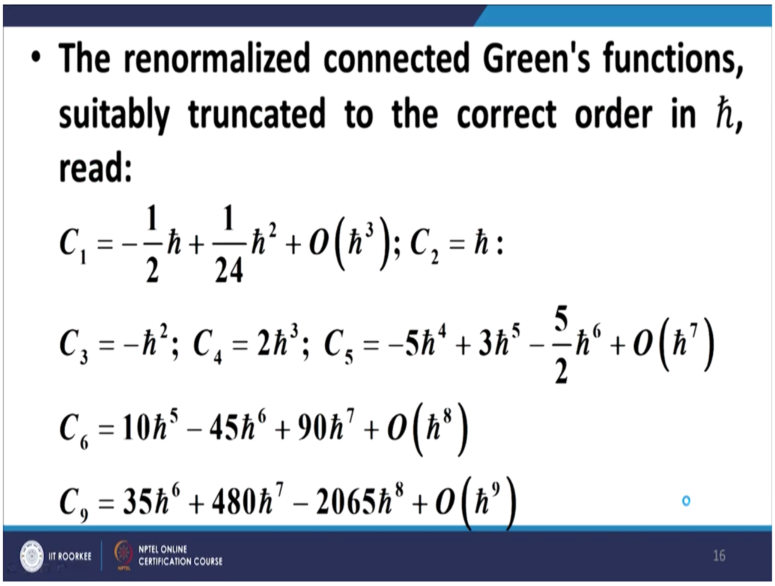


So, what we do is we need to revert back to C 2, C 3 and C 4, we need to rework them at the two loop level when we rework them at two loop level and then workout the values of mu lambda 3 and lambda 4, the values that we get is given at the bottom of the slide.

Let me repeat and C 2, C 3 and C 4 values we had obtained at the one loop level and those were the values that we saw here, here in the green box. These were the one loop values. But we then estimated using this one loop values we estimated the Green functions at the two loop level. And we found that C 2, C 3 and C 4 do not coincide with the experimental results as is to be expected.

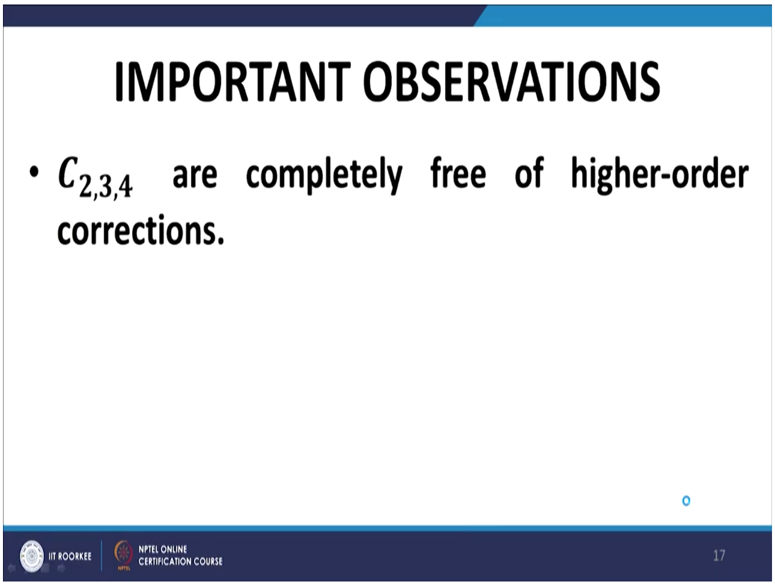
Therefore, we would revisited C is C 2, C 3 and C 4, we worked out the C 2, C 3 and C 4 are the two loop level. Using the revised working of C 2, C 3 and C 4 at the two loop level we rework the parameters at the two loop level that is we rework mu lambda 3 and sigma 4 at the two loop level, and then we use these expression to work out C 1, C 2, C 3 and so on.

(Refer Slide Time: 08:33)

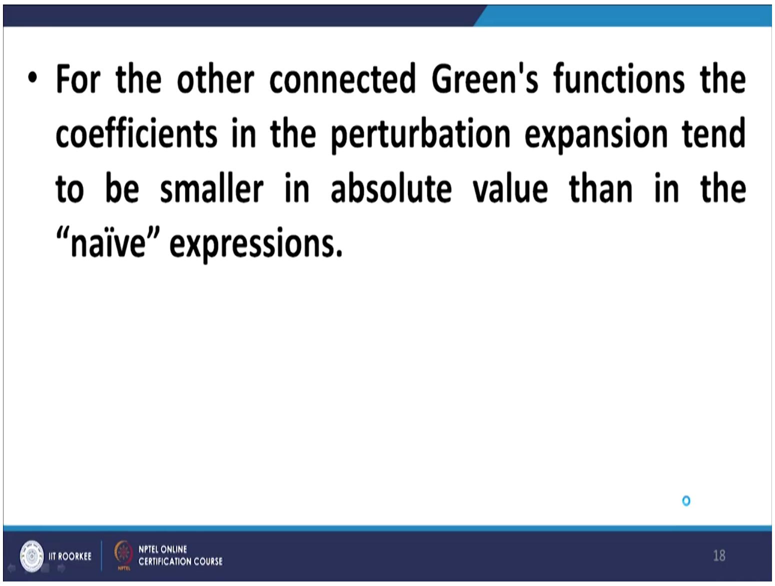


And we find now that C 2, C 3 and C 4 do coincide with the experimental result that is one part of the story.

(Refer Slide Time: 08:48)

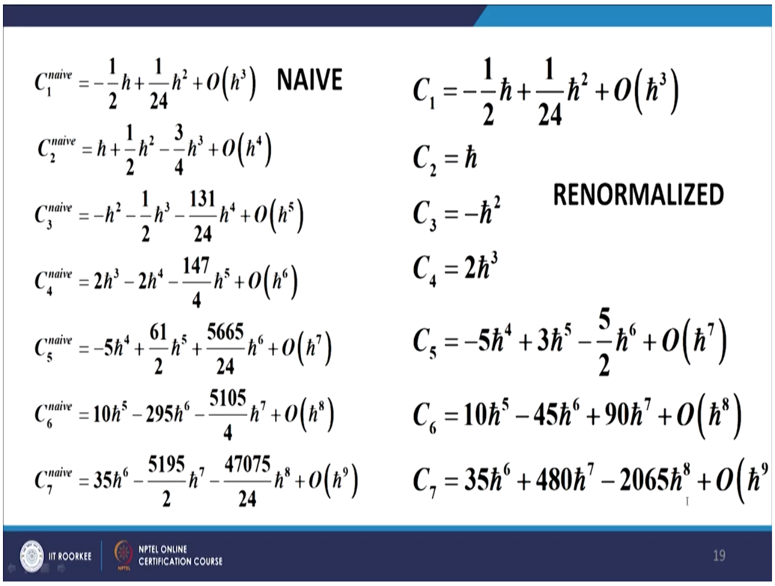


(Refer Slide Time: 08:51)



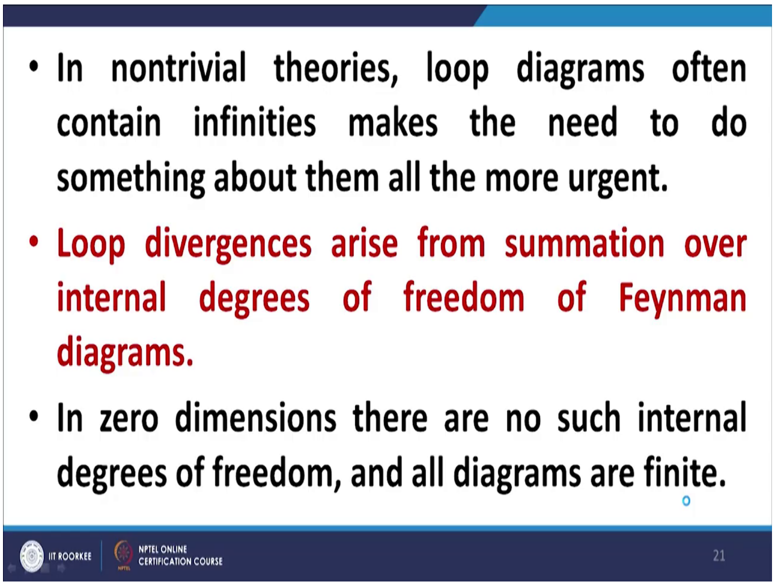
The second part of this story is in that if you look at this if you look at the relative values, the naive values naive values that we use without renormalization and the values that we used with renormalization you find that the differences have are much less in the case of renormalization compare vis-à-vis, the values that are worked out on the naive basis.

(Refer Slide Time: 08:52)



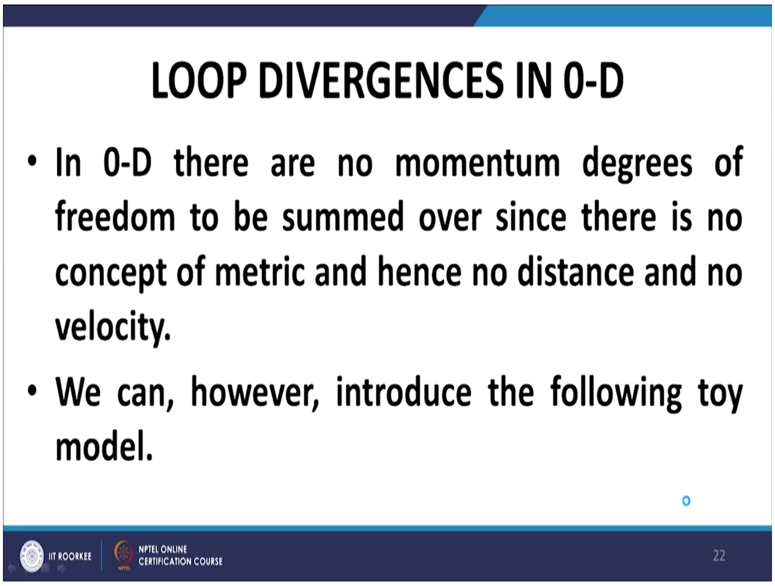
So, that is an important, this is an important topic. This of course, would be carried on towards the when we talk about field theory in Minkowski space but fundament at the conceptual level this is what renormalization is all about. We now talk about loop divergences.

(Refer Slide Time: 09:41)



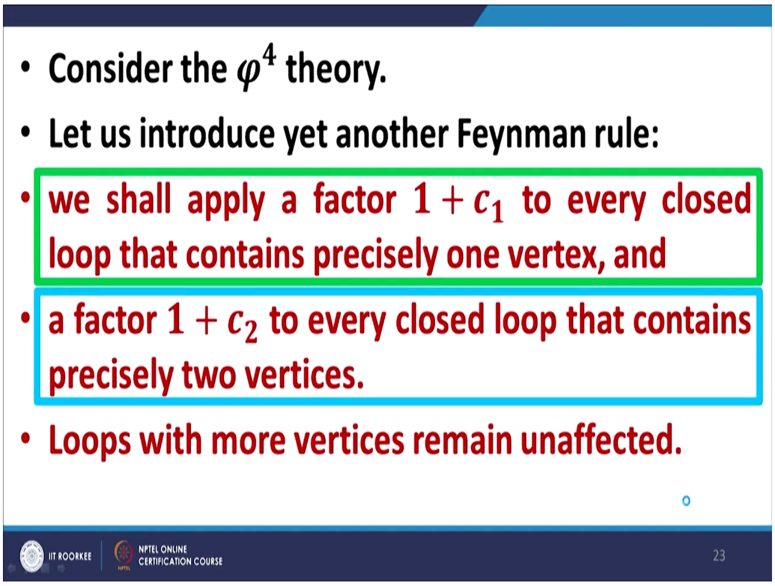
Now, where do, what do we mean by loop divergences? Feynman diagrams for certain field theories or certain types of actions contain diagrams which take the form of loops. These loops arise from summation over internal degrees of freedom in the in these diagrams. When the when the summation is to be done over internal degrees of freedom for example, internal momentum and the integration is to be done from 0 to infinity, divergences may arise.

(Refer Slide Time: 10:19)



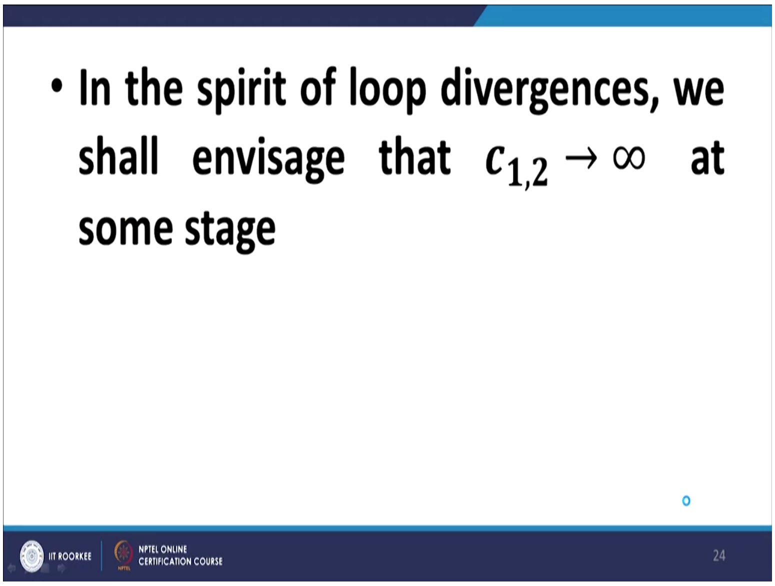
Although, in the case of 0 level, 0-dimensional field theory the concept of loop divergences is redundant because there is no concept of momentum as such, there is no concept of a metric, no concept of a distance and no concept of velocity. And therefore, there is no concept of momentum and as a result of which loop divergence is do not exists in 0-dimensional field theory. But, by way of illustrating the phenomenon we introduce certain new features into our into our phi 4 theory.

(Refer Slide Time: 10:41)



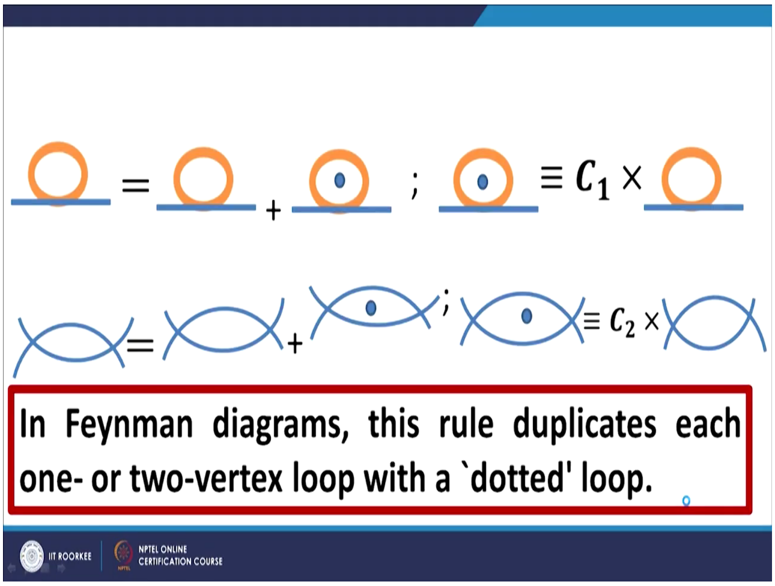
We introduced new rule two new Feynman rules. We apply a factor of 1 plus C 1 to every closed loop that contains precisely one vertex, and we apply a factor of 1 plus C 2 to every closed loop that contains precisely two vertices. Loop with more vertices remain unaffected. So, these are two rules that we add to the existing refer to of rules to externally introduced the concept of loop divergence in our phi 4 theory in 0-dimensional space time because a priori in 0-dimensional space time loop divergences would not exist.

(Refer Slide Time: 11:44)



And we will then later on we will make C 1 and C 2 approach infinity to explore the implications of loop divergences.

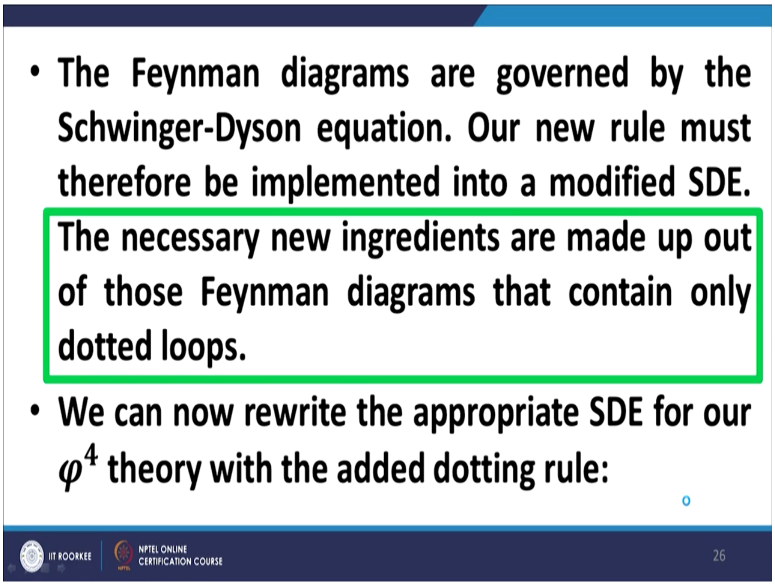
(Refer Slide Time: 11:56)



Diagrammatically in terms of a Feynman diagrams the two rules translate to this to what is shown in your figure in this figure, if you have a loop with one vertex that is the upper diagram it translates to a loop and a dotted loop, where a dotted loop is C 1 into the original loop.

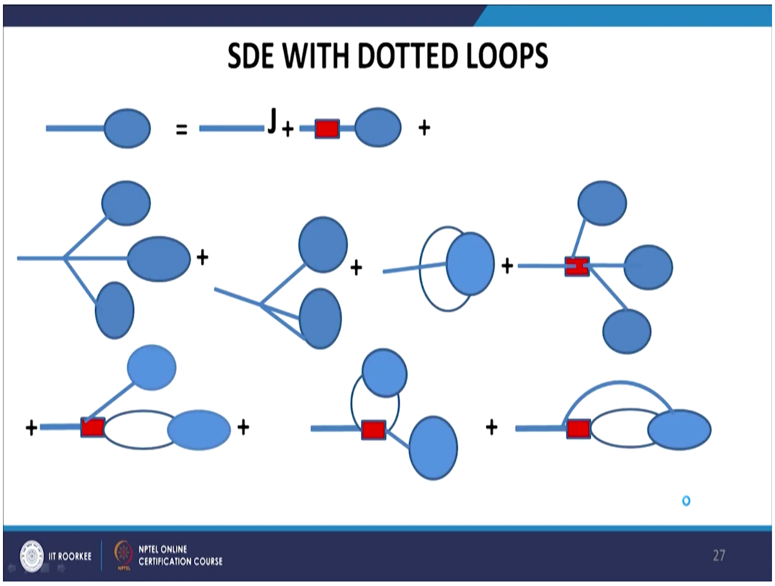
Similarly, if you have a loop with two vertices and the it translates to a loop with two vertices and a dotted loop with two vertices, the dotted loop with two vertices is nothing, but C 2 into the original loop. Recall that in due course C 1 and C 2 would be made to approach infinity.

(Refer Slide Time: 12:43)



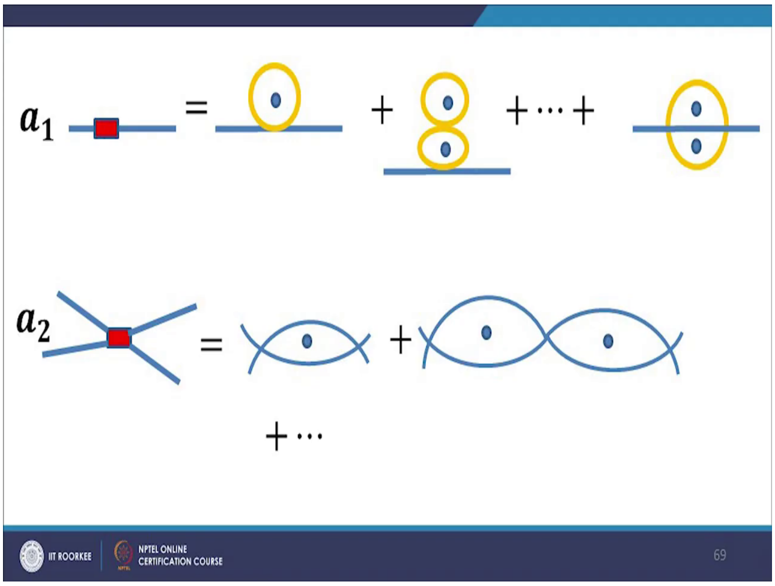
Now, we will look at the Schwinger Dyson equation. Having introduced the new Feynman rules to take cognizance of the possibility of loop divergences we introduced this Schwinger Dyson equations. Let us write down the set of Feynman diagrams, and then write down the Schwinger Dyson equation on the basis of those diagrams.

(Refer Slide Time: 13:03)



The with the introduction of dotted loops the Feynman diagrams red face and the red coloured boxes figures or figures with red coloured boxes or the new additions to the to the Feynman diagrams. And these are mandated by the introduction of dotted loops. Let us look at the definition of these red coloured boxes.

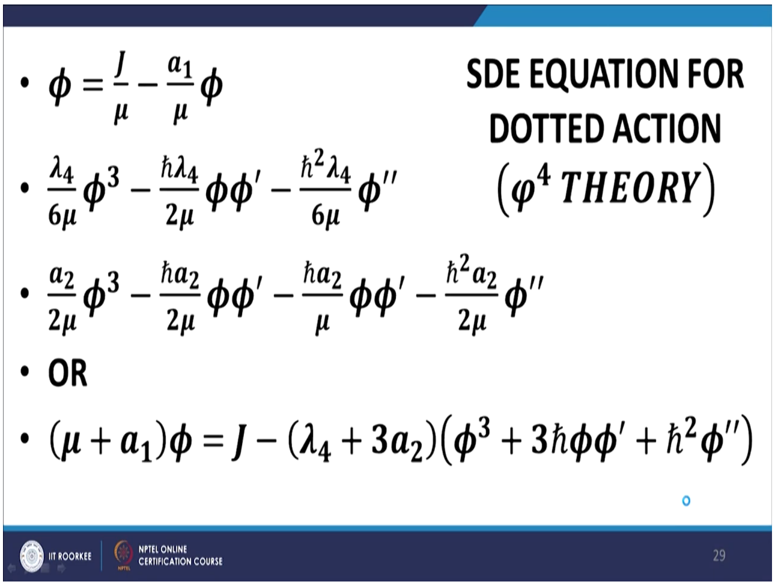
(Refer Slide Time: 13:36)



The red coloured boxes with two lines represent a single vertex with a with the dotted loop in this form which is given here. This is nothing, but C 1 into the original loop, this is C 1, C 2 in original loop, then we will have C 1, C 2 square into the origin and so on, and this diagram is the extra diagram here. And if we have a box with 4 vertices, this is equal to the two what two vertices dotted diagram and then the two vertice dotted diagram in parallel, and then 3 and so on. So, this is how we define the boxes that are shown here.

And we let us call this a 1 and let us call it a 2. And the upper box is the upper diagram on the left hand side let us call it a 1 the lower diagram on the left hand side let us call it a 2. Then, making use of this set of Feynman diagrams and we can write down the Schwinger Dyson equation including these loops in the form of this expression.

(Refer Slide Time: 14:54)

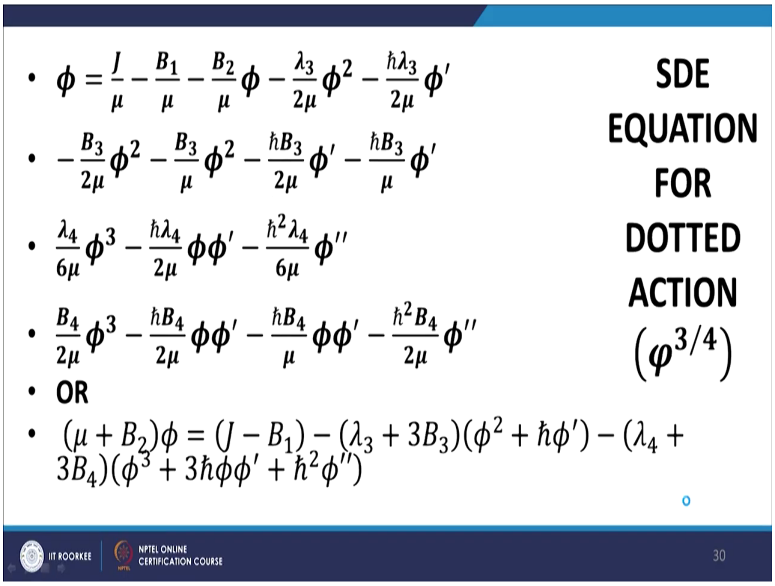


This is the Schwinger Dyson equation for the phi 4 theory including the various loops and it can be written simplified and written in the form which is given in the lower equation, the bottom equation here.

The upper equation corresponds to a up diagram wise evaluation of each diagram. For example, the first diagram here is evaluate evaluated as J upon mu. The second diagram if you note is evaluated as minus a 1 upon mu into phi. And this part is phi, the this loop is phi, the box here is minus a 1 and we have one line, so we have 1 upon mu; so, 1 upon mu, a 1 into phi.

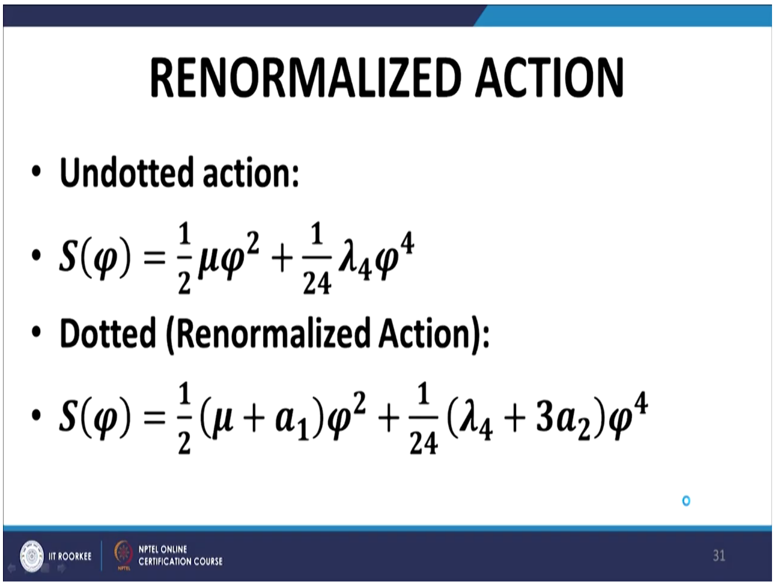
And that is in that is the way the evaluation is done for all the other diagrams here. And this whole equation can be written simply in a simplified manner in the second equation that is given at the bottom of your slide.

(Refer Slide Time: 16:04)



For the phi 3 4 theory the equation becomes a little bit more involved. It is the form that is given on this slide.

(Refer Slide Time: 16:13)



Now, if you look at the action that corresponds to the Feynman diagrams that include the dotted diagram, that in other word that include the red boxes. Then, the dotted renormalized action takes the form of the bottom equation, the equation that is given below.

This is the this was the original equation, the upper equation was the original equation and the bottom equation is the renormalized equation action, renormalized action, the upper action is the original action as you can very well fine see. And the bottom action is the renormalized action which gives rise to this particular Schwinger Dyson equation, the bottom Schwinger Dyson equation.

Now, the it is a very interesting observation here. In this action if you look at it the bare parameters of the action that is mu and lambda 4 these were the bare pair parameters as you can see from the upper expression for the action. These were the original parameter. They do not appear alone anywhere. In the action because they appear in the action itself together with a 1 mu appears with a 1 and lambda 4 appears with 3 a 2.

So, it follows as a corner a that wherever in whether it be an experimental measurement or be it be a theoretical computation of any level or any computation of Green functions are connected green function or any other any other parameter or any other attribute of the theory these two expressions will appear as one unit, as a complete unit. Mu plus, mu will always appear with a 1, and lambda 4 will always appear with 3 a 2, if we take account of the loop divergences represented by the dotted in diagrams.

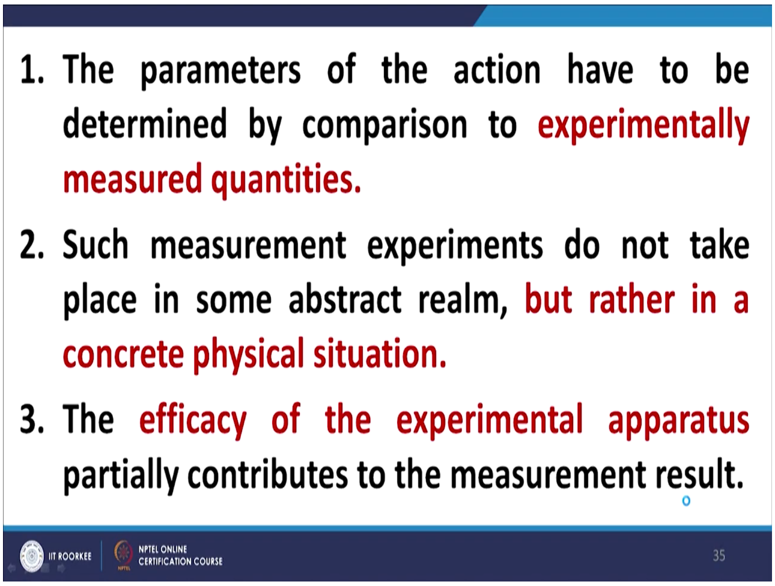
Well, there is an another fallout of this. The fallout is that, if now if now C 1 and C 2 and are allowed to approach infinity or allowed to move infinity representing divergences, then obviously, a 1 and a 2 approach infinity a 1 and a 2 also tend to infinity. So, in some sense it would trigger a reaction that, the action becomes a unbounded and therefore, the theory fails.

But what recovers the theory or what saves the theory if the fact that we can have mu also, the bare parameters also infinite and as such the infinities could very well cancel each other. And if the infinities cancel each other then we are left with finite quantities and therefore, the action becomes finite at every level and all computations arising from the action, in fact, the theory itself becomes a finite and violable theory.

So, in a sense this enables us the fact that mu and a 1 lambda 4 and 3 a 2 occur in pairs invariably essentially enables us to do a kind of renormalization to get out of a situation where the loops become a 1 and a 2, the collection of loops represented by a 1 and a 2 show divergent behavior, right.

(Refer Slide Time: 20:02)

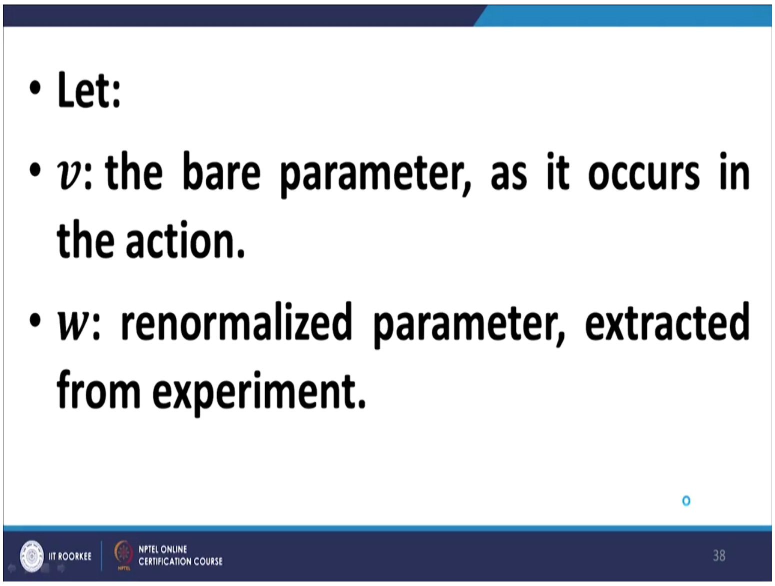
(Refer Slide Time: 20:04)



Now, let us let us explore the concept of renormalization a bit further. Now, we look at the issues scale dependence. When we when I talk about scale dependence, I am essentially talking about the various factors that influence the measurement experimental measurement of the Green functions or the connected Green functions. This is the fundamental thing is that these quantities are usually measured by scattering experiments.

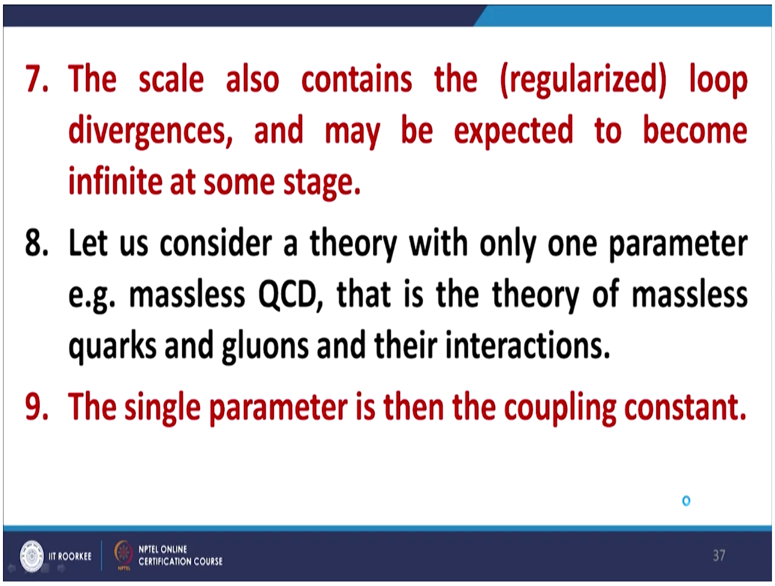
Now, scattering experiments or the outcome of scattering experiments really depends on the on the energy scales at which the experiment experimental apparatus is able to operate. And therefore, it becomes relevant to explore the impact of such factor as the level as the energies involved in this scattering experiments on the measured quantities of attributes or parameters of the theory. So, that is the next exercise, right.

(Refer Slide Time: 21:11)



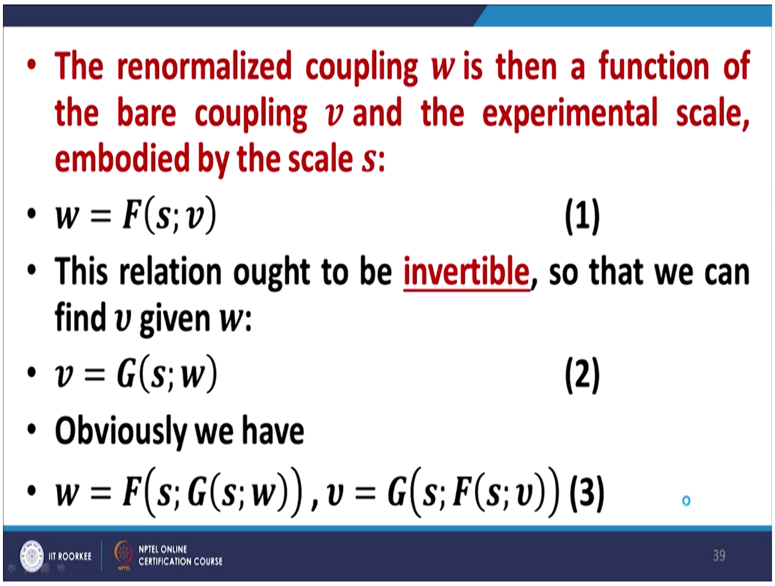
So, let v with the bare parameter as it occurs in the action, v is the original parameter that occurs in the action.

(Refer Slide Time: 21:24)



We were now for the moment we are confining ourselves to an a theory with only one parameter. For example, it could be a massless quarks and gluons theory with their interaction. So, there is only one parameter there is no mass term and there is only one parameter that that is involved in the action that we call v. v is the bare parameter, w is the renormalized parameter which is extracted from the experiment. Repeat, v is the bare parameter as it occurs in the action and w is the renormalize parameter extracted from the experiments.

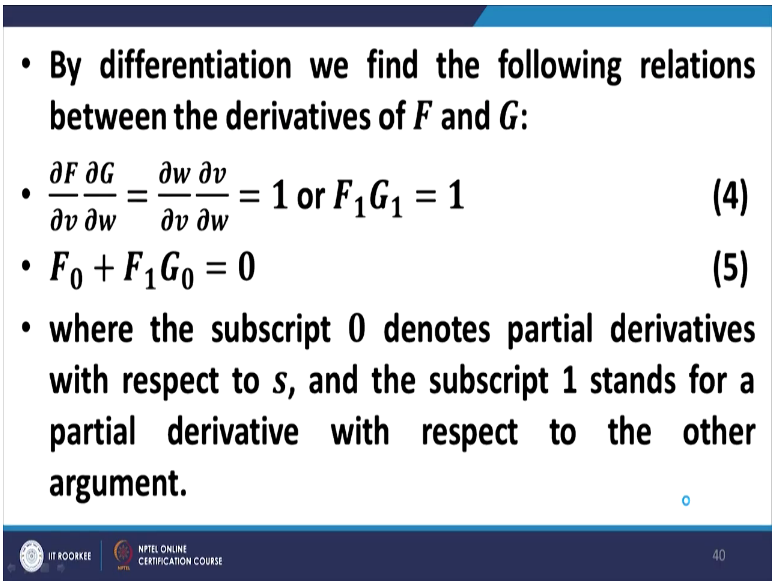
(Refer Slide Time: 21:58)



Now, the renormalized parameter, renormalized coupling or the renormalized parameter w would be a function of not only the original parameter the bare parameter v, but also the scale of energy or such similar factors that we are clubbing together as a scale. So, it they it would be a function number 1, of v which is the bare parameter the true parameter of the theory and number 2, as the scale of operation of the operators by which this determination is taking place. So, we can write w as a function of s and v.

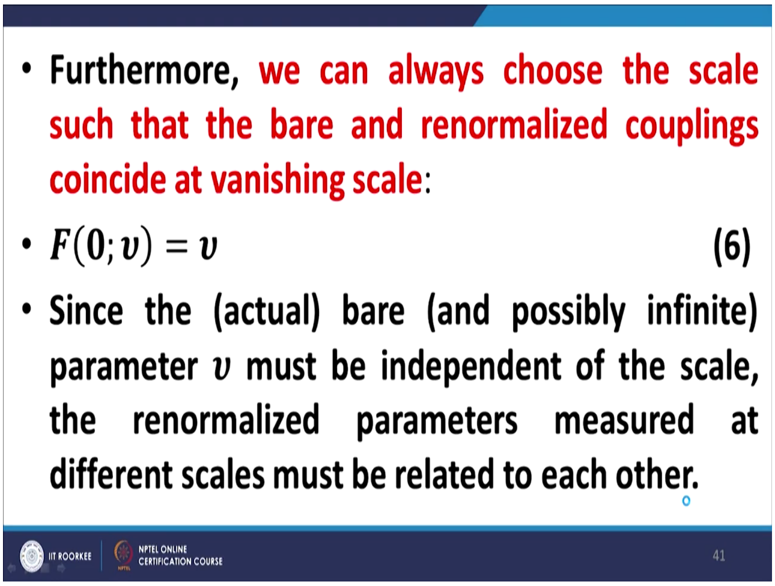
Now, we need this relation to be invertible because we want to be able to determine the bare parameter out of the renormalized outcomes or the renormalized computations or the renormalized observations. And therefore, we must have v is equal to G s and, or G G of s and w which enables us using 1 and 2, we can simply write equation number 3.

(Refer Slide Time: 23:10)



Now, by differentiating by differentiating these two expressions, equation number 1 and equation number 2 we can arrive at these two fundamental results. F 1 G 1 is equal to 1, F 0 plus F 1 G 0 is equal to 0, where the suffix 0 relates to differentiating will differentiating with respect to s, and 1 represents to differentiation with respect to the other parameter, other independent variable that the a 4 G depend on. For example, F depends on two parameters s and v. So, when I write F 1 it is the derivative of F with respect to v and when I write F 0 it is a derivative of F with respect to s.

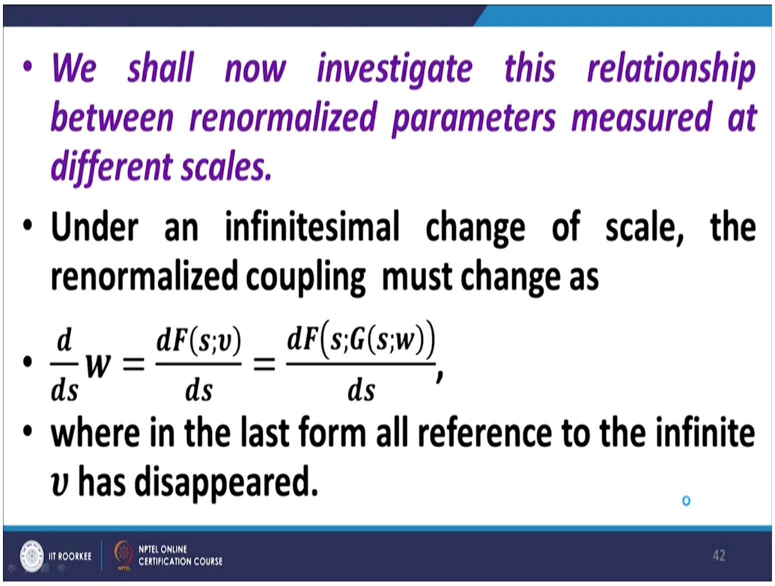
(Refer Slide Time: 24:09)



Now, with we can choose the scale such that the bare and renormalized couplings coincide when the scale is 0. In other words, we can define our scale in such a way, define s in such a way that at s equal to 0 the renormalized and the renormalized value v and the bare value renormalize value w I am sorry; renormalize value w and the bare value v coincide.

Let me repeat we can choose the s equal to 0 at that point, at that scale at which the bare value or the bare parameter v equals the renormalized parameter or the value of the parameter through renormalization that is w. So, I can write F 0 of v is equal to v, because remember F 0, F s of v was equal to w if you recall, F s of v was equal to w. We are defining the 0 of s such a way that at s equal to 0, w is equal to v.

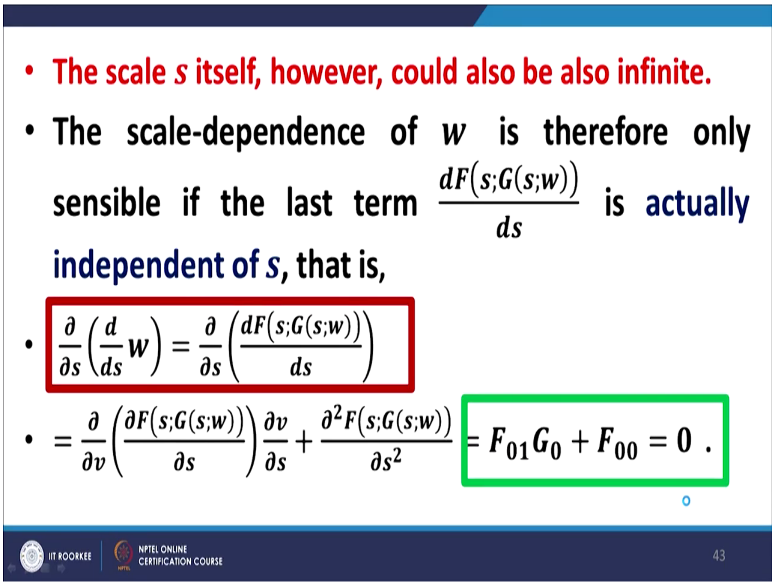
(Refer Slide Time: 25:29)



Now, we investigate the relationship between the renormalized parameters that is w and the various scale that is the energy scales or such other scales. So, under an infinitesimal change of scale the renormalized coupling will change as ds d w by ds, d w by ds is given by this expression, and w is nothing but F s v and that is nothing but this expression where I have substituted a v equal to G s or w.

So, interestingly if you look at this expression, the first and third equations, the first and third terms there is no mention of the parameter v at all here. There is absolutely no mention of the parameter v. Recall that the parameter v could also take infinite values in view of the concept of renormalization that I had explained just a few minutes back.

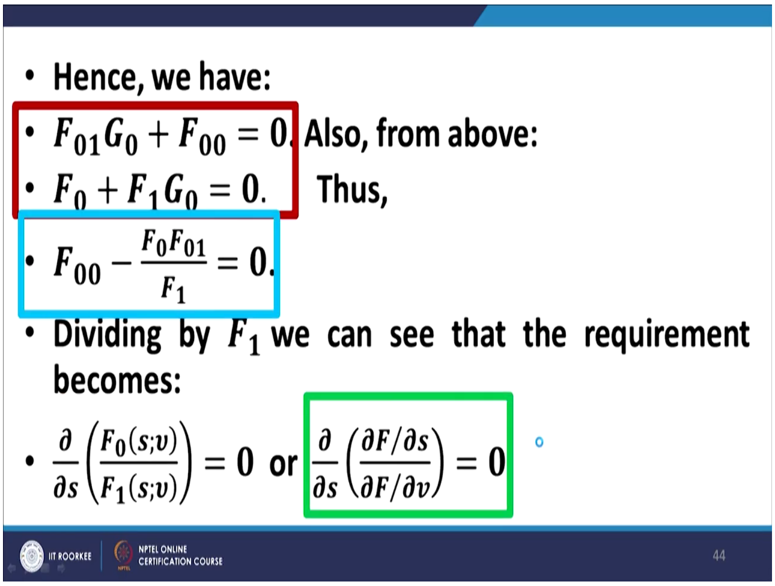
(Refer Slide Time: 26:21)



Now, however, the scale itself can become infinite and therefore, this expression d w upon ds can have a meaningful or a sensible value or a logical value only if the derivative with respect to s or this expression d w by ds is independent of s, because s can take values up to infinity. So, that being the case we must require that the derivative of w with respect to s must be independent of s.

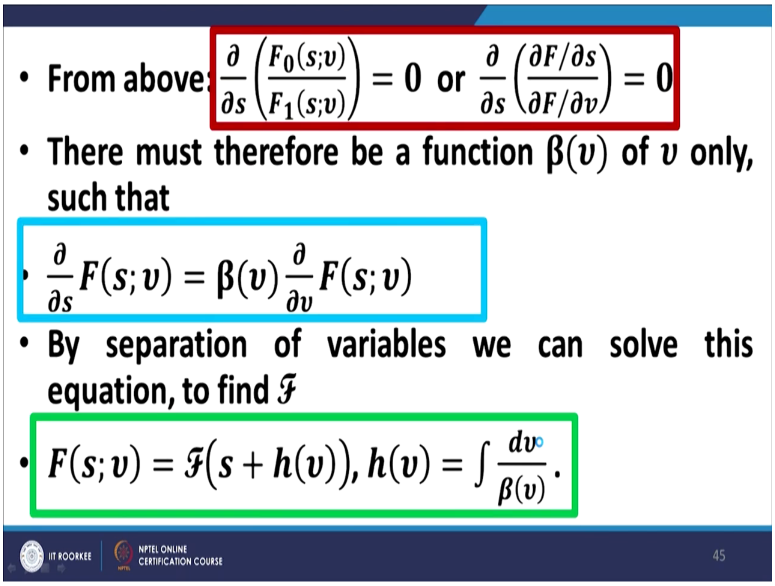
Now, if you simplify this expression, if you simplify this expression what we get is F 01, G 0 plus F 00 is equal to 0. There is quite obvious straightforward steps in simplification, nothing else. Simply taking the total derivative as the expanding the total derivative that is it and we get F 01, G 0, F 00. Recall F 00 is the is the second derivative of F with respect to s, F 01 is the derivative of F with respect to s and with respect to v, and G 0 is the derivative of G with respect to s.

(Refer Slide Time: 28:00)



Hence, we have two equations F 01 G 0 plus F 00 equal to 0 and F 0 plus F 1 G 0 equal to 0. Using these two equations for substituting for G 0 from equation the second equation in equation 1, I get the equation in the blue box. If I divide throughout by F 1 I can write this equation in the form that is given at the bottom of your slide. The first equation at the bottom of your slide which simplifies to the second equation, that is the equation given in the green box at the bottom of your slide.

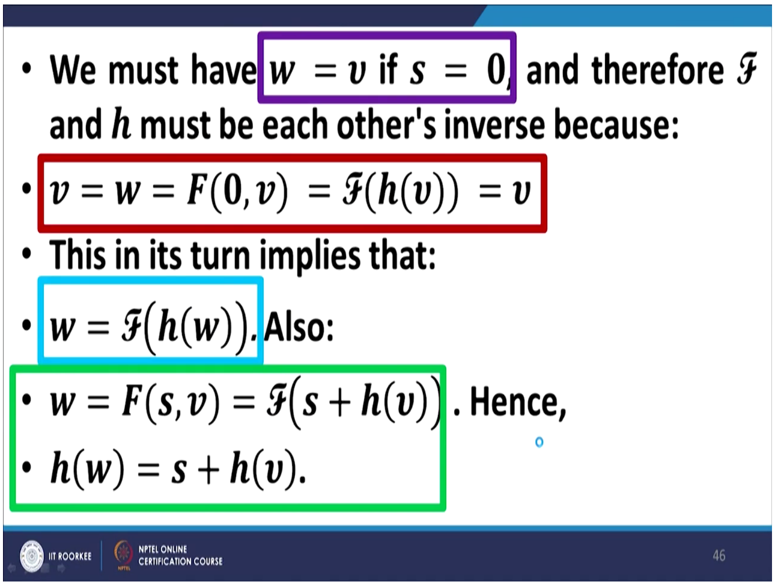
(Refer Slide Time: 28:40)



Now, from this expression we find from this expression if you look at it carefully del by d by ds, dF by ds, dF by dv is equal to 0. That means, what? That means, dF by ds is equal to some constant into dF by dv, some dF by ds is equal to some function of v into dF by dv. dF by ds is equal to some function of v into dF by dv. So, I can write it as this way dF by ds is equal to beta of v, beta of v some function of v into dv by d; dF by dv.

By separation of variables we can solve this equation and we find that F s, v can be written in the form of F script s plus h v, where h v is a new function which is obtained by this expression integral of dv upon beta v.

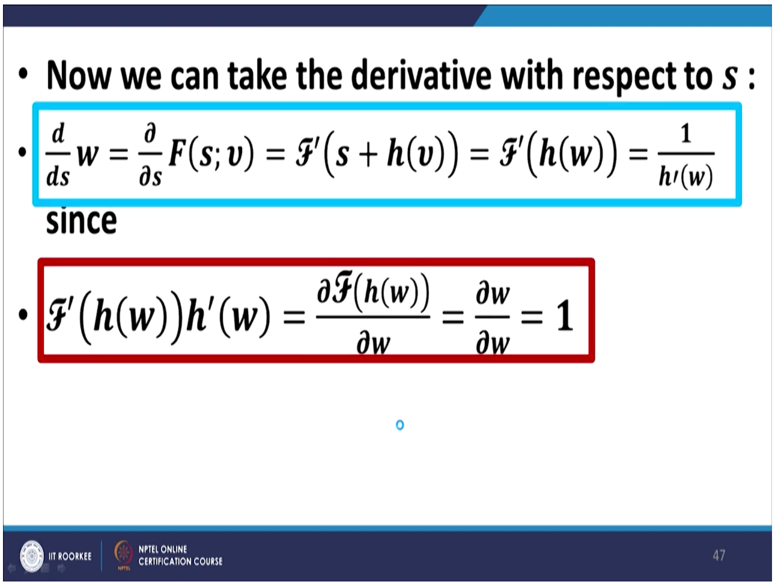
(Refer Slide Time: 29:48)



Now, w is recall, w is equal to v if s is equal to 0. By the definition of s, by the definition of our scale we have defined our scale in such a way that at s is equal to 0, w is equal to v. Therefore, what happens? v is equal to w is equal to F 0 v we have already shown that, is equal to F h of v because s is equal to 0 here. If I put h equal to 0, I get this F 0 v which is nothing but v, which is nothing but w. So, v is equal to w is equal to F 0 v is equal to F F script, s is 0, so F script h v.

And from this what do I infer? I infer from v is equal to F h v, that F and h are inverses of each other, and F and h are inverses of each other. And if h and F and h are inverses of each other, we automatically have w is equal to F of h v. And w, but w is also equal to F h v that is equal to F s plus h v. Now, from these two equations w is equal to F F script h v and w is equal to s script s plus h I am sorry, I am sorry; w is equal to F script h w and w is equal to s script s plus h v, that implies that h w must be equal to s plus h v.

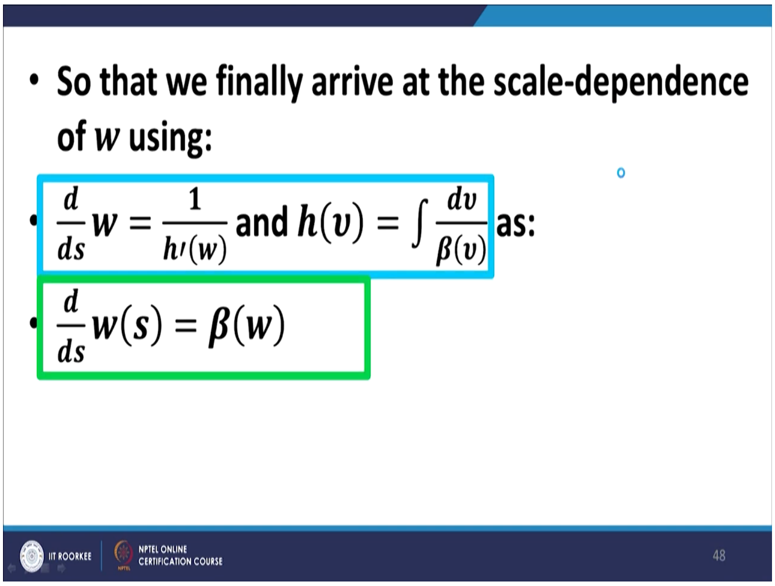
(Refer Slide Time: 31:30)



So, now we take the derivative with respect to s we get the d by ds of half w, d by d as of w is equal to d by ds of F s v that is equal to F s plus h v, that is equal to that is equal to F dash F script dash s plus h v that is equal to F script dash h w that is equal to 1 upon h dash w. This is quite obvious because w is nothing but s plus h v, it is here. Just a minute, yes is here; h w is equal to s plus h v. So, h w is nothing but s plus h v, so I have put this here.

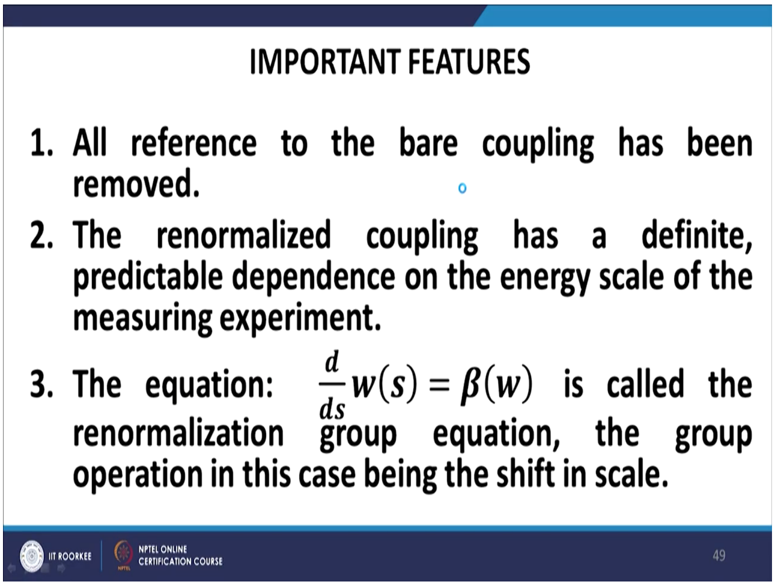
And then I have this is nothing but yeah F s v is equal to F script s plus h v, F h v is equal to, F script s plus h v the derivative takes the derivative here. And then we have s plus h v is equal to h w, so this becomes here and this is equal to 1 upon h dash w. Why is this shown in the expression on the red box? That F dash h w, h dash w is equal to 1 that is that is why F dash s w is equal to 1 upon s dash w.

(Refer Slide Time: 33:01)



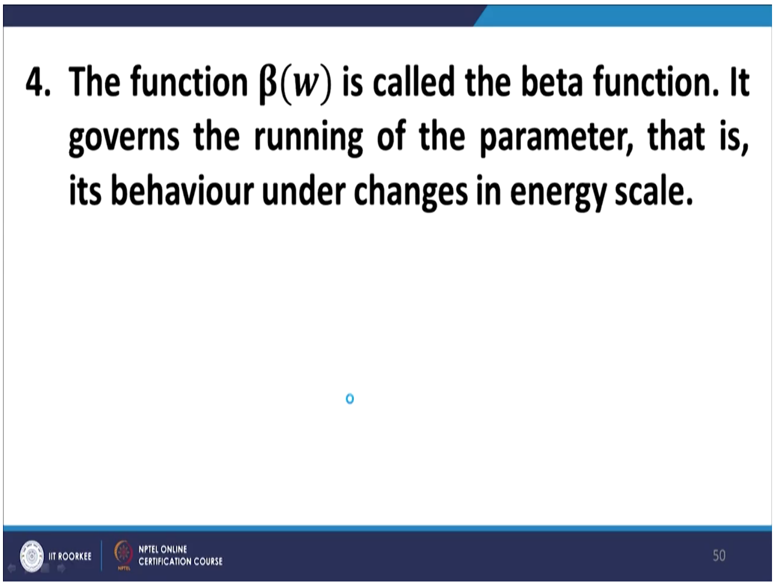
So, we finally, arrived at the scale dependence d w by ds is equal to 1 upon s dash w and, but h v is equal integral dv upon beta v and therefore, therefore, h dash w is nothing but beta w. When you differentiate h v is equal to this expression you differentiate and put v equal to w, what we get is beta w. So, d by ds is equal to w s, is equal to d by ds w s is equal to beta w.

(Refer Slide Time: 33:40)



So, all reference to the bare coupling has been removed number 1. The renormalized coupling has a definite predictable dependence on a energy scale of the measuring experiment that is given by beta w. The equation d by ds, w s is equal to beta w is called the renormalization group equation. And the group parameter is the shift in the scale, the group parameter or the group operation is the shift in the scale. So, I repeat the equation d by ds w, as is equal to beta w is called the renormalization group equation and the group operation is the shift in the scale. So, this is important.

(Refer Slide Time: 34:27)



And beta w is the beta function which governs the running of the parameter that is the behaviour under changes of the energy scale.

Thank you.