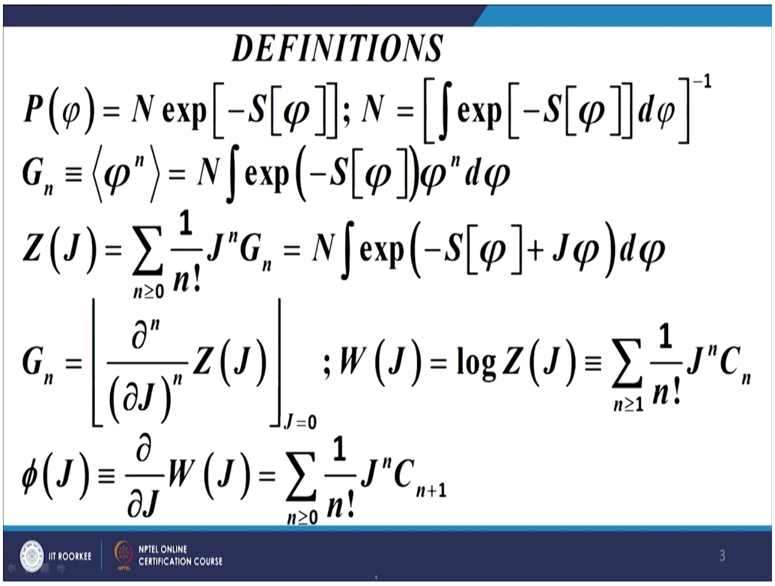
Path Integral Methods in Physics & Finance
Prof. J. P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee

Lecture - 37
Effective Action, Renormalization

(Refer Slide Time: 00:30)

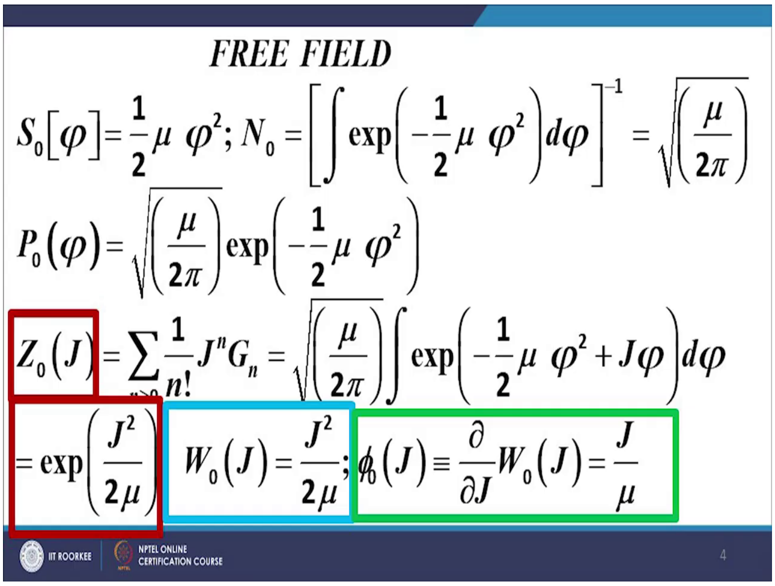


Welcome back. So, a quick recap before we proceed further. The various equations that we have arrived at or we have formulated in the course of our discussions are presented in your first slide. The free field expression for the probability distribution and exponential minus S phi. S is the action. Action is given by for the free field it is given by mu 1 by 2 mu phi square.

We also have the expression for the Green functions as the moments of the distribution. The path integral or the generating function of the Green function is given here. And also, how to extract the Green functions from the path integral by taking successive derivatives is also here.

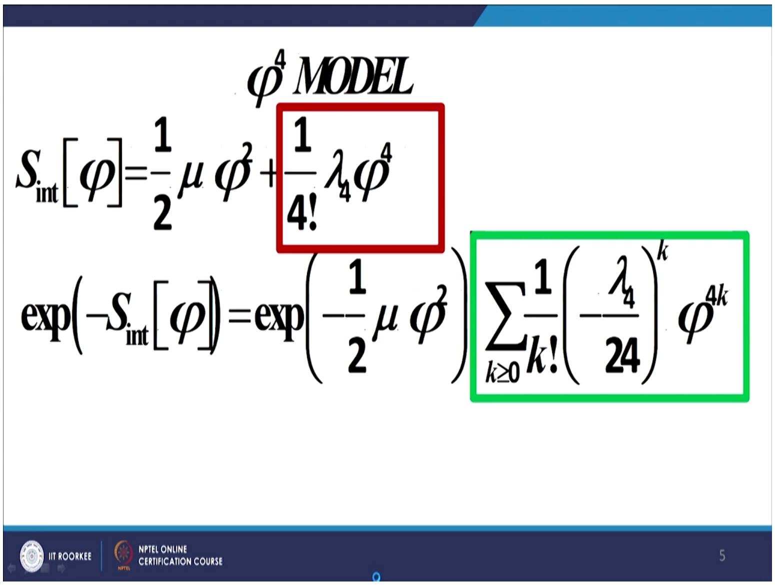
We also have the expression for the path in for the generating function for the connected Green functions as log Z of J. And we also have here the expression for the field function which is the first derivative of the generating function for the connected Green functions.

(Refer Slide Time: 01:32)



For the free field, as I mentioned the action takes the form 1 by 2 mu phi square. The normalization constant we found out is under root mu upon 2 pi and the generating function for the Green function works out to the expression in the red box. The generating function for the connected Green function is given in the blue box and the free field function is given in the green box.

(Refer Slide Time: 02:02)



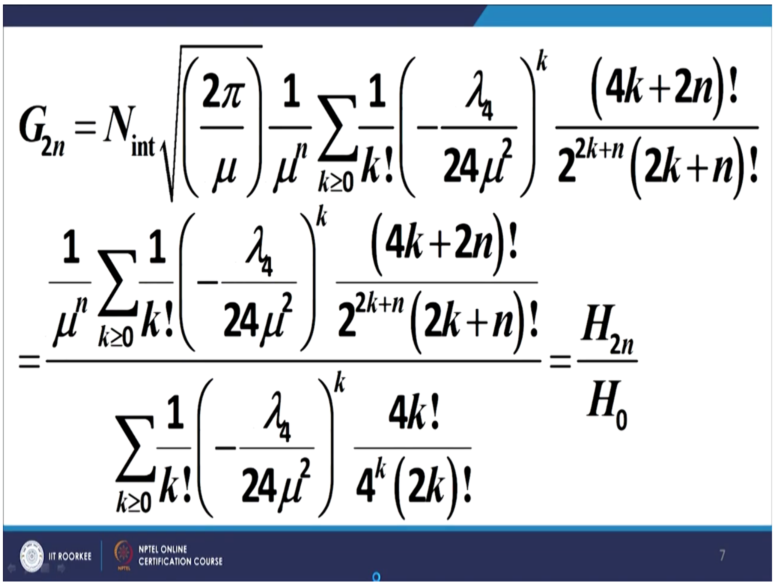
For the phi 4 model, we add an interaction term to the action given by 1 by 4 factorial lambda 4 phi 4. And then we expand this interaction term on the premise that the coupling constant lambda 4 is very small as a perturbation series.

(Refer Slide Time: 02:21)

$$Z_{\text{int}}(J) = N_{\text{int}} \sum_{k \geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24} \right)^k \int \varphi^{4k} \exp\left[-\left(\frac{1}{2} \mu \, \varphi^2 \right) + J\varphi \right] d\varphi$$

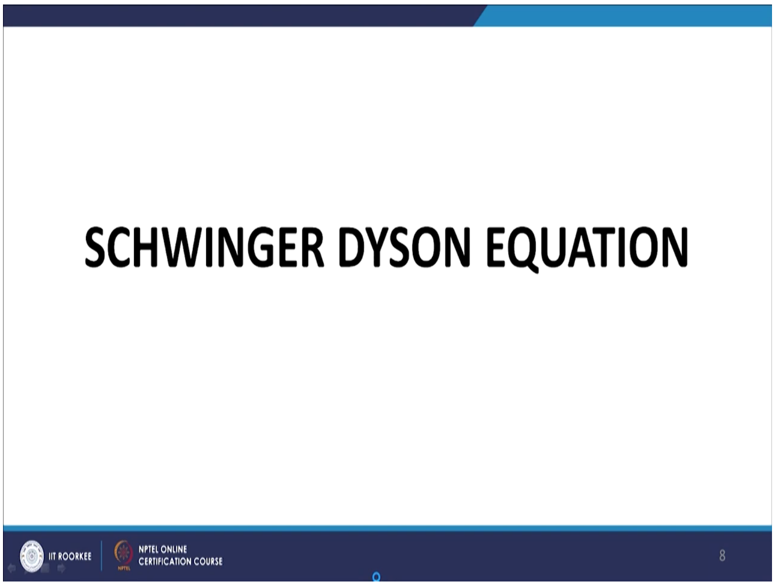
$$N_{\text{int}} = \left[\left(\frac{2\pi}{\mu} \right)^{1/2} \sum_{k \geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24\mu^2} \right)^k \frac{4k!}{4^k (2k)!} \right]^{-1}$$

(Refer Slide Time: 02:31)

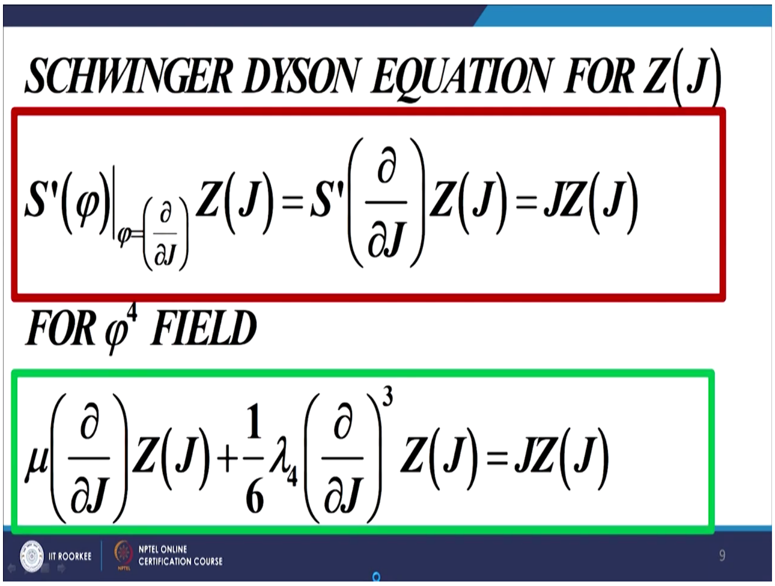


And on that basis, we get the expression for the normalization the new normalization given in the green box here. And we also get the expression for the 2 nth Green function 2 n point Green function as the expression given on the slide, that is represented usually as H 2 n upon H 0 where H 0 is connected to the normalization. It is the inverse of the normalization.

(Refer Slide Time: 02:48)

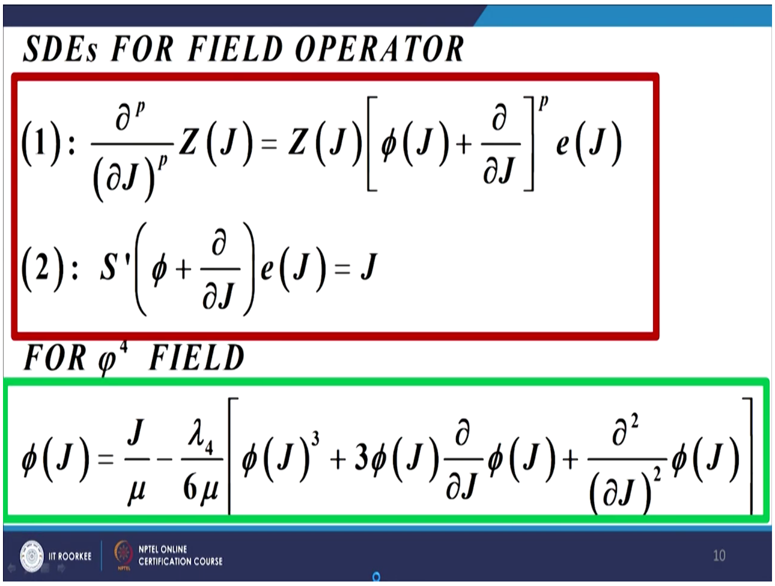


(Refer Slide Time: 02:51)



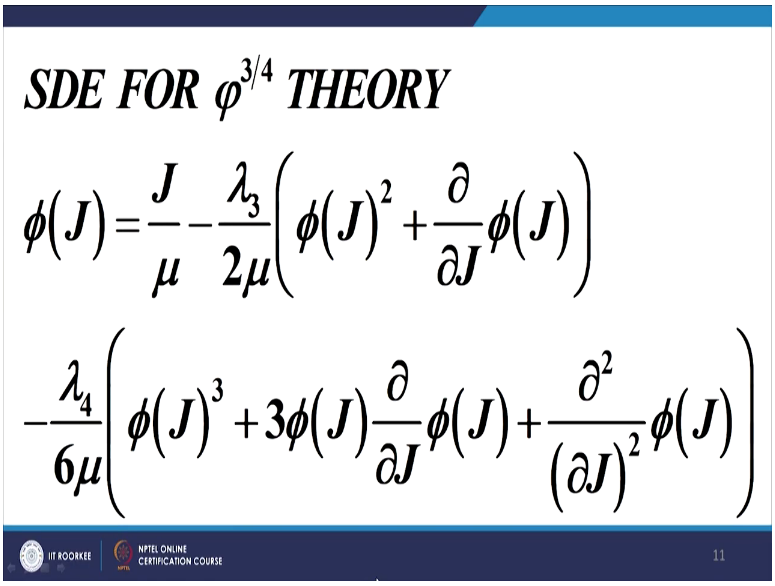
We then obtain the Schwinger Dyson equation. Schwinger Dyson equation for the path integral given in your red box. Schwinger Dyson equation for the specific case of the phi 4 field interaction given in the green box.

(Refer Slide Time: 03:02)



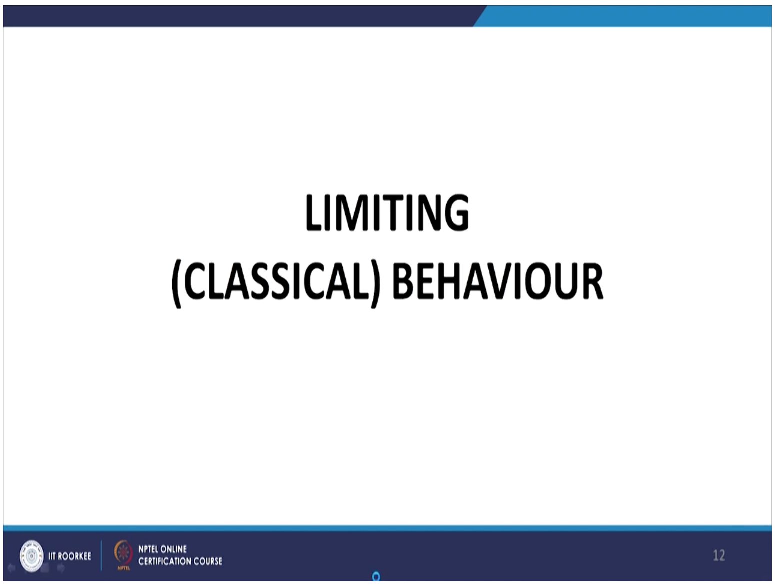
Then we obtained these Schwinger Dyson equation for the field operators. We obtained it in two forms given as equations 1 and 2 in the red box.

(Refer Slide Time: 03:18)



And for the specific case of the phi 4 field, we got the expression in the green box. This Schwinger Dyson equation for the phi power 3 by 4 field is given in this slide. We have that extra term with the coefficient lambda 3 which is representing the phi 3 field in the action.

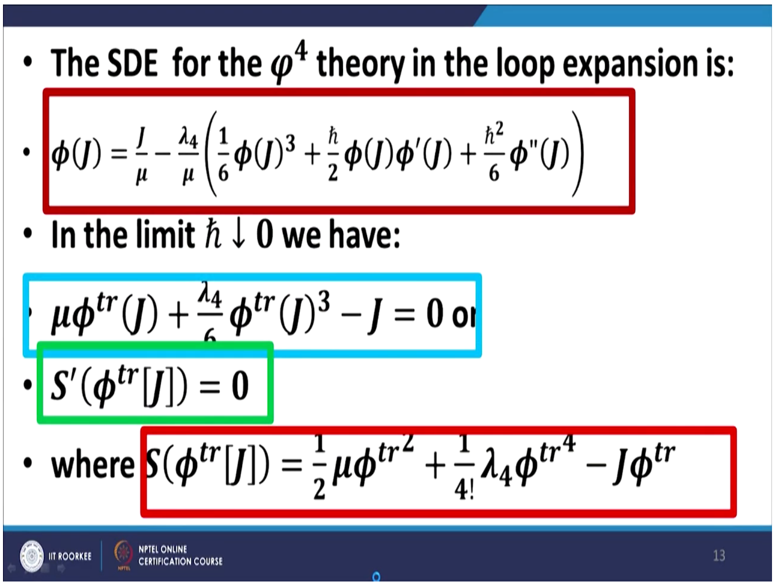
(Refer Slide Time: 03:35)



Now, we then of course, we also work through the Feynman diagrams. We arrived at the Schwinger Dyson equations using the approach of Feynman diagrams. while writing down the Feynman diagrams all possible Feynman diagrams excluding the vacuum bubbles. And then expressing them or evaluating each of the diagram. And we arrived at the Schwinger Dyson equation through that modulus operandi as well.

We found a very interesting conclusion. That the symmetry factors that were incorporated in the Feynman diagrams on the basis of combinatorics, turned out to be the precisely the correct symmetry factors which were dictated by the physics of the physics content of the Schwinger Dyson equation.

(Refer Slide Time: 04:25)



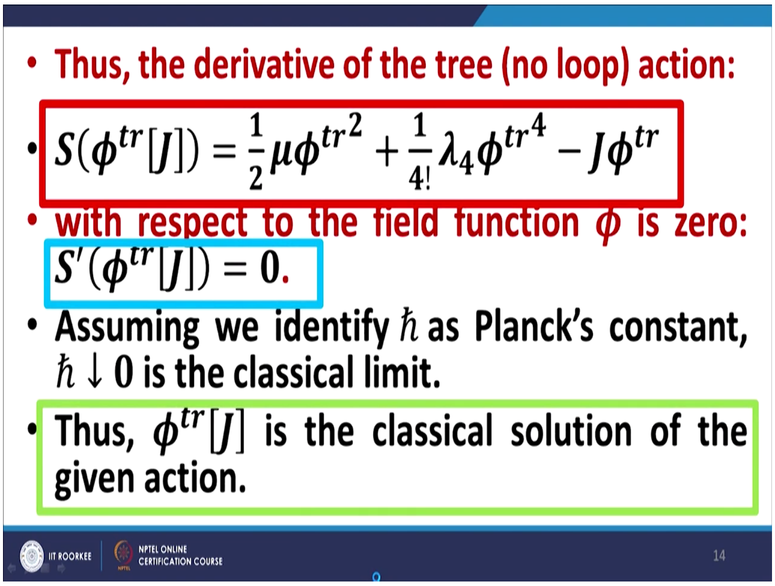
Now, we move to the limiting or the classical behavior of the phi 4 field. So, we start with the phi 4 field in the loop expansion. We introduce the concept of loops by incorporating the parameter h bar into our into the Schwinger Dyson equation. h bar represent powers of h bar corresponding to the number of loops in the Feynman diagrams.

So, the equation in the red box has two expressions with the h bar in it. One corresponding to one loop component of the Schwinger Dyson equation and the second corresponding to the two loop component of the Schwinger Dyson equation. Now, in the limit that h tends to 0 h bar tends to 0, we get the tree diagram the diagram which excludes all loop diagrams and consists only of tree diagrams.

In that situation this Schwinger Dyson equation of the red box translates to the equation that is given in the blue box. Now, if you look at it carefully, this equation in the blue box is nothing, but the derivative of the action or rather. If I use the action that is given in the red box at the bottom of your slide, take its derivative and evaluate it at the phi tree of J I get 0.

In other words, this equation in the blue box is representing nothing, but the condition S dash of phi tree of J is equal to 0. Where S phi the action is given by the expression in the red box.

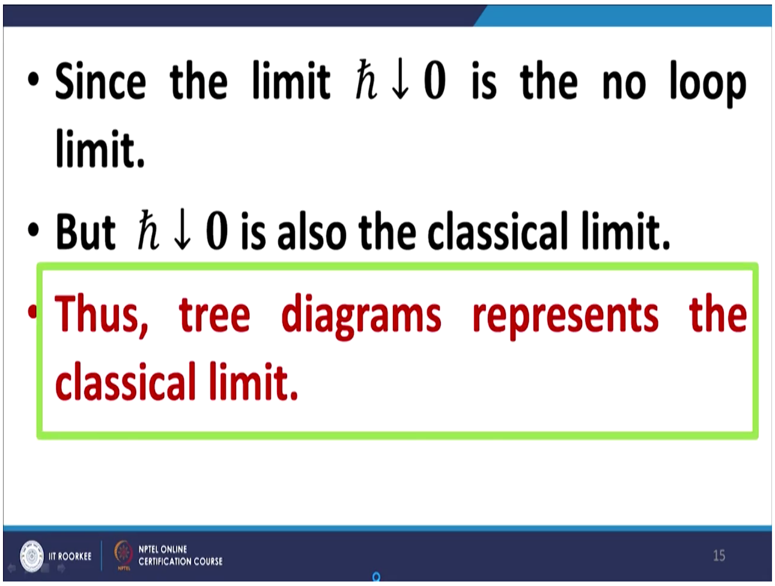
(Refer Slide Time: 06:15)



So, the no loop or the tree actions. This is the tree action the tree action the actions as represented that is shown in the red box, that we carried forward from the previous slide the red box slide here.

We and. If I take this derivative with respect to phi and equate it to 0, I get precisely the expression I get precisely the expression of the Schwinger Dyson equation in the limit that h bar tends to 0. So, this implies that this expression that we have obtained here is the solution the this expression in the blue box here the expression in the blue box here is the solution of the classical solution of the given action and which is here in the red box.

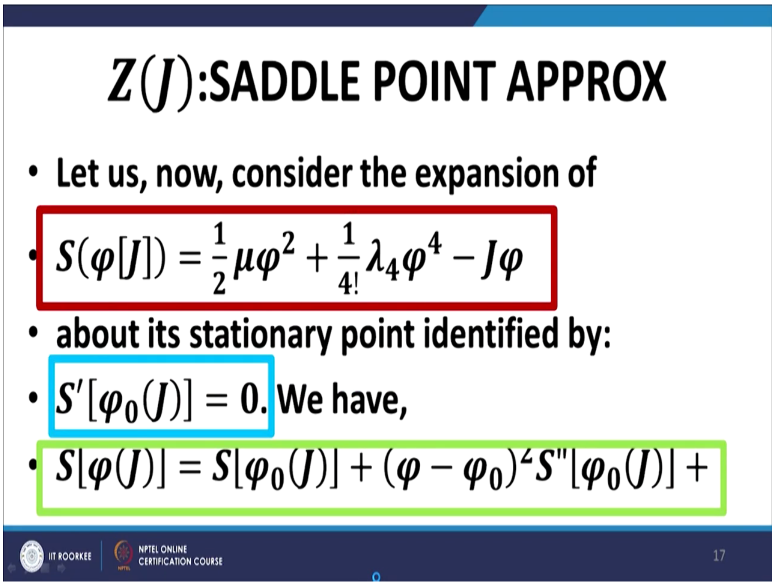
(Refer Slide Time: 07:08)



But, we took it. The important thing here is, why we are able to say this is? Because h bar tending to 0. If you identify h bar with the Planck's constant. And h bar tending to 0 then corresponds to the classical scenario or the classical limit as you may say. And therefore, when we take the limit as h bar tending to 0. On the one hand it corresponds to the no loop expansion and on the other hand, we find that h bar tending to 0 reproduces the classical limit.

And therefore, we can also infer that the tree diagrams represent the classical limit of the Schwinger Dyson equation or the tree portion of the Schwinger Dyson equation represents the classical limit of the complete Schwinger Dyson equation. Now, let us explore this a bit further.

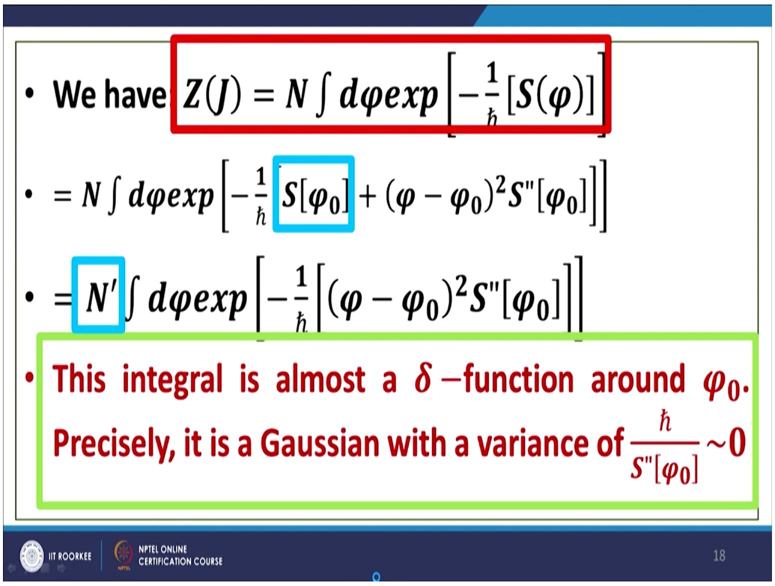
(Refer Slide Time: 08:00)



Let us do a Saddle Point approximation or an expansion about the saddle point of the given action. Our action is given in the red box here. We obtain the stationary point by solving the equation which is given in the blue box equating the first derivative of the action to 0. And we find. And then whatever by equating the first derivative to 0, the point that we get at the saddle point let us call it phi 0. We expand our action S of phi around this action this saddle point or the stationary point phi 0.

And it takes the form given in the green box. Please note that the term in the first derivative is absent, because of the condition imposed in the blue box here. The first derivative defines this saddle point. And therefore, when we are expanding around the saddle point we will get no term in the first derivative that term will be 0.

(Refer Slide Time: 09:08)



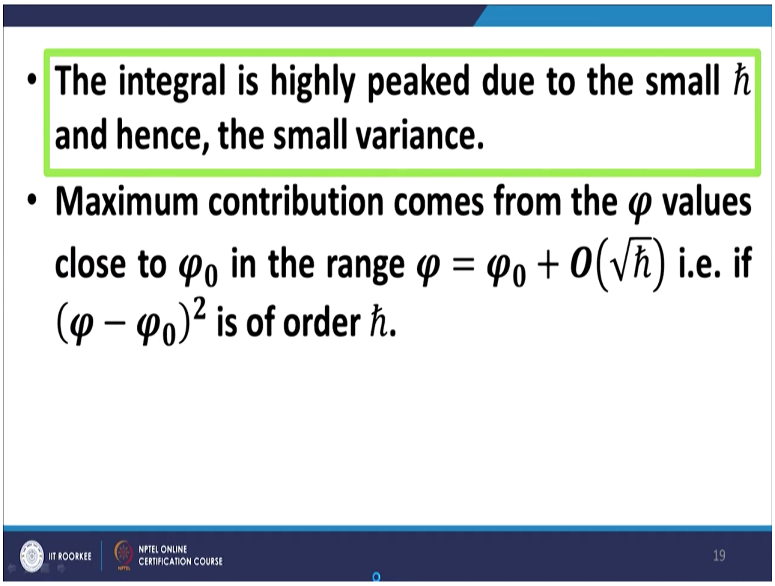
And then we have we start with the path integral and we write the path integral in this form. And now, we expand our S phi in terms of the expression that is given here in the green box. Using this expression expansion in the green box here, we write out the expression for S phi for the path integral given in the red box here.

And we absorb the first term S phi 0 with the normalization, because it is independent of phi. It is a constant and it can be absorbed in the normalization. We have a new normalization N dash and what remains inside is given here

The exponent is given by minus 1 by h bar phi minus phi 0 S double dash phi 0. Of course, we are ignoring higher derivatives of S at phi 0. Now, this integral is almost a delta function. Why do I say that its almost a delta function around the point phi 0?

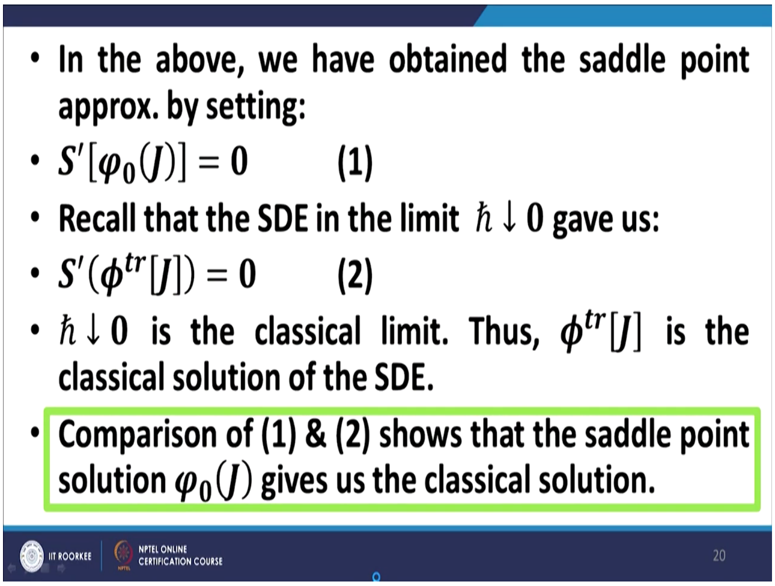
Because if you look at this the variance of. It is clearly a Gaussian integral; it is clearly a Gaussian integral. If you look at the variance of this integral the variance is, h bar upon S double dash of phi 0. And keeping in view that h bar is very very small we find that this variance of this Gaussian is very very small.

(Refer Slide Time: 10:40)



And in other words it approximate. So, it is very close to an impulse or a delta function. It is highly peaked due to the small h. And the maximum contribution therefore, comes from the phi values that are close to phi 0. And you can express them as in the range phi 0 plus order of root h, that is phi 0 plus root h to phi 0 minus root h which makes phi minus phi 0 whole square of the order of h bar.

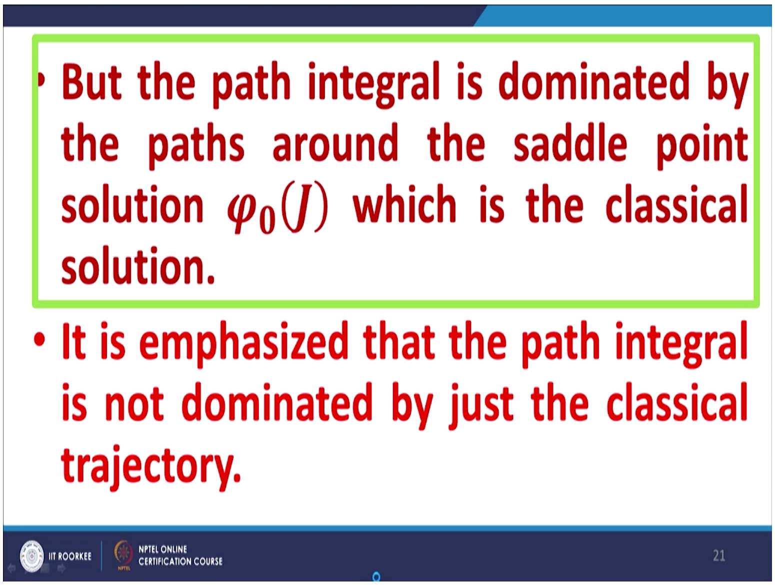
(Refer Slide Time: 11:11)



So, to compare, we have obtained two expressions. One expression we have obtained from starting from the action quantum action and we have obtained the expression expansion around the saddle point. We have got S dash of phi 0 equal to 0. And we also obtained that in the limit that h bar tends to 0 the tree portion of the Schwinger Dyson equation. This is under the condition of h bar tending to 0 which is the classical limit.

Therefore, phi trees J is the classical solution as I mentioned earlier as well of the Schwinger Dyson equation. If you compare 1 and 2 then what we find is this expression the expression number 1 also gives us the classical solution.

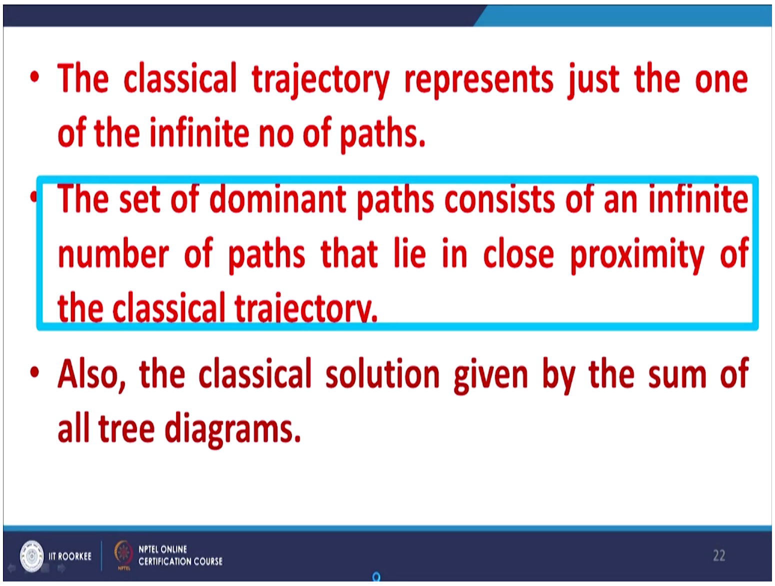
(Refer Slide Time: 12:01)



Now, the and then the important thing that needs to be emphasized is that the path integral. Therefore, the classical solution of the path integral or the classical trajectory would be determined by equation number 1. That is one part.

But this trajectory only is only one of an infinite number of trajectories that constitute the entire path integral set up. The various paths that contribute dominantly or with that dominate the contribution to the ultimate path integral is our paths that are in close vicinity of the classical path. The classic. You see the point is. As I mentioned in the context of quantum mechanics as well.

(Refer Slide Time: 13:14)

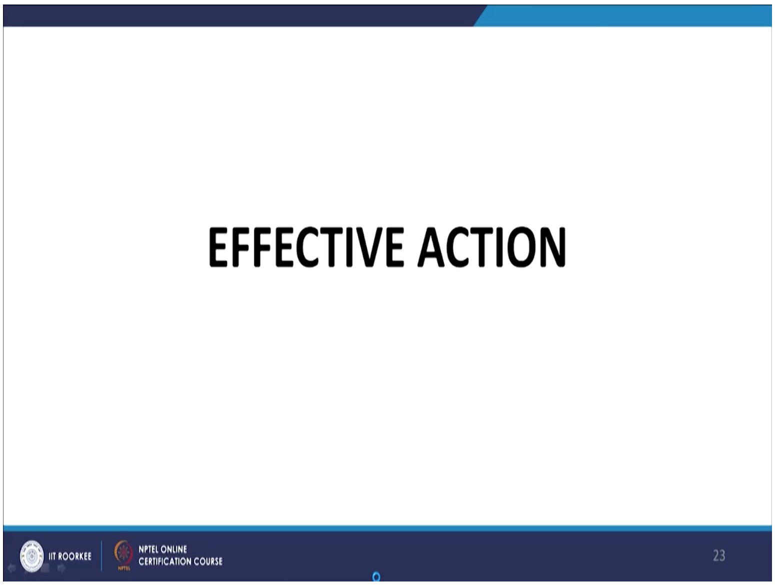


The weight of each path is the same. But the important thing is that the path integral is dominated by paths which are close to the classical path which are very very near to the classical path. It is just it is not one path, every path has the same weight factor.

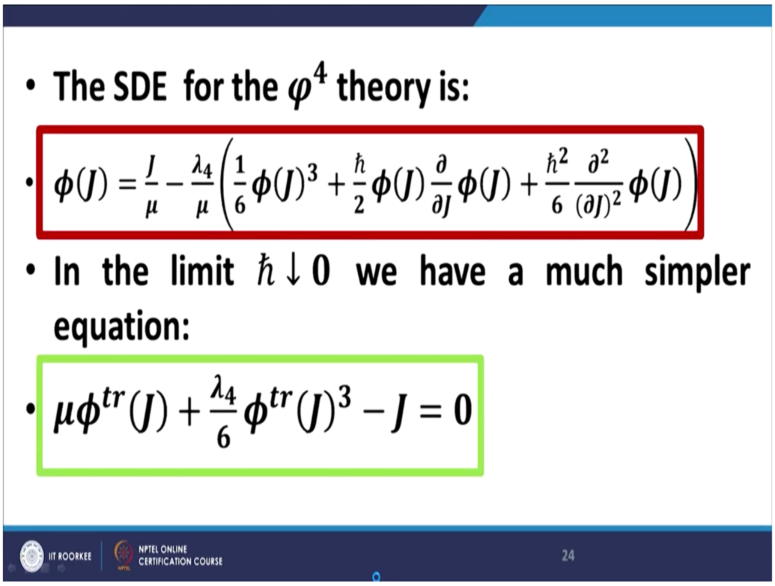
But there are a number of paths which are close to the classical path which interfere coherently constructively. That is the reason that the path integral seems to simulate to some extent or to a large extent the classical trajectory.

But I reiterate it once again, that the classical trajectory is just one path. And the set of dominant paths consists of an infinite number of paths that are in close vicinity of this stationary path or the classical path. And because they are in close vicinity of this stationary path minor change in the action does not cause any significant change in the phase. And therefore, they interfere constructively. That is the important part.

(Refer Slide Time: 14:13)

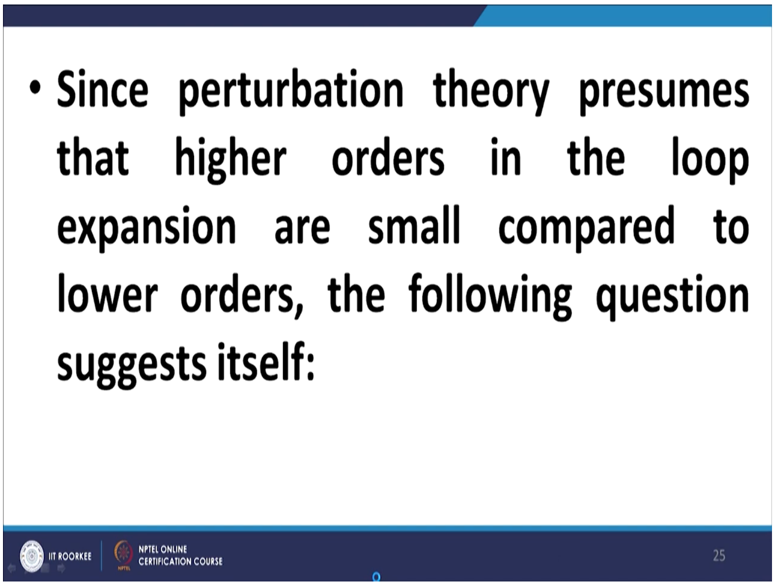


(Refer Slide Time: 14:17)

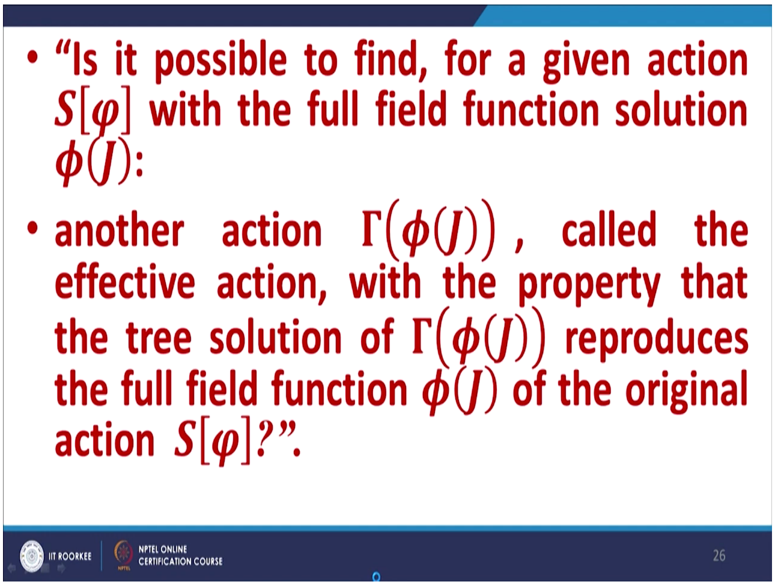


Now, we talk about a new concept which is the effective action. The Schwinger Dyson equation for the phi 4 theory is given in the red box here of course, incorporating there in the loop expansions. In the limit h bar tending to 0 we have already seen that this equation reduces to this quantity.

(Refer Slide Time: 14:35)



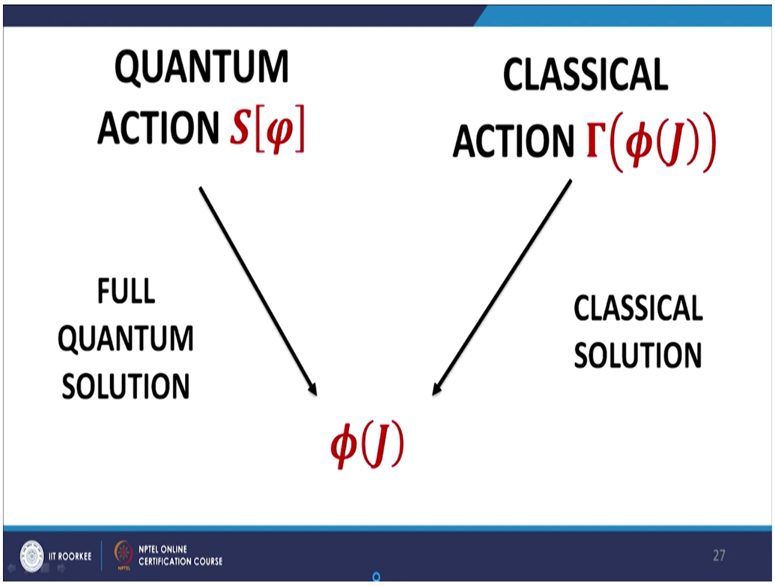
(Refer Slide Time: 14:39)



Now, we ask the question; we ask the question is it possible to find, for a given action you are given a certain action S of phi S phi of J or S phi that has a full field solution given by the symbol phi J: if this is the solution of corresponding to this action can we find another action?

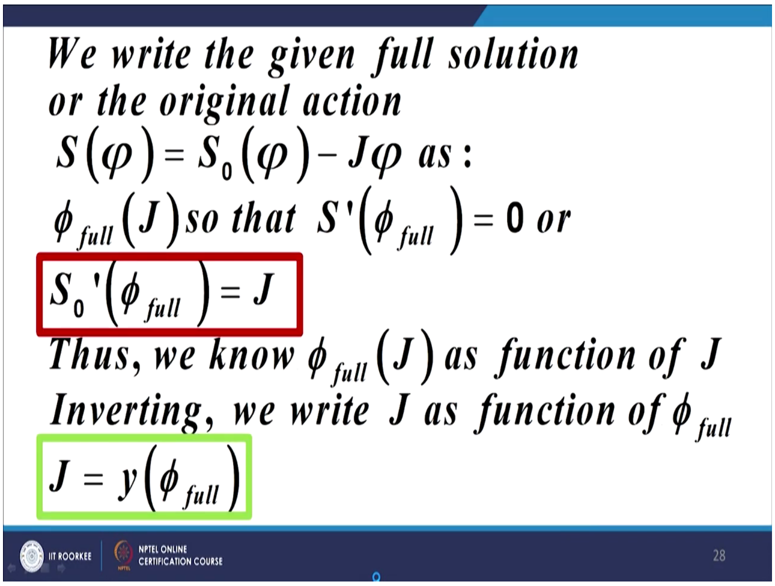
Let us say gamma phi of J. Can we find another action gamma phi of J, such that the classical solution the tree level solution to this gamma phi of J reproduces the full solution phi J of the original action S J? That is the question that we attempt to address. This is what the problem is depicted here.

(Refer Slide Time: 15:25)



You have the quantum action S phi. It has the full solution phi symbol of J. This is the full quantum solution of the quantum action, this is what is given to you. We want to find a classical action gamma phi of J; obviously, it will be a function of this phi of J the full solution. Such that the phi J that we have as the full solution of the quantum action represents the classical solution of this action gamma phi of J. So, that is the question we attempt to answer.

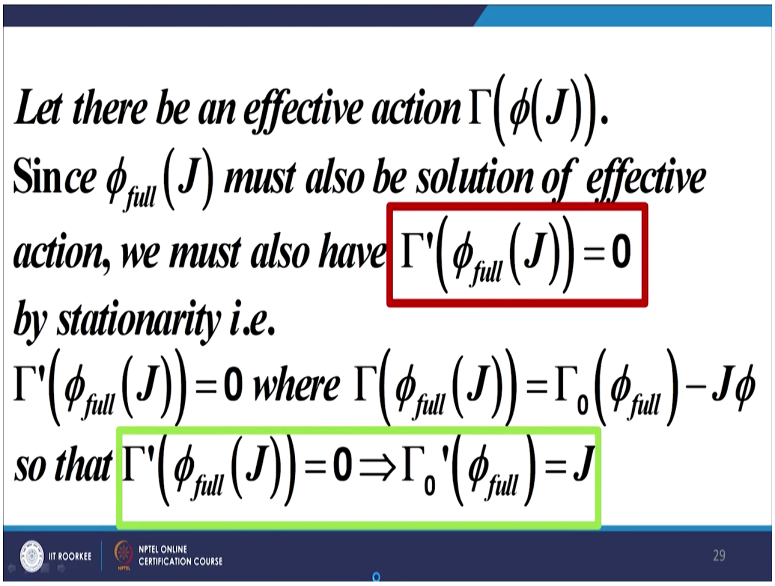
(Refer Slide Time: 16:04)



So, we write the given full solution of the original action as: S phi is equal to S 0 phi minus J phi, where S 0 is equal to the standard 1 by 2 mu phi square, in the case of phi 4 theory plus 1 by 24 lambda 4 phi to the power 4. And, because phi full J is the solution of this it is a solution of this. Therefore, we must have S dash phi 4 phi full of J equal to 0 or S 0 dash phi full of J is equal to J.

We arrive at the equation at the expression given in the red box, because of the phi full of J is a complete solution as to the field full field solution of S phi. That implies that S dash of phi full must be equal to 0 and that gives us this result. Therefore, knowing the expression given in the red box which expresses phi full as a function of J. We can invert it and we can express J as a function of phi full. Let it take the form given in the green box.

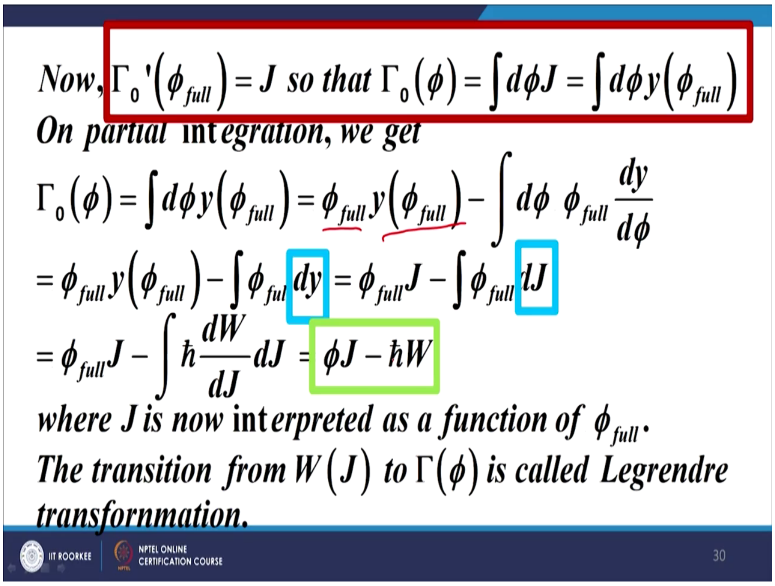
(Refer Slide Time: 17:12)



Now, we want to obtain an action gamma phi of J such that, phi full of J is the classical solution of gamma phi of J. Let us assume, because phi full of J has to be a solution of gamma phi of J. We again have by stationarity requirement that gamma dash of phi full of J is equal to 0, which gives us the expression given in the green box.

Gamma dash phi full of J is equal to 0 or gamma dash 0 of phi full is equal to J. This is what we get, because of this stationarity requirement imposed on the action gamma phi of J. Because phi J phi of J is a solution of this action gamma phi of J.

(Refer Slide Time: 18:09)



Now, from here what we get is from the previous slide what we get is. Gamma 0 dash phi full is equal to J and that gives us; that gives us gamma 0 of phi is integral of d phi of integral d phi of J. From this expression we immediately obtain in the expression given, but the integration is with respect to phi the integrand is J. We need to; we need to change variables to phi. We already have the expression with us we have this expression, because of this inversion given in the green box here.

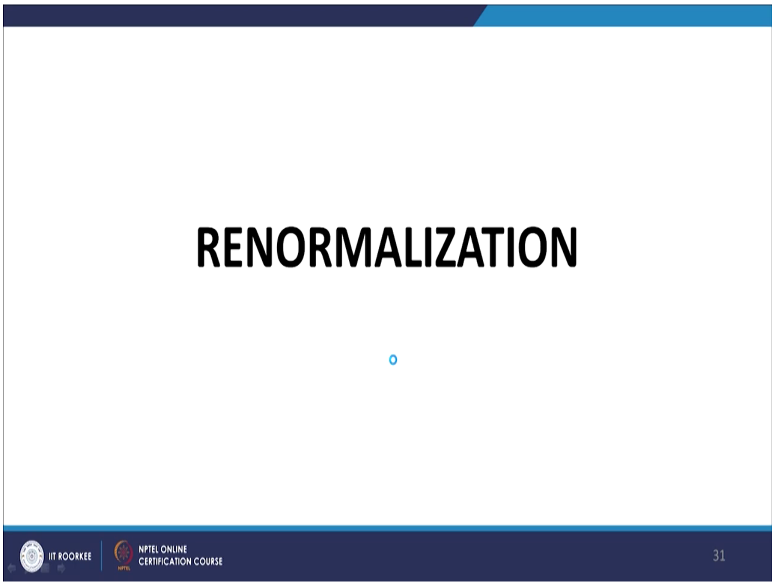
We make use of this and we write J as y of phi of full, where J is equal to y phi full which we had obtained by this stationarity of the quantum action right. So, now, we do an integration by parts. When we do an integration by parts. What we get is, the expression phi full y of phi full minus integral of d phi phi full d y by d phi. Taking y phi full as one function and one as the other function.

First function y as one function and one as the other function we do an integration by part. First function y into integral of the second function minus integral of derivative of the first function into integral of the second function. So, now, if you look at this. The first expression you taken as it is the second expression d y is equal to d J.

So, we simply substitute this expression integral phi full d phi d phi cancels we get integral phi full d y which is nothing, but integral phi full d j. But phi full is nothing, but phi is the field function and field function is the first derivative of the generating function for the connected diagram. So, we can write phi full as d W upon d J and of course, with the loop parameter h bar. And this whole thing simplifies to give us phi J minus h bar W.

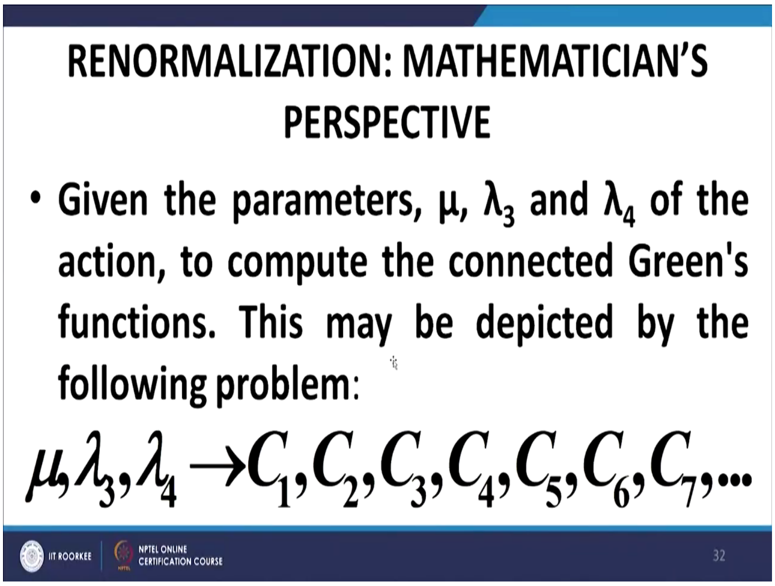
When you simplify this integral you get phi J minus h bar W where J is now a function of phi full. And this whole thing is called is a Legrendre transformation.

(Refer Slide Time: 20:56)



Now, we come to a new topic Renormalization. This is one of the most fascinating topics that we encounter in quantum field theory. In the context of zero dimensional quantum field theory the approach is somewhat abridged. But nevertheless, it brings out the nuances that are associated with renormalization. I will go through it slowly, because it is a bit technical, but it is very very fascinating.

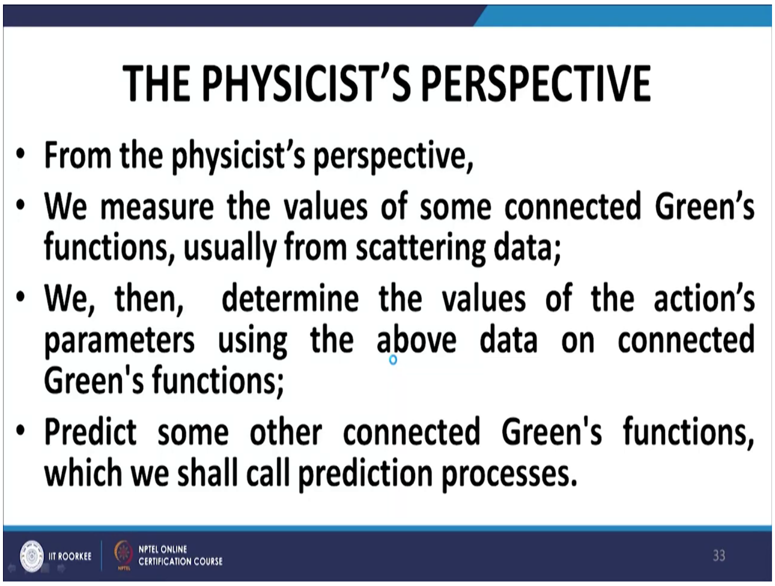
(Refer Slide Time: 21:30)



So, let us look at the mathematicians perspective of what we are what we try to do in quantum field theory. And what we have tried to do so far in the context of perturbation expansions for finding out the Green functions and connected Green functions of the theory.

The mathematician would view it as you are given certain inputs. The inputs would consist of the mass and the coupling constants of the theory. And using this mass and coupling constants the mathematician would like to arrive at expressions for the various Green functions various connected Green function C 1 C 2 C 3 and so on. So, that is the mathematicians problem and that is what by and large we have been addressing so far.

(Refer Slide Time: 22:15)



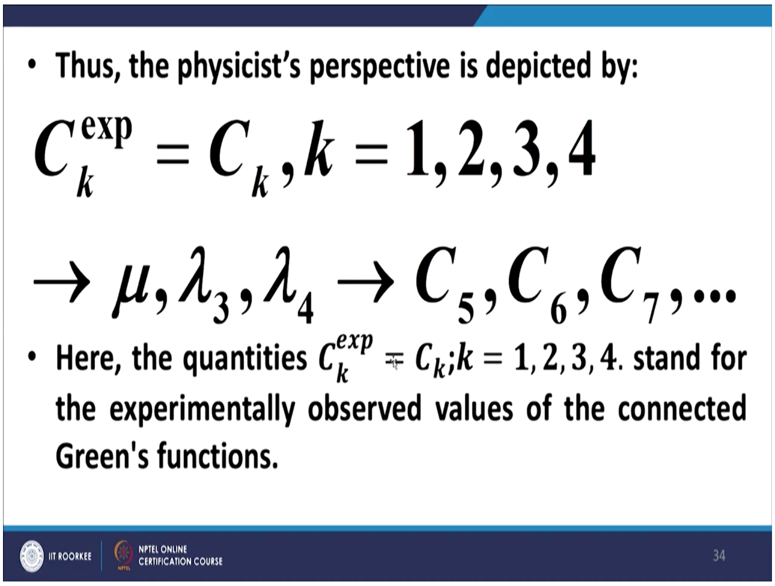
For the physicist; however, it becomes a slightly different perspective. From the physicist perspective what we have is we measure certain values of some connected Green functions using scattering experiments. On the basis of those scattering experiments, we get experimental values of certain connected Green functions.

Using those connecting connected Green functions. The next step is we work out the parameter values the bare parameter values of the inputs that go into the action. That is the mass and the coupling constants various coupling constants, that is the second step. And then on the basis of those mass and coupling constant that we have estimated using the experimental results, we try to predict try to work out the more connected Green functions and that is the.

And then of course, we may we may arrive at expressions for higher order Green functions. And experiments may then be devised to study or to work out to calculate these higher order Green functions and see, whether they relate to or whether they coincide with the Green function that we have worked out and find out whether the theory is correct or the theory needs some modifications or needs to be abandoned.

So, this is the physicists perspective on this issue. Having outlined the difference in perspective between the mathematician and the physicist.

(Refer Slide Time: 23:54)



We in the form it is a. The physicist perspective can be depicted here. We work out the experimental the certain number of connected Green function say, C 1 C 2 C 3 C 4 experimentally.

We get experimental values. They form our inputs to determine the various parameters the bare parameters of the action. And these bare parameters of the action which we worked out on the basis of inputs received from the experiments can are then used to work out more Green functions. And these Green functions are then compared with further experiments to arrive at the veracity accuracy of the theory.

Now, the issue of renormalization arises, because we do not we as we have seen we do not are or we are not able to exactly solve or solve the theory even the simplest phi 4 theory in a closed form. We invariably use the perturbation theory as a mechanism of estimating or approximating the Green functions up to a certain level.

That raises the issue of the point at which the truncation of the Green functions is turned. A truncation of the expansion of the series expansion is done at for loop level the various quantities are calculated. That is very important and that any inconsistency there would be reflected in accuracy. And that is where the role of renormalization comes into play. Let us see how? Let us suppose that we worked out C 1 C 2 C 3 and C 4 at a certain level of a loop certain loop level. Let us say at the p th loop level.

We have got C 1 C 2 C 3 C 4 values from experiment. Using the expressions for C 1 C 2 C 3 C 4 at say the p th loop level. We work out the expressions for the various bare parameters of the action at the p th loop level. And on that basis, we work out the values of the other connected Green functions say C 5 C 6 C 7 and so on at the p th loop level so far so good.

And of course, then it can be compared with experiments and so on. Now, let us say assume that it becomes possible or somebody exceptionally ingenious is able to work out C 5 C 6 C 7 C 8 at the p plus 1 th loop level work out expressions for C 5 C 6 C 7 C 8 at the p plus 1 th loop level.

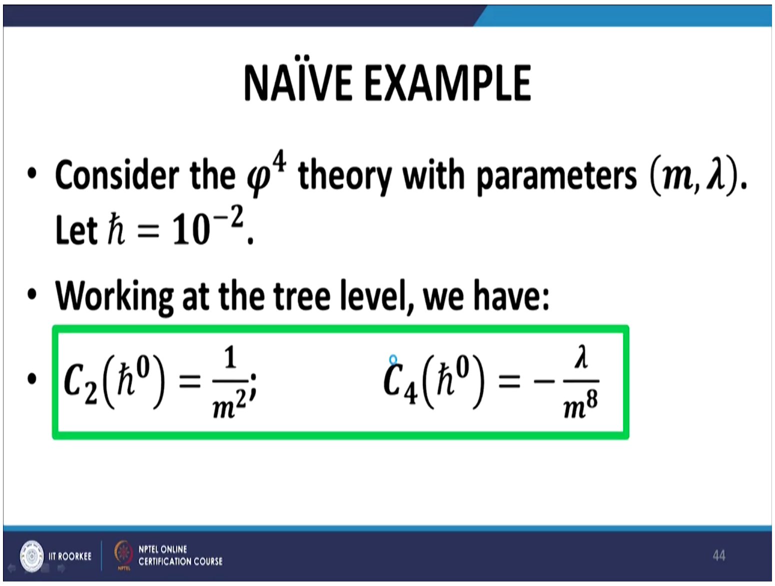
Now, the question then arises. Whether we can use the data at the p th loop level for mu the mu and the coupling constants as inputs to the formula at obtained for C 5 C 6 C 7 C 8 at the pre plus 1 th loop level? And that is where the question of renormalization makes its presence.

The answer is no. If we are working out the connected Green functions C 5 C 6 C 7 C 8 at the p plus 1 th loop level or if we have corrected, let us say C 5 C 6 C 7 C 8 from the p th loop level to the p plus 1 th loop level. It is absolutely imperative that we use as inputs to those p plus 1 th corrected C 5 C 6 C 7 C 8.

The values of mu lambda 1 lambda 3 lambda 4 and so on. That are also corrected to the p plus 1 th loop level. That is fundamental. The loop level must be consistent otherwise what will happen is, we will end up with divergences. And these divergences will distort the results of assessment or evaluation of the theory.

So, that is the how the problem arises. Now, let us see let us work practically work out through a very naive example and see what actually happens?

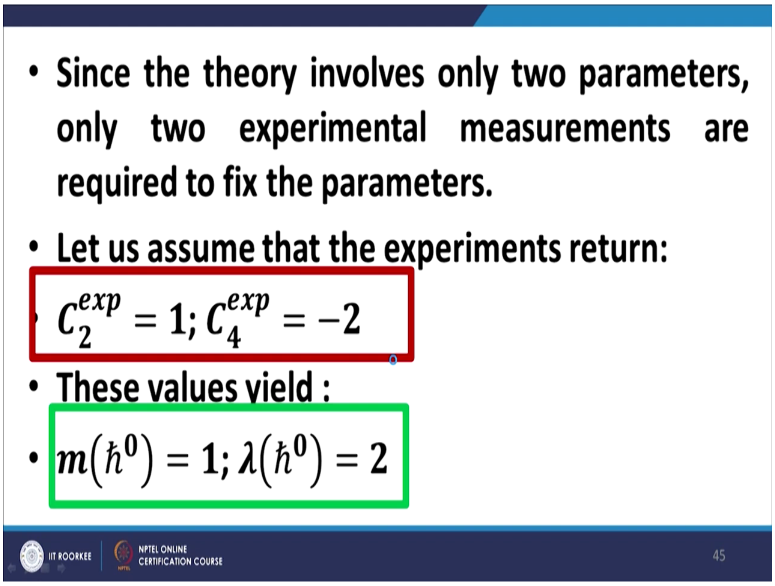
(Refer Slide Time: 28:48)



We have got the phi 4 theory. This is an example please note this. This is a. You got a phi 4 theory. There are only two parameters m and lambda m and mu are related h bar is taken as 10 to the power minus 2. Now, working at the tree level we have worked out the formula for C 2 h 0.

h 0 means that we were working at the tree level. So, we work the formula for C 2 is 1 upon m square. This can be done on the basis of the Feynman diagrams. C 4 we obtain as minus lambda upon m to the power 8. So, to repeat. We are having a phi 4 theory. The parameters are m and lambda. h bar we have taken as 10 to the power minus 2 and the formula for C 2 and C 4 we know, at the tree level are 1 upon m square and minus lambda upon m to the power 8.

(Refer Slide Time: 29:54)

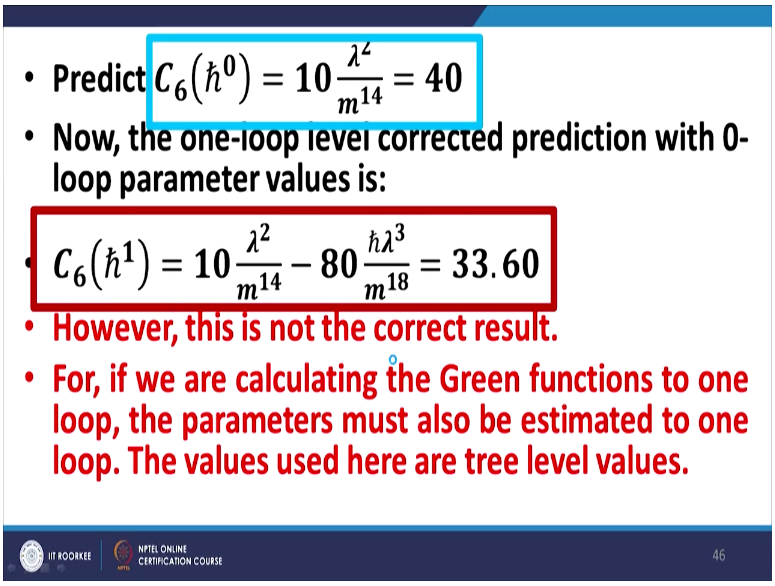


Now, because the theory involves two only two parameters, m and lambda 2. In principle at least two experimental measurements are sufficient to fix the parameters. We do the two measurements experimentally and what we find is that C 2 experimental.

The experimental value of C 2 is found to be 1 and the experimental value of C 4 is found to be minus 2. These are input. These are these are on practically worked out experiment scattering experiments that have given us these inputs as C 2 experiment is 1 and C 4 experiment is minus 2.

So, obviously, these will not change. These are actually worked out through experiments done in the lab. Now, on the basis of these values when we input into these equations in the equations that are given in the green box. When we input those experimental values. We arrive at m at the tree level is equal to 1, lambda at the tree level is equal to 2.

(Refer Slide Time: 31:03)



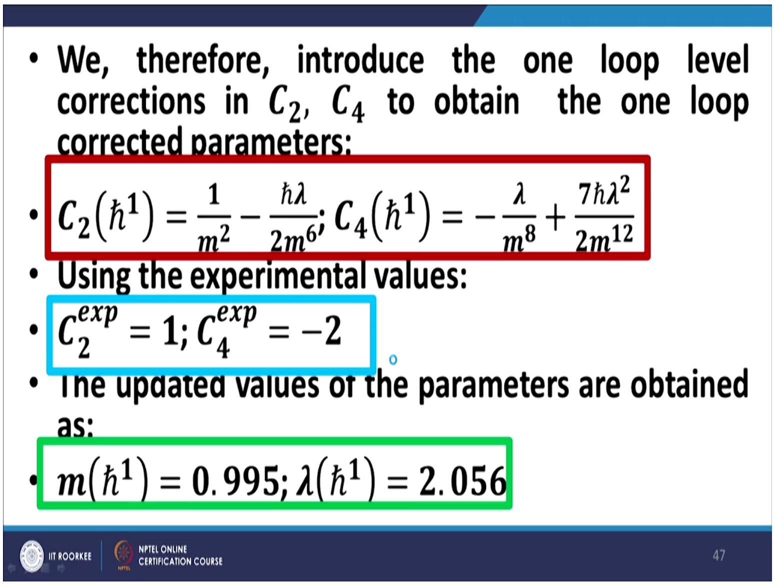
So, now we what we do is we predict C 6 at the tree level. And we find that it is given by 10 lambda square. Please note h bar is equal to 10 to the power minus 2. So, we predict C 6 and we find C 6 of at the tree level equal to 10 lambda square upon m to the power 14 which we find as 40. So, this is so far so good. Everything is done at the tree level there is no in consistency. It is fine.

Now, what happens is. Let us say, we work we want to work out or somebody has worked out the C 6 value at one loop level. When the C 6 values that the 6 point Green function is worked out at one loop level. The expression is found to be given in the red box here and the value on the basis. Now, this is important this is very important. Value of C 6 h 1 on the basis of the parameters m and lambda worked out at the tree level that is at h 0 level works out to 33.60.

This is. Please keep track of the figures. C 6 at h 0 was found to be 40 this was correct this was the value of C 6 at h 0. Then, we worked out C 6 at one loop level we use the formula for C 6 worked out at one loop level, but we inputted the bare parameters of the action into this formula which were worked out at the tree level and we found the value to be 33.60 right.

Now, what we did what we do is. We go right back and we go back to C 2 and C 4. And we correct our C 2 and C 4 formula which were earlier at the tree level.

(Refer Slide Time: 33:08)

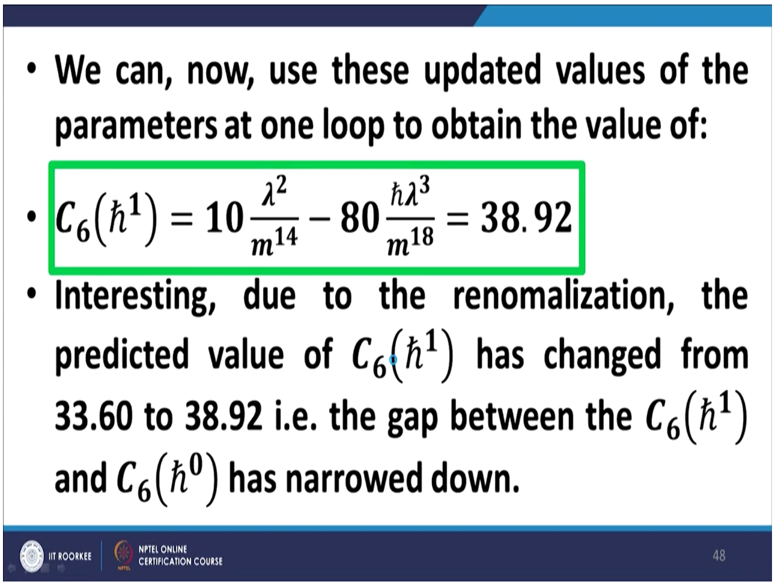


And we work them out rework them out at the one loop level. We work out C 2 and C. Please note these are not the experimental values. Experimental values continue to remain what they are.

The formula for the C 2 and C 4 are reworked and we incorporate there in the correction due to the first loop. And we find that the corrections are given by. For C 2 we find the correction as minus h lambda upon 2 m to the power 6. And for C 4 we find the correction to be 7 h bar lambda square upon 2 m to the power 12.

We retain this experimental values as 1 and minus 2 that we started with. We input them into this new formula that we have derived using the first order corrections in h. And we find m h 1 is equal to 0.995 and lambda h 1 is equal to 2.056. So, these are two values that we are we have now obtained the bare parameters to reaction corrected at the one loop level.

(Refer Slide Time: 34:23)



And when we input these new parameters the updated parameters the revised parameters at the one loop level into the expression for C 6, which is also at the one loop level we get the expression of 38.92 right. So, the discussion on this result set of results. I will take up after the break.

Thank you.