Path Integral Methods in Physics & Finance
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Lecture - 36
Feynman Diagrams & SDE

Right, so what we do now is, we work out the Schwinger Dyson Equation using the Feynman Diagrams. Let us C n represent the set of all connected graphs, connected diagrams with no source vertices and n external legs.

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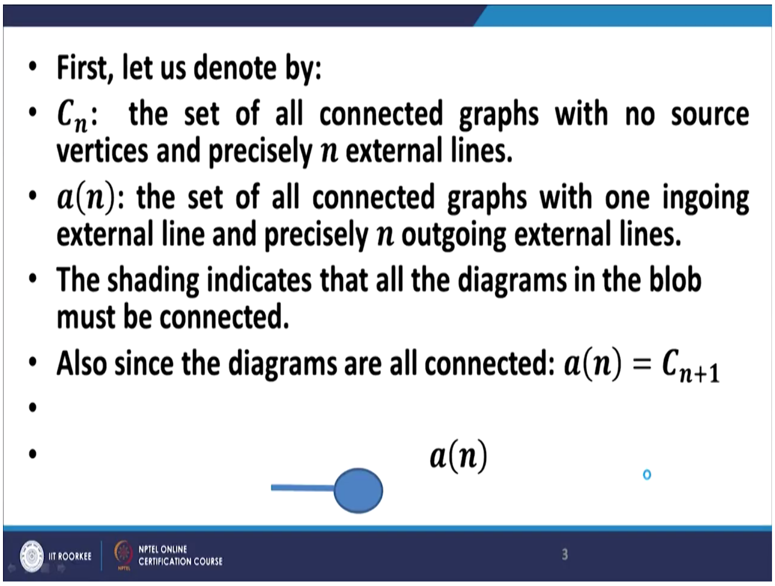

Now, and let a n represent the set of all connected graphs with one ingoing leg and n outgoing lines, precisely n outgoing external lines and one ingoing line. So, it is clearly obvious that C n plus 1 is equal to a n, because C n represents the total number of lines. Whereas, a n represents the outgoing legs and we correspond to one in ingoing lines.

So, a n is corresponding to C n plus 1 we represent a n and therefore C n plus 1 by this diagram this line with a blob.

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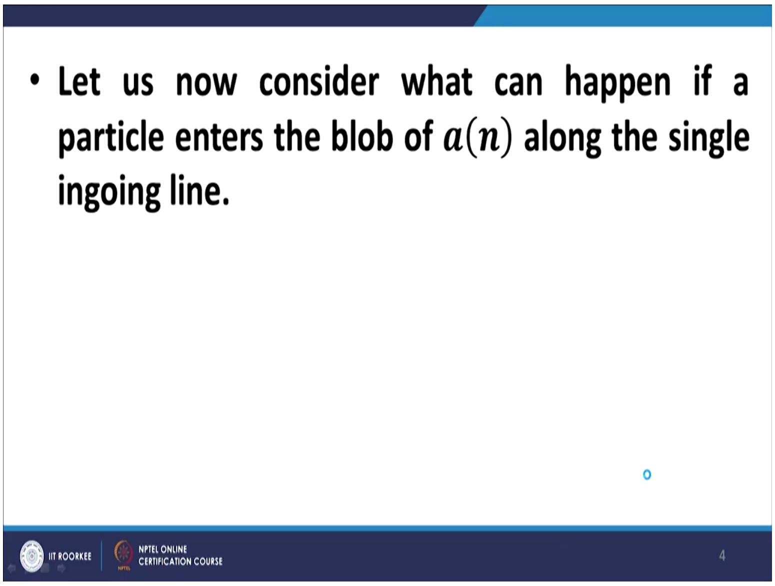

Now, let us see if a particle enters the system enters interaction chamber. Let us say which is which is represented by this blob; which is represented by this blob; which we have seen earlier and this represents the interaction and along a single ingoing line along the ingoing line. What are the various possibilities that can happen?

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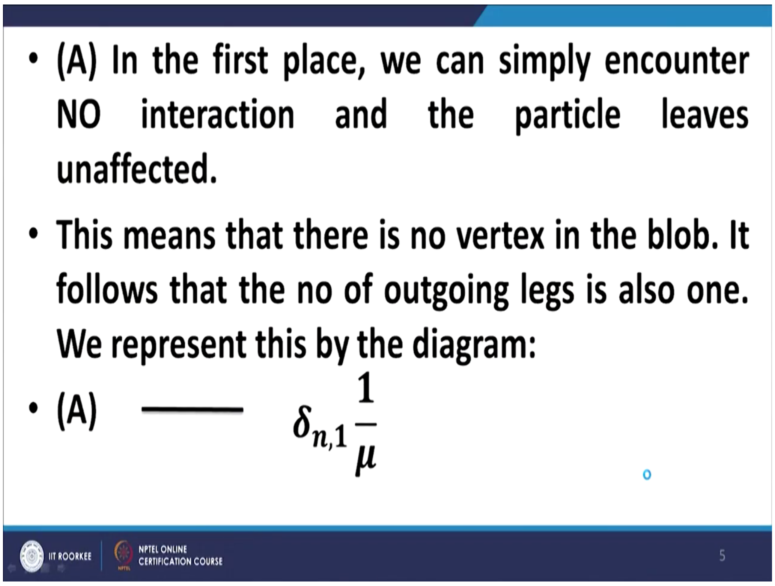

The first possibility is that the particle does not interact at all and if it does not interact at all it moves out or it leaves the system precisely in the manner that it was before it interacted or before it entered into the blob. So that therefore and that is represented by a straight line, this particular as possibility this particular possibility.

The possibility of no end interaction that is in a sense the free field is represented by a straight line and it evaluates to delta n comma 1, 1 upon mu. Why delta n comma 1? Because, in this case the number of outgoing legs has to be 1 by default. Because the number of incoming lines is 1 there is no interaction, therefore the number of outgoing lines also has to be 1.

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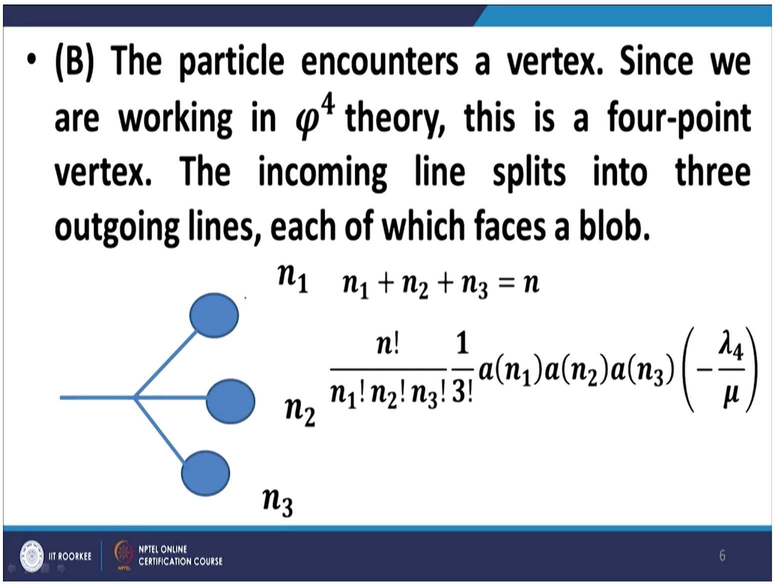

The second possibility is that the incoming particle encounters a 4 point vertex. Please note we are working in phi 4 field phi 4 theory. So, the particle interacts encounters a four-point vertex. And when it interacts the 4 point vertex the possibilities are that it could it could go along any of these 3 directions and then encounter another blob. With the first blob corresponding to n 1 external lines the second blob corresponding to n 2 external lines and the third blob corresponding to n 3 external lines.

In other words the incoming line at the vertex splits up into 3 outgoing lines, with each of which these outgoing lines ends up in a blob. The first one first blob relates to n 1 external legs the second one relates to n 2 external legs and the third one relates to n 3 external legs. So, clearly n 1 plus n 2 plus n 3 must be equal to n and number 2 the symmetry factor is 1 upon 3 factorial clearly, because they can be these 3 blobs can be interchange in 3 factorial ways without disturbing the diagram.

And the number of ways in which we can select n 1 identical lines in blob 1, n 2 identical lines in blob 2 and n 3 identical lines in blob 3 out of a total of n objects is given by n factorial divided by n 1 factorial n 2 factorial n 3 factorial. And a n 1 represents the value of this blob first blob upper blob, a n 2 represents the value of the middle blob and a n 3 represents the value of the third blob. And the value of this vertex is equal to minus lambda 4 and the value of this incoming line is equal to 1 upon mu.

So, the entire value of this diagram is equal to this whole expression. Let me recall because it is a this is the first one, the incoming line is giving me a factor of 1 upon mu. The vertex here is giving me a factor of minus lambda 4 and then when we have this this line and this blob. This line and this blob are giving me a factor of a n 1, this gives the first line and blob the top line and blob give me a factor of a n 1, the middle line and blob give me a factor of a n 2 and the bottom line and blob give me a factor of a n 3.

A symmetry factor because we can interchange between these blobs interchange across these blobs and therefore we have a symmetry factor of 1 upon 3 factorial. And the number of ways in which we can select n 1 lines or n 1 outgoing lines out of a total of n lines n 2 n 1 identical lines out of a total of n lines, n 2 identical lines out of a total of n lines n 3 identical lines out of a total of n lines. Subject to the constraint n 1 plus n 2 plus n 3 is equal to 1 n is given by n factorial divided by n 1 factorial n 2 factorial n 3 factorial.

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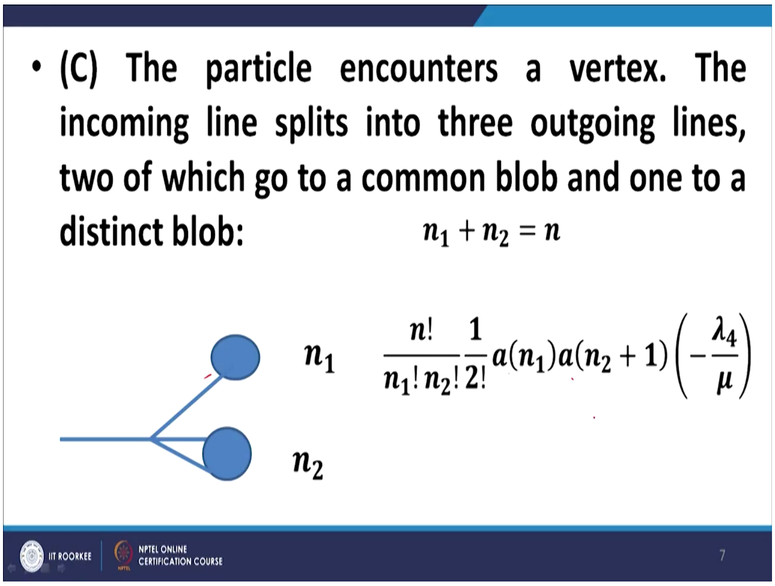

So, this is the whole expression corresponding to this Feynman diagram. The third possibility is that the particle encounters a vertex, it encounters a vertex. The incoming line splits up into 3 lines, one of the lines encounters a blob and the 2 other lines encounter a common blob. As shown in the diagram the incoming line splits into 3 lines, because we are we have a 4 vertex; 4 therefore the one incoming line we have 3 outgoing lines. The one of the outgoing lines has a separate blob and the 2 other 2 lines have a common blob.

So, in the value of this factor let us this diagram let us work it out again we have 1 vertex that gives me minus lambda 4, the incoming line gives me minus 1 upon mu. And now this line this blob and this line this upper line and the blob upper line and the blob give me a factor of a n 1 and the bottom lines give me a factor of a n 2 plus 1. Why n 2 plus 1? Because here we are having 2 incoming lines; if you look at the bottom blob we are having n outgoing lines, but not one incoming line we are having 2 incoming line.

So, instead of a n 2 it now has a factor of a n 2 plus 1, it now has an argument of n 2 plus 1, because now we have 2. Remember what was a n 2? a n 2 was 1 incoming and n 2 outgoing. Now we have 2 incoming and n 2 outgoing, therefore the argument of a is n 2 plus 1. And the upper one obviously evaluates to a n 1 and the symmetry factor because the 2 lines here can be flipped among each other. So, we get a symmetry factor of one 1 upon 2 factorial.

And the number of ways out of n objects we can pick up n 1 similar objects and n 2 similar objects is given by n factorial upon n 1 factorial into n 2 factorial. So, this explains the evaluation of this diagram. So this is the second case, which case we have one incoming line branching into 3 lines, one line going over to a separate blob 2. The 2 other lines coming back to a common blob.

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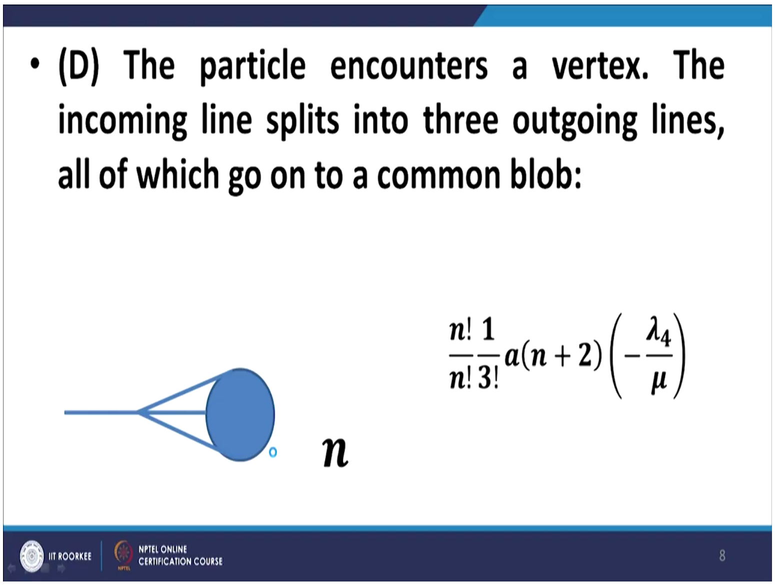

Now we come to the fourth case, in the fourth case you encounter all the 3 lines. You encounter a vertex you split into 3 lines and all the 3 lines reunite or enter into the same blob. Therefore, in this case what happens is and this this can happen in only one way. So, we have n factorial upon n factorial that is one way.

But the lines can be interchanged amongst each other in 3 factorial ways. So, the symmetry factor is 1 upon 3 factorial. Now we as far as this blob is concerned we have 3 incoming lines and n outgoing lines, because you have 3 incoming lines 1 incoming lines give me a n.

Therefore, if you have 3 n incoming lines you have a n plus 2, remember a n was 1 incoming line n outgoing lines. In this case we have 3 incoming lines n outgoing lines, so that corresponds to the argument n plus 2 for the function a. And of course, the as usual as in the previous cases the vertex gives me a factor of minus lambda 4 and the line incoming line gives me a factor of minus mu.

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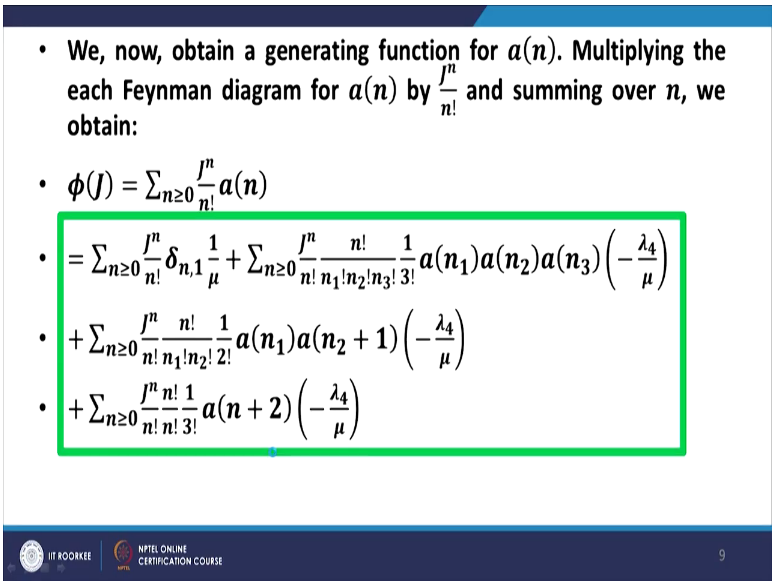

What we do now to obtain a generating function for a n, what we do now is, we multiply the entire expression that we have the sum of all the Feynman diagrams that we have case a plus case b plus case c plus case t. We multiply throughout by J to the power n upon n factorial we multiply n throughout and we sum over n.

Multiply everything by J to the power n by n factorial and we sum over n. The left hand side represents the field function phi J phi symbol J and let us see what we get for the right hand side. The first expression is quite straightforward, the first expression because of this delta function n comma 1 it will pick out n equal to 1 term. So, we will end up with J upon mu. So, the first expression evaluates to simply J upon mu it is independent of n.

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$$\bullet \sum_{n\geq 0} \frac{J^n}{n!} \frac{n!}{n_1! n_2! n_3!} \frac{1}{3!} a(n_1) a(n_2) a(n_3) \left(-\frac{\lambda_4}{\mu}\right)$$

$$= \sum_{n_{1,2,3}\geq 0} J^{n_1+n_2+n_3} \frac{1}{n_1! n_2! n_3!} \frac{1}{3!} a(n_1) a(n_2) a(n_3) \left(-\frac{\lambda_4}{\mu}\right)$$

$$= \frac{1}{3!} \left(-\frac{\lambda_4}{\mu}\right) \phi(J)^3$$

Now, we come to the evaluation of the second term. Let us look at the evaluation of the second term, the second term is given by the expression at a top of your slide. Remember the constraint is n 1 plus n 2 plus n 3 is equal to n, but please note n 1 plus n 3 n 2 plus n 3 equal to n with n being sum from 0 to infinity. So, we can write this as J n can be written as J n 1 plus n 2 plus n 3 J to the power, n can be written as J to the power n 1 plus n 2 plus n 3; the n factorials cancel out these terms are as it is. The 1 upon 3 factorial term is as it is.

Now, if you pick this one a n 1 together with 1 upon n 1 factorial together with J n 1 this whole term is nothing but phi of J. If you look at the definition of phi of J let us go back. The definition of phi of J is J to the power n upon n factorial a n, this is precisely what is happening is, J to the power n 1, n 1 factorial a n 1 this is nothing but phi of J.

Similarly for J to the power n 2, n 2 factorial a n 2 this gives me another factorial another factor of phi of J, J n 3, n 3 factorial a n 3 gives me another factor of phi of J. So, we have in all these whole thing condenses to phi and the summation is remember over n greater than equal to 0 to infinity.

So, we can write it as n 1 comma n 2 comma n 3 greater than equals 0 to infinity. And therefore, we can decompose this expression into phi J into phi J into phi J that is phi J whole cube. The rest is as it is minus lambda 4 upon mu is retained and so is the symmetry factor 1 upon 3 factorial. So, this second term in a nutshell evaluates to the expression that is given in the green box.

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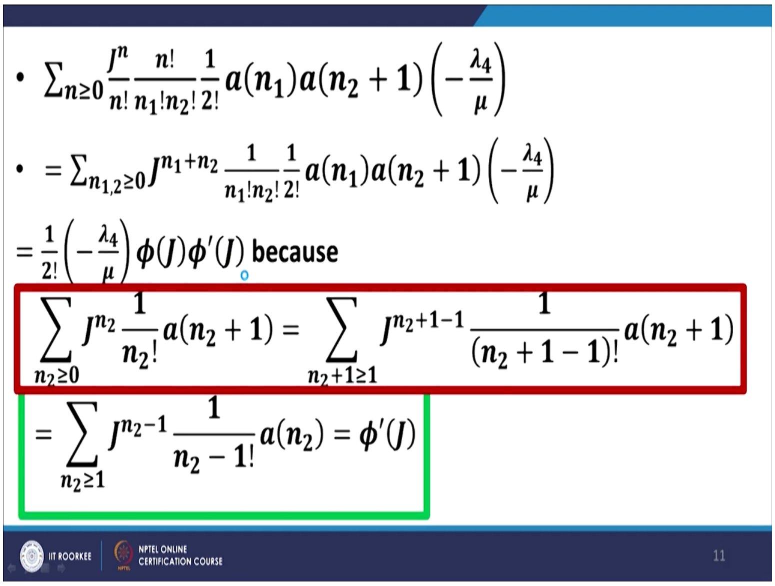

Let us look at the third term, the third term again adopting the same metrology as far as n 1 is concerned it gives me phi of J. If you look carefully we have J n 1 a J to the power n 1 here we have n 1 factorial we have a n 1 here.

So, this whole expression gives me phi of j, but the problem here is in the case of n 2 we have a n 2 plus 1. So we need to handle this, let us look at this term carefully this term is rewritten as isolated and rewritten in the red box.

Summation n 2 greater than equal to 0 J to the power n 2 1 upon n 2 factorial a n 2 plus 1. I can write it as J summation n 2 plus 1 greater than equal to 1 J n 2 can be written as J n 2 plus 1 minus 1 1 upon n 2 factorial can be written as 1 upon n 2 plus 1 minus one factorial and a n 2 plus 1 is retained as it is.

Now, simply changing the index from n 2 to n 2 plus 1 and renaming it as n 2, we get n 2 greater than equal to 1 J n 2 minus one 1 upon n 2 minus 1 factorial a n 2 and this is nothing but phi dash of J. The first derivative of J with respect for first derivative of phi J with respect to J; so, we substitute that here and we get the valuation for the case c, we have got valuation for case a we have got valuation for case b we have got valuation for case c.

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- Similarly
- $\sum_{n\geq 0} \frac{J^n}{n!} \frac{n!}{n!} \frac{1}{3!} a(n+2) \left(-\frac{\lambda_4}{\mu}\right)$
- $= \frac{1}{3!} \left(-\frac{\lambda_4}{\mu}\right) \phi''(J)$

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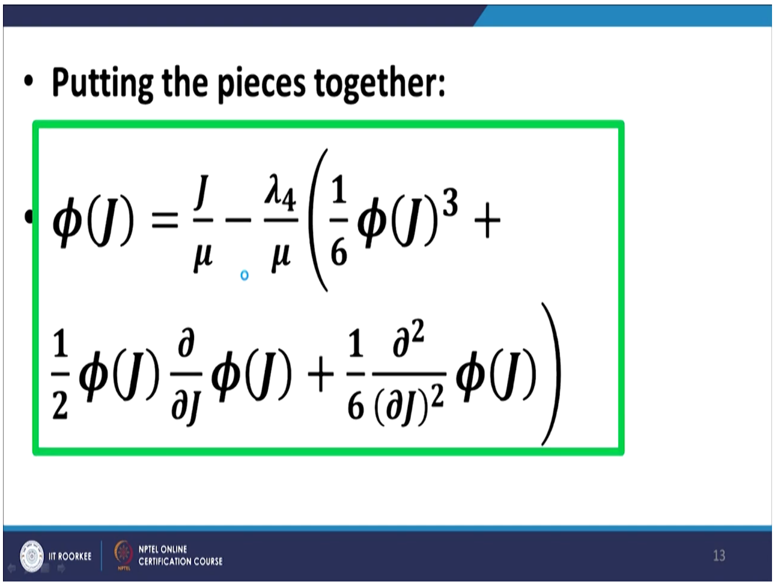

For case d we proceed similarly and we get this valuation for case d. Now let us substitute everything, when we substitute everything in the left hand side is phi J of course. The right hand side as per these various valuations that we have done we get the expression here in the green box. And the very interesting part is this is precisely the equation that we have obtained earlier in the previous derivation of the Schwinger Dyson equation for the field function, exactly the same equation we have obtained.

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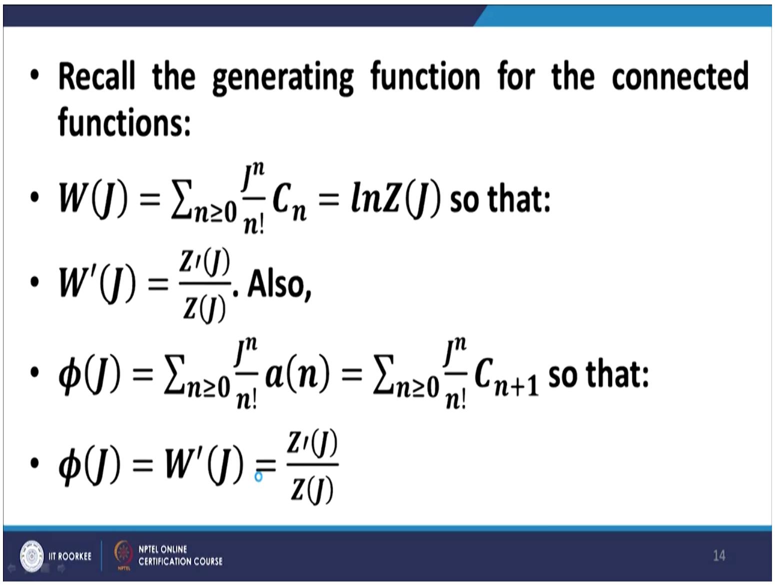

So, we come to a very fundamental inference a very important inference, but before that in just brief on the on the various other parameters. The generating function for the connected green functions are given by this expression that is log of Z J. Therefore, W dash W J is the is the generating function for the connected green function that is given by log of Z J w dash J is because W J is log of Z j.

So, W dash J is equal to Z dash J upon Z J and phi J is equal to this expression by definition and that can be written in the terms of as I mentioned a n is equal to J n plus C n plus 1 a n is equal to C n plus 1 I am sorry a n is equal to C n plus 1. So, we write C n plus 1 here and we find that phi J is equal to W dash J is equal to Z dash J upon Z j.

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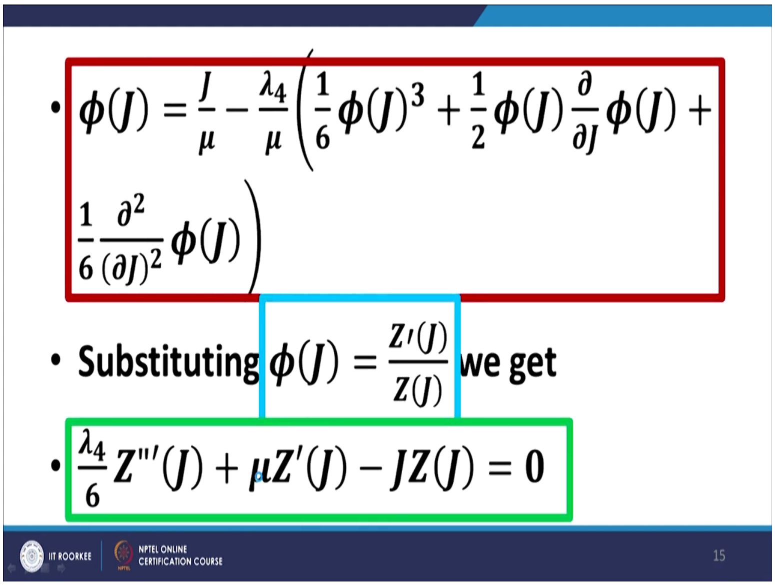

Now phi J is given by this expression, if I substitute phi J equal to Z J Z dash J upon Z J what do I get? I get this Schwinger Dyson equation for the path integral for the generating function which is given in the green box here.

Using the expression in the red box which we have derived just now from Feynman diagrams and using the definition of phi J that we just obtained here phi J is equal to Z dash J upon Z J. The last equation on your slide on this slide, we obtain the expression or the Schwinger Dyson equation for the path integral or the generating function.

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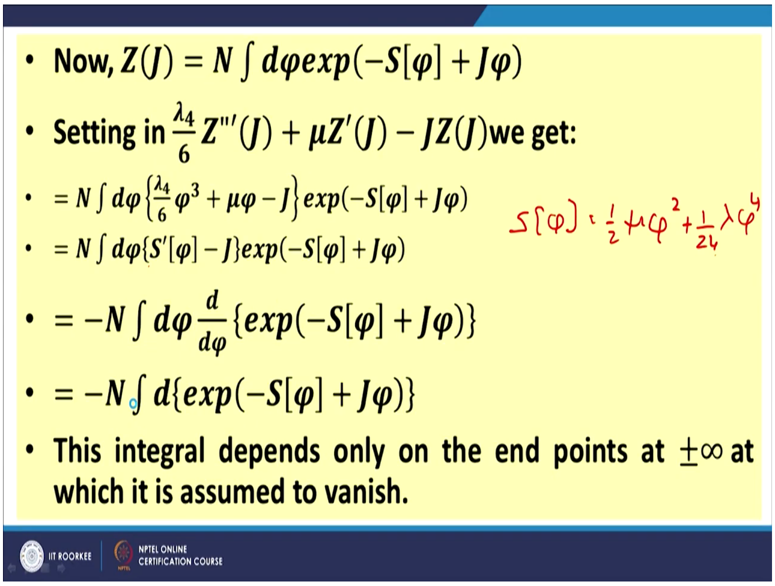

Now, Z J by definition is equal to this expression N integral d phi exponential minus S phi plus J phi, if you set if you use the Schwinger Dyson equation that we have just now put derived lambda 4 upon 6 Z triple dash J plus mu Z dash J minus J Z j.

If you operate if you substitute Z J equal to this, we get N integral d phi lambda 4 upon 6. Because you see Z J and the derivatives of Z J when they operate on this when you differentiate Z J with respect to J, every differential will pull down a factor of phi and bring it within the integral. So, that is precisely what is happening here, when you differentiate Z J 3 times with respect to J you pull down a factor of phi cube. When you differentiate once you differ you bring down a factor of phi and Z J as it is brings you J.

So, in the net result is when this this expression operates on Z J in a sense you bring down a factor of lambda 4 upon 6 phi cube plus mu phi minus J. Of course, this is within the integral with respect to d phi and if you look carefully if you recall the expression for the action what was the expression for the action? One e to the power if you recall the expression for the action s is equal to 1 upon 2 mu phi square plus 1 by 4 lambda sigma phi to the power 4 minus J phi.

Then, clearly this expression is nothing but S dash of phi minus J. Of course, if you include J within the action then it becomes a part of the action otherwise if you take J as separate, then it becomes S dash phi minus J let me write it down S phi is equal to 1 by 2 mu of phi square plus 1 upon 24 lambda phi 4.

Now, if you differentiate this, you get mu of phi mu of phi which we have here plus lambda 4 upon phi lambda 4 upon lambda 4 phi cube upon 6 which you have here and of course, J is there in both cases. So, we get this expression S dash phi minus J here. Now if you look if we write we can write this expression S dash phi minus J exponential minus S phi plus J phi as minus integral d by d phi exponential minus s phi plus J phi and that turns out to be integral of a total derivative d exponential minus S phi plus J phi which is equal to 0.

Because it depends only on the end points and the endpoints are plus minus infinity and we assume that the action at these two point is 0 or the action drops of sufficiently rapidly so, that at the two extremes minus infinity and plus infinity action vanishes. Now these the, these inferences are very interesting.

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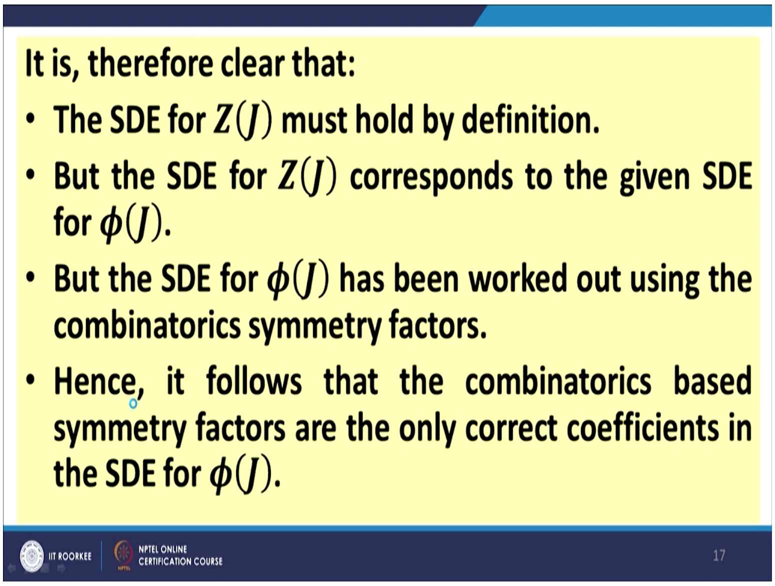

The Schwinger Dyson equation for Z J must hold by definition and we got it here, we got this expression at the Schwinger Dyson equation must hold by definition, but this Schwinger Dyson equation for Z J corresponding corresponds to the given Schwinger Dyson equation for phi J.

You see if you recall this expression this Schwinger Dyson equation, which we obtained directly from the first principle from Z J itself from the definition of Z J, if you go back this Schwinger Dyson equation is also obtained from this Schwinger Dyson equation for phi J that is up given in your red box. But this Schwinger Dyson equation for phi J has been worked out using the combinatorics symmetry factors.

Now, you see how do you do work this out? We got it from the Feynman diagrams and in the Feynman diagrams the symmetry factors that we plugged in were based on certain combinatorial rules or combinatorics of the diagram. The various nuances or the various topologies of the diagrams various diagrams.

So, what is the net inference? The net inference is that those very specific symmetry factors are the correct specific coefficient and the only correct specific coefficients that can be used in at those places because they give rise to this hierarchy of results. Let me repeat this is fundamental, the Schwinger Dyson equation for Z J can be directly obtained straightaway from the value of Z J as equal to n integral exponential minus S plus minus J phi d phi.

So, that you can obtain right away from Z J from this n exponential minus s phi plus J phi, this can be used to obtain the Schwinger Dyson equation. But this this Schwinger Dyson equation can also be obtained directly from phi J and phi J can has been obtained directly from Feynman diagrams and Feynman diagrams contain that specific symmetry factors.

Therefore, the specific symmetry factors must be the correct symmetry factors because the outcome is correct at the end of the day the Schwinger Dyson equation that we get for Z J coincides with the original Schwinger Dyson equation that we get from first principles.

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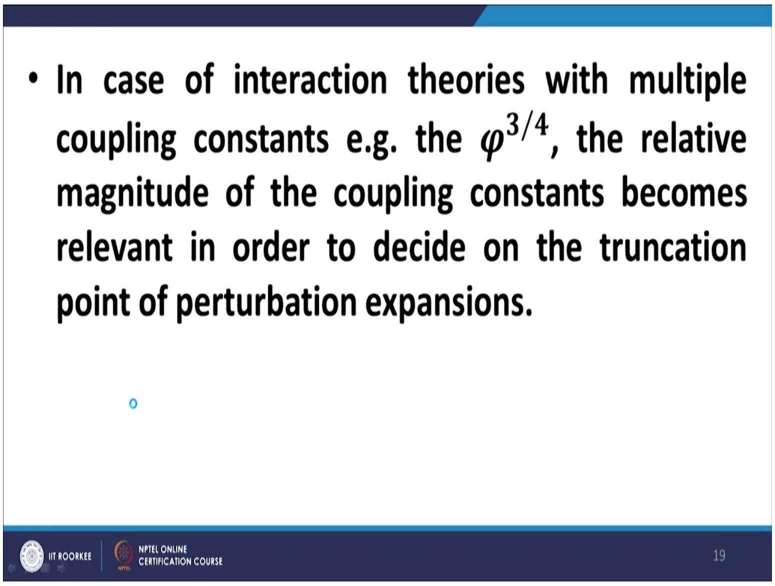

Now, we talk about loop expansions, this is another interesting topic. You see fundamentally this whole theory is perturbation theory. If you have one coupling constant to handle this issue of loop expansion does not become very significant because we have terms only the power series is only in one particular coupling constant. What happens if we have two coupling constant for example, if we have phi to the power 3 oblique 4 theories 3 by 4 theories.

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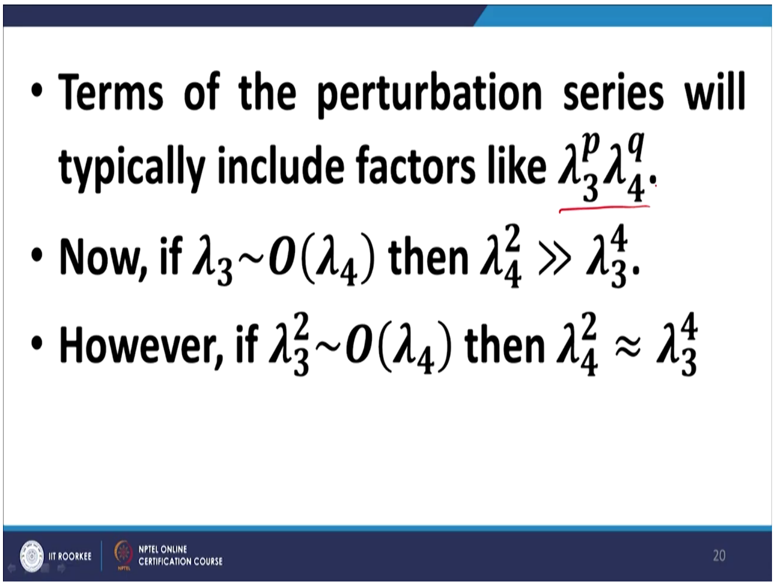

In that case we have two coupling constant lambda 3 and a lambda 4. So, therefore, the perturbative terms, the terms in the perturbative series will be of the form lambda 3 to the power p lambda 4 to the power q. Now to truncate or to determine to ascertain the point of truncation of the perturbation series, we need to have a relative assessment of the magnitudes of these two coupling constants, where exactly the truncation is to be done would depend on the relative magnitudes of the two coupling constant lambda 3 and lambda 4.

For example, if lambda 3 is of the order of lambda 4 then lambda 4 square would be much greater than lambda 3 to the power 4; please remember we both of them are very small. So, lambda 4 square would be much larger than lambda 3 to the power 4. But however, if lambda 3 square is of the order of lambda 4, then the situation would be different then lambda 4 square and lambda 3 to the power 4 would be comparable and would have to be taken cognizance of.

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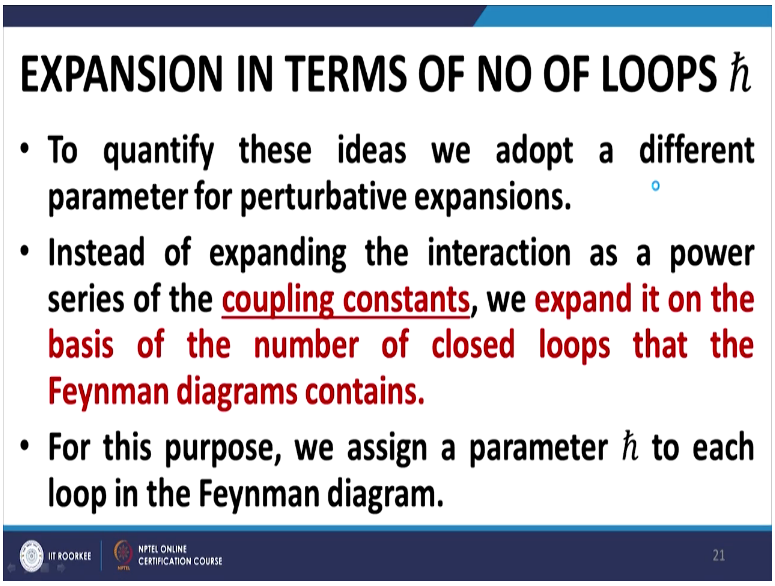

Now, to manage this problem you see so, far we have been doing perturbation theory and we have been expanding the perturbation series in terms of the coupling constant that is in terms of the lambdas, the power series in lambdas. We did it in case of lambda 4 theory in the powers of lambda. Instead of doing that we introduce a new parameter into the arrangement and that parameter does the expansion or does the perturbation expansion on the basis of the number of closed loops of the Feynman diagrams.

The perturbation or the Feynman diagrams of the perturbation series would contain more and more number of loops. So, the expansion can be identified in respect of or with reference to the number of loops that a particular term contains. So, we can decide upon the truncation on the basis of the number of closed loops up to which the perturbation terms are to be retained.

And therefore, in other words we have to now introduce a bookkeeping device; bookkeeping device in respect of the number of loops in a particular term. We identify this by introducing a parameter h bar into the Feynman diagrams. One h bar for corresponding to one loop; if a diagram has 1 loop we attribute a factor of h bar to the diagram to the loop and if a factor has 2 loops a factor of h bar square to the 2 loops and so on.

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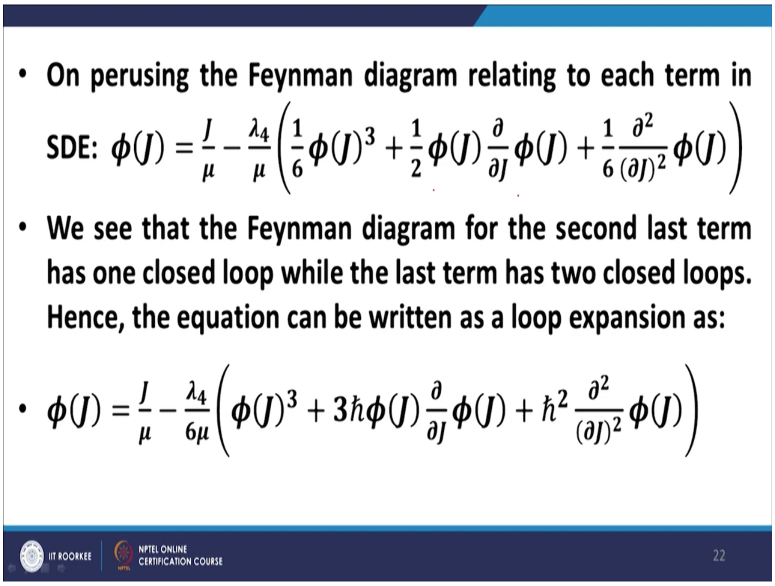

Let us see how it works. Now this Schwinger Dyson equation if you recall the Schwinger Dyson equation for the phi 4 theory the field function of the phi 4 theory is given at the top of your slide.

Now, if you recall the diagrams also if you recall the diagrams also it was these two diagrams this last diagram and the second last diagram that contained loops. This second last diagram phi J del by or the diagram corresponding to this term phi J phi dash J corresponding corresponded to a Feynman diagram with one loop and phi double dash J corresponding to a Feynman diagram with 2 loops. Therefore, if we insert a factor of h bar corresponding to each loop the revised Schwinger Dyson equation will take the form given at the bottom of your slide.

We will have a factor of h bar in the second last term and a factor of h bar square in the last term. Remember the last term had 2 loops. So, we had a factor of h bar square and the preceding term had a single loop and we have a factor of h bar.

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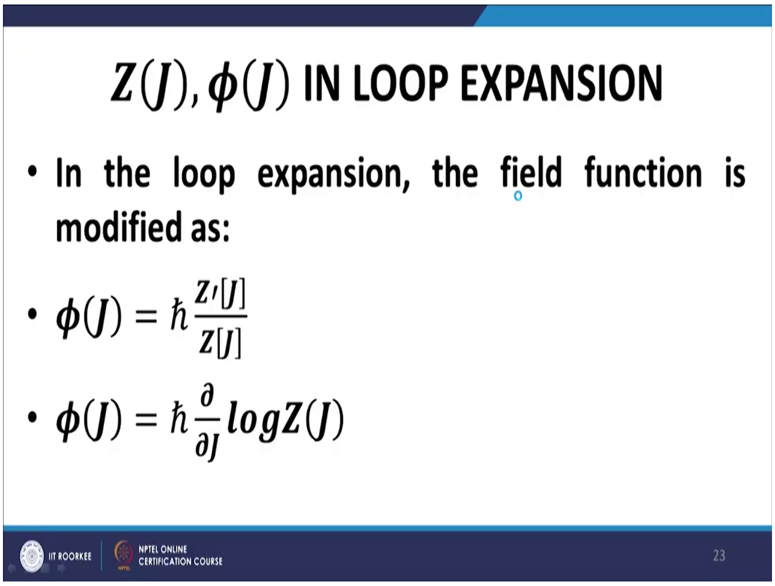

Now in the context of loop expansion the other quantities also need to be redefined. When we are introducing this factor h bar into our Schwinger Dyson equation it is quite natural that the other quantities need to be redefined.

The field function is now redefined as h bar Z dash upon Z J and that is equal to h bar del by and del by del J log of Z j.

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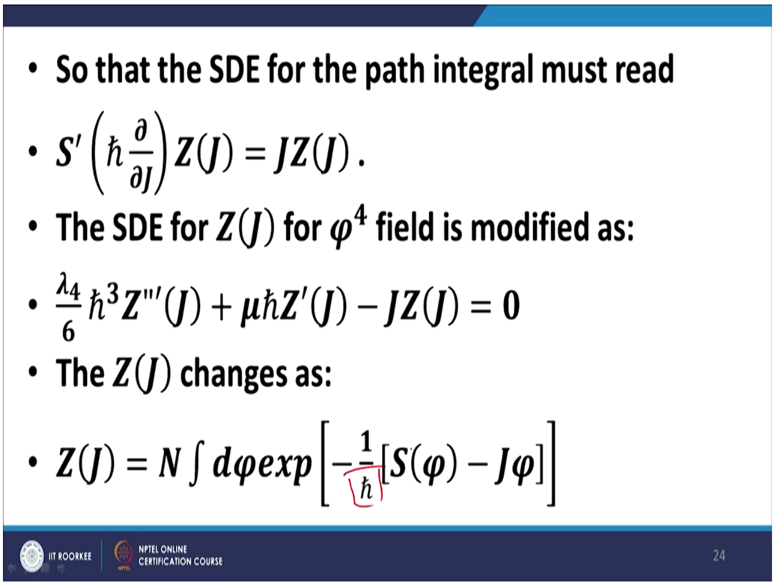

And the Schwinger Dyson equation for the path integral becomes S dash of h d by d J Z J is equal to J Z J the Schwinger Dyson equation for the phi 4 field gets modified here. The expression is given on your slide, lambda 4 upon 6 h bar cube Z triple dash J plus mu h bar Z dash J minus J Z J and the Z J changes to we have a factor of h bar coming into play together with the action.

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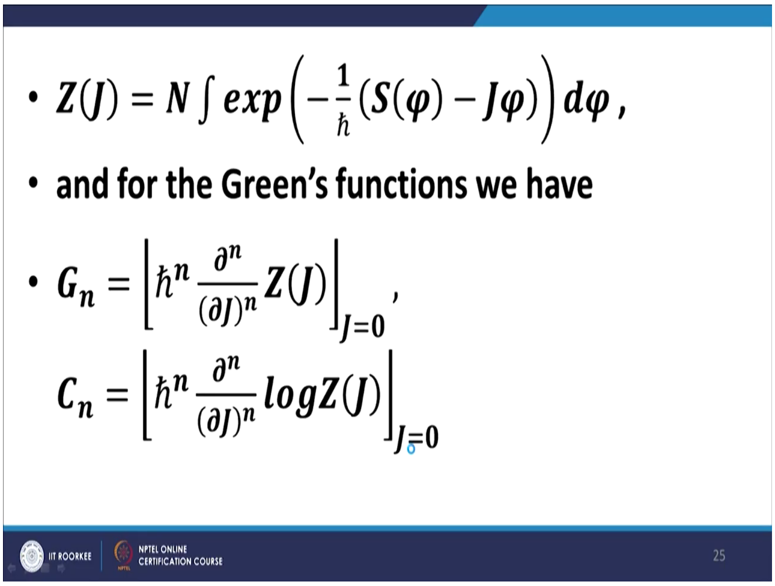

As you can see the green functions also get modified they contain a factor of h bar and the connected green functions also get modified and most importantly the to reiterate the path integral also gets modified by introducing a factor of 1 upon h bar.

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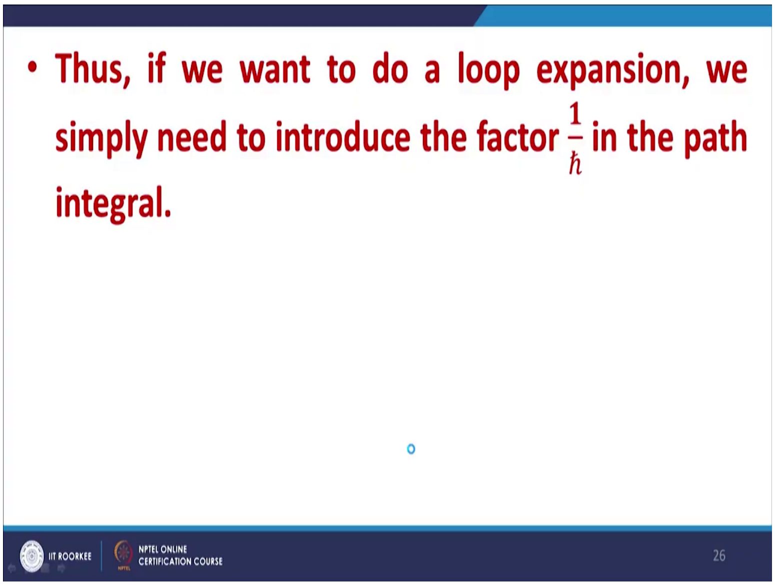

So, if we want to do a loop expansion we simply need to introduce a factor of 1 upon h bar in the path integral.

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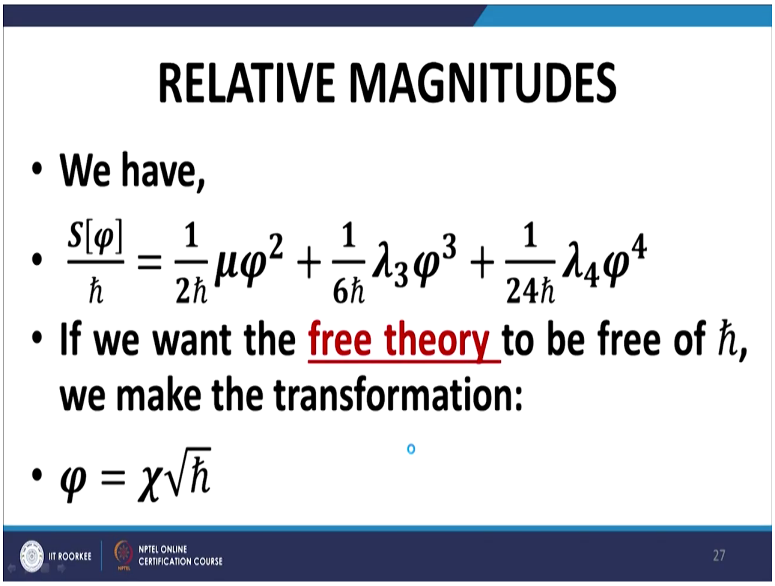

Now relative magnitudes; from on the expression for the action we have this simple expression. If we know that the action is given by 1 upon 2 mu phi square plus 1 upon 3 factorial lambda 3 phi q plus 1 upon 4 factorial lambda 4 phi to the power 4 dividing throughout by h bar on both sides, we get this expression. Now if we want that our free field action is independent of h bar then we said we make a transformation of variables phi is equal to chi root h.

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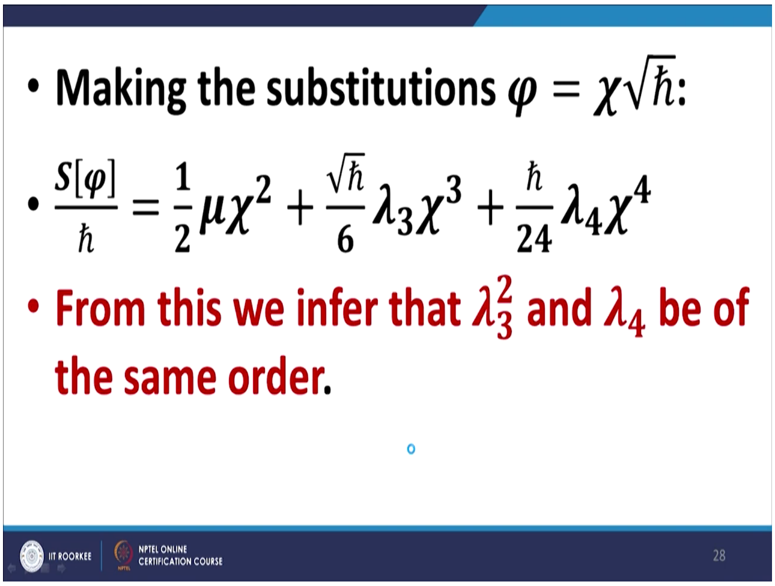

On this substitution the h bar term in the first term the free field term vanishes and we have the remaining terms under root 6 under root h bar by 6 lambda 3 chi q plus h bar upon 24 lambda 4 chi to the 4. Comparison of these two comparison of powers of h clearly show that we have lambda 3 square and lambda 4 are of the same order.

So, that is what we infer that is what we infer or what we can interpret in terms of or by introducing this factor of h bar. It is interesting that that the same inference can be drawn by the use of the of the Feynman diagrams.

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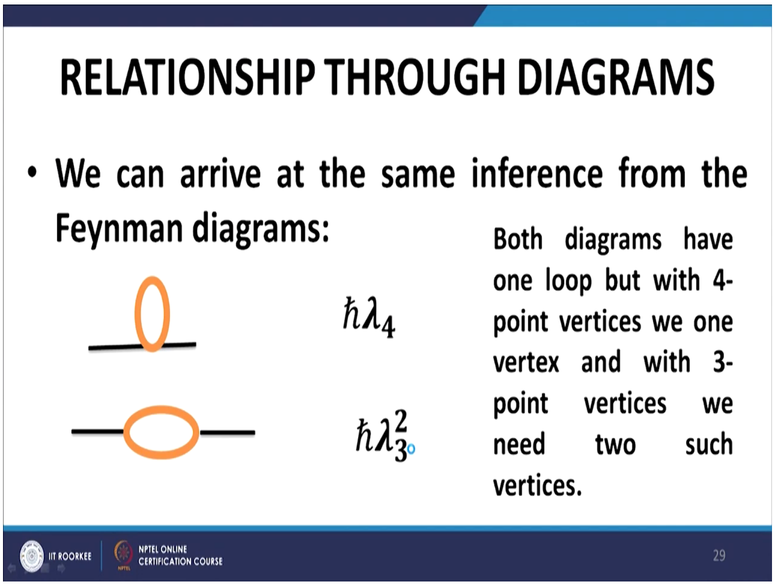

For example, in the case of the 4 vertex when you have a 4 vertex you have one loop that is the upper diagram that you have here. This is interpreted as h bar lambda 4 lambda 4 because we have a 4 point vertex here and h bar is because of the one loop.

Now, to reconstruct the one loop using 3 vertices, we need two 3 vertices two 3 point vertices with one 3 point vertex it becomes to construct a loop with 3 point vertices, you need two 3 point vertices. So, as a result of which because you have two 3 point vertices, you have lambda 3 square and because of the one loop you have h far. So, in a sense if the loops represent equivalence in some sense then we have lambda 3 square is equal to lambda 4.

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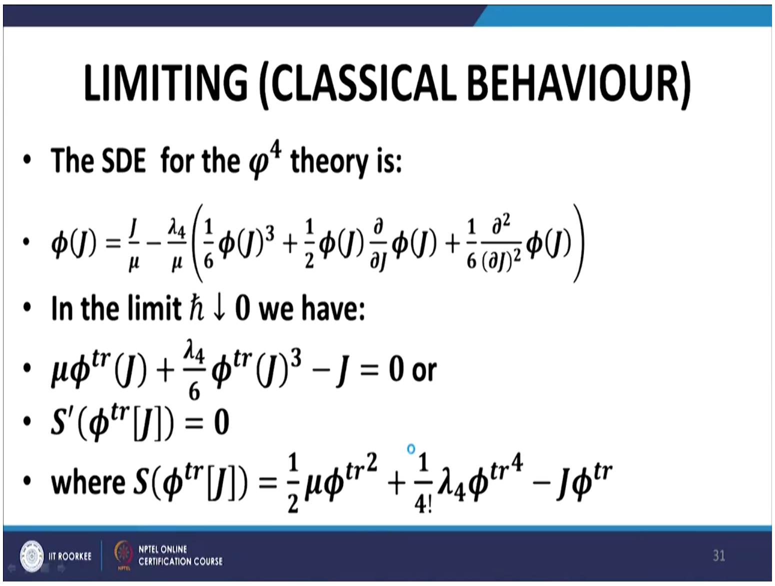

So, now we talk about the classical behaviour or the tree level behaviour of the phi 4 theory. Ok. We will continue in the next class.

Thank you.