Path Integral Methods in Physics & Finance
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Lecture – 35
SDE, Feynman Diagrams

Welcome back. So, in the last lecture we developed the theory at the quantum field theory using the path integral framework, we developed the Schwinger Dyson equation for the phi to the power 4 field, let us quickly recap for where we were last time.

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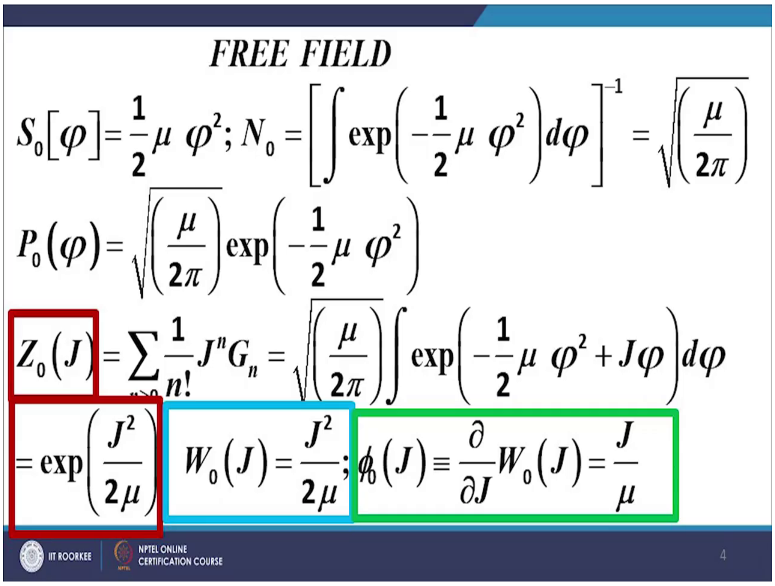

And then proceed further and the free field parameters we have the equation, we have the action functional 1 by 2 mu phi square, the normalization we worked out by and doing a Gaussian integration as under root mu upon 2 pi the probability density function was therefore, became under root 2 pi mu upon 2 pi I am sorry exponential minus 1 by 2 mu phi square.

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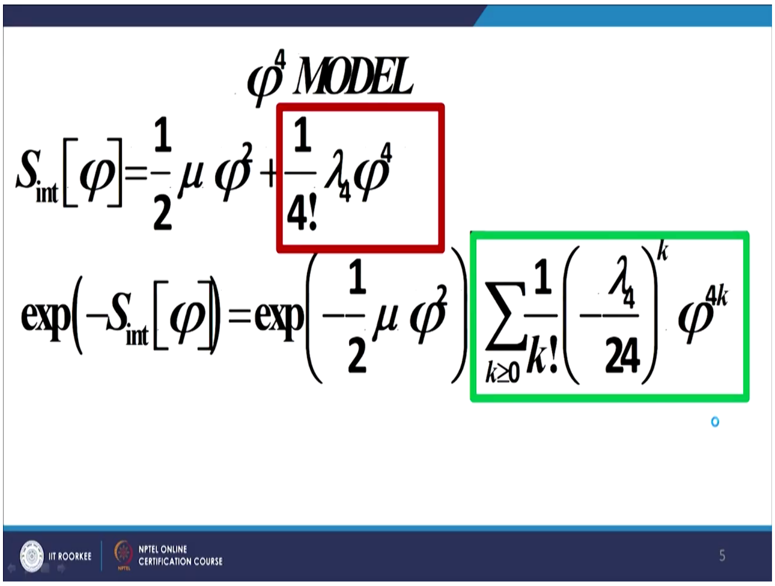

So, this was for the free field, then we talked about the interaction represented by phi to the power 4 term in the action. We added this term to the action and we then on the premise that the coupling constant lambda 4 is very small we expanded this action as a perturbation series in powers of lambda.

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$$Z_{\text{int}}(J) = N_{\text{int}} \sum_{k\geq 0} \frac{1}{k!}\left(-\frac{\lambda_4}{24}\right)^k \int \varphi^{4k} \exp\left[-\left(\frac{1}{2}\mu\, \varphi^2\right) + J\varphi\right] d\varphi$$

$$N_{\text{int}} = \left[\left(\frac{2\pi}{\mu}\right)^{1/2} \sum_{k\geq 0} \frac{1}{k!}\left(-\frac{\lambda_4}{24\mu^2}\right)^k \frac{4k!}{4^k (2k)!}\right]^{-}$$

So, and we found the expressions for the revised normalization, the normalization of the interaction field which is given in the Green box. And the expression for the revised generating functional for the Green's functions is given in the red box.

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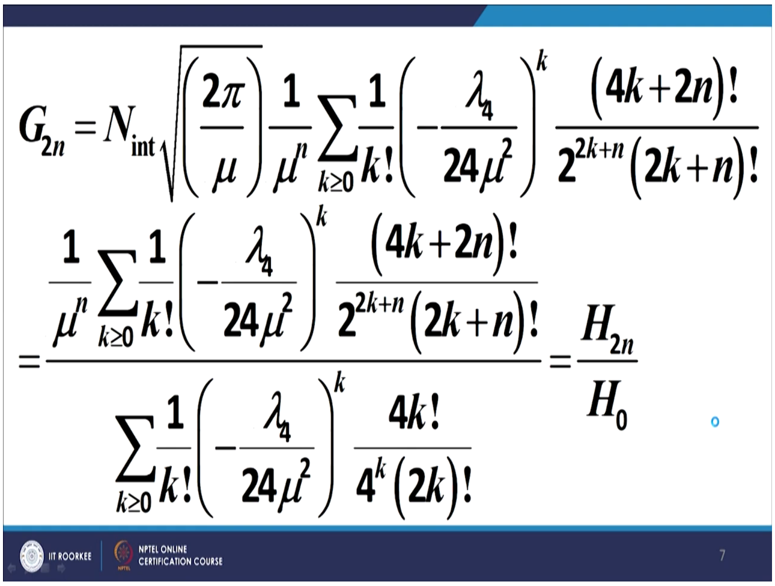

The explicit expression for the Green functions 2 n point Green functions is given in terms of the perturbation series as on the slide. And then we obtained the Schwinger Dyson equation.

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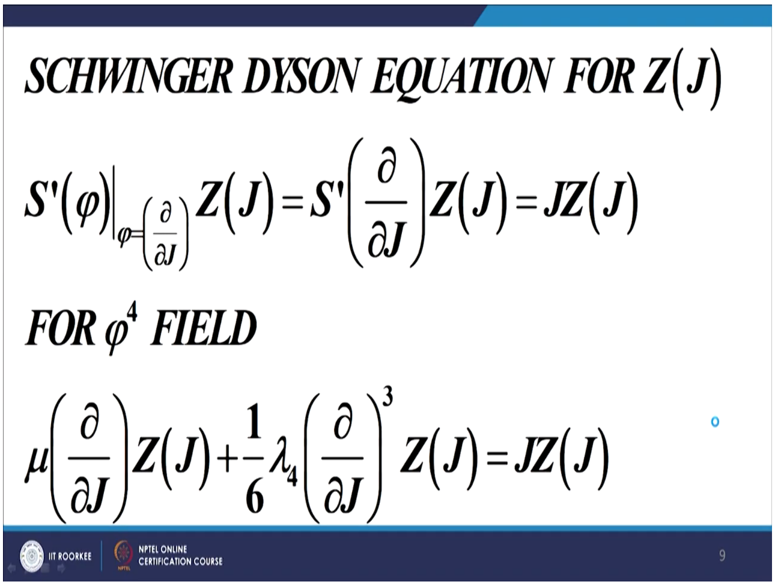

The general Schwinger Dyson equation for the generating functional or the path integral as you may call is given in the first equation on your slide and for the specific case of the phi 4 field, this equation takes the form of this second equation right at the bottom of your slide.

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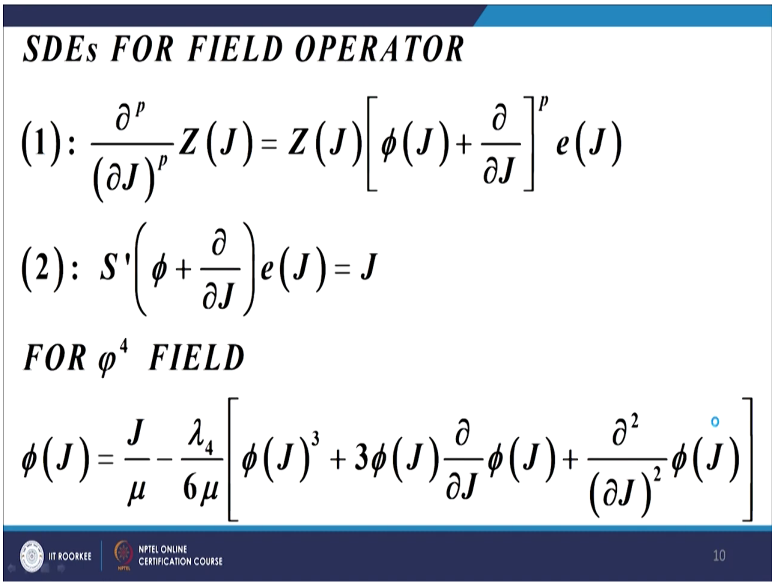

The Schwinger Dyson equation for the field operator we found it in two forms two expressions were obtained for this Schwinger Dyson equation for the field operator; number 1 and 2 equation, number 1 and 2 given on your slide.

And for the field for the interaction field phi 4 we have this expression which is the bottom equation on your slide; this is the expression for the Schwinger Dyson equation for the field operator.

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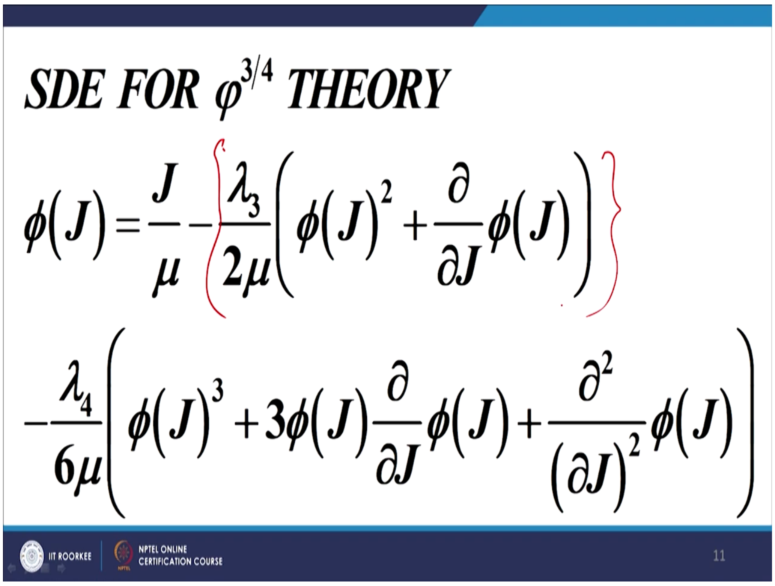

The Schwinger Dyson equation for the phi 3 4 field is given in this slide, you see the additional terms involving lambda 3; which is the coupling constant for the phi to the power 3 term to the action, which is included here as an additional term to the original expression for the phi to the power 4 field which is given in this last equation bottom equation of the slide.

The additional terms comes from this additional term comes from the phi to the power 3 component or the coupling to the free field action.

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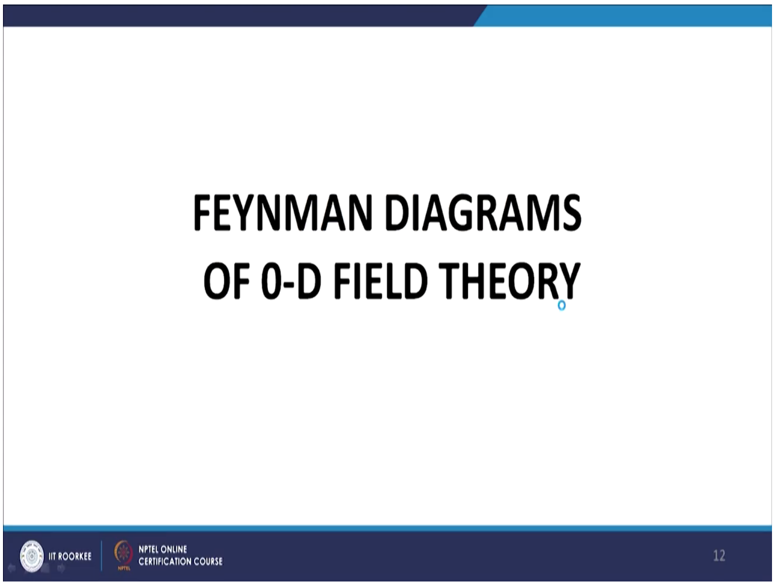

Now today we started our discussion of the Feynman diagrams.

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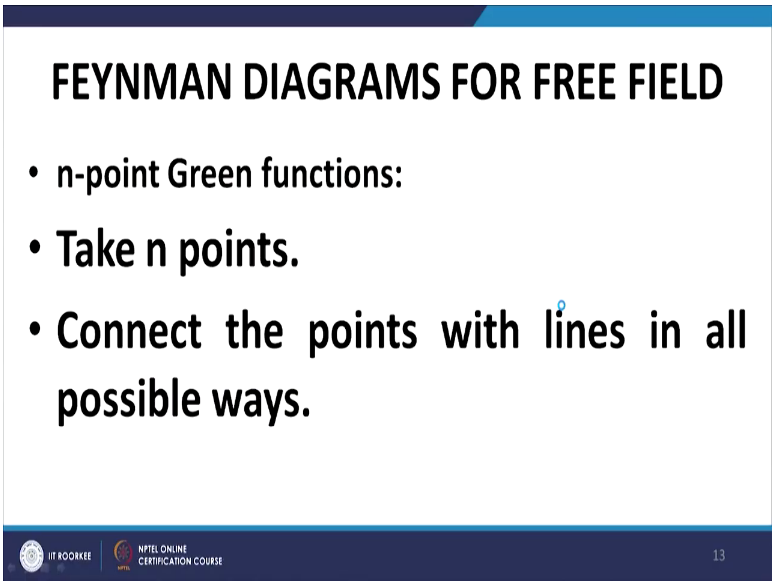

Let us start with a very simple straightforward example. We use the Feynman diagrams for developing for determining the Green functions n point Green functions of the free field.

What we do is, we simply take n points and connect these n points with lines in all possible ways. And we take pairs of points and we connect every possible pair of points and then we see how we can develop or we can extract the various expressions for the Green functions from this particular diagrammatic representation for the free field take n points and connect these n power in pairs by straight lines in as many possible ways as can be done.

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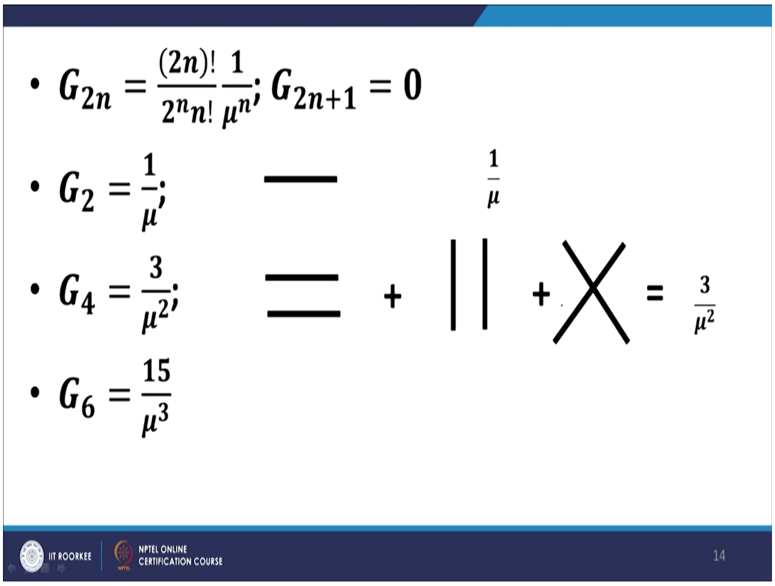

For example, a for G 2 we have only 2 points because we are talking about G 2 we are talking about 2 points and 2 points can be connected by only one line in only one way. So, we have just the one line being represented here and we attach a weight because you see if you look at it the expression for G 2 n is given by 2 n factorial upon 2 to the power n, n factorial and there is an additional expression of 1 upon mu to the power n.

To accommodate this additional expression of 1 upon mu to the power n and there will be n pairs of lines n lines rather n pairs of points connected by n lines. And therefore, we attach a weight of 1 upon mu to each line and that is precisely what is done here, we have got just the 1 line here and that 1 line carries a weight of 1 upon mu. So, we have G 2 is equal to 1 upon mu. Let us look at a more involved case of G 4.

Here we have n equal 2 n equal to 4 that is n is or we want to determine the 4 point functions. So, we talk about 4 points and these 4 points can be connected 2 with 2 lines, those 2 lines can be represented in the form 1, 2 or the third one. Please note the third one the third line is not intersecting its moving one the two lines are independent of one other.

There is no vertex is connect which is the point of intersection of the line 1 and the line 2, they are one is placed independent of the other, but the point is there are 2 lines here there are 2 lines vertical here and then we have this cross in this form here and each of them carries now each line carries a weight of 1 upon mu.

So, because we have two lines we have a weight of 1 upon mu square attached to each diagram now because there are three possible diagram. So, we have a total weight of 3 upon mu square which is the value of G 4 similarly we can compute G 6 and so, on.

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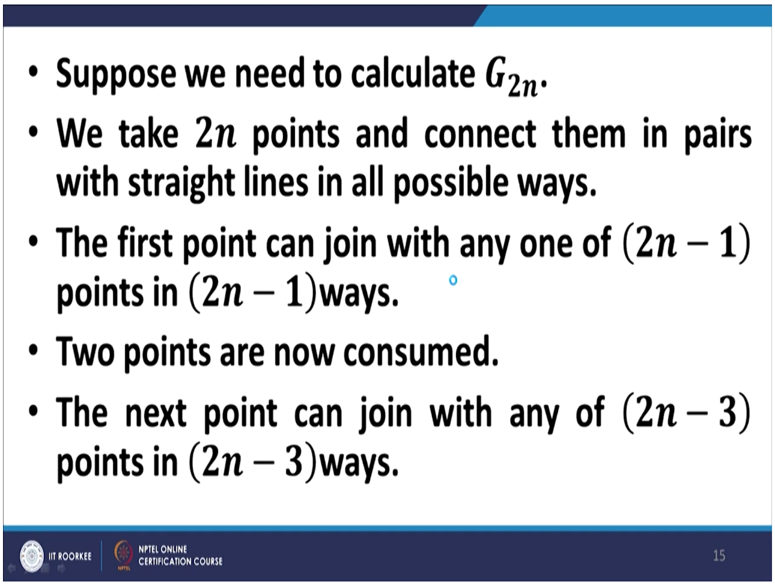

Let us see the logic behind it. Suppose we need to calculate G 2 n. Then we take 2 n points and them in pairs with straight lines in all possible ways. The first point can the first point let us say we take any point as the first point and that first point can be connected to any of the remaining 2 n minus 1 points in 2 n minus 1 ways.

Now, with the first point and being connected to any one particular point out of the 2 n minus 1 points, we are now left with 2 n minus 2 points. We select the 2 n minus 2 th point and the 2 n minus 2 th point can be connected again to any one of the 2 n minus 3 points that are remaining free that are not connected. So, in other words we now have 2 n minus 1 factorial into 2 n minus 1 into 2 n minus 3 possible combinations.

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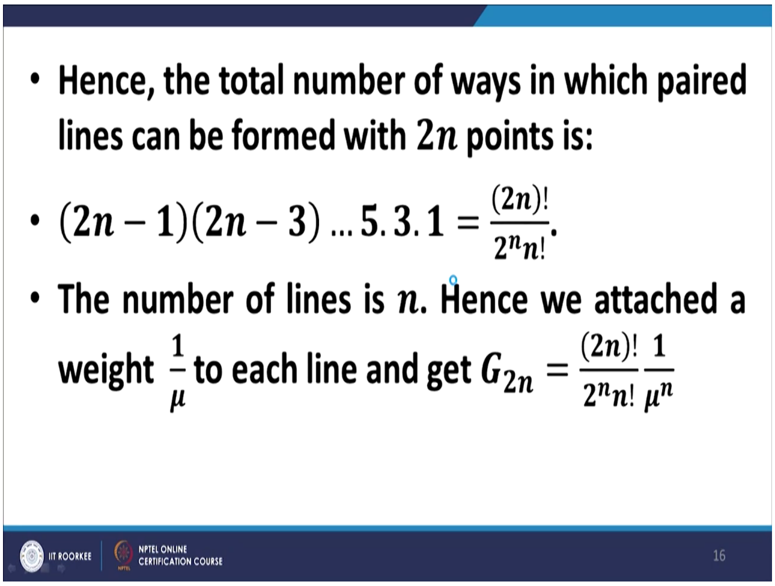

Similarly, we can continue and we have a total of the total number of ways in which we can form pairs we can connect pairs of lines with point and connect pairs of points to form lines as 2 n minus 1 into 2 n minus 3 into 5, 3, 1. First point connect it to 2 n minus 1 points, then the 2 of the total 2 n points are exhausted, we now have 2 n minus 2 points left, we select the 2 n minus 2 th point that is connected to any one of the 2 n minus 3 th point.

So, we have 2 n minus 3 ways in which that can be done and then this process continues until we reach when we reach the last quantity of last factor of 1 and this factor can be simplified or put in a compact notation as 2 n factorial upon 2 n 2 to the power n into n factorial. Now and this is the number of this is the number of lines, now we attach a weight of 1 by mu to each line.

So, we have 2 n factorial 2 to the power n, n factorial into 1 upon mu to the power n and that is precisely what is the value of G 2 n as worked out on the basis of the Gaussian distribution. So, this shows the correspondence at least which is quite simple and which is quite tractable in the free fields case.

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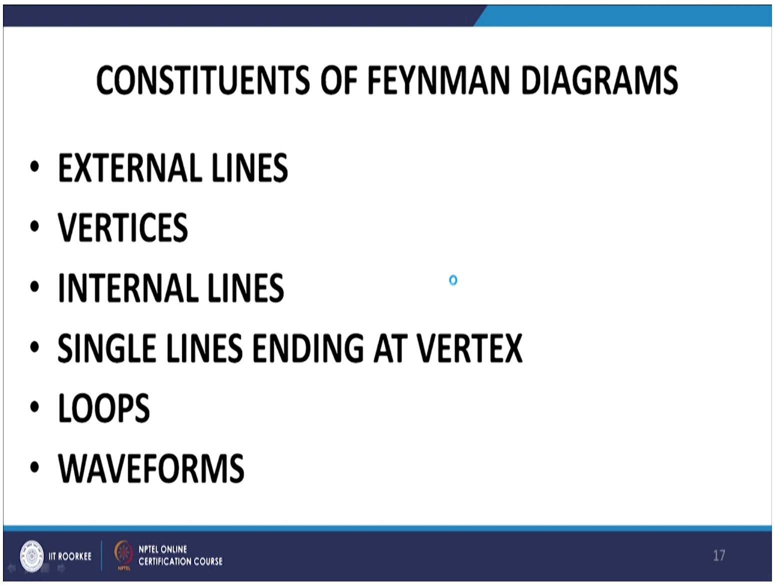

Now, let us look at the terminology of the Feynman diagrams. The Feynman diagrams it can consist of the external lines, we have the vertices, we have the internal lines, we have single lines that are ending at a vertex, we have loops and we have waveforms.

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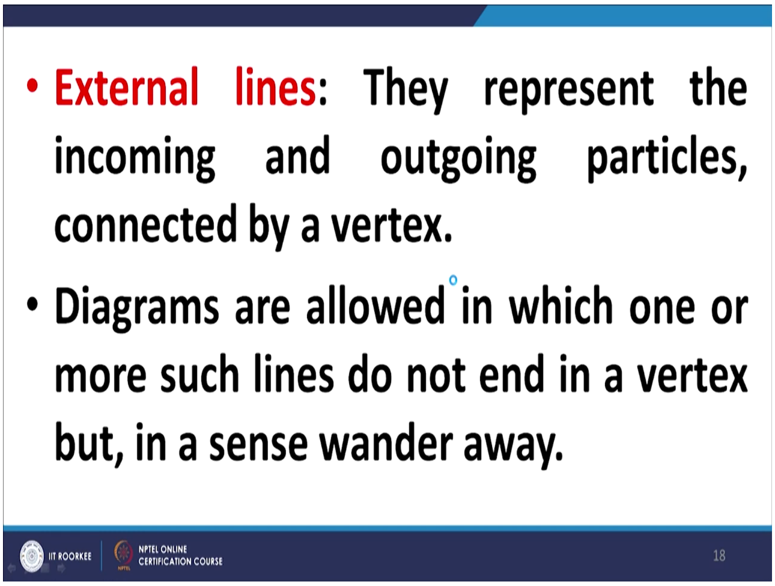

The external lines represent incoming and outgoing particles and they are connected by a vertex. Diagrams are allowed in which one of those lines do not end in a vertex and they in a sense they simply wander away endlessly.

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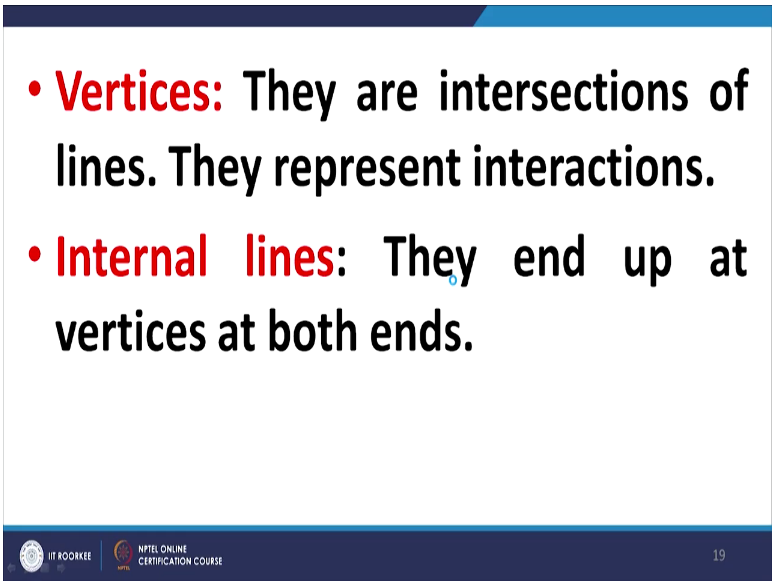

Vertices are in vertices represent the interaction points, they are intersections of lines they represent the points of interaction.

Internal lines are lines that are connected by vertices at both the ends. So, you have external lines that may be connected to a vertex at one end. And the internal lines have are connected to vertices at both the ends and then you have loops.

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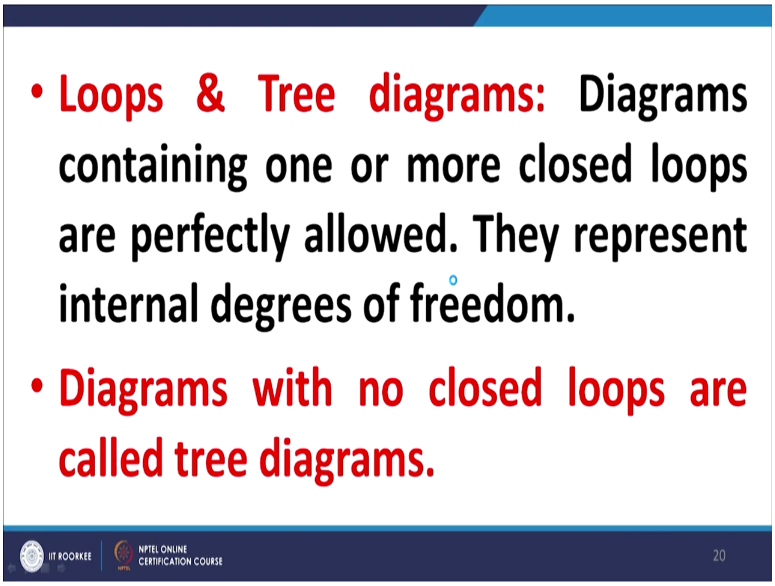

Loops see loops are closed loops are represent internal degrees of freedom, they start and end at a at the same vertex and they do not have any bifurcations there they simply start at one point at a vertex and they terminate at the same vertex.

They basically represent internal degrees of freedom that may that may take infinite values also we will talk about that. Now tree diagrams are those diagram those Feynman diagrams which do not include any loops. In other words, if we take out from a set of Feynman diagrams, we take out the diagrams that consists of loops or that contain one or more loops the rest what we get is a tree diagram. Then the source vertex is a vertex at which a single line ends and carries the Feynman rule factor of plus J, this is the source vertex.

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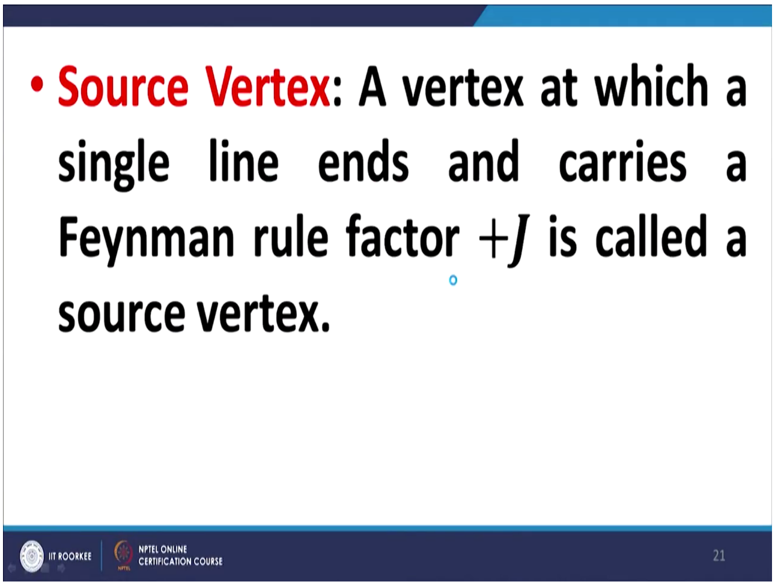

So, this is some terminology that the viewers need to be accustomed to; gradually as you proceed in this course we shall get more and more conversant with the use of this terminology and the and the diagrammatics involved with the terminology. So, let us proceed. Time flow can be taken either from left to right or from bottom to up as at the choice of the analyst. Connected diagrams are those diagrams where you can move from one point in that diagram to any other point following the lines of the diagram.

And obviously, disconnected diagrams are those diagrams which are not connected they consist of disjoint pieces that are that are themselves connected, but they are not totally connected to form 1 composite piece in other words from 1 point in the diagram you cannot move to every other point through the lines of the diagram. Now what are the Feynman rules for the phi 4 theory?

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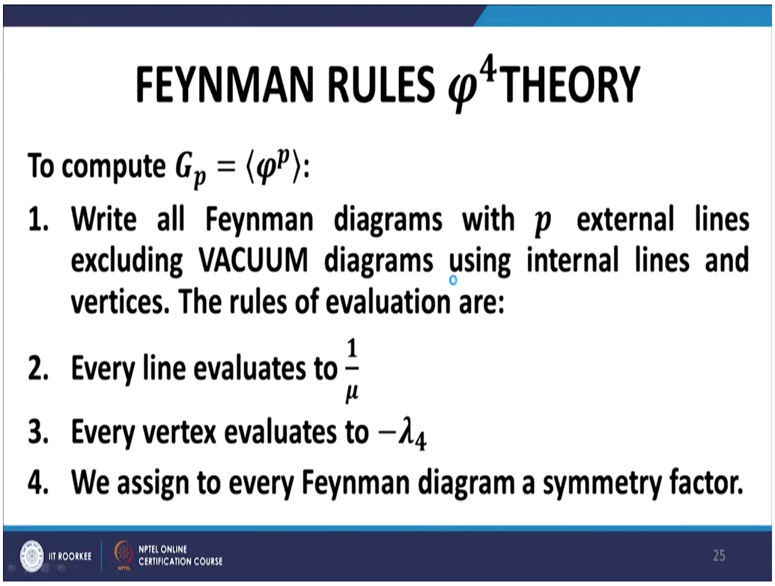

For example, as an example we shall be using the phi 4 theory. To compute now we talk about the interaction recall that in the earlier case we have just talked about the free field theory, now we talk about the interaction theory. We write all Feynman diagrams with p external lines. Suppose we have to compute G p that is the p point Green function we write all Feynman diagrams with p external lines excluding vacuum diagrams.

What are vacuum diagrams? We will come back to in a minute, but using internal lines and vertices we write all possible Feynman diagrams with p external line. The number of external lines must match the order of the Green function. For a p th order Green function p point Green function we need to construct Feynman diagrams with p external lines. Every line as I mentioned evaluates to 1 upon mu and every vertex evaluates to minus lambda 4 that is the coupling constant minus of the negative of the coupling constant and then there is the issue of a symmetry factor which comes into play.

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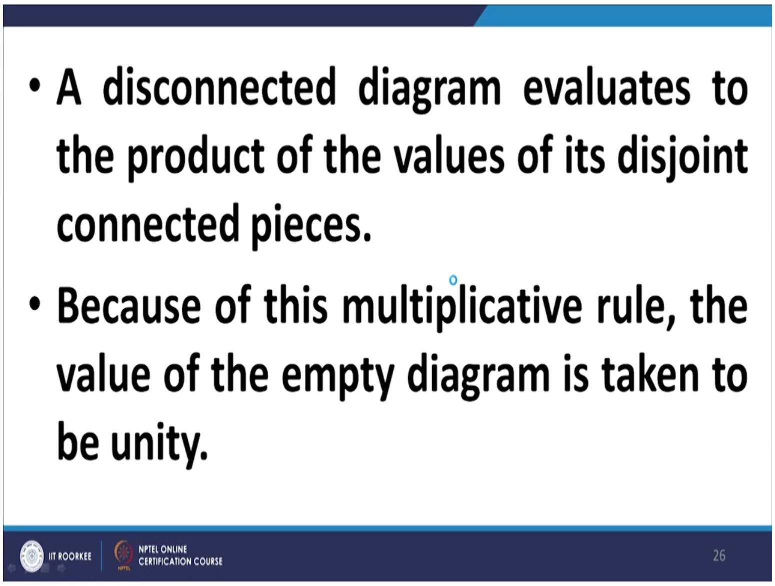

What is a symmetry factor we shall come back to? In the mean time at if we have a disconnected diagram that evaluates to the product of the values of its pieces. For example, if a disconnected diagram consists of two connected diagrams; then that value of that disconnected diagram is equal to the product of the values of the two connected diagrams and because of this multiplicative rule the value of the empty diagram is taken as the unity.

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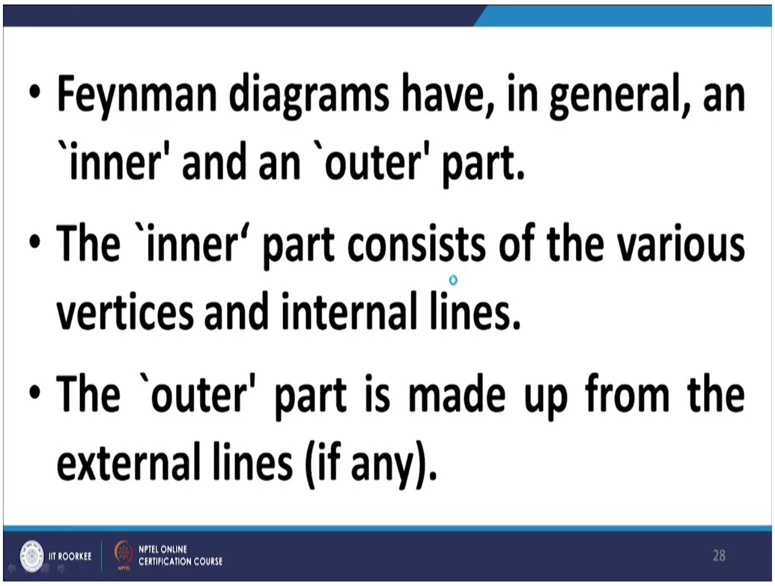

Now, we come to symmetries as I mentioned. The Feynman diagrams as can be separated into two parts the inner part and the outer part. The inner part consists of the vertices and internal line the outer part consists of the external lines. The inner part is related to the symmetry factor the outside outer part is related to what is called the multiplicity. Now what are the rules for the symmetry factor these are important.

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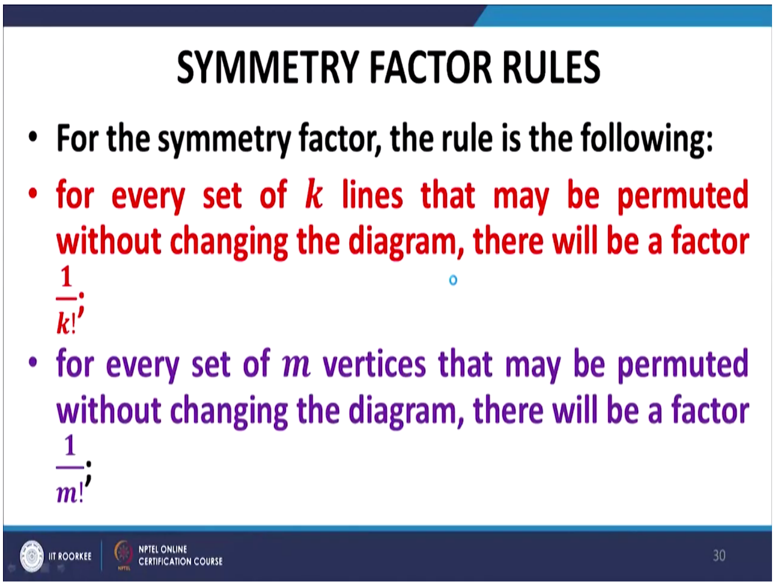

For every set of k lines that may be permuted without changing the diagram this is important. For every set of k lines which can be permuted without changing the diagram there will be add an added factor of 1 of 1 upon k factorial 1 upon k factorial. For every set of m vertices which can be permuted without changing the diagram a factor of 1 upon m factorial will be added to the value of the diagram.

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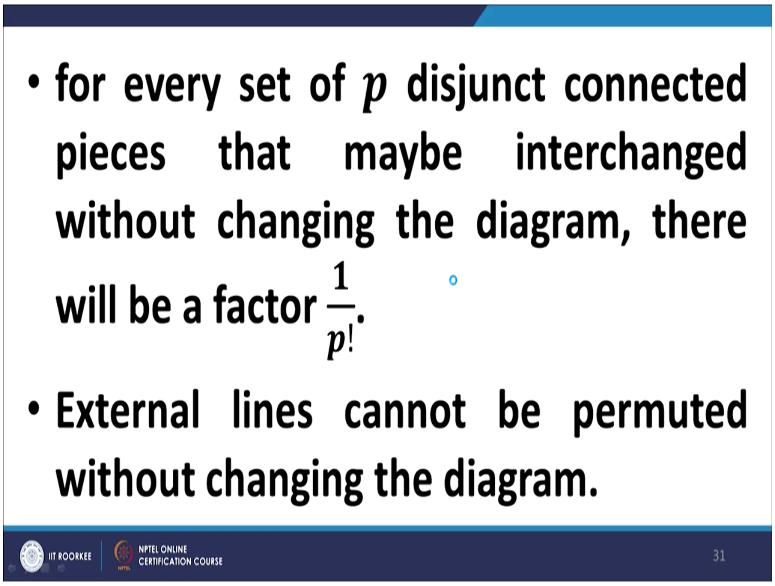

And for every set of p disjunct connected pieces, p disjunct connected pieces that maybe interchange without changing the diagram your p disjunct connected pieces. And you can interchange them without changing the diagram then you add a factor of p factorial to the diagram. External lines cannot be permuted without changing the diagram that is a condition.

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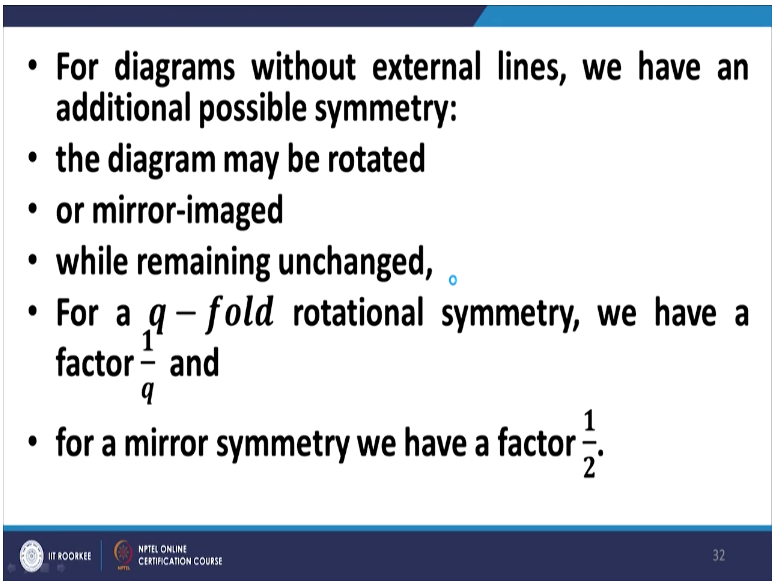

In this context the role of multiplicity will come into play, for diagrams without external lines we have additional possibilities a possible symmetry like in the case of loops. The diagrams may be rotated and the diagrams may carry a mirror images symmetry mirror image symmetries.

Now, if there is a q-fold rotational symmetry a factor of 1 upon q comes into play while evaluating the diagram and if there is a mirror symmetry a factor of 1 upon 2 comes into play while evaluating the diagram.

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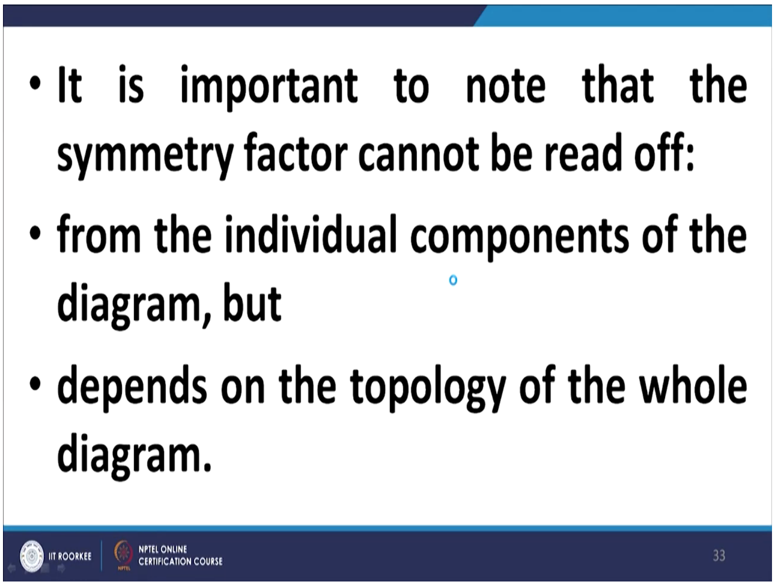

Now the symmetry factor has cannot be read off it really depends on the topology of the diagram. So, it cannot be read off from the individual components, it the composite set or the composite what should I say the topology of the diagram how the various constituents of the diagrams are being connected to form the diagram have a role to play in determining that the symmetry factor of the diagram.

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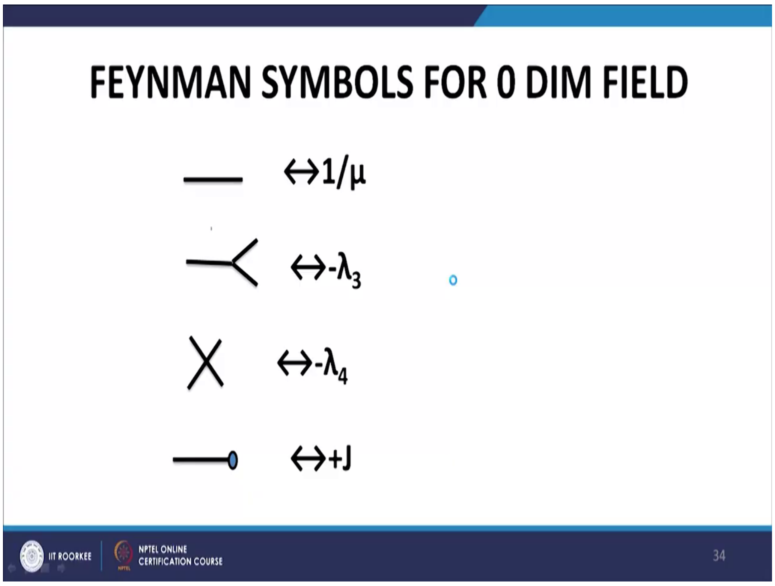

Now, a typical Feynman symbols for the 0 dimensional field that we are talking about. A straight line evaluates as 1 upon mu we have talked about that a 3 vertex in the context of a phi 3 field evaluates as minus lambda 3 of 4 vertex in the context of a phi 4 field evaluates as minus lambda 4 where lambda 3 and lambda 4 are the coupling constants and aligned with a blob evaluates as plus J we have already talked about this one.

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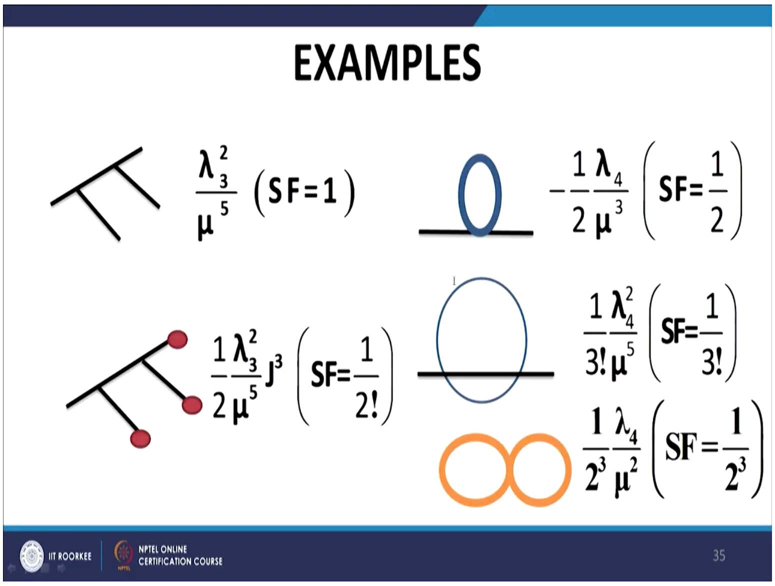

Let us take some examples. These are some illustrations I to explain them I have separately explained each and every one of them.

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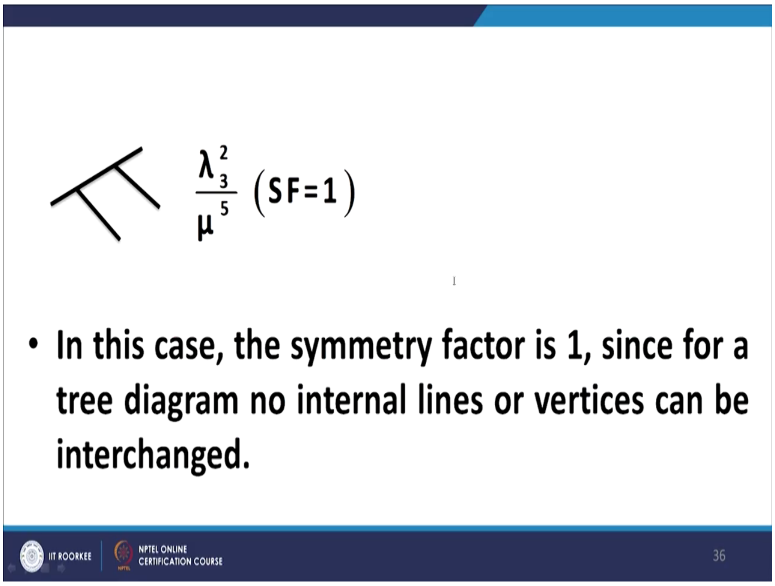

So, let us take the first one. In the first case as you can see there are 5 lines that that account for the factor of 1 upon mu to the power 5; 1, 2, 3, 4, 5 there are 5 lines and each line carries a factor of 1 upon mu.

So, this evaluates as 1 upon mu to the power 5, in addition there are 2, 3 vertices. Please note the vertices are 3 vertices this is 1 3 vertices and the other is this one is the other 3 vertex. So, the 2, 3 vertices evaluate at lambda 3 squared. Now there is this symmetry factor is 1 because there is no internal line which or vertices which can be interchanged. No internal line exists which can be interchanged now vertex exists which can be interchanged and therefore, the symmetry factor in this case in this particular diagram is 1.

In fact, in the case of tree diagrams the symmetry factor is 1 because internal lines and vertices cannot be interchanged.

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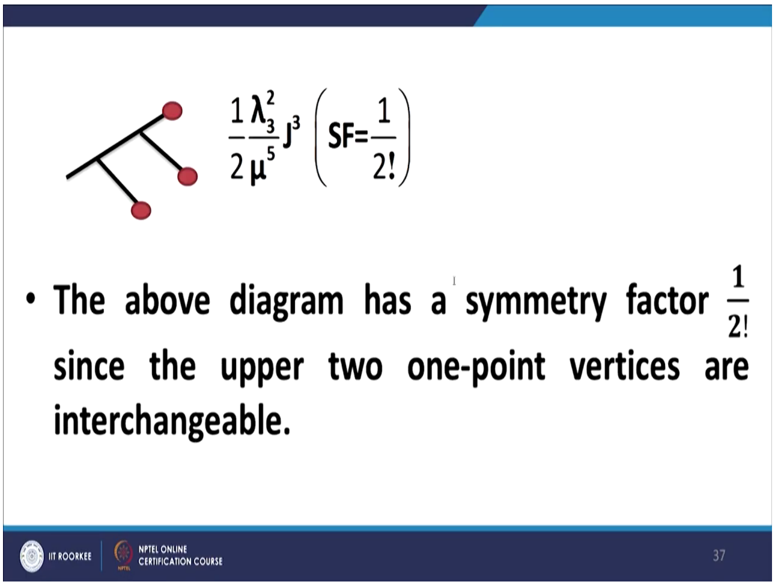

Now let us look at this diagram in this diagram again we have 5 lines. So, that accounts for mu to the power 5, 1 upon mu to the power 5, we have 2 vertices as in the previous case. So, we have lambda 3 and both of them are 3 vertices.

So, we have lambda 3 square we have 3 blobs. So, we have J and in 3 lines ending with blobs. So, we have J cube and look in this symmetry if you look at the symmetry the upper 2 branches the top 2 branches can be interchanged without disturbing the diagram and therefore, the symmetry factor is 1 by 2.

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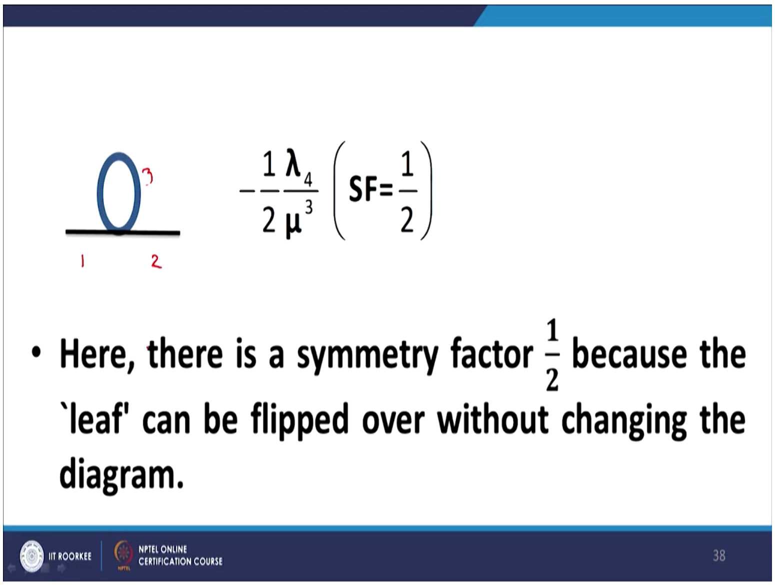

Here we have a loop with a with a with a line, now if you look at it carefully this is the 4 vertex the point at which the line and the loops are connected is a 4 vertex. Where 4 outgoing 4 lines emerge from that vertex and therefore, it is the lambda 4 vertex. So, we have it evaluates as minus lambda 4. In addition we have 3 lines 1 let us call it 1, this is 2 and this is 3.

So, we have 1 upon mu to the power q and of course, the symmetry represents the symmetry of the flipping of this leaf. So, the symmetry is 1 by 2 you can have the leaf like this or you can have the leaf like this. So, the symmetry is 1 by 2, symmetry factor is 1 by 2 and the net result is minus 1 by 2 lambda 4 upon mu cubed.

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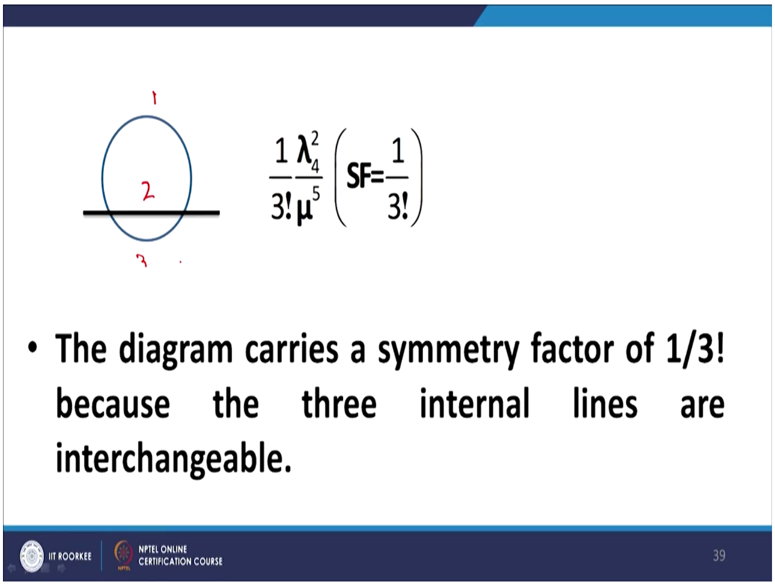

Let us look at this diagram here. In this diagram we have how many vertices? We have 1 vertex here which is again 4 vertex and we have 1 vertex here which is also a 4 vertex. So, we have lambda 4 square how many lines do we have? 1, 2, 3, 4 and 5 lines here. So, we have mu 1 upon mu to the power 5 and clearly these three lines can be interchanged without disturbing the diagram this line 1, 2 and 3 can be interchanged without changing the diagram and therefore, we have a symmetry factor of 1 upon 3 factorial.

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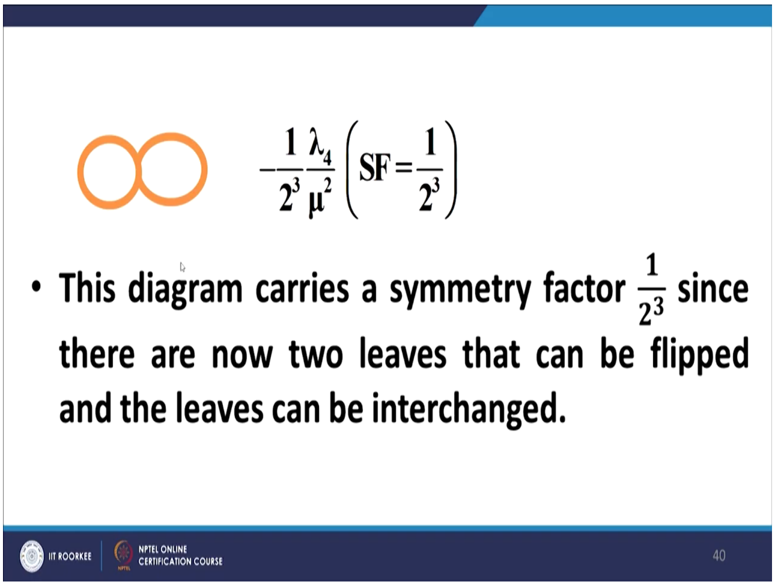

Nowo, in this case we have 2 lobes connect 2 connected lobes, we have 1 vertex and that vertex is a 4 point vertex as you can see there are 4 lines being joined at this point. So, its a 4 vertex. So, we have 1 lambda 4 the number of lines is 1 and 2.

So, we have mu square of 1 line with the first lobe and the 1 line with the second lobe. So, we have mus mu square and the diagram has a symmetry factor of 1 upon 2 to the power cube, the flipping of the first leaf the flipping of this flipping of the second leaf and the interchange of the leaves.

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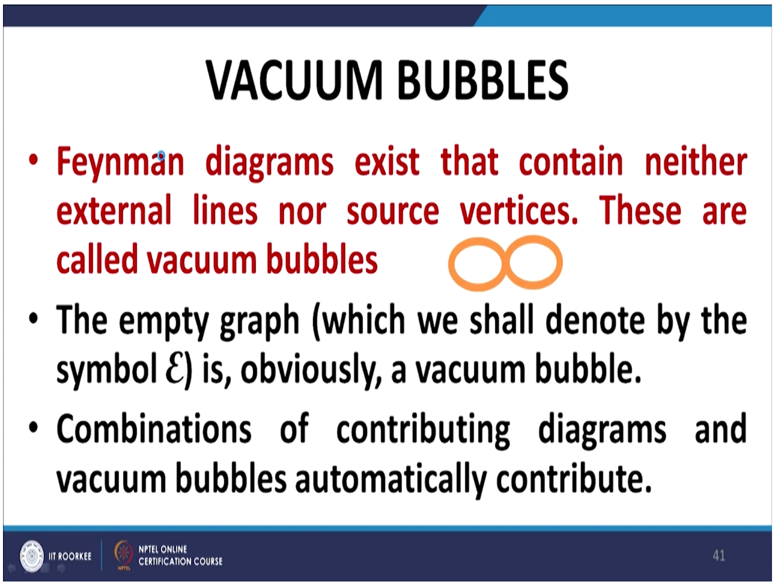

Now we come to the concept of vacuum bubbles. I have mentioned earlier the concept of vacuum bubbles the diagrams that contain neither external lines nor source vertices are called vacuum bubbles.

Now, the property of this of course, the empty graph is also a vacuum bubble.

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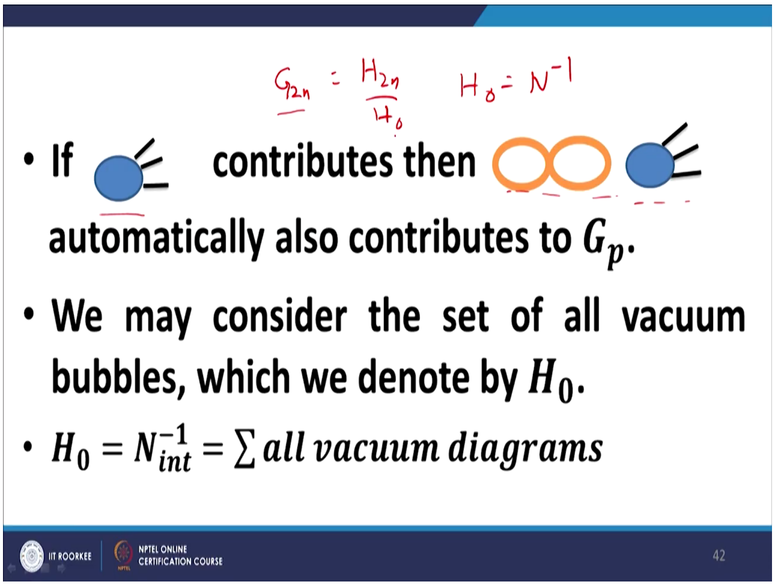

The property of this vacuum bubbles is this that if a particular diagram contributes to a Green function then the diagram together with a vacuum bubble will also contribute to the Green function. In other words, if this particular diagram contributes to a Green function, then this this diagram together with this constituent of the vacuum bubble also contributes to the Green function.

In in terms of interpretation they really relate to the normalization of the theory. And H 0 the denominator in the Green function recall the expression for the Green function. The expression for the Green function was G 2 n is equal to H 2 n upon H 0 where H 0 was related to the normalization.

So, H 0 is related to these norm on these vacuum bubbles, H 0 represent the sum of all these vacuum bubbles. So, when we work out the Feynman some of the Feynman diagrams to compute the Green function to compute the Green function say G 2 n, we first of all we include all the possible Feynman diagrams. And then extract away remove from them all those Feynman diagrams that constitute vacuum bubbles.

So, H 0 is equal to summation of all vacuum diagrams. All these vacuum diagram in other words in other words it amounts to what I am saying amounts to that when we work out the Green function using the technology of Feynman diagrams. We sum over all Feynman diagrams excluding all vacuum bubbles putting it either way either you sum over all Feynman diagrams and then remove the vacuum bubbles; because they represent the normalization or you sum over all Feynman diagrams excluding the vacuum bubbles.

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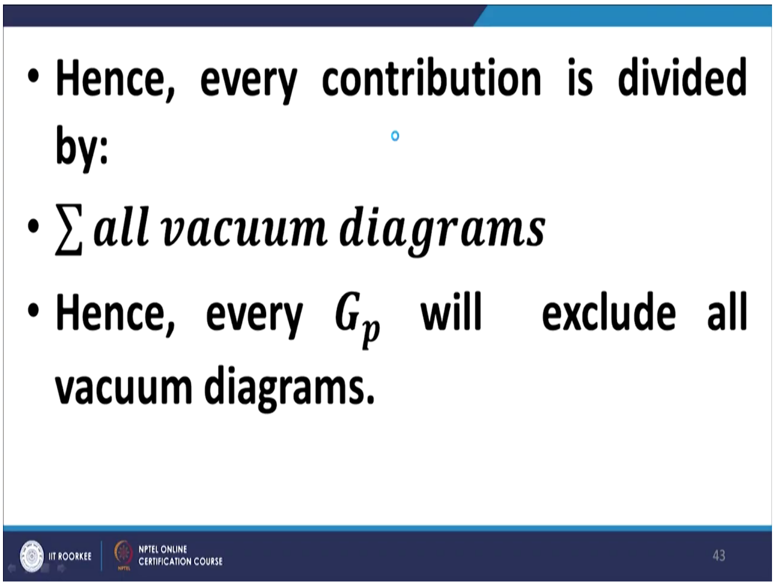

So, hence all can that is just what I mentioned, every G p will exclude all vacuum diagrams; because that vacuum diagrams constitute the normalization relate to the normalization.

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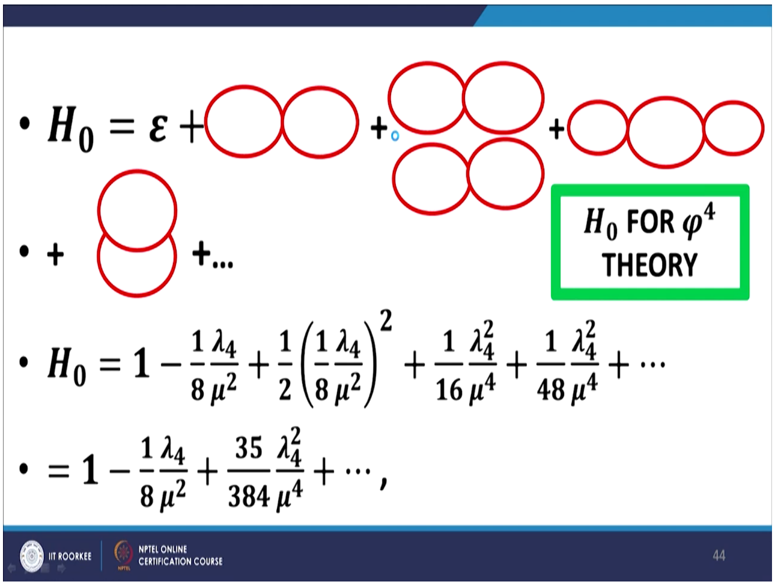

This is an example of what the vacuum bubbles would look like for the phi 4 theory and H 0 is equal to the empty diagram plus the 2 lobes plus the 2 lobes plus 2 lobes and so, on.

The evaluation of this diagram past the empty diagram values at unity. And this the evaluation of the second diagram we have already discussed its minus 1 upon 8 into lambda 4 upon mu square. And the third diagram evaluates to the product of the 2 diagrams first and second plus the additional symmetry due to transposition and so, on.

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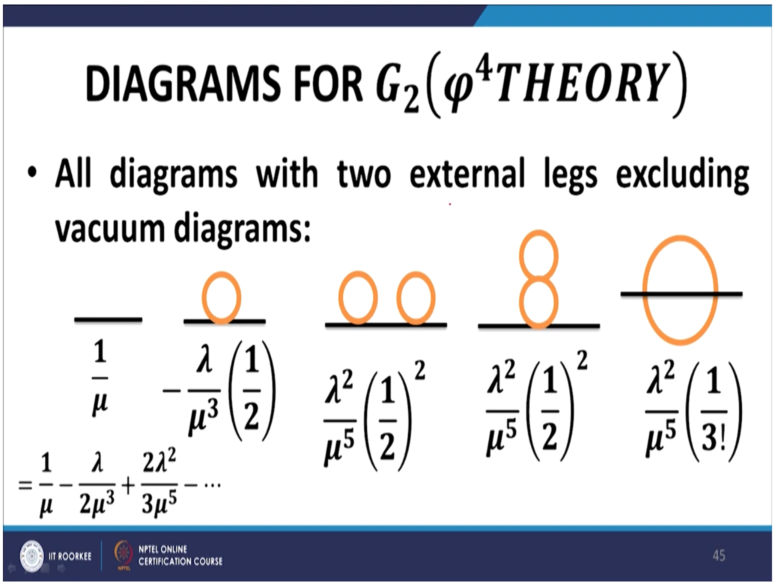

Now so, diagrams were G 2 for example, G 2 will consist of this G 2 will consist of the straight line first the free field propagator or the free field diagram that is 1 upon mu. Then it will consist of the 1 vertex or 1 vertex diagrams 1 4 point vertex and that is this one and the second diagram here that evaluates to lambda being this vertex and there are 3 lines. So, it evaluates as 1 upon mu to the power 3 the symmetry factor is due to the flipping of the leaf and similarly the other diagrams can be interpreted.

For example, this third diagram has how many lines? So, 1, 2, 3, 4 and 5. So, it evaluates as 1 upon mu to the power 5 there are 2 vertices here. So, it lambda square here and the symmetry factors arise due to the flipping of the leaves 1 upon 2 square because there are 2 leaves here. So, that is how the evaluation goes, it is worthwhile practicing the evaluation of these diagrams and one can become proficient through some practice.

So, for G 2 we need to start with diagrams with two external legs and excluding all vacuum diagrams. As I mentioned whenever we work out the Green function based on the Feynman diagrams, we need to exclude we need to exclude all vacuum diagrams then we need to start with those diagrams with the number of external legs equal to the order of the Green function.

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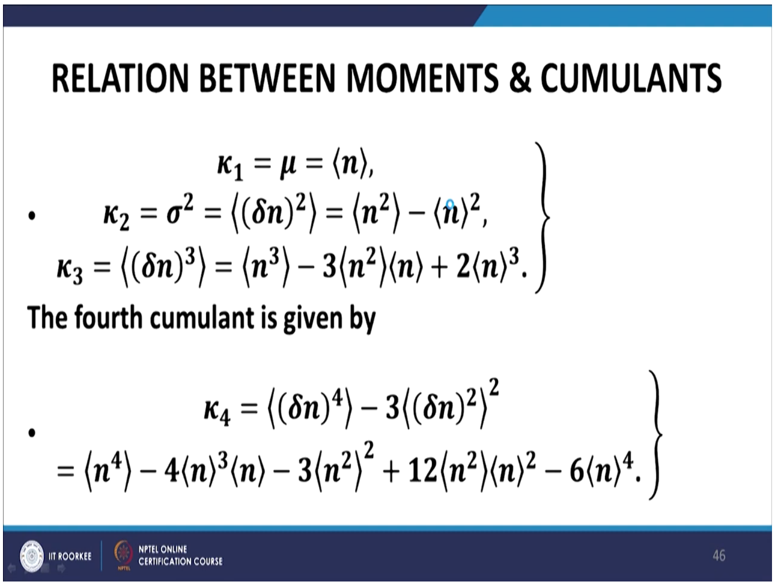

Now here is relationship between the movements and the cumulants. Recall that we very frequently migrate between the ordinary Green functions and the connected Green functions or the connected diagrams and the ordinary Green functions. So, this provides a reference for the migration between the movements and the cumulants. As you can see here the first cumulant is nothing but the mean, the second cumulant is the variance, the third cumulant is the skewness and the first fourth cumulant is the kurtosis.

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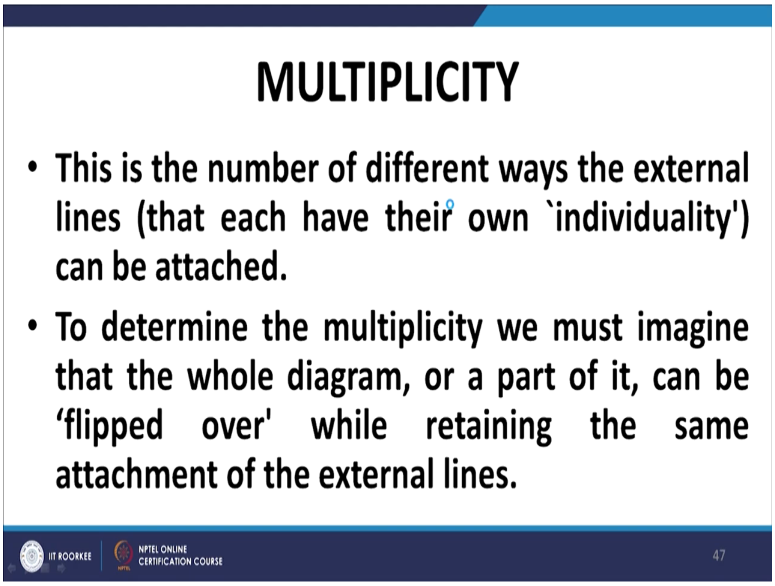

So, then we talked about the symmetry factor and the symmetric factor basically relates to the internal composition of the diagram. How you can manipulate the diagram, how many ways you can manipulate the diagram without disturbing its original structure in a form original form its; in a sense, multiplicity relates to the external legs of the diagram. It is the number of ways in with the external legs external lines can be attached in the diagram.

So, the symmetry factor relates to the internals, internals of the diagram multiplicity relates to the externals of the diagram; for example, let us look at it let us look at the diagram which is shown here.

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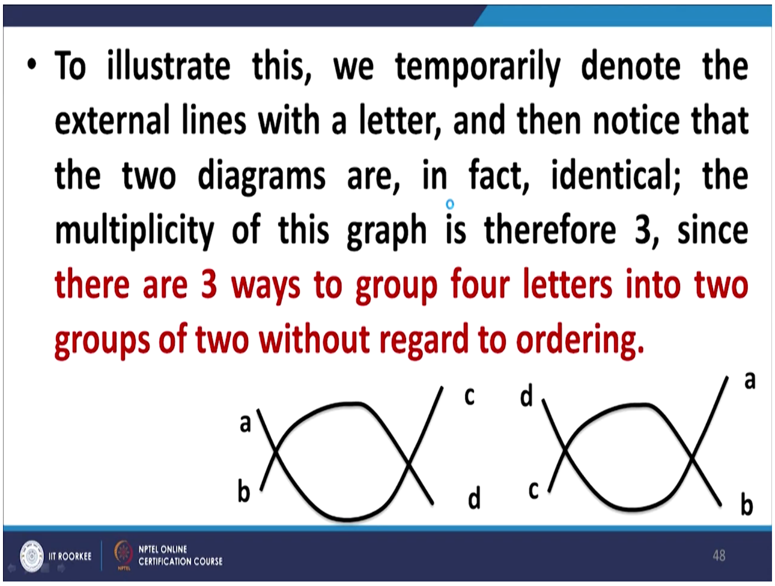

Now here we have these 1, 2, 3, 4 external legs. Let us enumerate them as a b c and d now we can group this 4 legs as a b, a, b c and d a, c b and d a, d b and c.

So, there are 3 ways in which these legs can be numbered or can be identified without disturbing the structure of the diagram and therefore, this diagram will carry a multiplicity of 3. So, that is how we need to work out the multiplicity of a diagram which is also relevant when we compute the value of a Feynman diagram.

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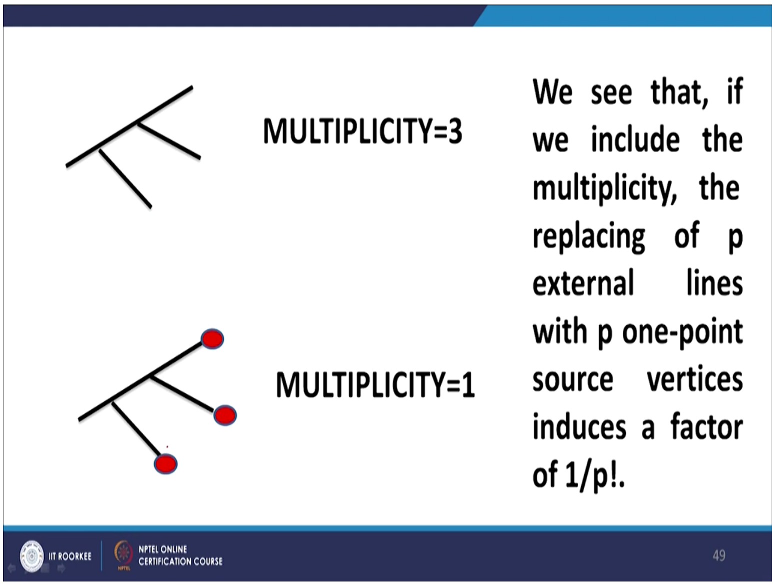

For example, in this case the multiplicity of this diagram is 3; however, when you attach a source vertex to each of them the multiplicity becomes 1.

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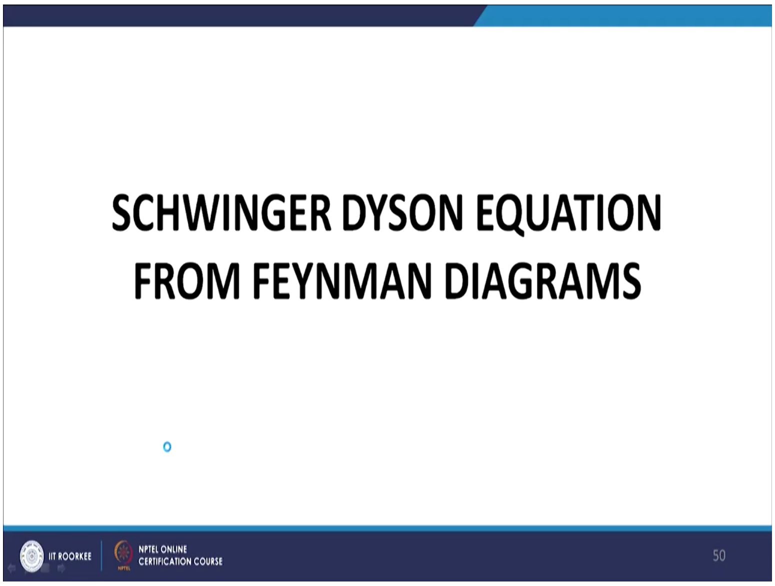

Now, we come to a very interesting analysis, we derive the Schwinger Dyson equation using Feynman diagrams.

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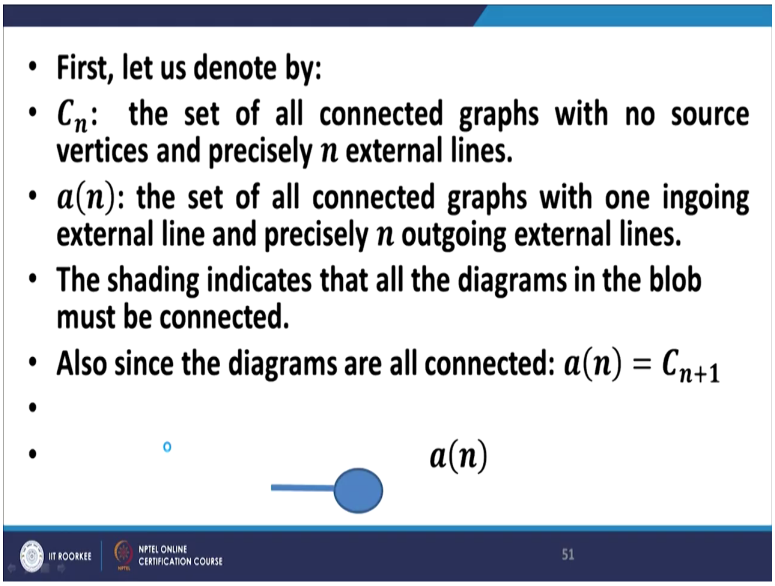

So, let us start. Let us denote by C n the number of connected graphs number of connected diagrams with no source vertices and n external lines no source vertex n external lines. a n these set now please note external lines include lines coming into as well as lines going out from the diagram. a n is the set of all connected graphs with one in going line and n outgoing lines. Now we represent this blob by a n and this is; obviously, it is equal to C n plus 1; we will continue from here.

Thank you.