Path Integral Methods in Physics & Finance
Prof. J. P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee

Lecture – 34
Schwinger Dyson EQS, Convergence of Integrals

Welcome back. So, let us continue from where we stopped before the break.

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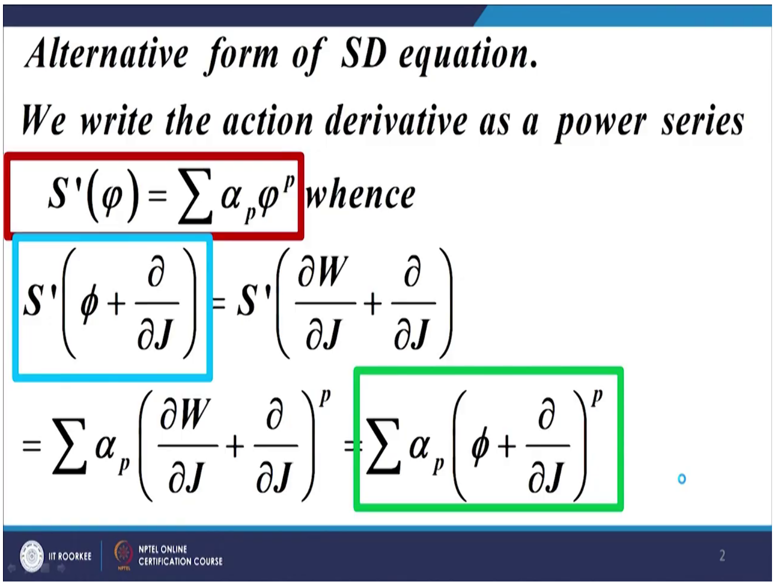

Now, we look at an alternative form of the Schwinger Dyson equation for the generating functional. For this purpose we write the action derivative as a power series. We write the action function as a power series of the field variables.

When that is the expression given in the red box at the top of your slide and that gives us; that gives us immediately that S dash of phi plus d by dJ will be equal to the expression that is, because phi is nothing but the first derivative of the generating function for the connected Green functions W. So, it follows that my state phi d by dJ is nothing, but S dash dW by dJ plus d by dJ.

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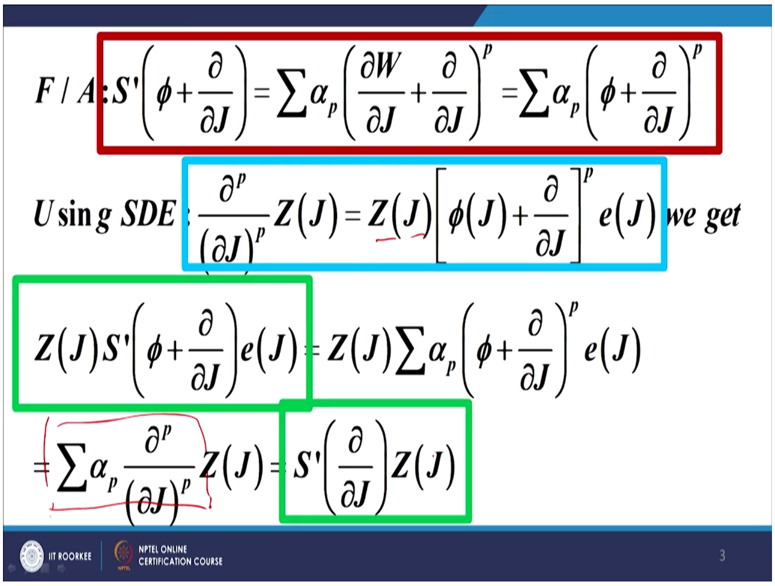

So, from the previous slide we have what the expression that is there in the first in the red box above. Now, we also have for the from the Schwinger Dyson equation that we worked out earlier for the field function, the expression that is given in the blue box d the P-th derivative of the Z J is equal to Z J into phi J plus d by dJ to the power P e J, where e is the unit operator.

When we you make use of this and you see the point is this holds for every P, this expression holds for every P and remember we have expanded the action derivative that is S dash as a power function as a power series in phi plus d by dJ. In other words, what I am trying to say is that for every term of the expansion of the derivative of the action this expression in the blue box would hold.

For every term of the of the power series expansion of the actions derivative which is in fact, which is the third term on your red box, in this expression or the blue box term would hold the blue box equation would hold. In other words, if you make a summation of series with respect to P, then you should get the expression that you have at the bottom of your slide by comparing the two side, by compiling the two sides.

Because if you are if you multiply both sides by alpha, alpha P and then sum over p, if you multiply both sides by alpha P and then sum over P. Then summation then summation can go inside this Z J, summation can go inside the Z J and you sum over alpha sum over P multiply this by alpha and sum over this P and this is nothing, but this is nothing, but the expression on the left hand side. So, that is precisely what we have done here. That is precisely what we have done.

Putting it very simply we have multiplied both sides by alpha P and then summed over P and we have got the expression that is at the bottom of your slide. And when we have that summation, when we have the summation alpha P this and this expression becomes S dash d d by dJ, because this expression when multiplied by alpha P and summed over P will give me this expression I write at the bottom of your slide. I am encircling it as a box.

And this expression if you see, if you compare with the definition of S dash as the power series this is nothing, but S dash d by dJ of Z J, Z J is this Z J.

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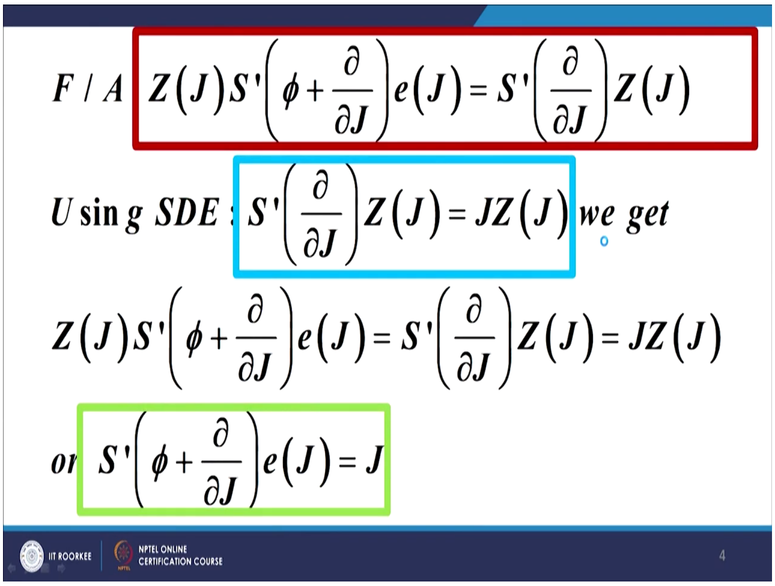

So, combining all the things together what do I get? I get Z J S dash phi d by dJ into e J is equal to S dash d by dJ into Z J.

Now, again I use the Schwinger Dyson equation. Now, I use the Schwinger Dyson equation for the generating function for the Green functions this was the Schwinger Dyson equation for the field function. Now, I am using the Schwinger Dyson equation for the green function. What does it give me? It gives me the right hand side is equal to J Z J; the right hand side of the expression in the red box, the right hand side of the expression in the red box gives me J Z J that means, what I get is Z J S dash phi d by dJ e J is equal to this expression. And this expression is given by this is equal to this expression from the Schwinger Dyson equation.

This Z J because the J and the Z J commute with each other, I can write this expression as Z J into J and then the Z J can be eliminated or multiplying both sides by Z J inverse what I get is

S dash phi and d by dJ e J is equal to J. This is another expression for the Schwinger Dyson equation for the field function, Schwinger Dyson equation for the field function.

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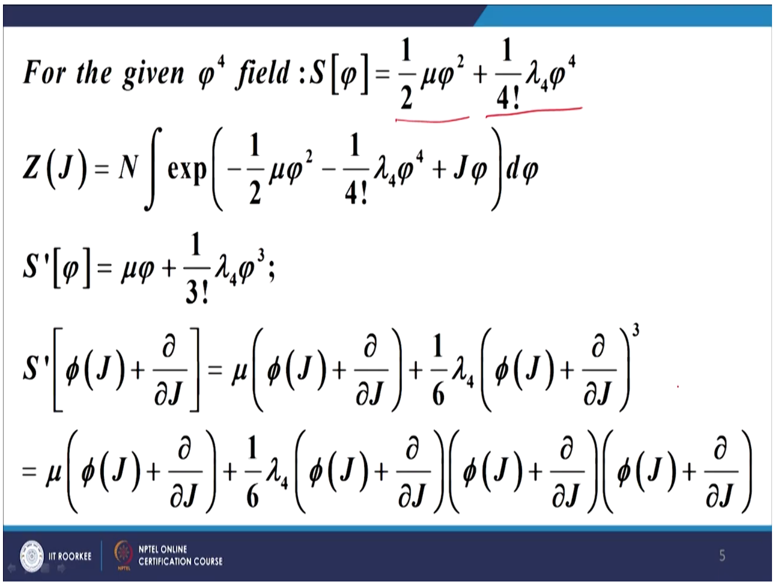

Now, let us see what we get phi 4 field which is given by this expression, 1 by 2 mu sigma mu phi square plus 1 by 4 factorial lambda 4 phi to the power 4. This is the action that we are considering as the interaction action. The interaction term is this one and the free field term is the first one. The first term is the free field term and the second field is the interaction term.

Now, we get S dash of phi is equal to this expression, mu phi plus 1 by 3 factorial lambda 4 phi cubed. And therefore, S dash of phi J, phi J the phi symbol that is the phi field function plus d by dJ comes out to be in this whole expression on the right hand side.

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$$= \mu\left(\phi(J) + \frac{\partial}{\partial J}\right)$$

$$+\frac{1}{6}\lambda_4\left(\phi(J) + \frac{\partial}{\partial J}\right)\left(\phi(J)^2 + \frac{\partial}{\partial J}\phi(J) + \phi(J)\frac{\partial}{\partial J} + \left(\frac{\partial}{\partial J}\right)^2\right)$$

$$= \mu\left(\phi(J) + \frac{\partial}{\partial J}\right)$$

$$+\frac{1}{6}\lambda_4\left(\begin{array}{l}\phi(J)^3 + \phi(J)\frac{\partial}{\partial J}\phi(J) + \phi(J)^2\frac{\partial}{\partial J} + \phi(J)\left(\frac{\partial}{\partial J}\right)^2 \\ + \frac{\partial}{\partial J}\phi(J)^2 + \frac{\partial}{\partial J}\frac{\partial}{\partial J}\phi(J) + \frac{\partial}{\partial J}\phi(J)\frac{\partial}{\partial J} + \left(\frac{\partial}{\partial J}\right)^3\end{array}\right)$$

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$$
=\mu\left(\phi(J)+\frac{\partial}{\partial J}\right)+\frac{1}{6}\lambda_4\left(\begin{aligned}&\phi(J)^3+\phi(J)\frac{\partial}{\partial J}\phi(J)+\phi(J)^2\frac{\partial}{\partial J}+\phi(J)\left(\frac{\partial}{\partial J}\right)^2\\&+2\phi(J)\frac{\partial}{\partial J}\phi(J)+\frac{\partial^2}{\partial J^2}\phi(J)+\frac{\partial}{\partial J}\phi(J)\frac{\partial}{\partial J}+\left(\frac{\partial}{\partial J}\right)^3\end{aligned}\right)
$$

$$
=\mu\left(\phi(J)+\frac{\partial}{\partial J}\right)+\frac{1}{6}\lambda_4\left(\begin{aligned}&\phi(J)^3+3\phi(J)\frac{\partial}{\partial J}\phi(J)+\phi(J)^2\frac{\partial}{\partial J}\\&+\phi(J)\left(\frac{\partial}{\partial J}\right)^2+\frac{\partial^2}{\partial J^2}\phi(J)+\frac{\partial}{\partial J}\phi(J)\frac{\partial}{\partial J}+\left(\frac{\partial}{\partial J}\right)^3\end{aligned}\right)
$$

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The rest of it work is simplifying this expression and simplifying this expression is a slightly tedious job. I have done it in the presentation step by step, and the viewers can go through it and maybe work it out themselves as well.

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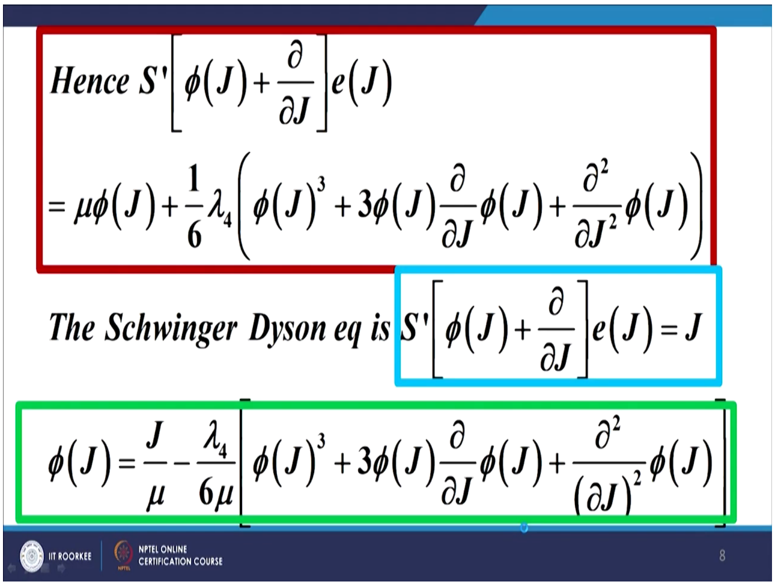

The net result that we arrive at is the expression that is on the slide, this bottom expression on the slide. Now, when this expression operates on the on the unit function unit function, when this expression operates on the unit function the only terms that remain that survive are the expressions that are given on the right hand side.

That is the terms that have d by dJ operating on the unit functions will not contribute anything. For example, the this term will not contribute anything, this term will not contribute anything, similarly this term will not contribute anything and this term will not contribute anything.

So, the terms that contribute are this and this term contributes, this term contributes and this term contributes. So, these are the terms that will contribute and the rest of the terms will not contribute because they are acting on the unit function.

The derivative acts on the unit function and the terms that involve the derivative acting on the unit functions will not contribute because the unit function is a constant. And therefore, what we are left with is the expression that is given in the red box; this represents the Schwinger Dyson equation for the field equation for the phi 4 field.

The expression in the green box the S dash was given by this. Now, the Schwinger Dyson equation is S dash is equal to J, so we put the left hand side equal to J and phi J is equal to J is equal to this whole expression on the right hand side of the red box. And by transposition and simplification we get an expression for the field function as this J upon mu minus lambda 4 upon 6 mu, phi J cube plus 3 phi J phi dash J plus phi double dash of J. So, this is this is now, the Schwinger Dyson equation for the for the field function for the phi 4 theory.

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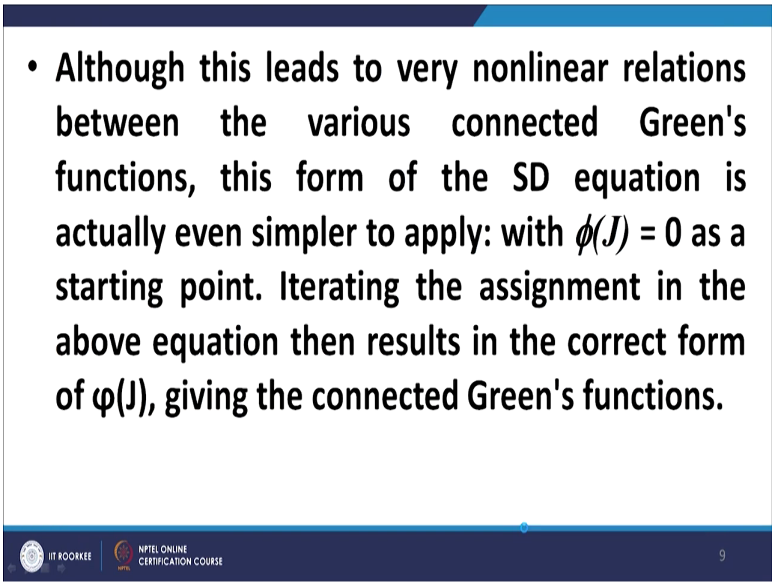

Now, this can obviously be solved by iteration. And as you will see later there is an intimate connection between what we have done what we have arrived at this particular equation is very significant, very important it completely describes the phi 4 theory. And it also carries the direct one to one correspondence with the Feynman diagrams. So, we shall discuss that soon. But for the moment this equation is very very important, very significant.

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For the $\phi^{3/4}$ theory, the Schwinger Dyson eq reads:

$$\phi(J)=\frac{J}{\mu}-\frac{\lambda_3}{2\mu}\left(\phi(J)^2+\frac{\partial}{\partial J}\phi(J)\right)$$

$$S(\varphi) = \frac{1}{2}\mu\varphi^2 + \frac{1}{3!}\lambda_3\varphi^3 + \frac{1}{4!}\lambda_4\varphi^4$$

$$-\frac{\lambda_4}{6\mu}\left(\phi(J)^3+3\phi(J)\frac{\partial}{\partial J}\phi(J)+\frac{\partial^2}{(\partial J)^2}\phi(J)\right)$$

For the phi 3 by 4 theory that is when we have two interaction terms we have the cubic interaction and the by quadratic interaction the Schwinger Dyson equation for the field function takes this form.

You can see there are two additional terms the rest of the equation is same as before except for these two additional terms, one involving phi J square and the second phi dash J. Even this has a direct nexus to the Feynman diagrams in terms of 3 vertices and we shall be in discussing that as well.

But, this is basically the 3 4 theory where the action is given by let me write it down the action is S phi is equal to 1 by 2 mu sigma square mu phi square I am sorry, plus 1 by 3 factorial lambda 3 phi cube plus 1 by 4 factorial lambda 4 phi to the power 4. So, there are two interaction terms, one involving lambda 3 and the other lambda 4 these are the coupling constants lambda 3 and lambda 4, and of course mu is the parameter for the free field.

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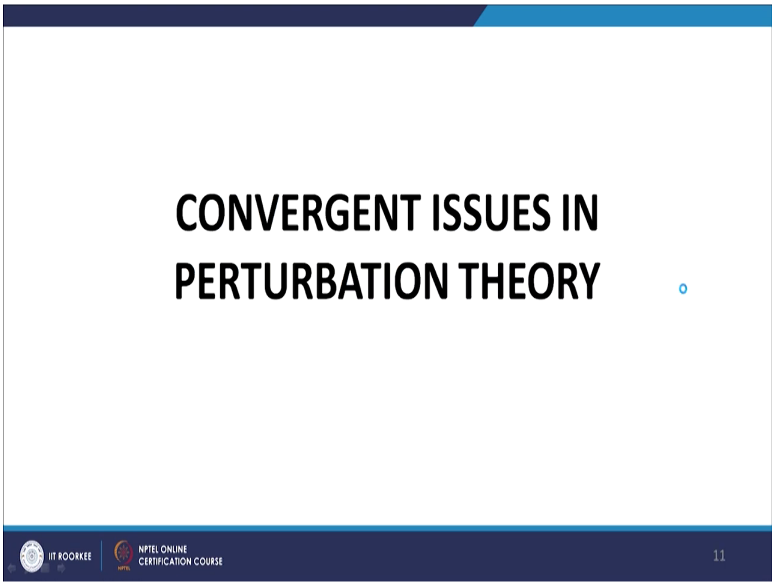

Now, let us go back, let us revert to the issue that I have raised in my earlier lecture also, and even in this current lecture at the beginning when we flipped the summation and the integral sign. Let us go back to that expression.

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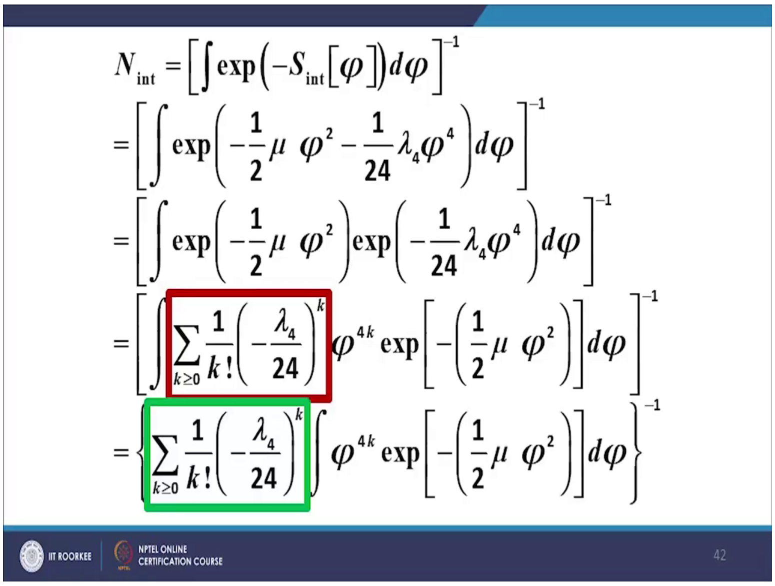

If you look at this what we had done was that we had expanded you see we had two terms in the interaction, these were two terms, so the first was the free field term and the second was the interaction term. We said that lambda 4 is very small and therefore, we did an we did a series expansion in the second term, the exponential series of the second term that is exponential minus 1 by minus 1 by 4 factorial I am sorry; 1 by 4 factorial. Lambda 4 phi 4 was expanded as an exponential series and we got we got this expression in the red box as an exponential expansion is exponential expansion as an exponential series of this term minus 1 upon 4 factorial and lambda 4 phi to the power 4. This thing was expanded to get the expression in the red box.

And then we did a new then we did a gimmick. What we did was we flipped the integral sign with the summation sign. In other words, we took the summation outside and retain brought the integral inside of the summation. Now, this was this is a very technical issue, a delicate issue and it has certain nuances. Let us explore that and before we proceed further to the case of Feynman diagrams.

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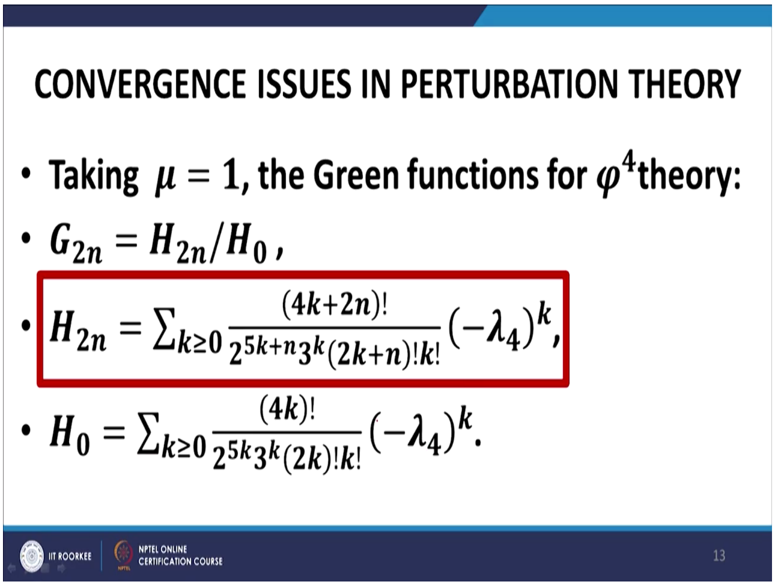

Now, if you take mu equal to 1 for simplicity because that is not really relevant at the moment, and then we can write G n is equal to H 2 n upon H 0 and H 2 n is the expression that is given in the red box and H 0 is the expression that is given below in the second line. H 2 n is the given expression given the red box, and H 0 is putting n equal to 0 you get the expression for H 0.

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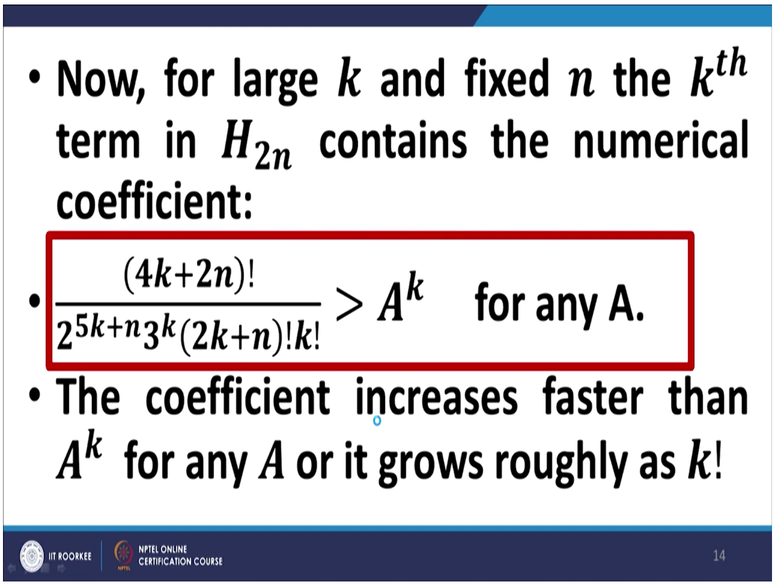

For large k, but for fixed n let us see what is the k-th expression for in H 2 n. The k-th expression in H 2 n is the expression that is here in the red box. The interesting feature is if you analyze this expression this expression diverges.

Diverges in the sense it increases or it is it exceeds faster than, it increases faster than a to the power k, the increase of this expression is the order of k factorial you can say. It roughly grows as k factorial. It is in a sense what we call super exponential. It increases and the rate of increase of this term is of the order of k factorial.

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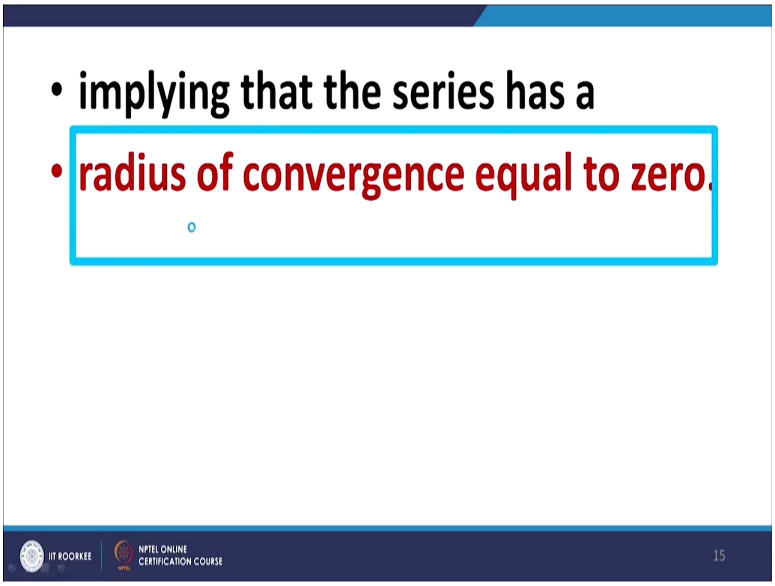

So, that clearly the series is diverging and the radius of convergence of the series is zero. Let us see what is the cause of this problem.

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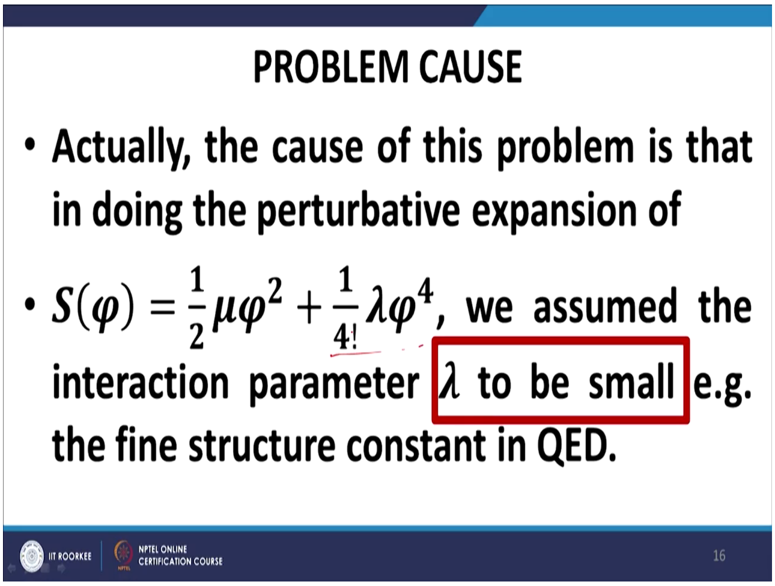

You see we started with the action 1 upon 2 mu phi square plus 1 by 2 lambda phi to the power 4 and we assumed fundamentally that lambda was small, we assumed lambda to be small. And only then we did the perturbative expansion.

We expanded the second term exponential 1 by 4 factorial lambda phi 4 as a exponential series summation series, while retaining the exponential in the first term that was on the premise that lambda was small and therefore, we could expand around lambda as in as a perturbative or a expansion in powers of lambda.

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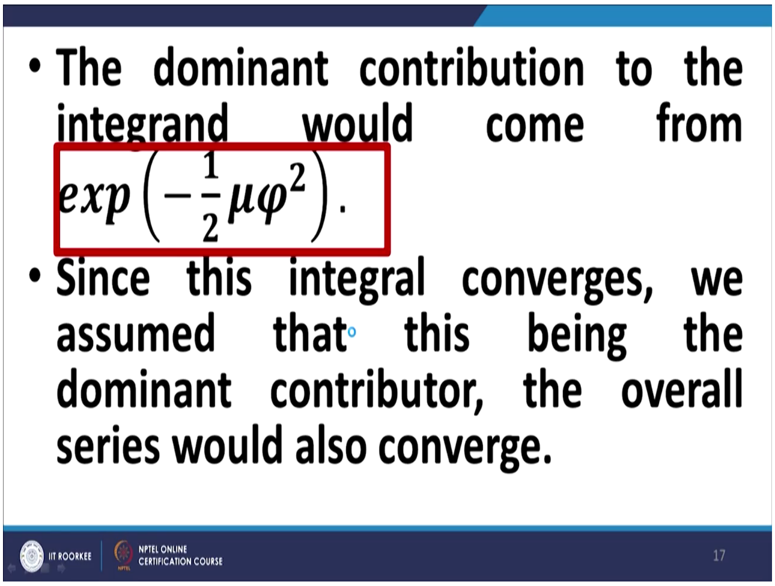

You know we believe that the dominant contribution would come from this expression, the expression in the red box because mu was in orders of magnitude larger than lambda. And therefore, it was believed that the integral, this integral because this integral converges the character of the overall expression would be dominated by this. And as a result of it the series would also converge with the flipping of the integral with the summation.

In other words, the resultant series that would be formed when we flip the integral with the summation would continue to converge because of the strong convergence of the first expression there by its convergent dominated by this expression; and because of the series being dominated by this expression and this expression converging. This expression converging, series dominated by this expression, therefore notwithstanding the fact that we flipped us the integral and the summations and the series continues to converge that was the argument propounded.

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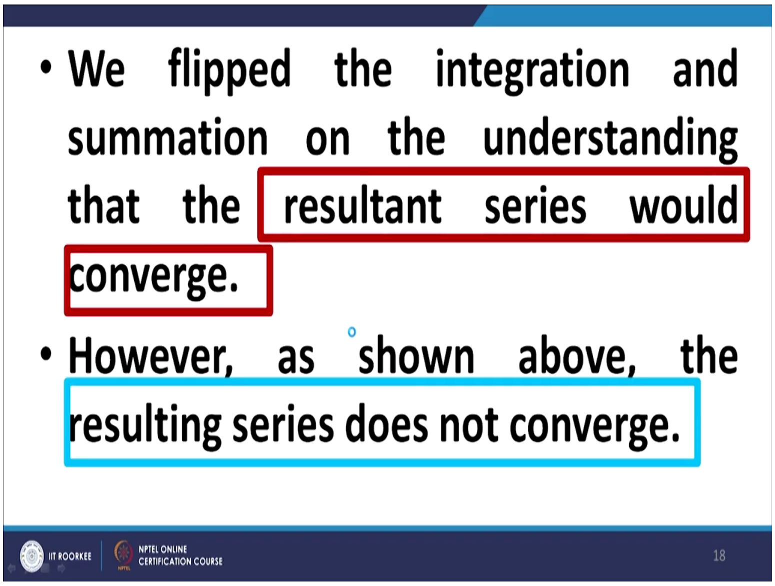

However, it does not it is clear that it does not happen that way and we have a radius of convergence of 0. Now, why is that? We see we are integrating from minus infinity to infinity.

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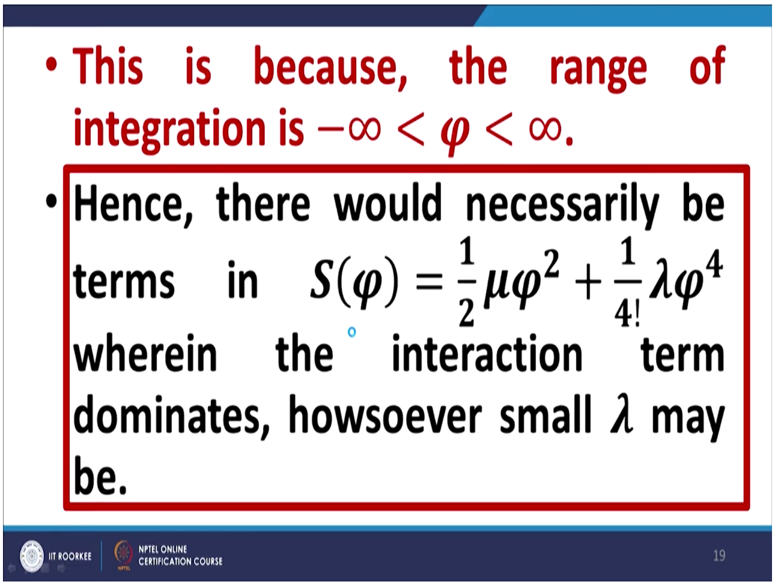

Therefore, if you look at this expression there would be some terms irrespective of the relative orders of magnitude of mu and lambda. Irrespective of the relative orders of magnitude of mu and lambda, there would be some terms where the lambda term would dominate the mu terms.

Some permutations would definitely arise where the lambda terms dominate the mu terms. And that is bound to happen. Howsoever, small lambda would (Refer Time: 17:32).

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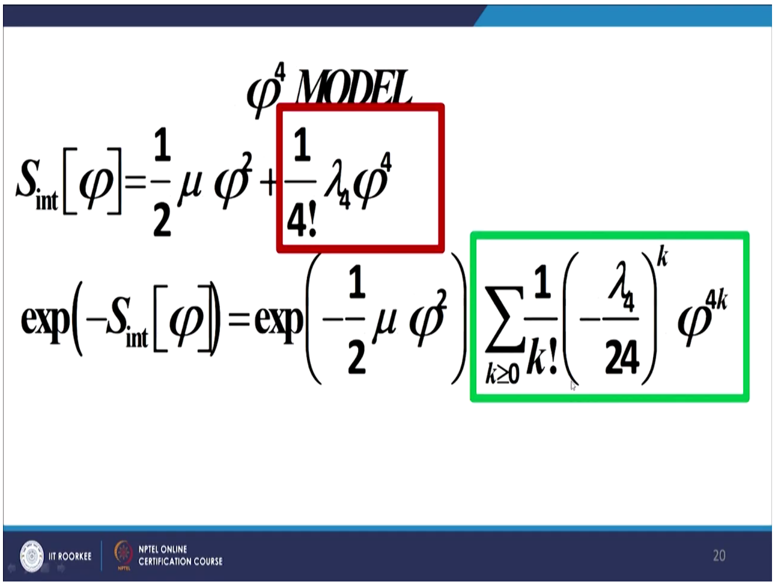

And as a result of which, as a result of which this flipping of the integral in the summation sign does not necessarily imply, does not necessarily imply that the flipped series would also converge, right.

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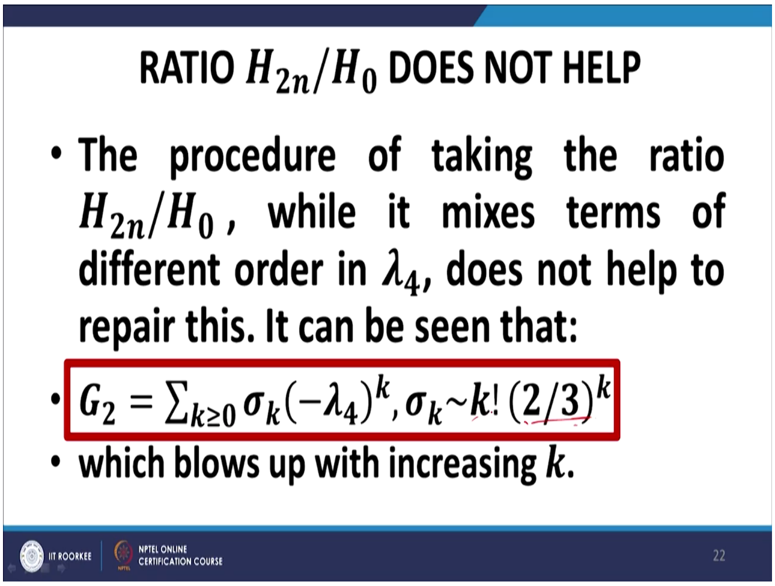

So, even if we work out the ratio H 2 n upon H 0, H 2 n upon H 0, we find we find that radius of convergence is of the sigma case of the order of k factorial. And therefore, sigma G 2 for, G 2 for n equal to 1 that is the radius of the if you work out G 2 you find that sigma would sigma k, it is of the order of k factorial and it clearly in blows up it will not withstand, not withstanding this number. Although this number is converging, but this k the presence of k factorial makes the sigma k blow up. So, that is the problem.

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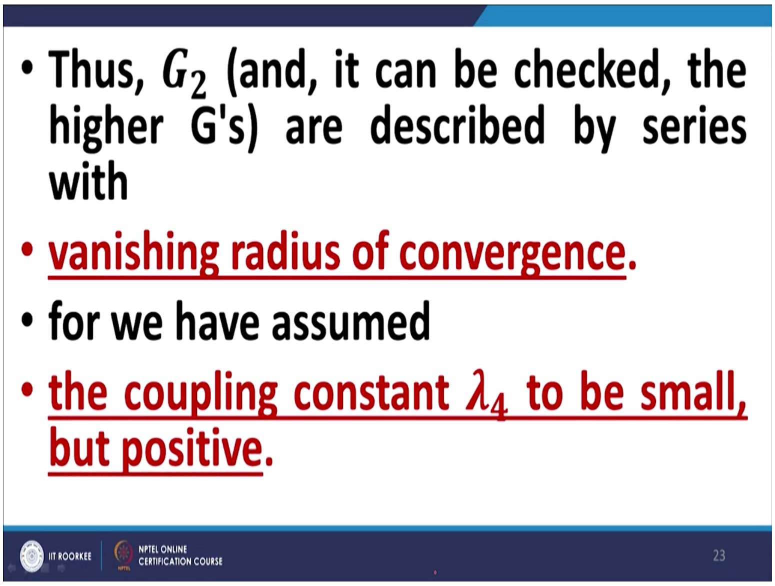

Thus, the net result is that there is a singularity at the origin, at this point 0 and the series has a vanishing radius of convergence.

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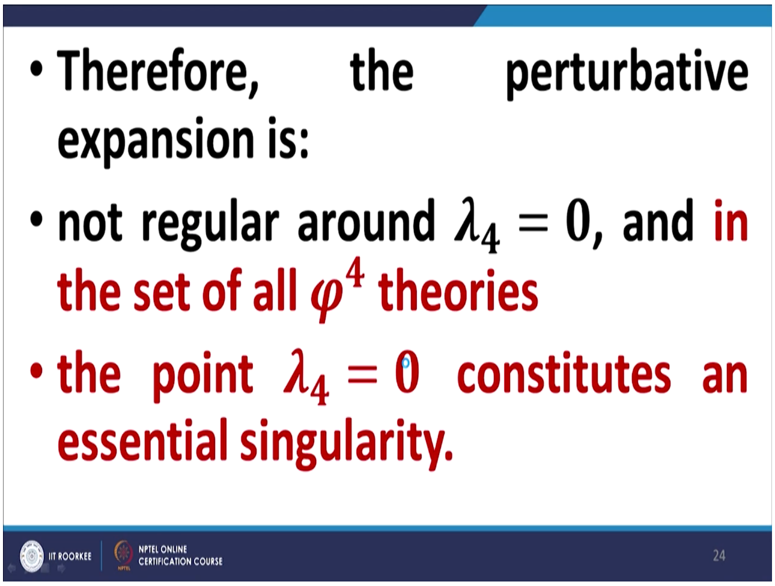

Now, another point; please note that we have assumed that lambda is positive, lambda 4 is positive. We assume that lambda 4 is positive indeed we had lambda 4 been negative the path integral itself would not have been defined as well at all.

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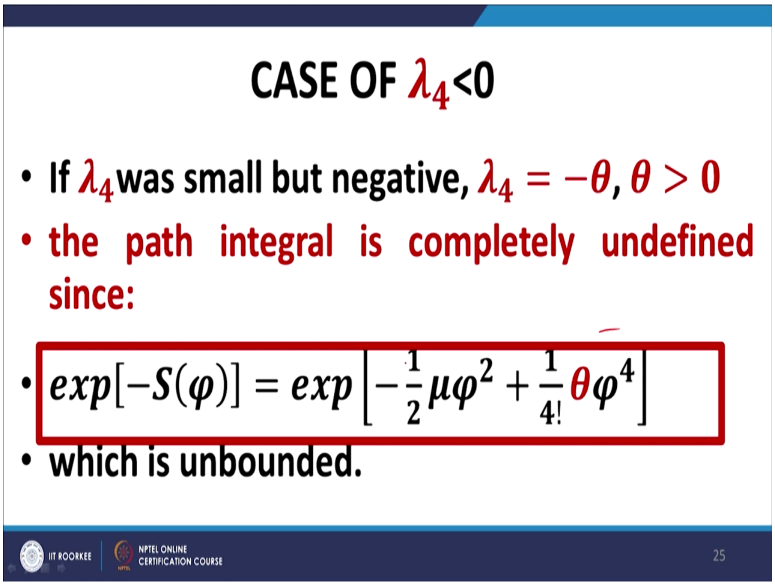

If lambda 4 were negative the path integral would not have been defined because then this part you see let us say lambda is equal to minus theta with theta positive, then the action takes this form minus 1 by 2 mu sigma square e to the power minus S phi takes the form exponential minus 1 by 2 mu sigma square plus this. And there are definitely values where the this value dominates and therefore, the sales becomes unbounded.

And therefore, when the case when lambda 4 is negative of course, the path integral become becomes undefined; but even in the case even in the case where lambda 4 is positive and small, the series is not regular at lambda 4 equal to 0 and the point lambda 4 equal to 0 represents a point of essential singularity. So, that is what disturbs the character of the series, that is what you call relates in the divergence of the series that is created by flipping the summation and the integral.

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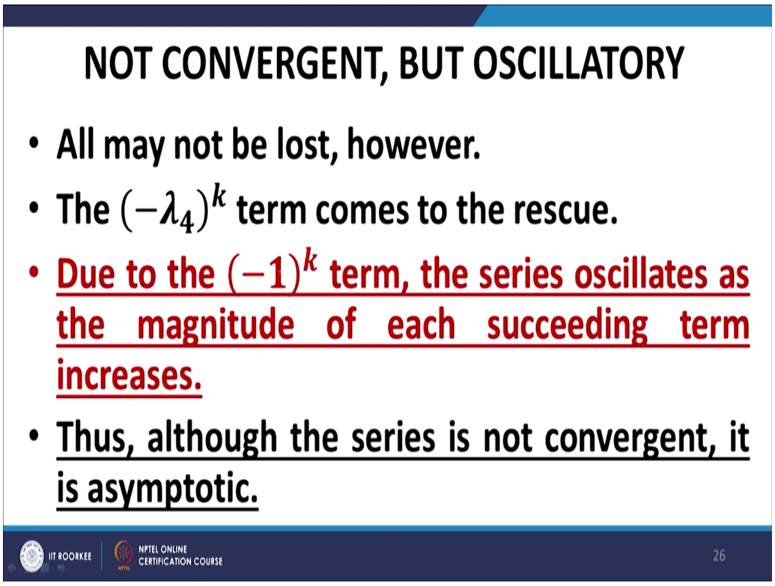

Now, but there is some rescue, there is some hope. That hope arises from the minus lambda 4 to the power k term, this minus factor the presence of this minus 1 factor comes to the rescue and because of the presence of the minus 1 factor although the magnitude of the series is increasing, the sign is; the sign is changing though.

So, actually the series notwithstanding the fact that the magnitude is changing the series is oscillatory and this enables us to do some something about the divergence. And, that would otherwise also not have been possible at all had this minus sign not been there, at the series not been oscillatory.

The presence of the oscillatory character of the series enables us to develop a framework, to manage a framework and by which we can we can ascribe a mathematical value to this particular series. Let us see how we do it.

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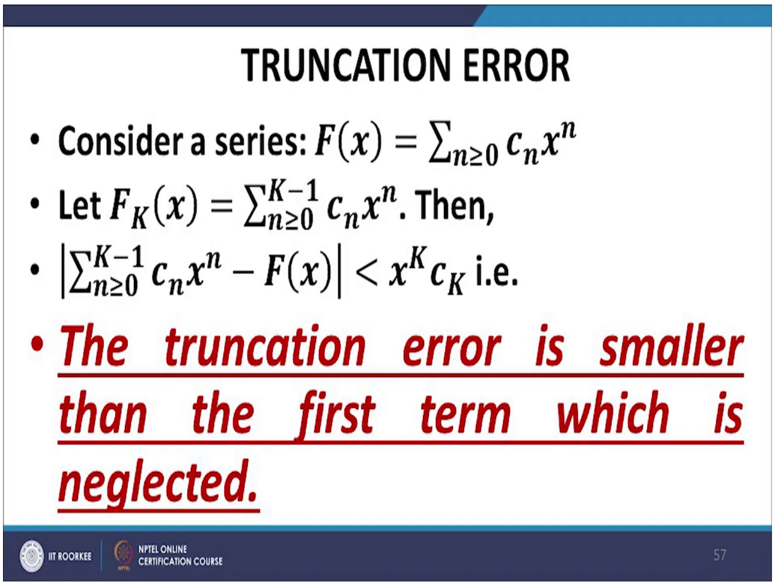

Consider the series c is summation c n, x n, F x is equal to this and this series. And let us say we truncate the series, we this is the series, this is a infinite series. We truncate the series at a certain value k, then it can be shown that the truncation error and that is the error representative represented due to this truncation that is this is the truncated term and this is the total series.

So, the error is the modulus value of the of the of the difference between what is the actual series and what is the series up to the truncation point that is less than x K, c K that is less than the term the next term which follows the truncation.

So, to repeat the error that is introduced by cutting the series at an earlier point, the series is infrared, but you are truncating it at a particular point so naturally you are creating an error you are not representing the entire series. The error that is introduced by truncating the series at a particular point is less than is less than the value of the next following term after the truncation, that is the theorem or that is the lemma.

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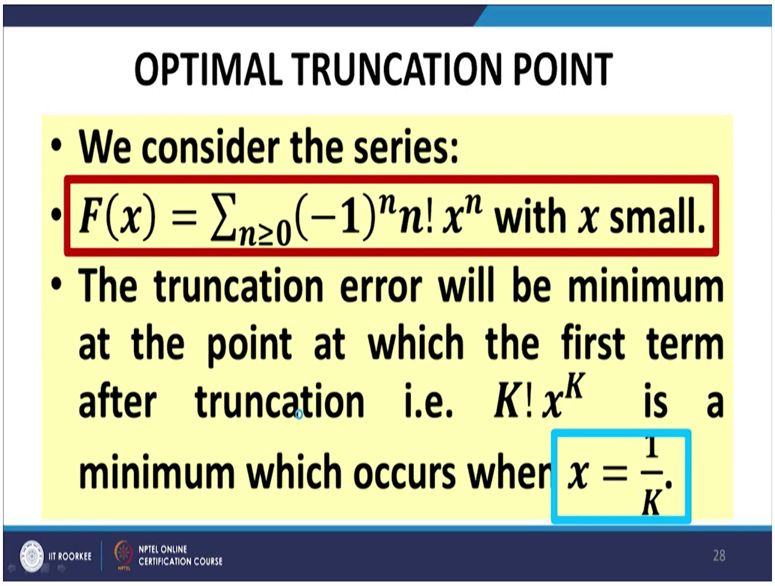

So, that being the case let us consider as an example. Let us consider the series F x is equal to summation minus 1 to the power n, n factorial x to the power n with x being small. The truncation error will be small; will be minimum at the point at which the first term after the truncation.

Let us say the truncation is after K terms the K minus 1-th term, so the K-th term is the term that is neglected. What is the K-th term here in magnitude? That is K x to the power K, K factorial x to the power K this is the K-th term. So, this is the immediately following term after the truncation and the truncation error will be less than this term.

So, we have to minimize this particular term K x to the power K, and if you minimize this particular term, the minimization condition occurs to be x is equal to 1 upon K. In other words, x equal to 1 upon K gives the optimal truncation point.

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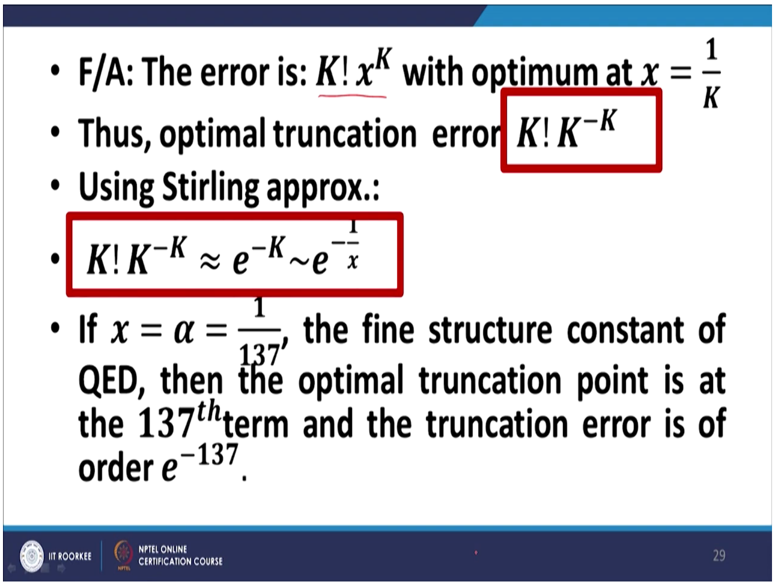

And the truncation and the truncation error is given by K factorial K to the power minus K that is obtained by substituting x equal to 1 by K in this expression. And recall this is the K-th term the term in which immediately follows the truncation.

Now, if we use stirling approximation apart from a factor of 2 pi, 2 pi K under root the expression K factorial, K to the power minus K is of the order of e to the power minus K that is of the order of e to the power minus 1 upon x. Now, it is suppose for the QED theory where we use the x as alpha which is the fine structure constant as a 1 upon 137, we find that the truncation error would be of the order of e to the power minus 137 and would be the optimal truncation point will be 137th term.

So, these are things which you know are so small, so small to be of insignificant. And in any case it would not be practical to work out the corrections up to 137 term in a QED series, QED theory. So, the point is that the point is, that notwithstanding the fact that we have a problem there when we flip the integral and the summation things are not that bad, and by making use of a satisfactorily or a reasonable truncation point of the series we are still able to obtain very good approximations.

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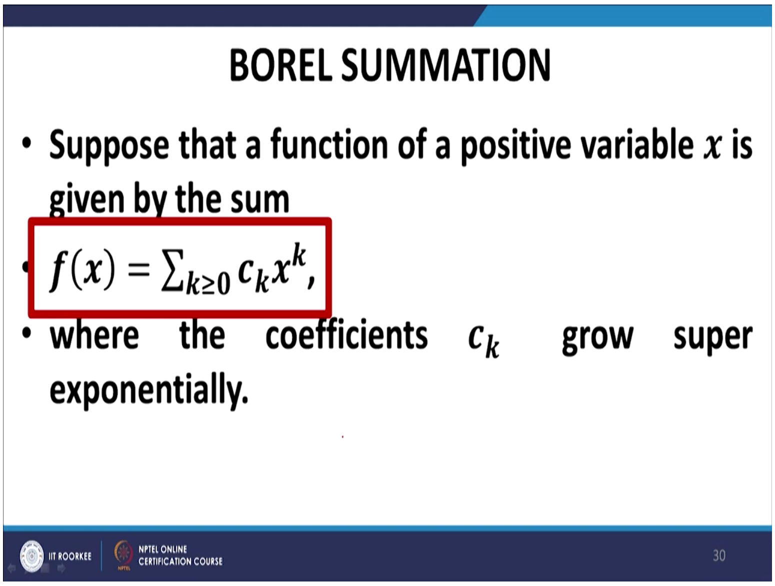

Let us take another example of a technique that that remedy is the issue in another way then that is called Borel summation. Let us assume that we have the series f x is equal to summation, k greater than equal c k x k, this is the series, infinite series again we want to work out the summation. These coefficient ck are growing super exponentially.

We consider the sum as an alternative or as a tool we consider another series which is defined by; remember our original series was what?

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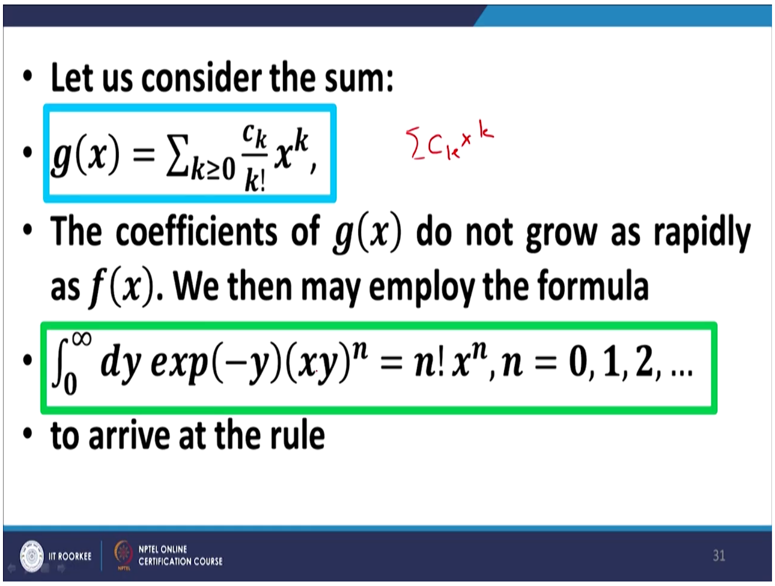

Original series was c k, x k of course, summation and we may be had a problem with the coefficient c k, they were going super exponentially that is they were growing of the order of k factorial and now what we do is we divide c k by k factorial. So, naturally the coefficients of g k the that is c k upon k factorial grow relatively slowly compared to f k.

Now, we work out the make use of the formula integral dy exponential minus y xy to the power n is equal to n x to the power n. And this formula is quite straightforward, x to the power n can be taken outside the integral and we get the in gamma function which is easily worked out.

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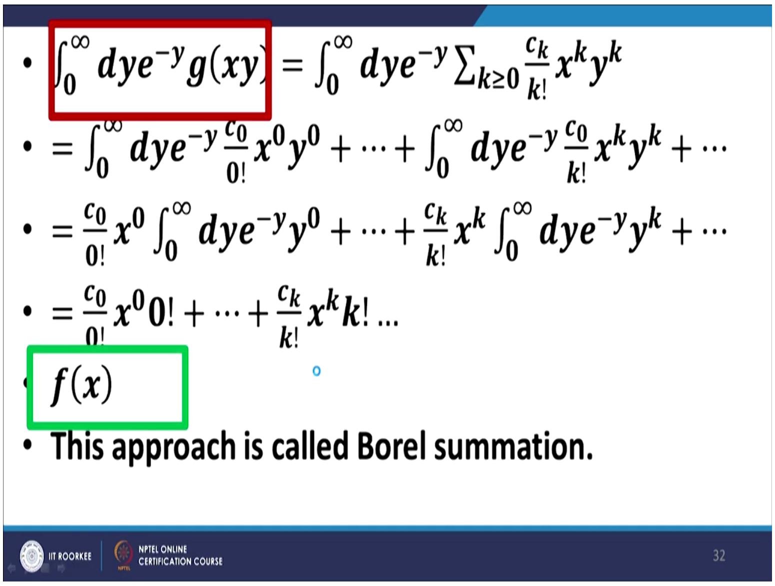

If you evaluate this integral dy e to the power minus y g xy, let us see what. We get dy e to the power minus y g xy, let us see what we get. On simplification, in simplification what we get is we end up with after all the manipulations straightforward manipulations we end up with F x. So, we have in other words, been able to write f x as an alternative integral dy e by g xy. And this is this process is called Borel summation.

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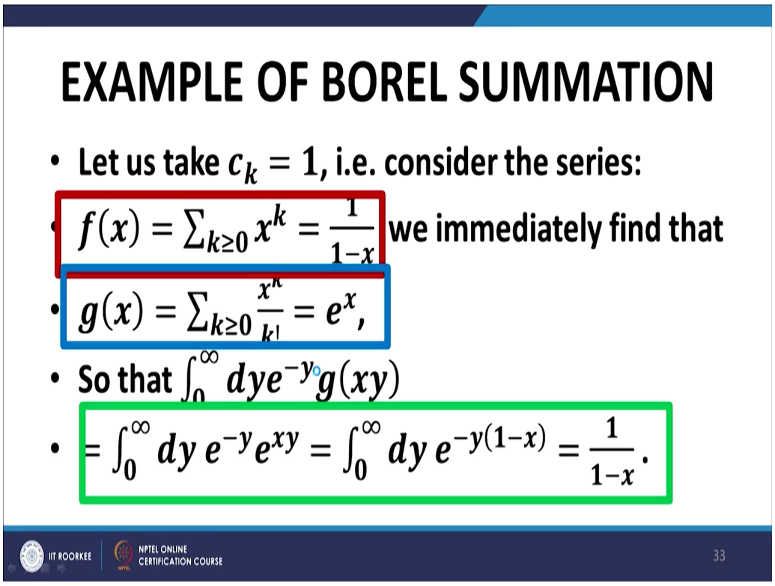

And as an example of Borel summation let us look at let us look at the series f x is equal to summation x scale in other words let us take c k equal to 1 for all k. So, we look at this summation this is clearly a geometric series and with the summation is equal to 1 upon 1 minus x g x works out to x k upon k factorial that is equal to e to the power x. The integral dy e to the power minus y g xy if you work it out it works out to 1 upon x.

So, our Borel summation gives us the same answer as we would expect in the normal course of events. However, the Borel summation has given us a certain leeway a certain advantage. Let us look at that.

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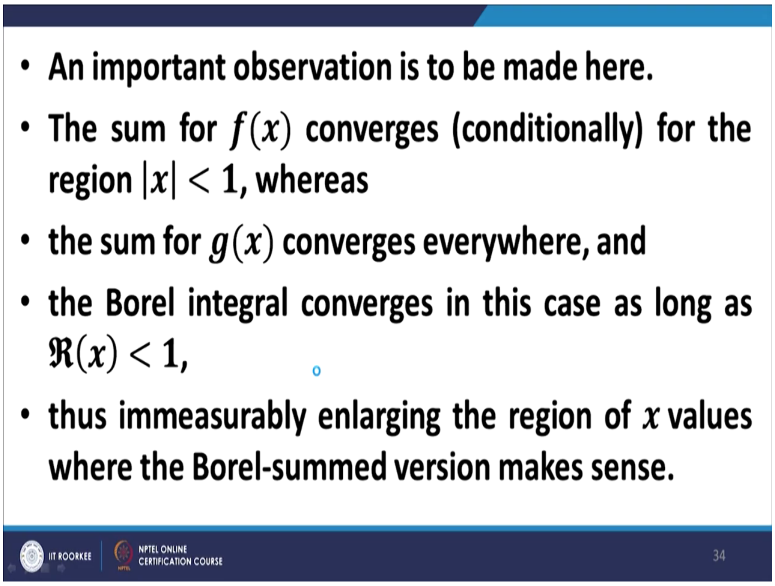

The f x converges conditionally for mod x less than 1, it is quite clear that this formula, the first formula the formula in the red box holds only if mod x is less than 1. And g x of course is convergent for all values of x. So, and that is not a problem. And this integral, this integral that we have now obtained integral dy e to the power g xy is equal to 1 upon x, this converges, this holds or this exists for real x less than 1.

So, by making use of this Borel summation, we have enlarged the region of x values over with the Borel summation can make sense. From mod x less than 1 we have increased the domain where this integral can hold to real x less than 1.

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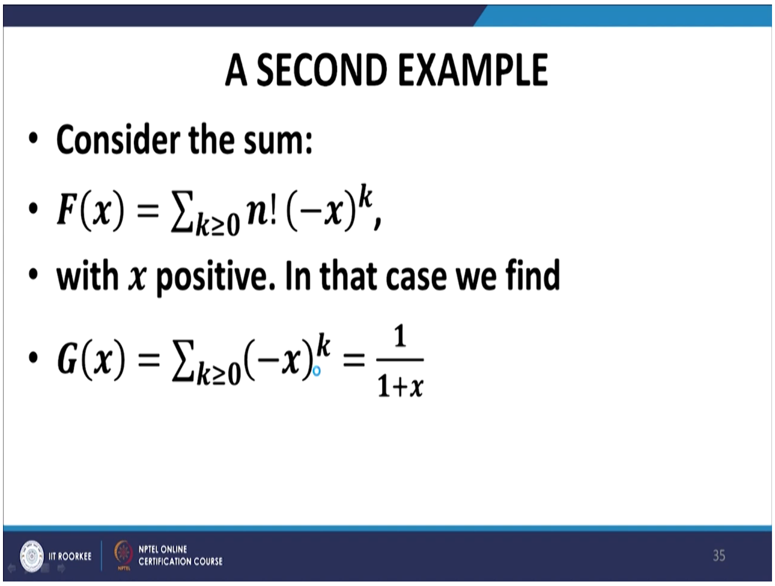

So, this is an important and the important part is that, of course, there is a second example in the presentation due to sparsity of time I could not cover. But the important thing is that this issue of flipping of the summation and integral is not a trivial issue. It can be saved in certain situations. It may not be possible to save in or it justify in other situations. I have just presented an example where it can be done.

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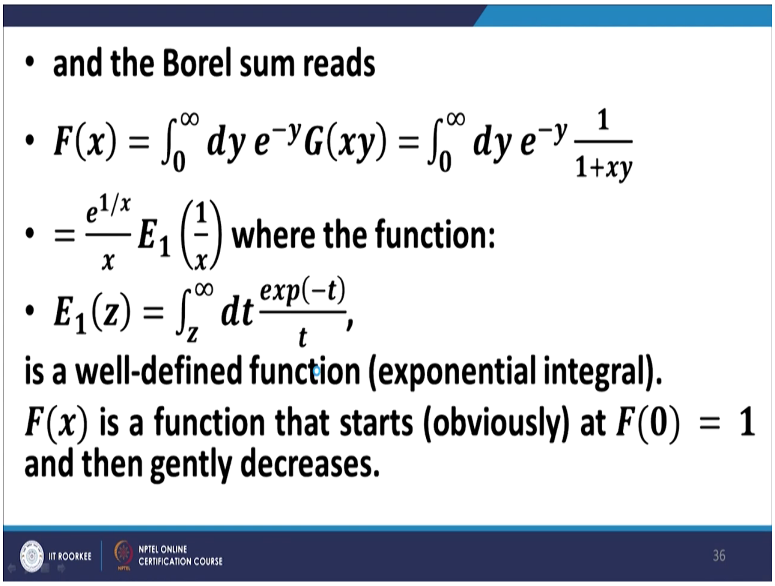

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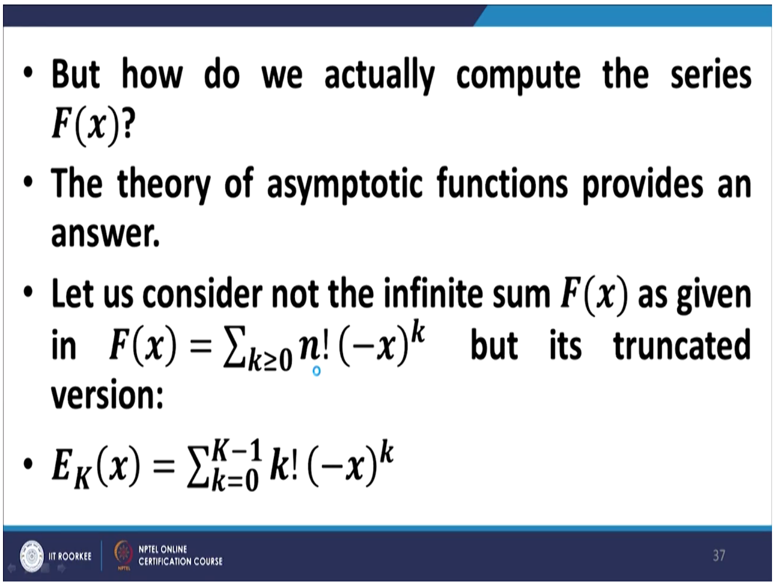

There is another example here in which it cannot be done and that example I will leave it as for the viewers. I will put it in the presentation in any case and we will continue with Feynman diagrams.

Thank you.