Path Integral Methods in Physics & Finance
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Lecture – 33
Field Theory in Zero Dimensions (2)

Welcome back. So, in the last lecture, we started talking about the Field Theory, Quantum Field Theory in a spacetime of zero dimensions. Now, in zero dimensional space time, it will consist of a single point and therefore, the theory can or the theory can be specified by the value of the field at that particular point. Now, the field value at that particular point can take any random real number.

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DEFINITIONS

$$P(\varphi) = N\exp\left[-S[\varphi]\right];\ N = \left[\int \exp\left[-S[\varphi]\right]d\varphi\right]^{-1}$$

$$G_n \equiv \langle \varphi^n \rangle = N\int \exp\left(-S[\varphi]\right)\varphi^n d\varphi$$

$$Z(J) = \sum_{n\geq 0}\frac{1}{n!}J^n G_n = N\int \exp\left(-S[\varphi] + J\varphi\right)d\varphi$$

$$G_n = \left[\frac{\partial^n}{(\partial J)^n}Z(J)\right]_{J=0};\ W(J) = \log Z(J) \equiv \sum_{n\geq 1}\frac{1}{n!}J^n C_n$$

$$\phi(J) \equiv \frac{\partial}{\partial J}W(J) = \sum_{n\geq 0}\frac{1}{n!}J^n C_{n+1}$$

And therefore, because randomness scripts in, we introduce the concept of probability. And the probability distribution for the random variable representing the field is given by the expression N exponential minus S of phi, where S of phi, phi is there a field variable and S of phi is the action. The action itself in this particular case will consist only of will not consist of any derivative terms because we cannot define a derivative in a zero-dimensional space time because there is no metric involved here.

We cannot define a metric and therefore, we cannot define a derivative and therefore, the action will not consist of any derivative terms. The normalization constant is given by the second expression here.

Now, being a probability distribution because of the randomness of the field variable, we introduce the probability distribution and the probability distribution, when it comes into play we can identify or we can de market or define the probability distribution in a sense by its various moments or its cumulants.

This moments of the probability distribution constitutes the green functions of the field. The generating functional or the generating function of the green functions is given by the expression here; 1 summation over n 1 upon n factorial J to the power n G to the power n from which the G n's can be recovered by differentiating and then, first differentiating and then substituting J equal to 0.

The term J is usually referred to as the source term. The W J is the generating function for the connected green functions and which is defined as the log of the Z J which is the generating function for the green functions and then, we have the field function and that is defined as the first derivative of W J and that is also that can also be expressed as a series or a power series in J, with the with the coefficients representing the connected green functions.

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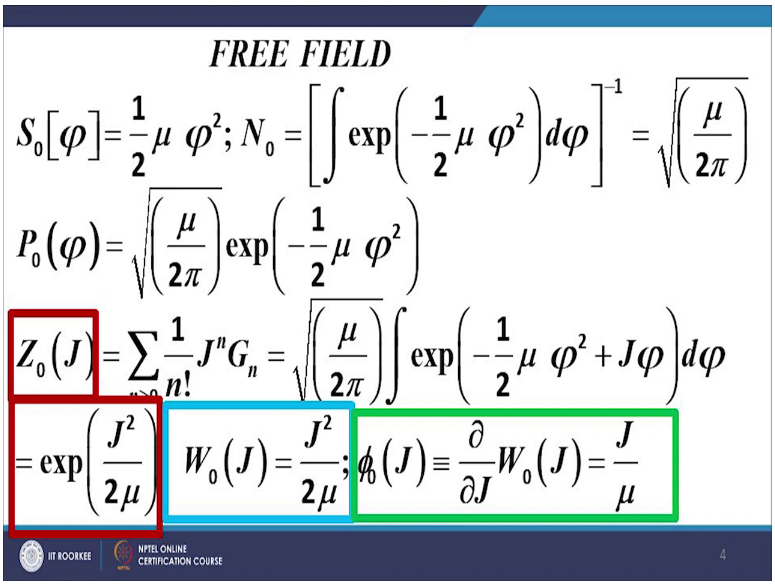

Then, I introduce the concept of a free field free field at 0 plus 0; that means, zero space time point, zero-dimensional space time and we in we saw that the field would consist, the free field would consist of the Gaussian action 1 by 2 mu sigma square.

The normalization constant works out to under root mu upon 2 pi, simple Gaussian integration and therefore, the probability distribution of the field variable turns out to be under root mu upon 2 pi exponential minus 1 by 2 mu sigma square. This is clearly a normal distribution or a Gaussian distribution.

The generating functional for the green functions, when we work it out, works out to the expression that is given in the red box and the generating function for the connected green functions works out to the logarithm of this, logarithm of the expression given in the red box, the field function on the other hand. So, this is the data that was derived that was obtained in the previous lecture with regard to the free field.

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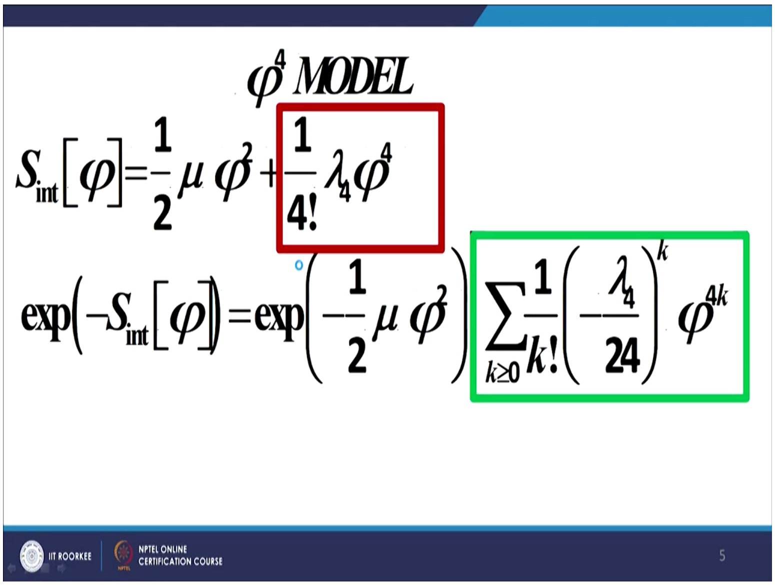

We then, introduced the interaction. We introduced the interaction as a phi to the 4 power 4 model where the interaction term consists of 1 upon 4 factorial that is 1 upon 24 lambda 4 phi to the power 4. This is the interaction term, the term that is given in the red box constitutes the interaction term. Then, we made an assumption.

We made an assumption that lambda 4 is much small compared to mu and therefore, when we exponentiate the action or the exponentiate the negative of the action, we can read we can expand this interaction term that is 1 upon lambda 4, 1 upon 4 factorial lambda 4 phi to the power 4 as a power series in phi the field variable.

So, let me repeat. We have retained the while exponentiating the negative of the action, we have retained minus mu square as the dominant term, as the dominant term and we have not expanded it as a power series. We retained it as an exponential, but we the other the interaction term on the premise that lambda 4 is relatively small, we have expanded a e the second term that is 1 upon 4 factorial lambda 4 phi to the power 4 as a power series in phi; exponential series in phi.

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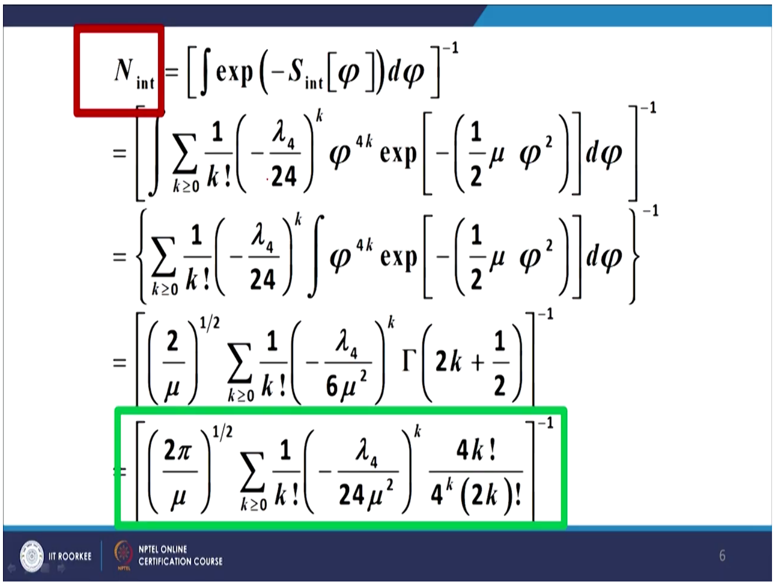

The normalization constant for this interaction theory is slightly more involved. If you work through it, the normalization the calculation of the normalization factor, there is one important step that needs that warrens mentioning that warrens mentioning.

In fact, very very prominently and that will also come back to revert to in a later part in this lecture today and that is the flipping of this non-flipping of this interaction of the of this summation with the integral. If you look at this carefully, in before we read the green box, there is there is from the first step to the second step, we have flipped the summation, we have taken the summation outside the integral.

Initially, the summation was naturally inside the integral because the exponential was inside the integral. So, thereafter the summation was taken outside the integral and then, the integral was done as it is a normal coat. This represents a 4 k moment of the Gaussian distribution and therefore, we got the expression in the green box.

So, this is the normalization constant. But please note, I reiterate that there is one step involved here which involves the flipping of the integral sign with this summation sign.

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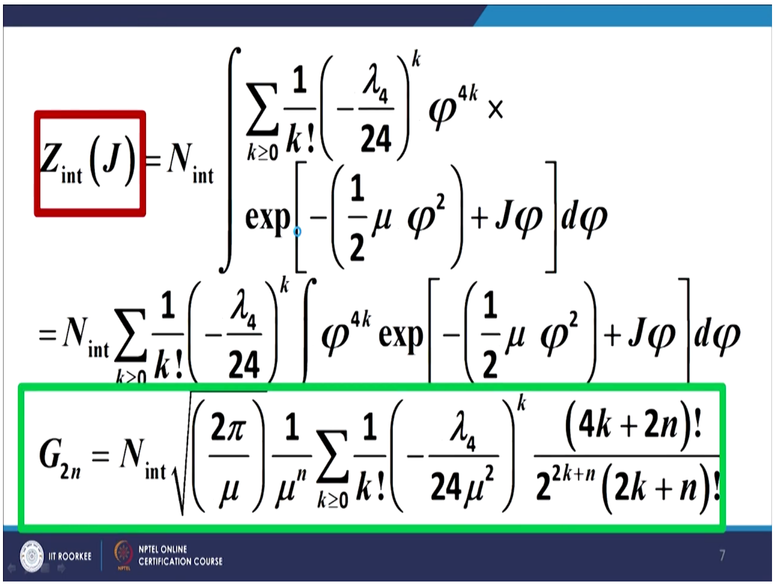

And as far as the generating function for the greens functions are concerned, we proceeded more or less on similar lines. We again got, we again did the same trick of flipping of the summation and the integral and we got the expression that was given in the green box right at the bottom of your slide.

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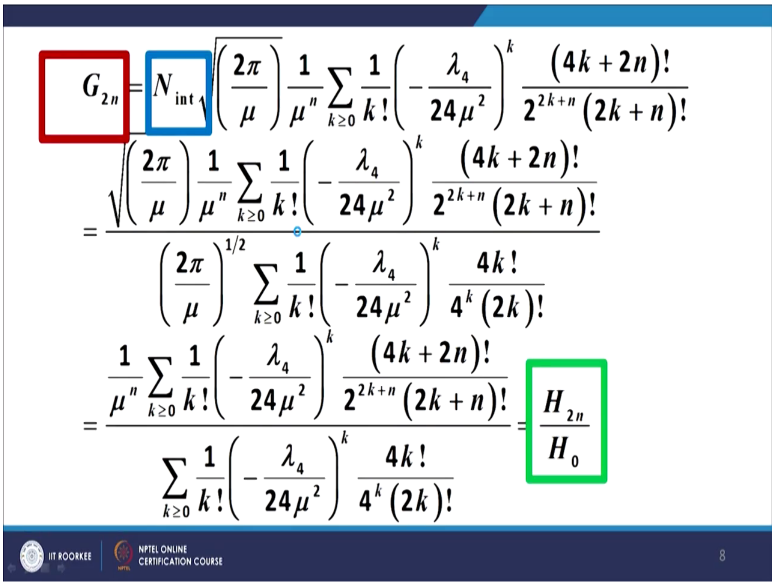

So, this was this was the expression for the generating function of the green functions. We worked introducing the normalization, the value of the normalization we got the green functions as the expression, as the expression given in the bottom equation.

The two factors 2 pi upon mu square root cancel out between the numerator and the denominator which represents the normalization in a sense and the remaining as it is can be represented as H 2 n upon H 0. H 0 emerges from the normalization and H 2 n is related to G 2 n, which is the 2 nth green function of the theory.

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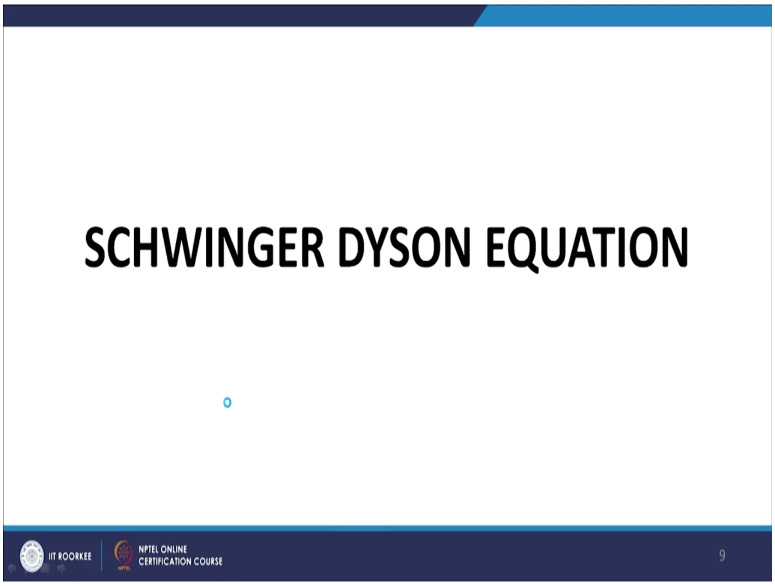

Now, we come to the Schwinger Dyson equation. So, that was a re brief recap of, where we concluded in the last lecture.

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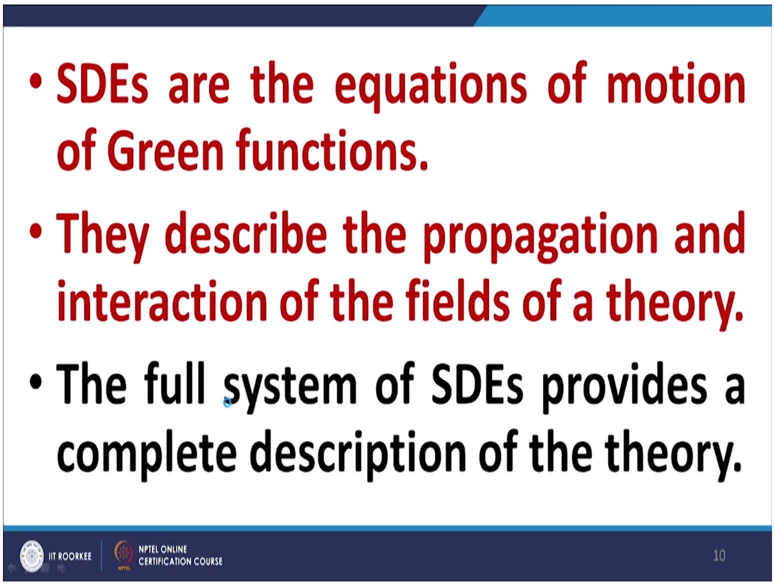

Now, we will take up the Schwinger Dyson equations, this Schwinger Dyson equations are the equations of motions of the green functions and they represent the propagation of the field interactions of a theory. The full system of this Schwinger Dyson equation completely describe the theory.

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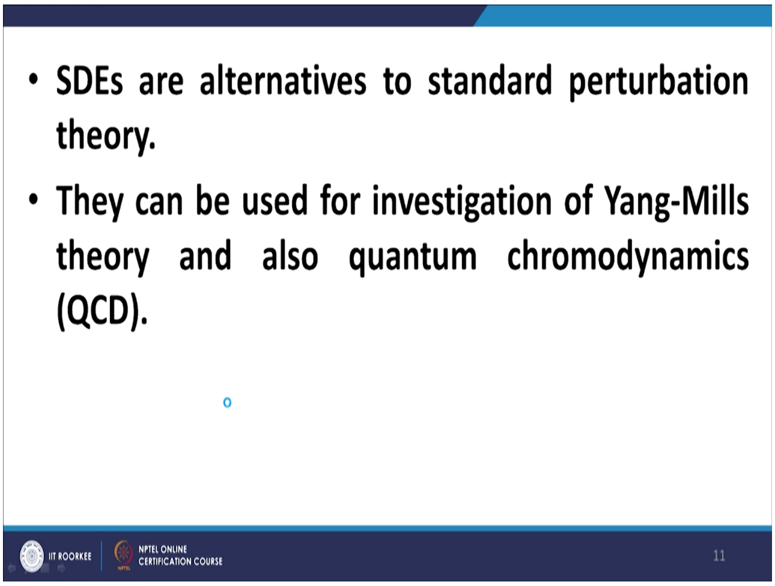

And they are in the sense there are alternatives techniques for solving the theory. So, they are alternatives to the standard perturbation theory. They are more compatible for use for investigation of the weak interaction and the strong interaction that is the Yang-Mills and the Quantum Chromodynamics environments.

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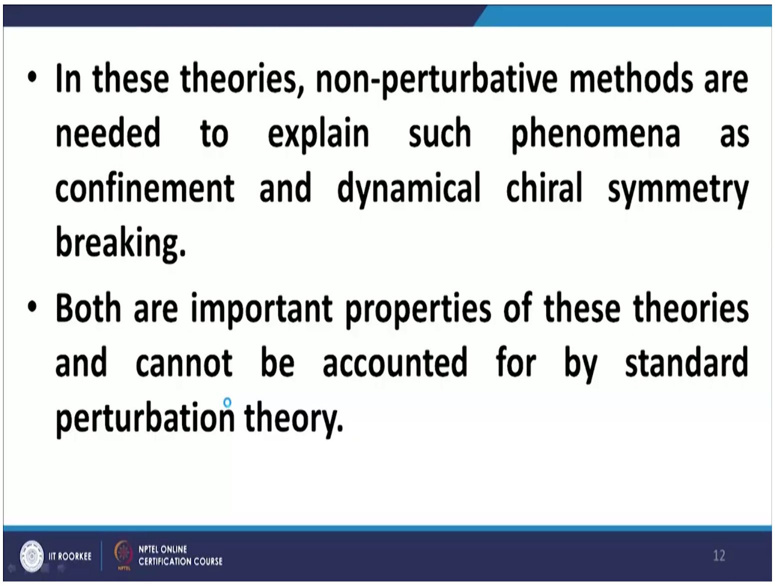

In these theories, you need non perturbative methods to explain certain phenomena like confinement and like chiral symmetry breaking and so on. So, here at this, the use of the Schwinger Dyson equation becomes more important although it is used in QFT, conventional QFT is also cannot be denied provided, we are able to solve the Schwinger Dyson equations.

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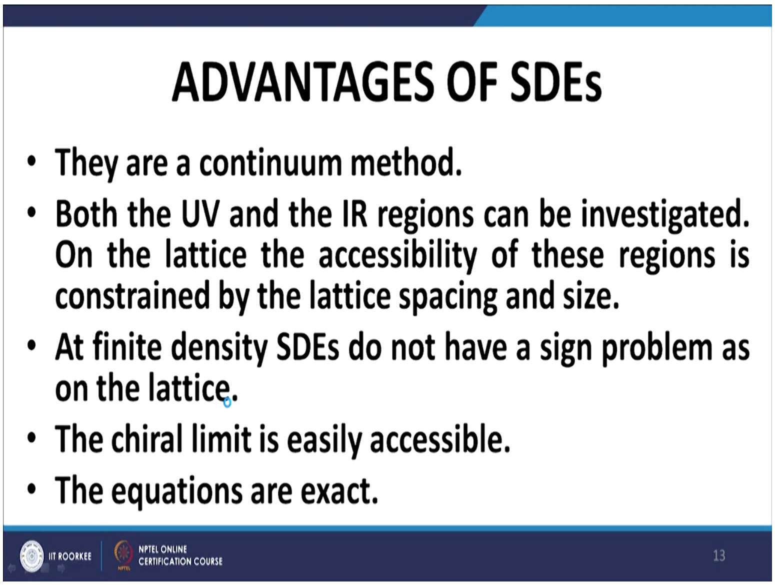

So, these are some advantages of the Schwinger Dyson equations. They are continuum in a sense and therefore, they can be used for the ultraviolet and infrared regions also. Lattice theories on the other hand, pose a problem when we try to investigate the ultraviolet and infrared regions. The chiral limit is easily accessed in this theories, which is again a problem in lattice theories.

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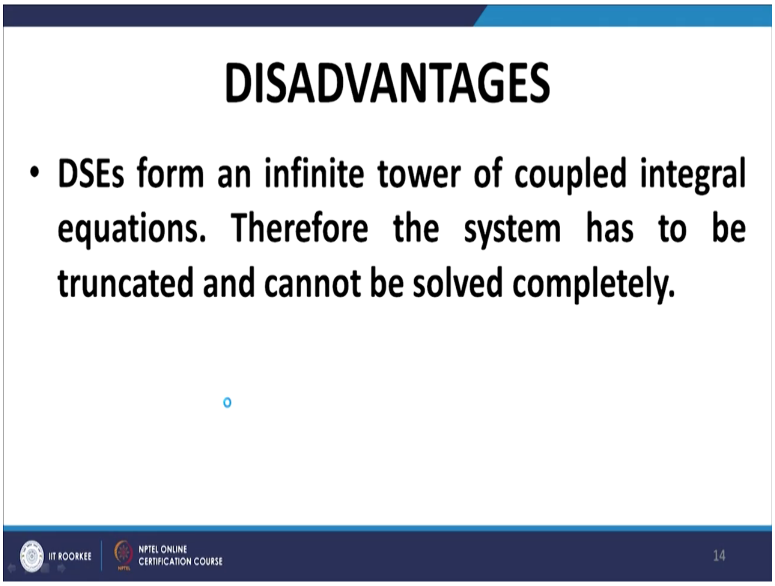

And these the SD equations, the Schwinger Dyson equations form an exact theories in then themselves. So, that is another advantage compared to perturbation theory.

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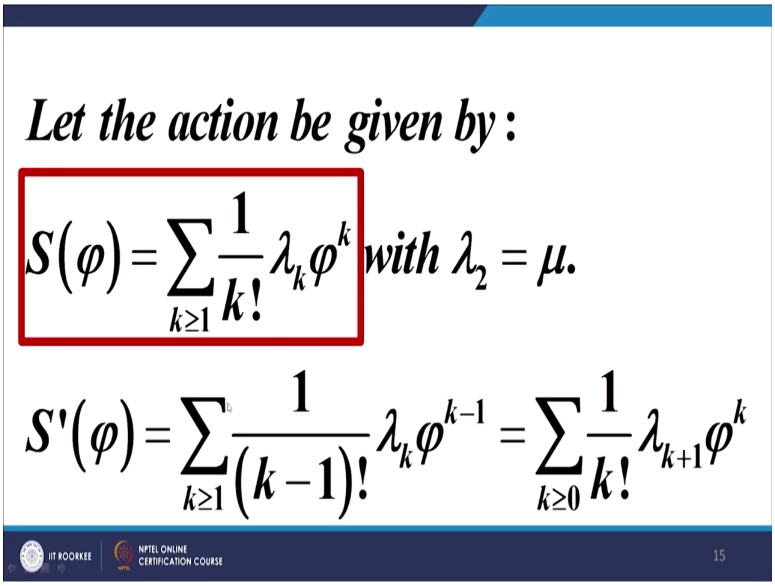

To arrive at the Schwinger Dyson equations, we consider the action as a power series in the field variable. We general take a general approach and we write the action as a power series in the field variable phi that is given in the red box, taking the derivative and redefining the index, we get the expression that is given in the bottom right hand side of your slide; summation 1 upon k factorial lambda k plus 1 phi to the power k.

This is the first derivative of the action corresponding to the action as far as equal to summation 1 upon k factorial lambda k phi to the power k.

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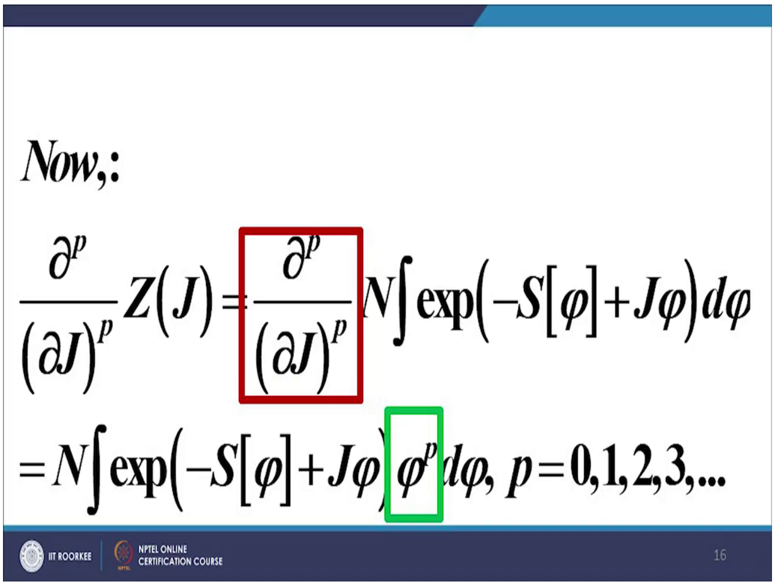

Now, if you take the derivative the pth derivative of Z J, what happens? The pth derivative of Z J acting on Z J, I have written Z J explicitly here, N integral exponential minus S phi plus J phi d phi and this is the value of Z J. So, if you differentiate this with respect to J, what happens is, this differential acts on the J phi term. This differential goes inside the integral because the integral is with respect to phi and the differential is with respect to J.

So, they we can transpose the integral or we can flip the integral inside the we can flip the differential and the integral operators and the differential can go inside the integral and then, it can operate on the J phi. So, it pulls down a factor of phi.

The net result is that if I differentiate Z J with respect to J, once I pull down 1 factor of phi due to the; due to the derivative operator acting on J phi. So, operating once, I get phi;

operating twice, I get phi square and so on. So, this is the way you generate the green functions by operating, by differentiating the generating function or the path integral.

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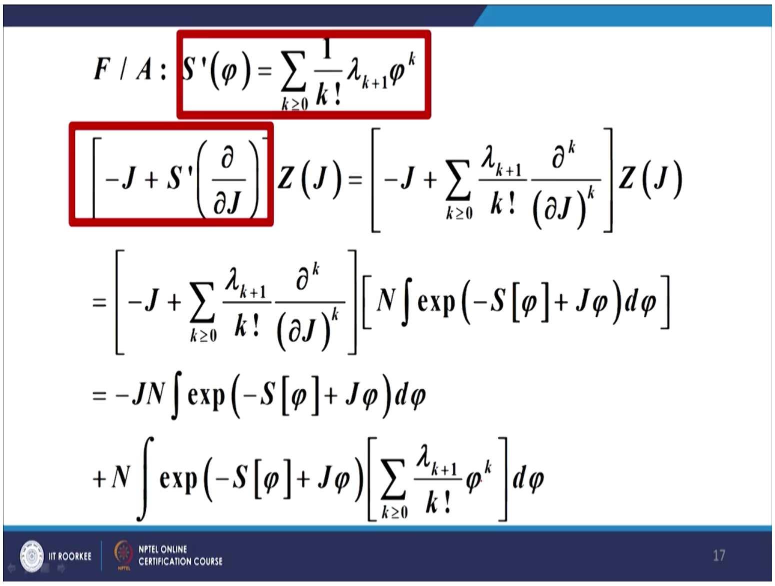

So, from the previous slide, we have got S dash phi is equal to this expression in the green box; there in the red box I am sorry. Therefore, minus J plus S dash del by del J of Z J. Now, S dash of phi is given by this expression. So, clearly S dash of d by dJ, who will be given by the expression that is in the square bracket of the second equation on your slide and this operates on Z J.

Now, when the first expression which Z J is what is given in the first term of the third equation. So, that is not an issue and the second expression when you look at it this d upon dJ acting on Z J, as we have shown in the previous slide, you can see it here d upon dJ operating on the Z J pulls down appropriate number of factors. The order of differentiation pushed on the appropriate number of factors of phi into the integral. So, that is precisely what is being done here.

We are pulling down appropriate numbers of phi and then of course, this is summation. This summation can be taken inside the integral, there is no variable involved here and therefore, I can attach the summation to the field variable and I get this second equation on the bottom slide.

So, just to reiterate my this S, S dash you see S dash d by dJ is a power series in d by dJ when it operates on Z J, each of the power to power terms or terms of the series contains Z by d by dJ to a certain power and that one operates on Z J pulls down the same power of phi into the integral, that is precisely what is happening in the bottom term, the last term on the slide.

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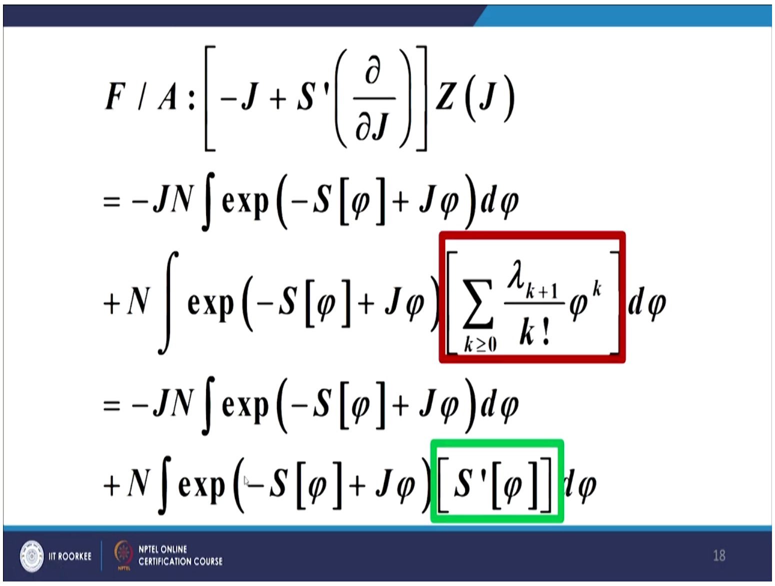

So, this is where we this is what we have from the previous slide. Now, if you look at this expression in the red box, if you look at this expression in the red box, it is nothing but S dash of phi, it is nothing but S dash of phi, simply S dash of phi nothing else and there is no other change this. The rest of the expression has been repeated as it is, only the expression in the red box has been substituted by S dash of phi.

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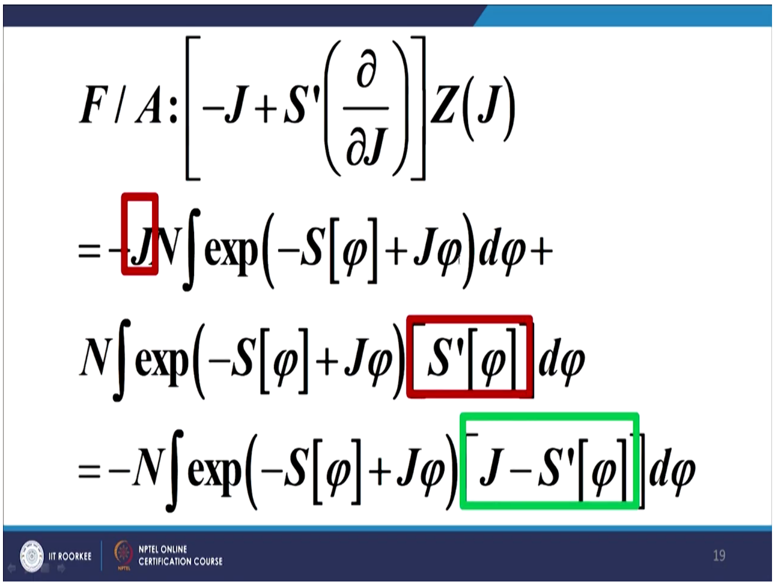

So, keep taking these two terms together, what I get is and taking this minus sign outside, what I get is J minus S dash of phi inside the integral, I can take this J inside the integral or S dash phi is already inside the integral. So, I can collect the two terms J and S dash of phi, the rest is nothing but Z J. So, this is what we have with the minus N of course outside.

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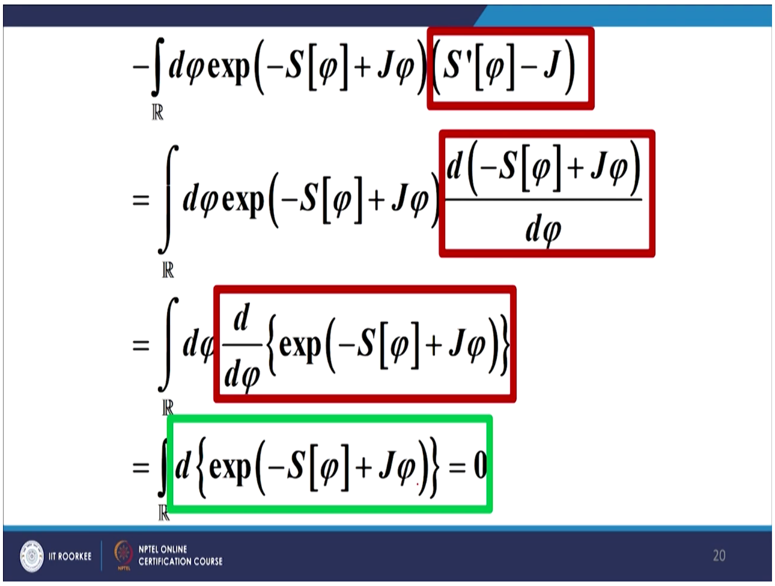

Now, comes a very important step. Now, comes a very important step. This is what we have from the previous slide, the first expression that expression in the red boxes what we have from the previous slide.

Now, I can write that in the form of the expression that in the second red box d by d phi minus S phi because this is S dash phi this minus sign, when you take it, take it together with S dash, you get minus S phi. The two thing this first of all this minus sign going inside and S dash being written as d by and d phi nothing else and J is being written as d by d phi of J phi.

So, the two expressions are equivalent, they have substituted the second one for the first one. Now, if you look at this exponential of this minus S phi plus J phi d minus S phi plus J phi upon J phi can be there is nothing but there is nothing but the derivative of d by derivative of exponential minus S phi d by d phi.

In other words, if you take the derivative of the expression that is given in the third red box, you simply get the expression in this second equation. d by d phi of exponential minus S phi plus J phi is nothing but exponential minus S phi plus J phi d by d phi minus S phi plus J phi and that is precisely what is the expression, we have in the second equation on your slide.

So, now this expression which is there in the red box can be written as a total derivative, total derivative d of exponential minus S phi plus J phi. And being a total derivative when you integrate it, the integral and integrate it within the limits minus infinity to infinity, clearly the integral will depend only on the limits of integration and because we assume that the integrand vanishes sufficiently fast enough so that its value at minus and plus infinity both or 0 or are negligible. Therefore, the value of this integral also k is also approximately 0.

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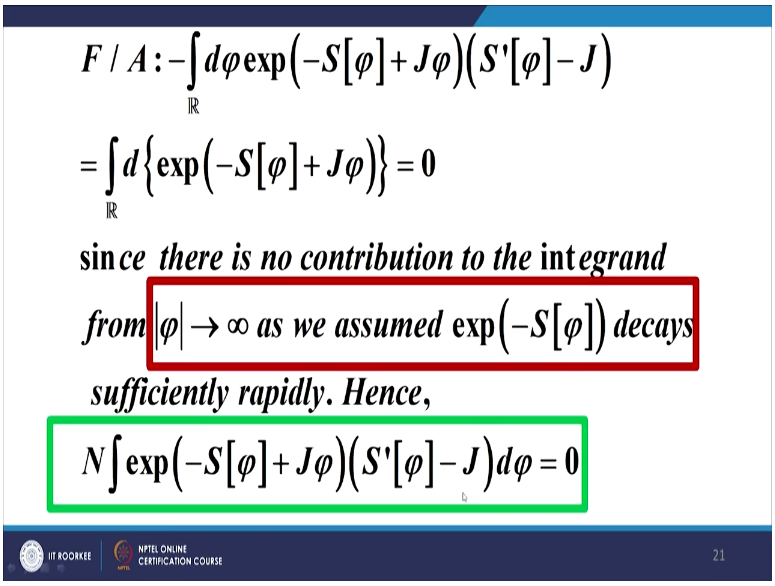

We have the expression on the first expression as equal to 0, the first expression minus integral d phi this expression, this first exponential minus S phi plus J phi into S dash phi minus J. If when you integrate this, you get 0 which is shown in the green box at the bottom of your slide.

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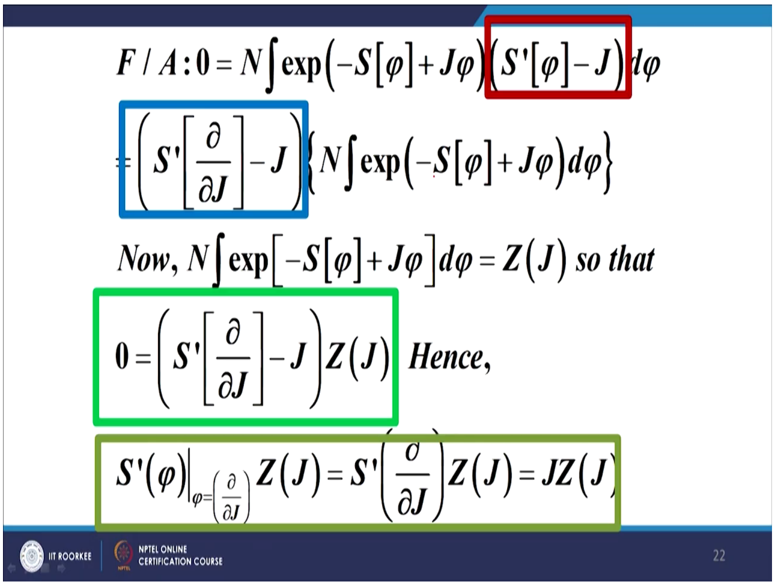

So, we now revert to where we started, we started with S dash d by dJ minus J into Z operating on Z J. Let me go back and show you. We moved quite a bit. It is here in this red box, second red box minus J plus S dash d by dJ operating on Z J.

So, this whole expression we find that this whole expression is equal to 0 and thus, that gives us the equation which is the Schwinger Dyson equation which is written in the dark green box right at the bottom of your slide.

When you simplify it, the expression at which you replace the integral with Z J and you simplify the expression a bit, you get the result which is shown in the dark green box at the bottom of your slide and which is the Schwinger Dyson equation for the generating function for the green function for Z J.

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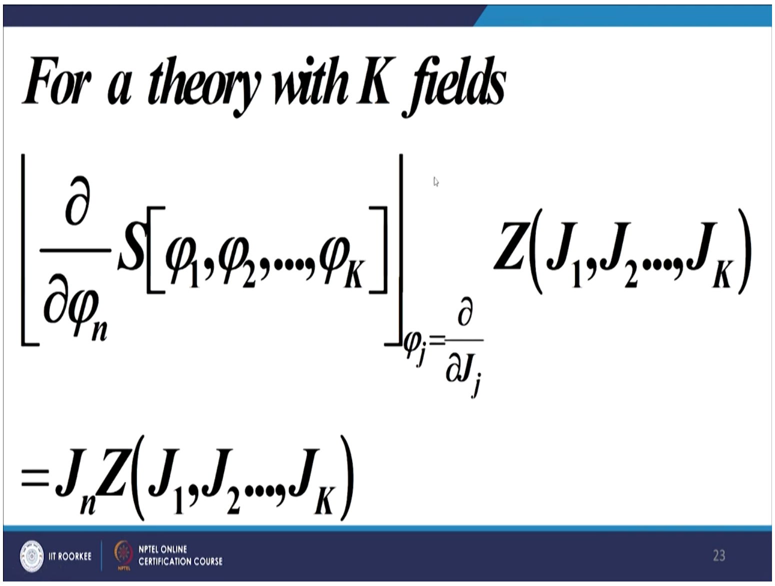

Of course, in the case in the theory with k fields, the same thing can be generalised extended and we get the result which is shown in your in the slide which is here.

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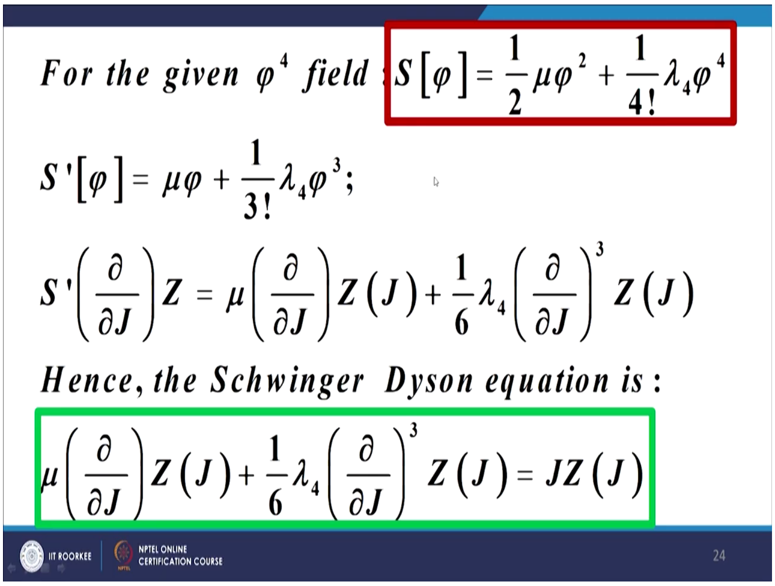

Now, for the given phi 4 field, for the phi 4 field that we are considering S phi is equal to 1 by 2 mu with the action 1 by 2 mu sigma square plus 1 upon 4 factorial lambda 4 phi 4. This is the action; this is the interaction field that we are considering for the moment. This is called the phi to the power 4 field.

So, we have S dash phi is equal to this mu of phi plus 1 by 3 factorial lambda 4 S simple differentiation and this gives us S dash d by dJ of into Z J is equal to mu into for phi, we substitute d by dJ and then, operating on Z J plus 1 by 6 lambda 4 d by dJ whole cube into Z J.

Hence, this is our Schwinger Dyson equation for the phi 4 field, for the phi 4 field. This is the earlier equation that we got was for a general field and general field which was expanded as a power series in the field variable, the action of that field was a power series expansion in the sealed field variable. Now, we are considering the specific case of the phi 4 field and phi 4 field has the Schwinger Dyson equation that is given in the green box at the bottom of this slide.

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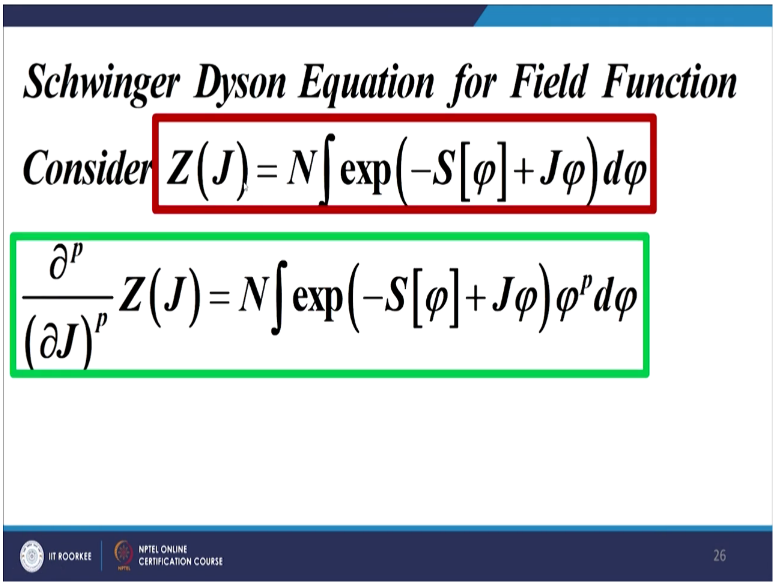

Now, we come to the Schwinger Dyson equation for the field function. Recall the field function is given by the derivative of the W J, where W is the generating function for the connected green functions.

In other words, it is given by Z 1 upon Z J into Z dash J or log of, the derivative of log of Z J also; either all the three expressions are equivalent in fact. So, we have Z J, this we have already seen. This expression, we have seen. This will be required in the next few slides. So, we I have rewritten it.

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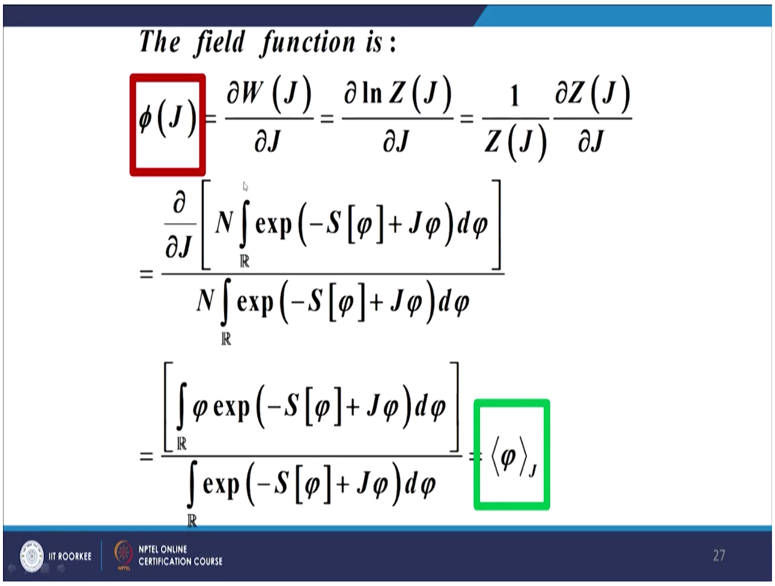

Phi J, as I mentioned just know is the first derivative of the first derivative of the generating function for the connected green function. So, and the connected green function is given by the log of Z J. So, clearly phi of J is given by d of dJ log of Z J which is equal to 1 of Z J into dZ J upon dJ.

So, differentiating this expression, if you differentiate this expression, dZ J upon dJ, you pull down a factor of phi as we have discussed a number of times, you pull down a factor of phi. And if you look at this expression now, so in a sense what we have is simply the expected value of phi.

So, the expected value of phi, but the difference is that the expected value of phi is calculated in an environment that is in the presence of a factor J which represents the source. So, what we can say here is simply that the field function is nothing but the expected value of the field variable in the presence of sources.

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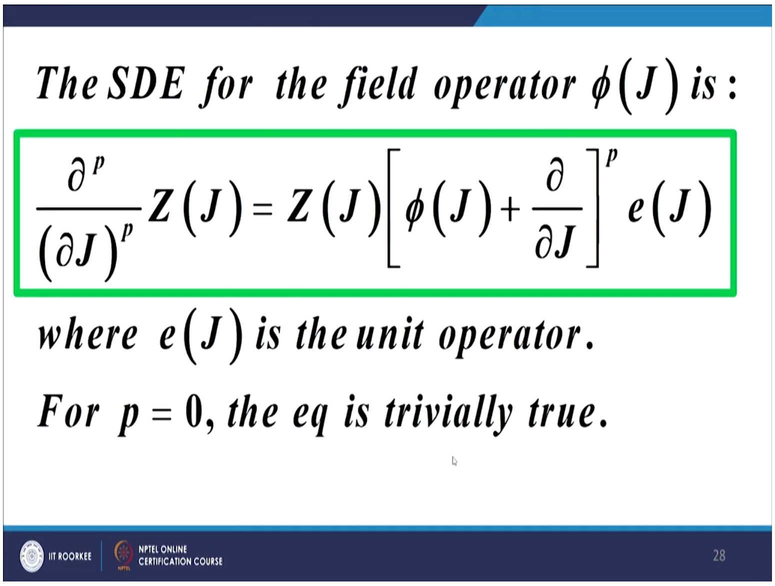

The Schwinger Dyson equation for the field operator phi J, the field function phi J is given by the expression in your green box, first green box; where, e J is the unit operator. e J here is the unit operator. For P equal to 0, if you can clearly see the equation is trivial because this it becomes 1. So, the right hand side is Z J, the left hand side is also Z J because the no derivative will operate with P equal to 0 and therefore, Z J is equal to the right hand side also which is also Z J.

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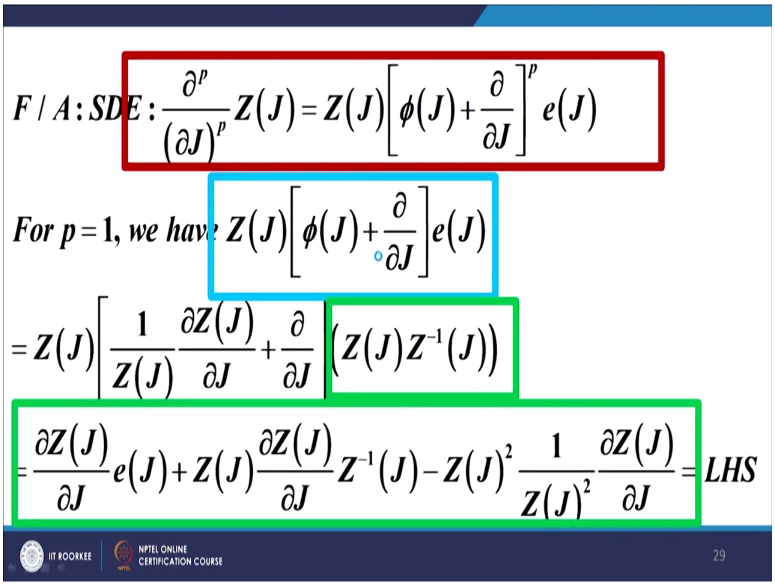

Let us see what happens for p equal to 1. For p equal to 1, the right hand side becomes the expression in the blue box. The right hand side becomes an expression in the blue box which the I can write the unit operator as Z J, Z inverse J and then, I proceed to operate the expression in the square brackets on Z J, Z inverse j. Let us see what I get.

The first term that I get is dZ J upon dJ into e J that is the first term. When I multiply Z J into 1 upon Z J into dZ J upon dJ the into this expression in the green box that is what I get. The second expression I get, Z J operating with d by dJ operating on Z J, Z inverse j.

So, that gives me derivative of Z J with Z Z inverse J minus this Z J and this Z J Z J square and the derivative of Z inverse J with respect to J is 1 upon Z J squared derivative of Z J upon Z J. Now, the net result of this expression is clearly derivative of Z J upon Z J which is the left hand side for p equal to 1. So, the equation holds for pequal to 1 as well. Let us see what happens for p equal to 2.

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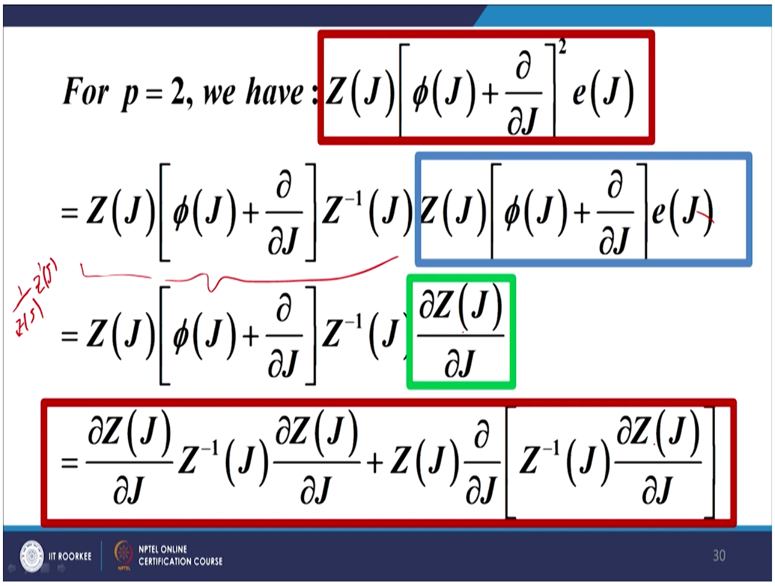

For p equal to 2, we can write this expression. We get the expression or we start with the expression given in the red box and we can write this expression as the second equation, simply by introducing Z inverse J, Z J which is the unit operator in between the two factors of p, out of two factors of phi J, two factors which are there in the square brackets. I repeat the first factor in the square bracket and then, I get the second factor in this square bracket; in between the two, I have imposed Z inverse J, Z J which is nothing but the unit operator.

Now, I pick out the term in the blue box Z J phi J plus del by del J e J. This expression is nothing, if you look at it this expression is nothing but the Schwinger Dyson equation for p equal to 1 and that gave us that gave us the result del Z J upon del J.

In other words, what I am left with is the whole first term first part of the term Z J into phi J plus d by dJ Z inverse J this is the first part of the term. This whole term, I have got, still pending and this term in the blue box gives me dZ J upon dJ from the previous example for previous proof of for p equal to 1.

Now, let us simplify this expression. Z J into phi J into Z inverse J operating on this will give me what? Will give me derivative of Z J with respect to Z J because this is this phi J is nothing but 1 upon Z J into the derivative of Z J. So, I can write this phi J as 1 upon Z J into derivative of Z J.

So, this 1 upon Z J into derivatives Z J, when this Z J we get the unit function here and Z dash or derivative Z J, this expression we have it the first term and Z inverse J, I get from here and this derivative of Z J I get from this expression.

So, this accounts for the first term, the first term of the red box and as far as the second term is concerned, when you multiply Z J into derivative of Z inverse J, derivative of Z J upon derivative of Z J.

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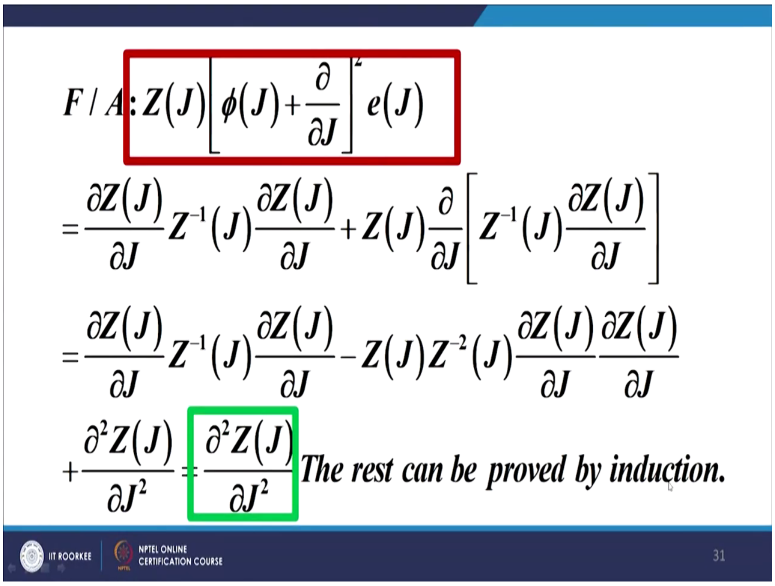

So, this term in the red box is what we get for p equal to 2. Little bit more of simplification, when I simplify this further, when I simplify this further, what I get is the first expression is as it is retained as it is. The second expression, I carry out the derivative the second, I carry out the derivative of the second expression this a derivative of Z inverse J gives me 1 upon Z J square derivative of Z J into Z J and this derivative of Z J remains as it is. This is the second term.

If this will drop out Z J and Z inverse Z J will drop out and I will have derivative of Z J acting on derivative of Z J giving me the second derivative of Z J. Now, these two terms cancel out, I get second derivative of Z J with respect to J. So, at the expression holds or the equation holds for p equal to 2 as well. The rest you know we can prove by mathematical induction, the rest we can prove my mathematical induction. We will continue from here after the break.

Thank you.