Path Integral Methods in Physics & Finance
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Lecture – 32
Field Theory in Zero Dimensions (1)

Welcome back. So, we were talking about the free field action before the break. The free field action is given by this expression in your green box.

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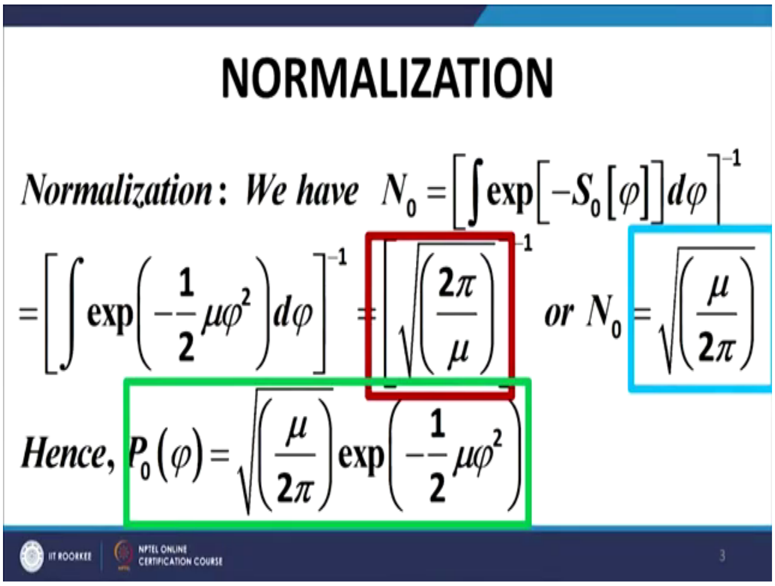

And the probability distribution is the normalization for the probability distribution we have worked out, it is nothing but a simple Gaussian integral. In fact, this distribution happens to be a Gaussian distribution, and the normalization factor is under root mu upon 2 pi therefore, the complete distribution is given by the expression in your green box right.

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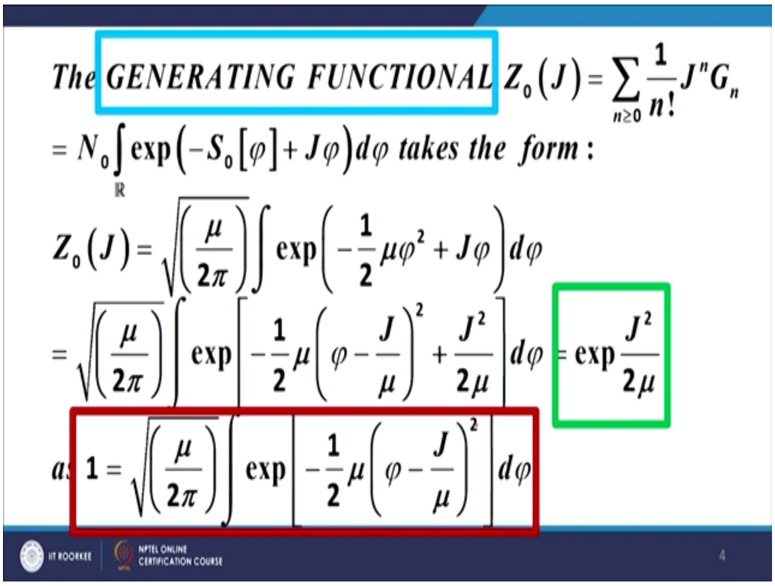

So, the generating functional we worked out, generating functional is again can be explicitly worked out can be worked out in closed form because of the simplicity of the distribution. And we have the generating functional as exponential of J square upon 2 mu, that is again obtained simply by a Gaussian integral.

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$$Also \quad Z_0(0) = N_0 \int_{\mathbb{R}} \exp(-S_0[\varphi])\, d\varphi = 1$$

And this is the normalization that I just talked about Z 0 of 0 the value of Z 0 for J equal to 0 is equal to 1.

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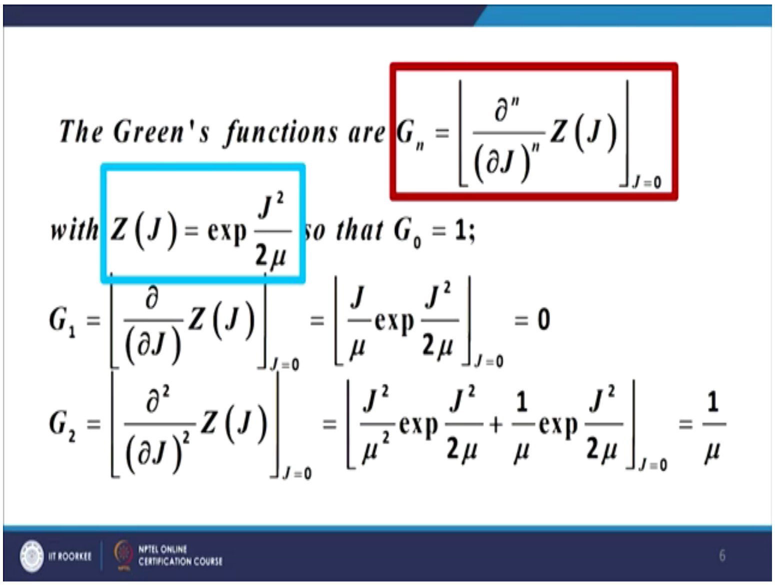

Now, we recover the Green functions from the generating functional let us see what we get you remember the generating functional is exponential J square upon 2 mu; so, G 1 will be equal to the first derivative.

G 0 therefore, you can check G 0 is equal to 1 here. G 1 if you work out; G 1 is the first derivative of this expression or the generating functional Z J, and the first derivative happens to be J upon mu exponential J square upon 2 mu put J equal to 0. This expression gives me 1 and this expression becomes 0. So, the net result is 0.

In fact, you will see that the all the odd Green functions odd point Green functions vanish. Let us see what happens to G 2; G 2 when we work out we take a second derivative of the generating functional we get on substituting J equal to 0, we get the expression 1 upon mu.

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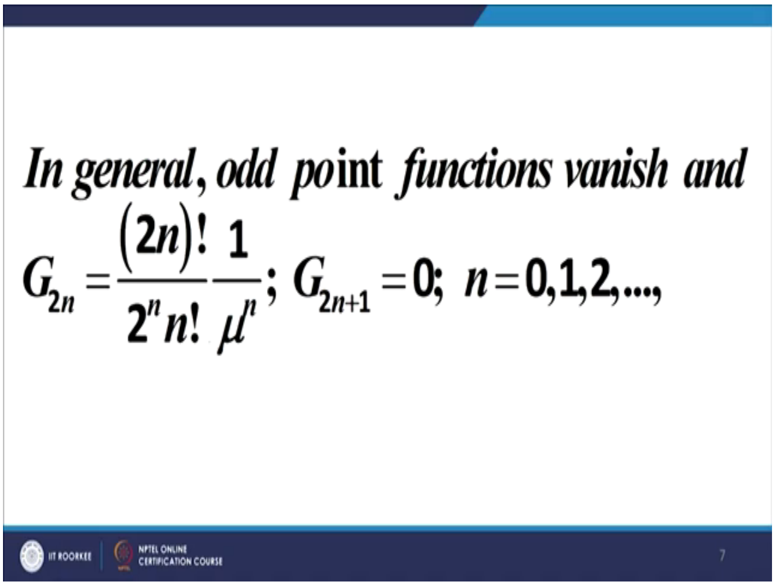

The odd point functions will vanish, and the even point functions will be given by this expression. And this is the this is obtained from the general formula or the formula for the general moments general even moments of the Gaussian distribution. So, G 2n is equal to 2n factorial upon 2 to the power n n factorial 1 upon mu to the power n; and all odd Green functions will be 0.

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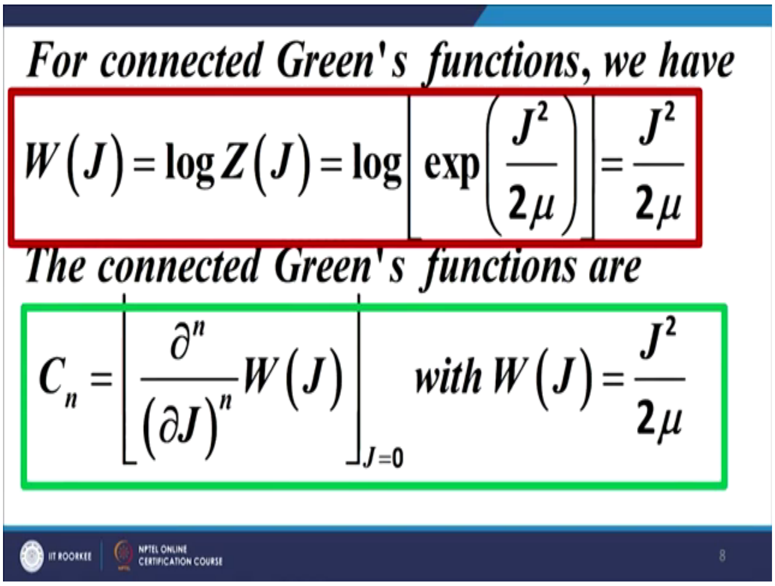

For the connected Green's functions, we have W Z is equal to square upon 2 mu remember Z J is equal to exponential of J square upon 2 mu therefore, log of Z J is J square upon 2 mu and therefore, the connected Green functions we get from taking derivatives of J square upon 2 mu which gives us the following.

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$$So\ that\ C_0 = 0;$$

$$C_1 = \left[\frac{\partial}{(\partial J)} W(J)\right]_{J=0} = \left[\frac{J}{\mu}\right]_{J=0} = 0$$

$$C_2 = \left[\frac{\partial^2}{(\partial J)^2} W(J)\right]_{J=0} = \frac{1}{\mu}.$$

$$Also\ C_n = 0\ \forall n > 2$$

C 0 is equal to 0; C 1 the first derivative again if you put J equal to 0, you get 0 and C 2 is equal to 1 upon mu. Remember C 2 is the variance of the distribution.

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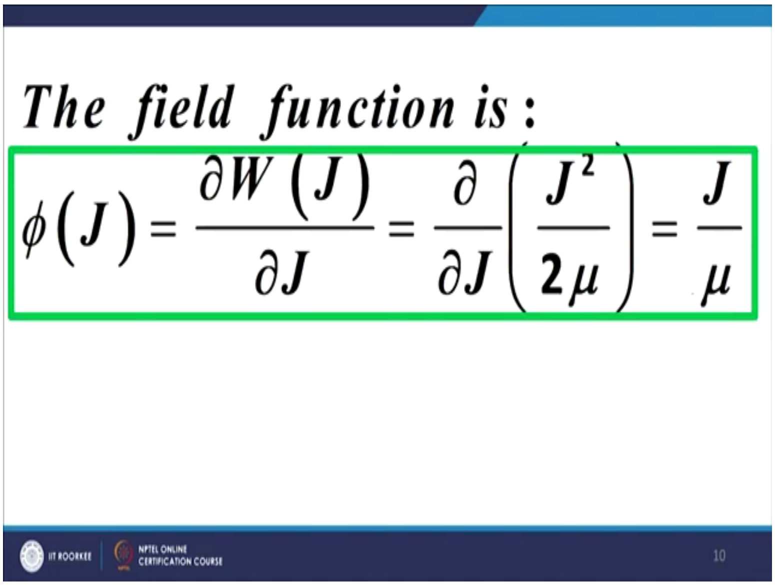

The field function, well the field function is given by phi symbol of J which is the first derivative of the generating function for the connected Green functions.

And in that when we substitute the value we get that is, J upon mu. So, the field function is given by J upon mu also recall that this field function is the expectation value of the field variable in the presence of the source term.

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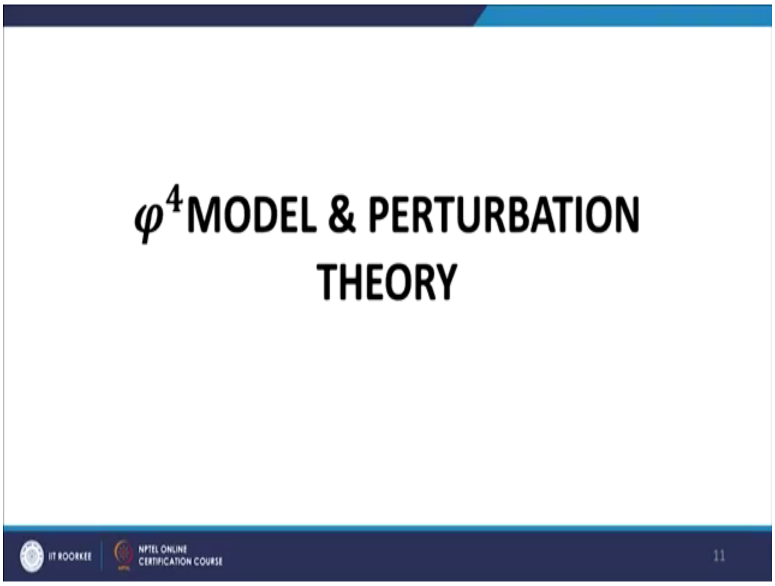

Now, we come to interaction models, we have talked about the free field theory of the zero-dimensional QFT. Now, we talk about an interaction model of zero-dimensional QFT where we have a interaction term in the action.

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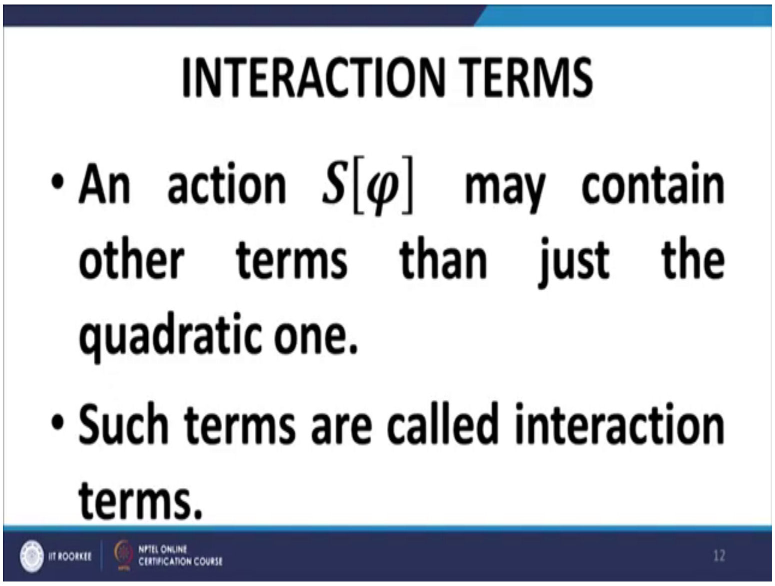

If there is any other term, then the quadratic term we call the other term as the interaction term.

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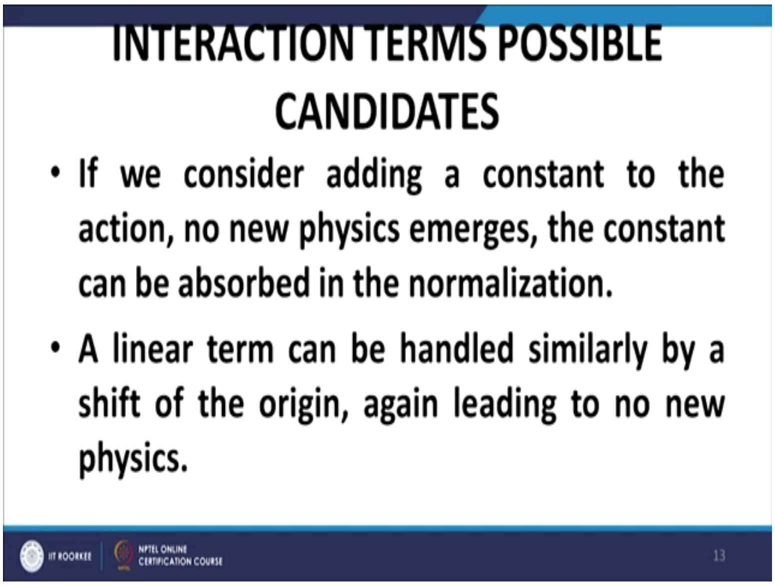

Now, if you consider adding a constant to the action, it does not give us any new physics. Why it does not give us any new physics is because it can simply be absorbed into the normalization.

For example, if I have instead of the action minus S phi plus say theta then, e to the power theta can be absorbed theta or minus theta can be absorbed into the normalization factor. A linear term can be treated similarly. If you have a linear term add a linear term to the action what happens is by changing the by shifting the origin we can retain the Gaussian structure of the action and therefore, it does not contribute to any new significant new physics.

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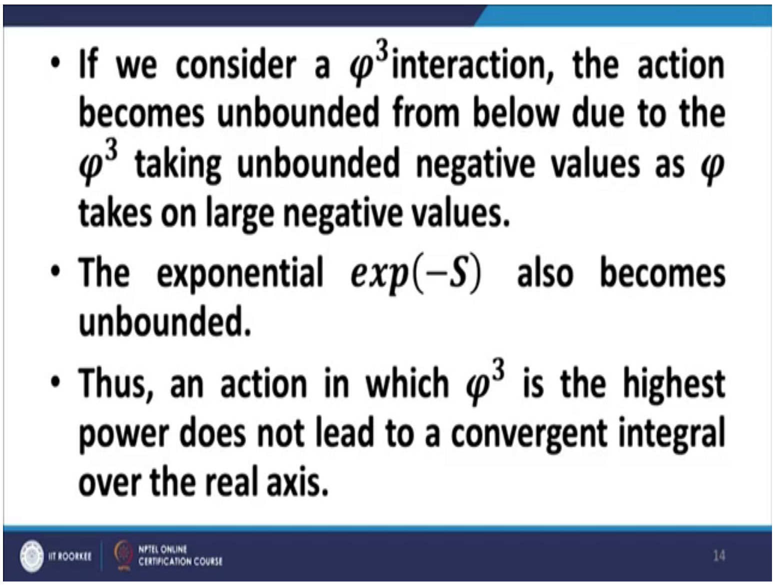

Now, what about a phi 3 term a cubic term? Now, if you have a cubic term and the situation becomes interesting. The situation does become interesting, but the action becomes unbounded from below because for negative values of phi, which are definitely permitted for negative values of phi this phi 3 term will blow up.

And the action will become unbounded from below and because the action will become unbounded from below due to the phi taking negative values and therefore, phi cube taking negative values.

The exponential minus S will become unbounded and as a result of which the action does not lead to a convergent integral. And therefore, the phi 3 interaction or the phi 3 interaction in itself becomes redundant exercise.

However, one can definitely have a phi 3 for theory; that means, you can have a phi cube interaction when you include a phi 4 interaction as well. We shall investigate a little bit about it in a later lectures, but for the moment because we have in this because of this problem of convergence associated with phi cube interaction the next candidate for an interaction term is the phi 4 interaction.

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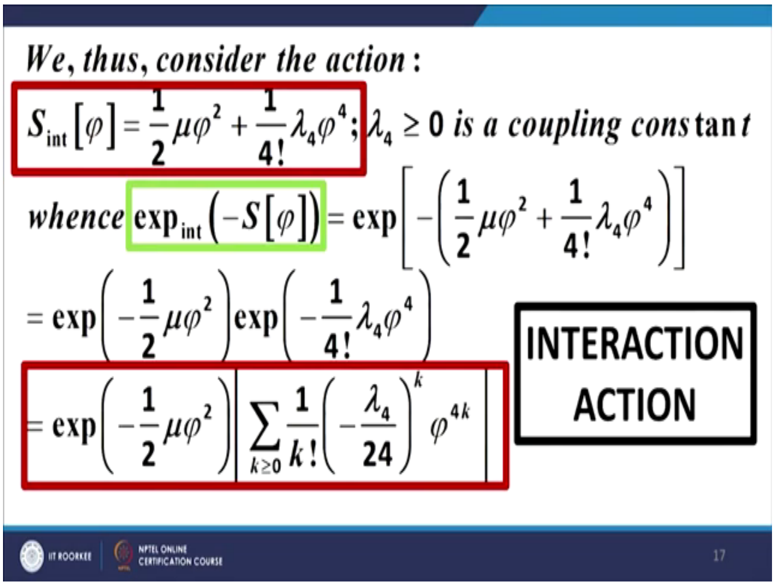

Now, lambda 4 is a interaction constant, a coupling constant it is called and we assume that lambda 4 is positive and it is small, small enough to facilitate a perturbation solution to our problem. So, we have the exponential of minus S phi S interaction please note this way.

Here the action with interaction action not the original free field action. It is the interaction action. So, we have it exponential minus 1 by 2 mu phi square plus 1 by 2 lambda 4 phi 4. This is the interaction action. And this is the interaction action.

So, if you simplify this; if you simplify this we can split the exponentials and we retain a exponential minus 1 by 2 mu square mu phi square and the other exponential we expand as a power series in lambda; the interaction term that is this interaction term we expand as a power series in lambda.

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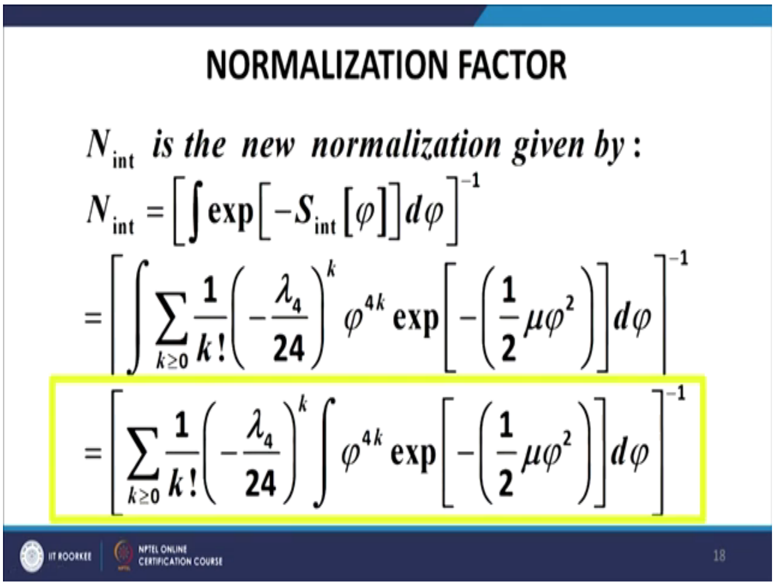

Now, obviously the normalization factor will also change. Once the interaction terms comes into play or comes into the action becomes a part of the action the normalization also changes.

The interaction the normalization in the case of the new action can be written in this form, this whole thing in this whole thing inverse.

See, we have simply as simply reproduced what was there in the earlier slide, this expression in the red box I have carried forward and this expression in the red box I have reproduced here.

Now, comes a very important step. The important step is if you look at this expression the summation is within the integral; in other words first we are doing the summation and then we are doing the integral. But, if you look at this the next step that is the step in the light green box then this summation has been taken outside the integral, in other words now what we are saying is, that we first do the integral and then we do the summation.

Now, there are a lot of technical issues associated with this transposition. It is not blatantly permitted, it is not blindly permitted, it is not universally permitted; it can only be allowed under certain specific circumstances, certain specific behaviour of the underlying variables.

Now, I shall come back to it will be a digression, but I shall address this issue in our current context at the end of this particular theory zero-dimensional theory and we shall talk more about it.

For the moment we take it as granted we take it as granted for the moment only, but we shall read address this particular situation. We assume that this is we are assuming, it is not a default option please note this it is not a default option we are assuming that this is permitted.

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$$F/A: N_{\text{int}} = \left[\sum_{k \geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24} \right)^k \int \varphi^{4k} \exp\left[-\left(\frac{1}{2} \mu \varphi^2 \right) \right] d\varphi \right]^{-1}$$

$$\text{Set } x = \frac{1}{2} \mu \varphi^2 \text{ so that } dx = \mu \varphi d\varphi \text{ and } \varphi^4 = \left(\frac{2x}{\mu} \right)^2 \text{ so that}$$

$$N_{\text{int}} = \left[\sum_{k \geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24} \right)^k \int \left(\frac{2x}{\mu} \right)^{2k} \left(\frac{\mu}{2x} \right)^{1/2} \left(\frac{1}{\mu} \right) \exp(-x) dx \right]^{-1}$$

$$= \left[\left(\frac{1}{2\mu} \right)^{1/2} \sum_{k \geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{6\mu^2} \right)^k \int (x)^{2k - \frac{1}{2}} \exp(-x) dx \right]^{-1}$$

Now, if this is permitted then things do simplify, then the integral if you look at it carefully the integral becomes a moment of the Gaussian distribution phi 4k exponential minus 1 by 2 mu this is nothing but the, 4 kth moment of the distribution represented by the PDF given by exponential minus 1 by 2 mu phi square PDF in the probability density function.

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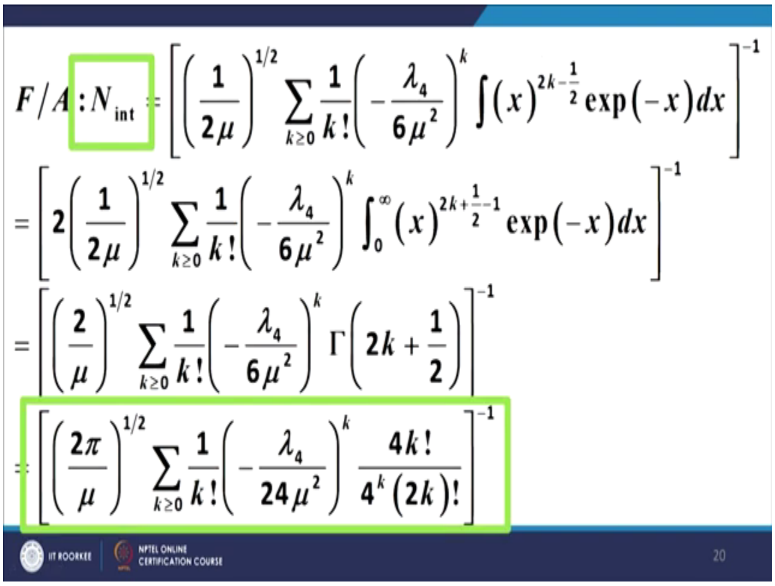

And therefore that being the case what we have is that, the result is that this whole expression; this whole expression the this is this whole this integral as a whole this only the integral part the integral part on simplification, because it is a it is an even moment of a normal distribution; it can be simplified exact result can be obtained and the exact result is 2 pi upon mu to the power 1 by 2 4 k factorial 4 to the power k 2 k factorial.

This is the expression for this the integral that I was talking about in the previous slide. The rest of the expression remains that is as it is for the moment and it is retained here in the current form.

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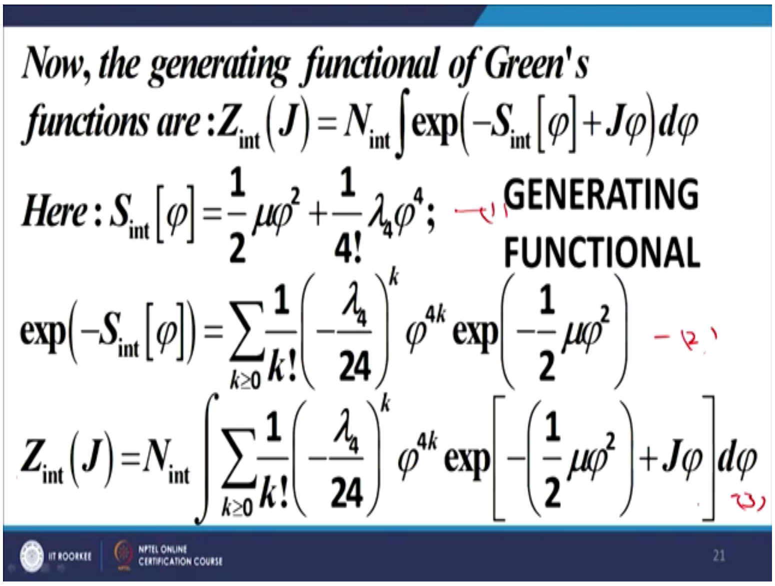

Now, this is; now this is the expression for the normalization constant, please note this. The expression in the green box is the expression for the normalization constant. Please note this inverse here also.

Now, we work out the generating functional. Now, let us work out the generating functional. The generating functional for the interaction theory is given by this expression Z interaction J is N interaction, where N interaction we have just worked out, it is a normalization factor.

Exponential mind now we have a source term added in source term included in J phi and putting in the expression for exponential, and the for the action that we worked out earlier we get this expression, this expression and now we have the additional expression of J phi here.

The expression for the interaction action is given in the in equation 1. So, the interaction action is given by equation 1 here and if you take the exponential we get this expression where we have expanded the perturbation or expanded the interaction as a power series as we have done earlier.

And the Z J the interaction generating functional is now given by this expression plus the additional J phi, which will help us to work out the Green functions for the various orders.

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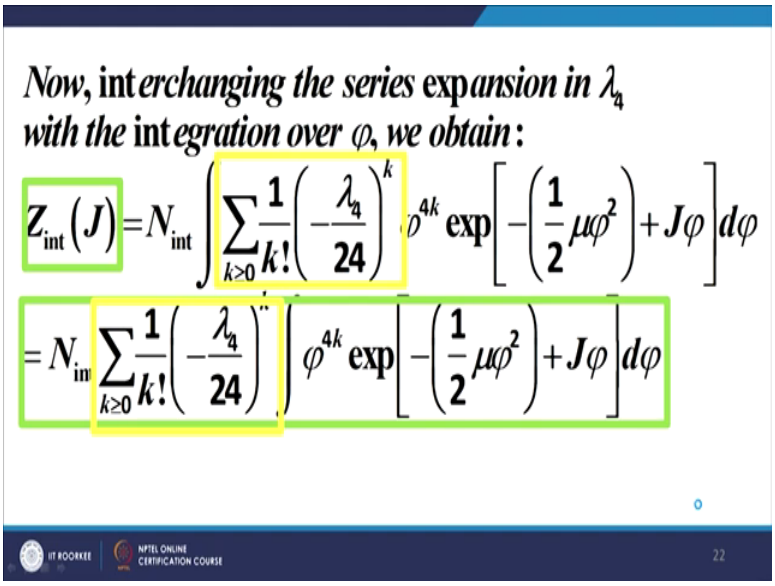

So, Z interaction J is what we have here. Now, again we have adopted this particulars trick or this particular manipulation I should say, that we are shifted the summation which is within the integral; to summation which is outside the integral.

As I mentioned strictly speaking it is not permitted by mathematics there are certain requirements that need to be met technical requirement that need to be met and only then we can make this transposition, but we will get back to it for the moment we assume that this is permitted. And we made this substitution and we transposition and we get this expression.

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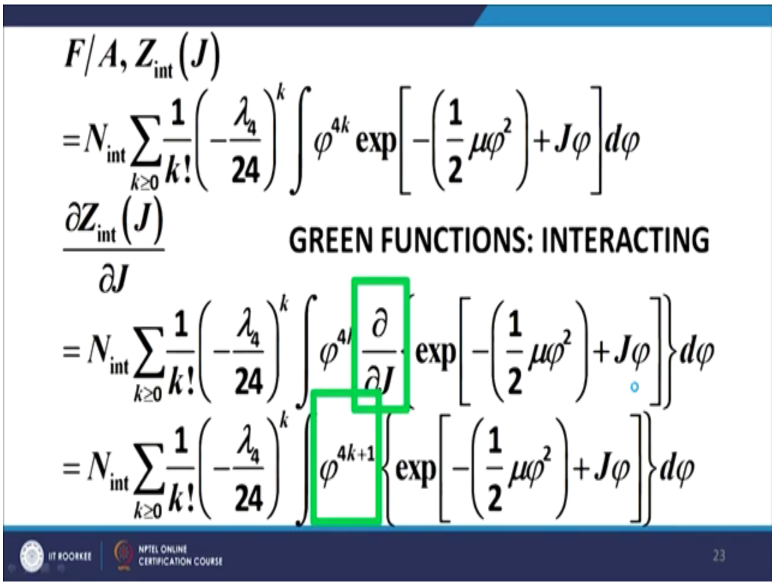

Now, we are now going to simplify the integral. When we simplify the integral and for the first derivative, let us say we want to work out the first Green function, first order Green function one-point Green function we take the first derivative of the generating functional with respect to the source term that is J.

Now, when we take the derivative with respect to J, this J attaches itself to the exponential term, because there is no other J here in fact. So, this J attaches and the integration is with respect to phi, the derivative with respect to J so, you can take J inside the integral and it attaches itself it operates on J phi and brings back or pulls out pulls back a factor of phi from here.

And when it pulls back a factor of phi from here, we get phi 4 k plus 1 here. And, this the job of this derivative is now over. So, what we are left with is integral of phi 4 k plus 1 exponential of this term.

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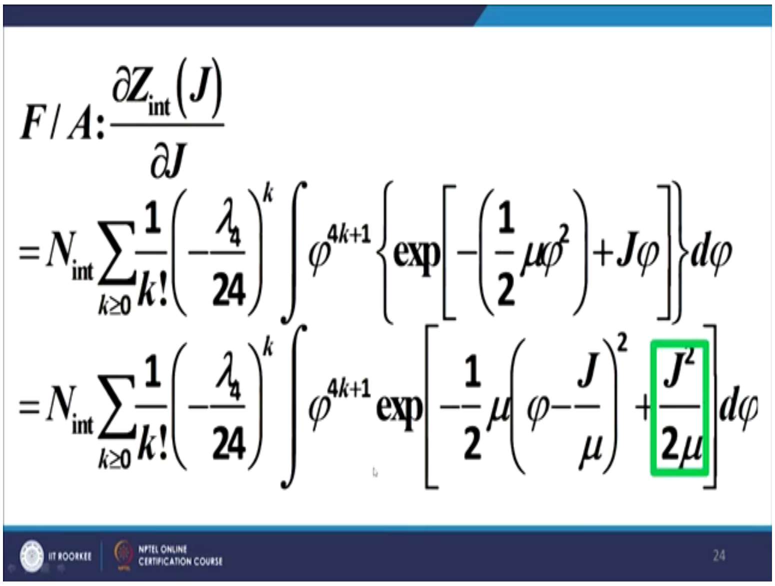

Now, if you look at this carefully this is again nothing but the phi; but the 4 k plus 1th moment of a Gaussian distribution. This is the whole thing can be converted to a Gaussian distribution by the user process of completing the square and you get this extra term outside J square upon 2 mu, which can be taken outside the integral and whatever remains inside is nothing but the odd moment or the 4 plus 4 k plus 1th moment of a Gaussian distribution which will happen to be 0.

Therefore, an indeed this is 0 for every value of k why? Because for every value of k we get odd for a we get a odd moment. For every value of k we get the odd moment and therefore, for every value of k this integral will vanish.

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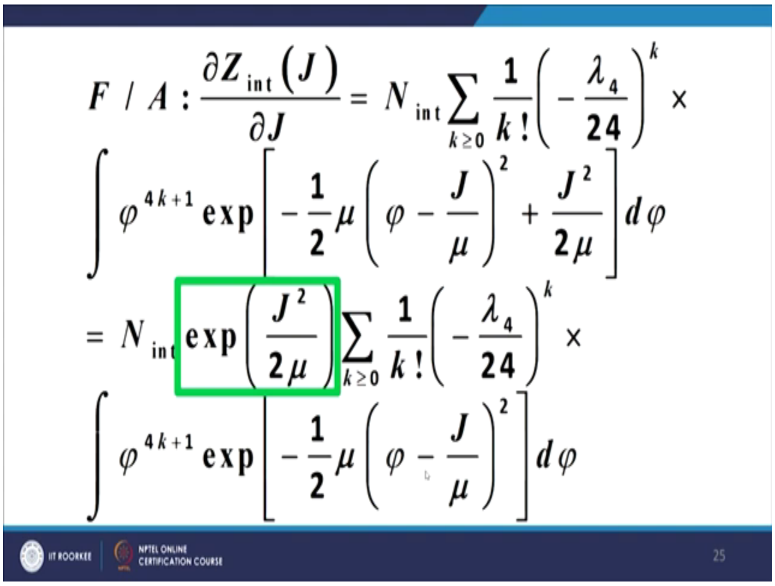

And the net result is that the first derivative is equal to; first derivative is equal to 0.

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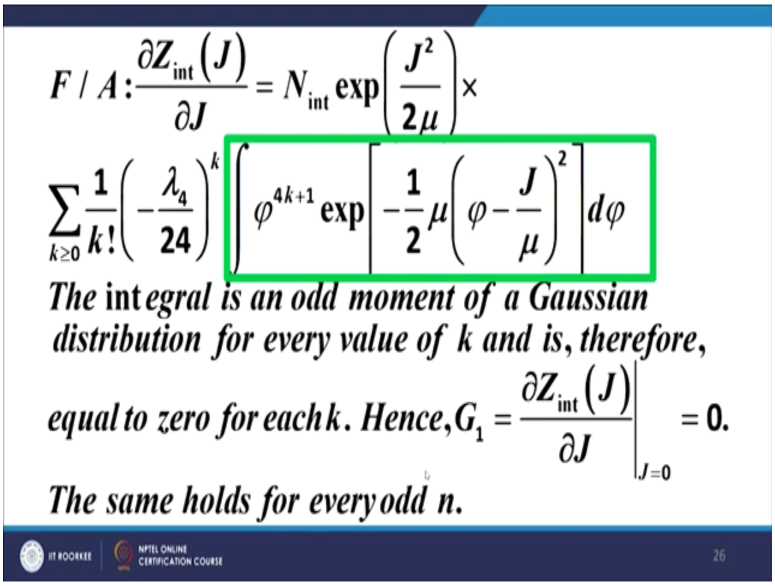

And therefore, on the one-point Green function of this theory turns out to be 0. This is all the working of the theory working of the calculations, how we have worked out or simplified this calculation, as I mentioned we can take this outside the integral because there is no phi involved in this.

You take this term outside this integral inside this integral what we have? It is the phi 4 k plus 1th moment, and for every value of k this happens to be an odd moment. For every integral value of k this happens to be a odd moment. And therefore, for all this all the case, that are covered by the summation here, this integral will carry a 0 value. And therefore, this whole summation will be a summation of 0's resulting in 0, and the whole thing will vanish.

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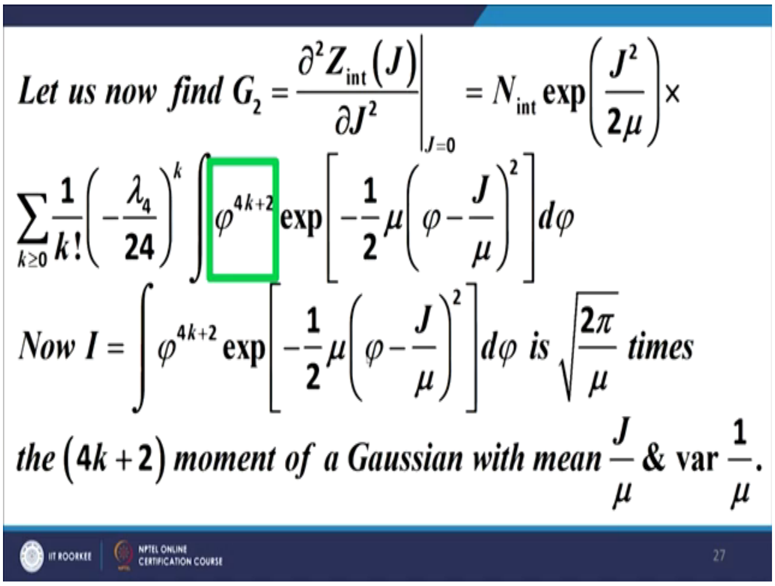

Now, let us look at this second two-point function. Let us look at for a two-point function we need to do a second derivative of Z interaction with respect to J, second derivative. Now, the first derivative we have already done, when we do the second derivative in this the derivative operator will again operate on the J inside the integral and will pull back another factor of phi.

Another factor of phi; that means, the phi inside the integral will have the power of 4 k plus 2. Now, that is even; now that is even therefore, it will not vanish. We make let us call this integral the old term under the integral. Let us call it I, now we have to simplify this.

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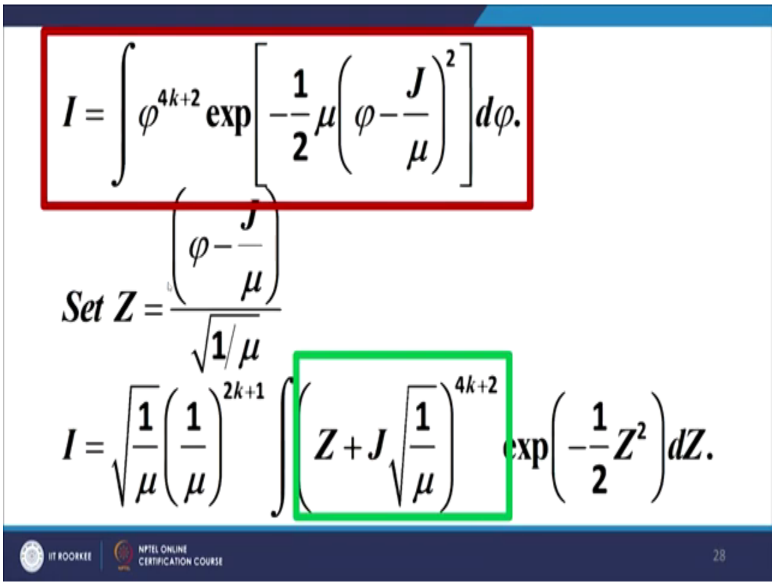

To simplify this, what we do is we substitute Z equal to phi minus J upon mu upon under root 1 upon mu. When you do this substitution, the factor I remember I is only the integral part it is not this entire part; it is not this entire part; it is not the entire value of G 2, it is only the value of the integral.

So, I you make this substitution Z equal to phi minus J upon mu upon under root 1 upon mu and we get this expression and this is non translation, the exponential term within the integral translation minus 1 by 2 Z square d Z. Please note this is the standard normal distribution PDF. Of course, subject to normalization.

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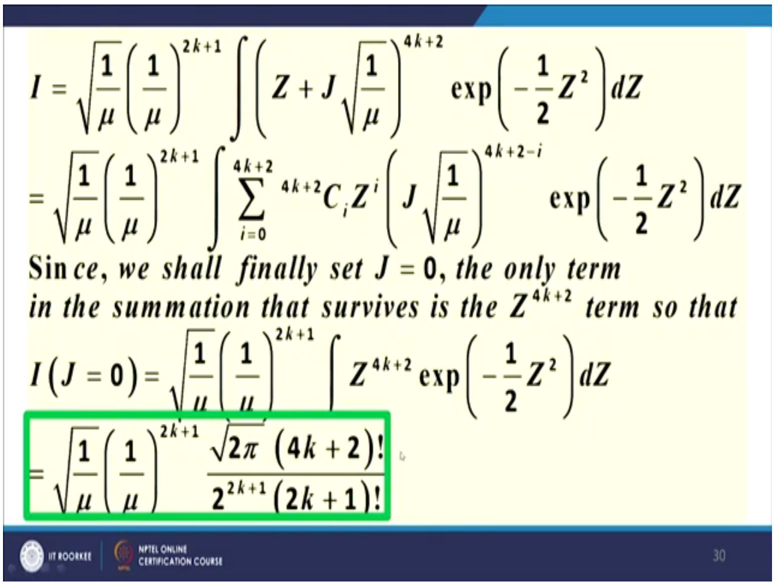

Yeah. And let us look at the expression in the green box. This is interesting; this is interesting when we look at the expression in the green box when you expand it as a binomial series or a binomial series what we get is the expression summation 4 k plus 2 C i this is nCr combination, Z to the power i J to the power this whole expression.

Now, and this is to be this is summation from i equal to 0 to 4 k plus 2. The important thing is that every term in this summation will contain a factor of G. In this summation every term will contain a factor of G except, the one factor one term where we have Z is equal where z has the power 4 k plus 2. Z has the power 4 k plus 2 all other terms will have some factor of J and, some of Z; Z some power and J some power with the other values of under root 1 upon mu of course.

But, the point is when I put J equal to 0 here, because ultimately my objective is to put J equal to 0 in this expression; when I put J equal to 0, all the terms will vanish, except that one term which has Z to the power 4 k plus 2 J to the power 0 which will be 1.

In other words, this summation will give me just the one term, which is 4 k to the power 4 k plus 2 C 4 k plus 2; that is equal to 1 Z to the power 4 k plus 2. So, the net result of this whole summation when I substitute J equal to 0 will be this expression, Z to the power 4k plus 2 e minus 1 by 2 Z square. This is the 4 k plus 2th moment of a normal distribution, with a normalization of course. And this on simplification taking care of the various normalization, I get the expression in the green box.

To repeat this is the 4 k plus 2th moment of a standard normal distribution. Standard normal distribution which is given by 4 k plus 2 factorial 2 to the power 2 k plus 1 2 k plus 1 factorial of course, with normalization. And when all those normalizations are taken care of, I get the expression in the green box.

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$$\textit{Hence},\; G_2 = \left.\frac{\partial^2 Z_{\text{int}}(J)}{\partial J^2}\right|_{J=0}$$

$$= N_{\text{int}} \exp\left(\frac{J^2}{2\mu}\right) \sum_{k\geq 0} \frac{1}{k!}\left(-\frac{\lambda_4}{24}\right)^k I(J=0)\Bigg|_{J=0}$$

$$= N_{\text{int}} \sqrt{\left(\frac{2\pi}{\mu}\right)} \sum_{k\geq 0} \frac{1}{k!}\left(-\frac{\lambda_4}{24}\right)^k \left(\frac{1}{\mu}\right)^{2k+1} \frac{(4k+2)!}{2^{2k+1}(2k+1)!}$$

$$= N_{\text{int}} \sqrt{\left(\frac{2\pi}{\mu}\right)} \frac{1}{\mu} \sum_{k\geq 0} \frac{1}{k!}\left(-\frac{\lambda_4}{24\mu^2}\right)^k \frac{(4k+2)!}{2^{2k+1}(2k+1)!}$$

Putting this value of I in the original expression; putting this value of I in the original expression what I get is the expression in the green box. So, this is the expression; this is the expression for G 2, the general expression for G 2, this is the what I get for G 2.

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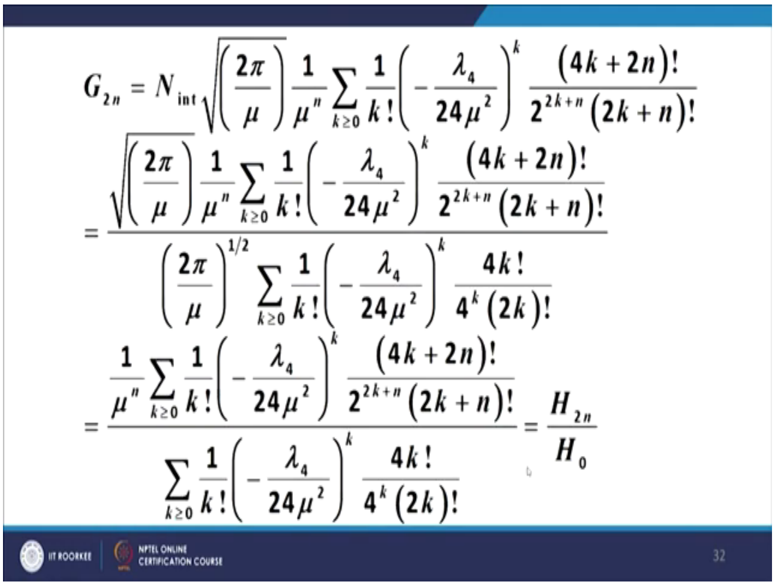

And working similarly, you can get an expression for G to the power G to n the by this is G 2 and working similarly, we can get an expression for G 2 n and putting the normalisation back also we can write G 2 n as H 2 n upon H 0, where H 0 is in a sense of factor connected with the normalization with the cancellation of that is 2 pi upon mu factor which is common to both the numerator and the denominator. The rest of the term in the denominator is H 0; the rest of the term in the numerator is H 2n.

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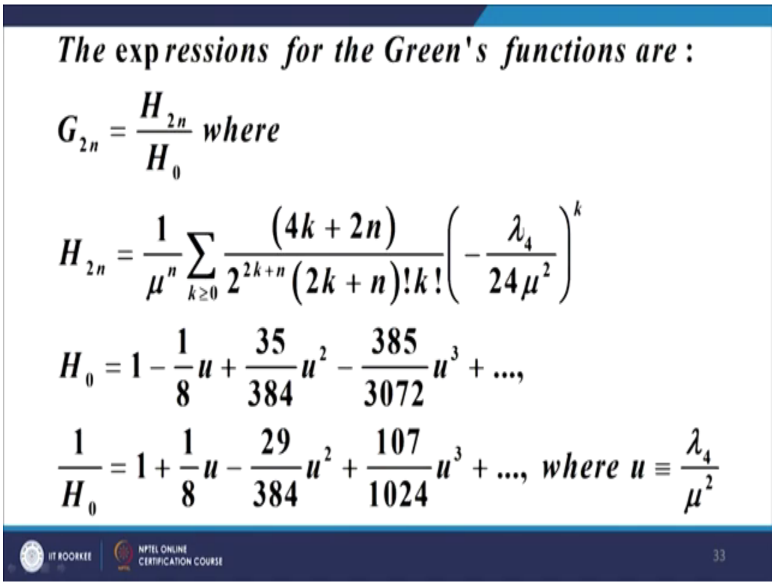

And the expression for the Green functions, where G 2 n is equal to H 2n upon H 0; are is obtained by the this expression and when you expand them; when you expand them, you get these expression for H 0 and 1 upon H 0 these are the normalization issues, where u is equal to 1 upon mu squared.

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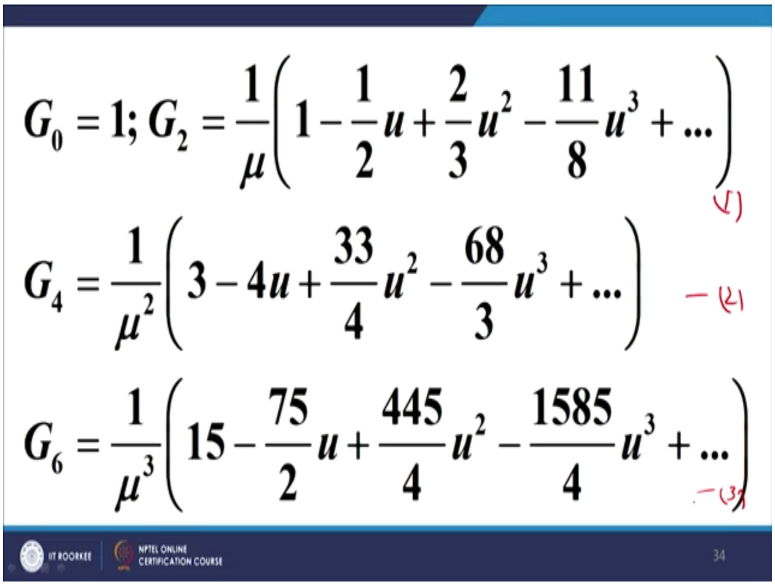

Therefore, this gives us G 0 is equal to 1; G 2 is equal to this whole expression, equation number 1 I say, G 4 is equal to equation number 2, G 6 is equation number 3 and so on.

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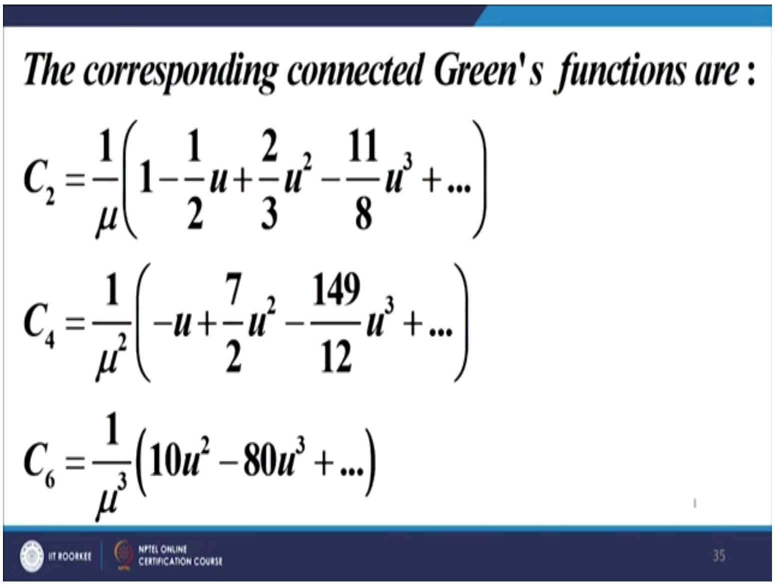

And the connected Green functions can be evaluated just in the way we have done earlier, and we get these expressions for the connected Green functions C 2, C 4 and C 6. This can be obtained directly from the generating functional of the connected Green functions which is nothing but the logarithm of the generating functional for the original Green function.

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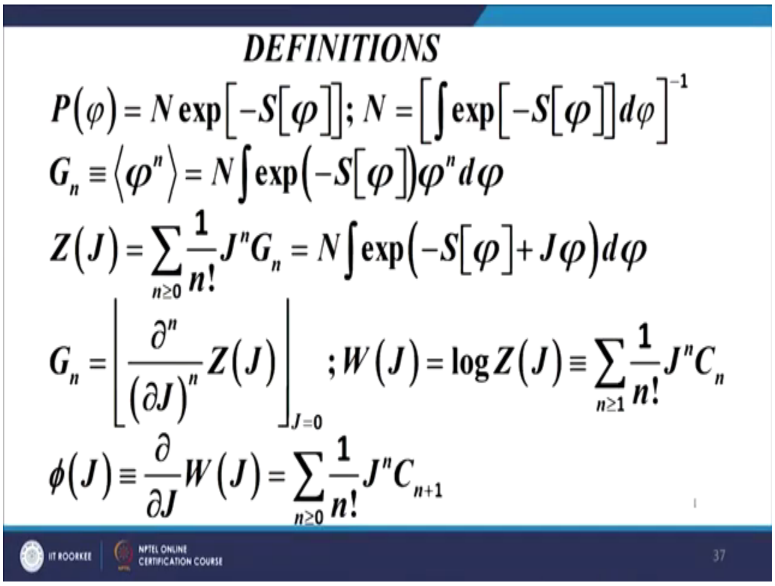

So, let us summarize what we have done quickly. The free theory is represent first of all this slide gives us the basic terminology in a nutshell. The probability is given by N exponential minus S phi, where S is the action normalization is done in this way, in the equation shown on the top right hand; top right hand of your slide, the end point Green functions are defined as the moments of the distribution.

Z J the generating functional of the Green function is given by introducing a source J and then G n is obtained by taking derivatives with respect to J of Z J and then putting; and then putting J equal to 0; the generating functional for the connected Green function again by log of Z J, this expression and the field function is given by the derivative of the generating functional of the connected Green functions.

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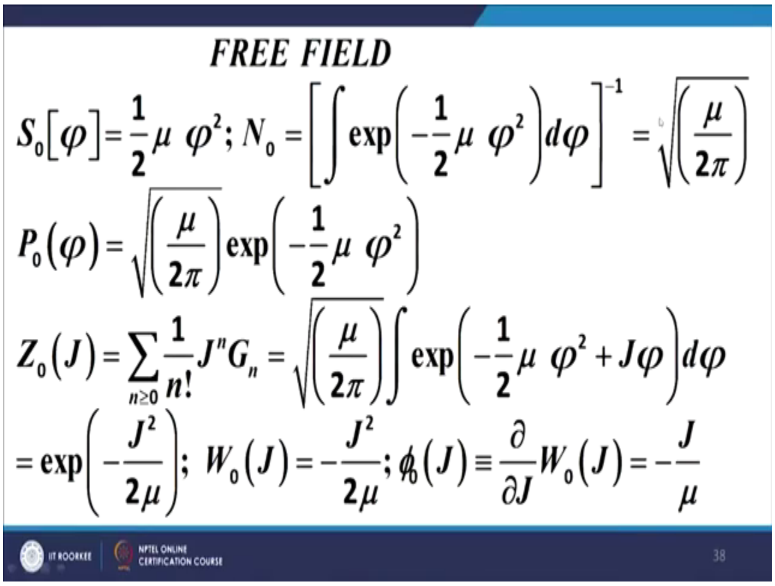

So, this was the terminology for the free field we have a very simple Gaussian action S phi is equal to 1 by 2 mu phi square and the rest of the terms have been evaluated. In this case of course, we can get closed form solutions of all the relevant terms, of all the relevant parameters its quite straightforward simply Gaussian integration.

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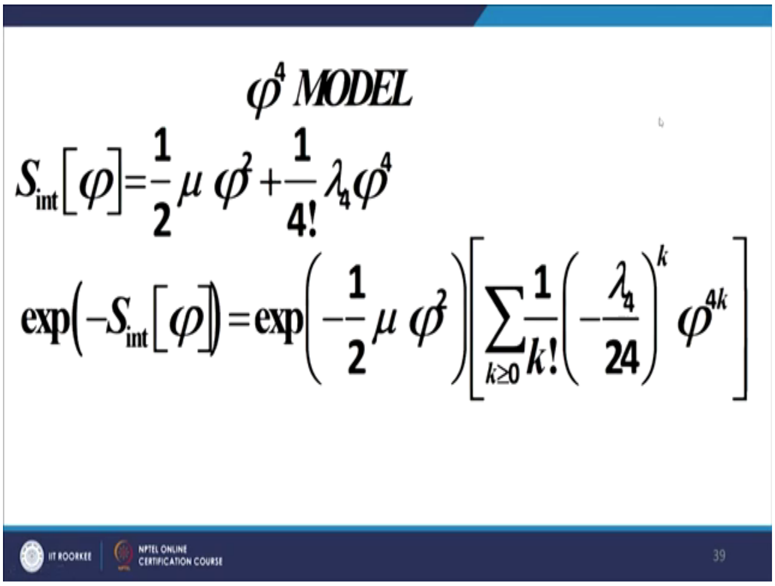

In the phi 4 model we have little bit of we introduced perturbation, the concept of perturbation the procedure of perturbation and making use of the perturbation we work out.

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$$N_{int} = \left[\int \exp\left(-S_{int}[\varphi]\right) d\varphi \right]^{-1}$$

$$= \left[\int \sum_{k\geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24} \right)^k \varphi^{4k} \exp\left[-\left(\frac{1}{2}\mu \, \varphi^2 \right) \right] d\varphi \right]^{-1}$$

$$= \left\{ \sum_{k\geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24} \right)^k \int \varphi^{4k} \exp\left[-\left(\frac{1}{2}\mu \, \varphi^2 \right) \right] d\varphi \right\}^{-1}$$

$$= \left[\left(\frac{2}{\mu} \right)^{1/2} \sum_{k\geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{6\mu^2} \right)^k \Gamma\left(2k + \frac{1}{2} \right) \right]^{-1}$$

$$= \left[\left(\frac{2\pi}{\mu} \right)^{1/2} \sum_{k\geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24\mu^2} \right)^k \frac{4k!}{4^k (2k)!} \right]^{-1}$$

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The various parameters of interest we work out the normalization for the interaction theory, we work out the generating functional for the interaction theory.

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$$Z_{\text{int}}(J)=N_{\text{int}}\int\sum_{k\geq 0}\frac{1}{k!}\left(-\frac{\lambda_4}{24}\right)^k\varphi^{4k}\times\exp\left[-\left(\frac{1}{2}\mu\,\varphi^2\right)+J\varphi\right]d\varphi$$

$$=N_{\text{int}}\sum_{k\geq 0}\frac{1}{k!}\left(-\frac{\lambda_4}{24}\right)^k\int\varphi^{4k}\exp\left[-\left(\frac{1}{2}\mu\,\varphi^2\right)+J\varphi\right]d\varphi$$

$$G_{2n}=N_{\text{int}}\sqrt{\left(\frac{2\pi}{\mu}\right)}\frac{1}{\mu^n}\sum_{k\geq 0}\frac{1}{k!}\left(-\frac{\lambda_4}{24\mu^2}\right)^k\frac{(4k+2n)!}{2^{2k+n}(2k+n)!}$$

And we work out the using the generating functions for the interaction theory we get the Green functions for the end point we worked out the two-point Green function explicitly, we have the expression for the endpoint Green functions of the interaction theory.

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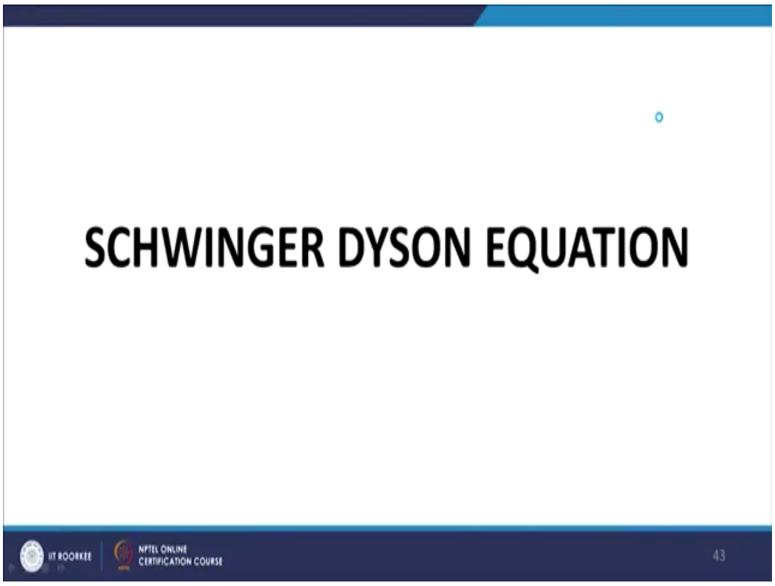

Then, we have expressions for the connected Green functions. Now, that is basically the main composition or the main part or the explicit part of the zero-dimensional field theory or a field theory on a underlying space of zero dimensions.

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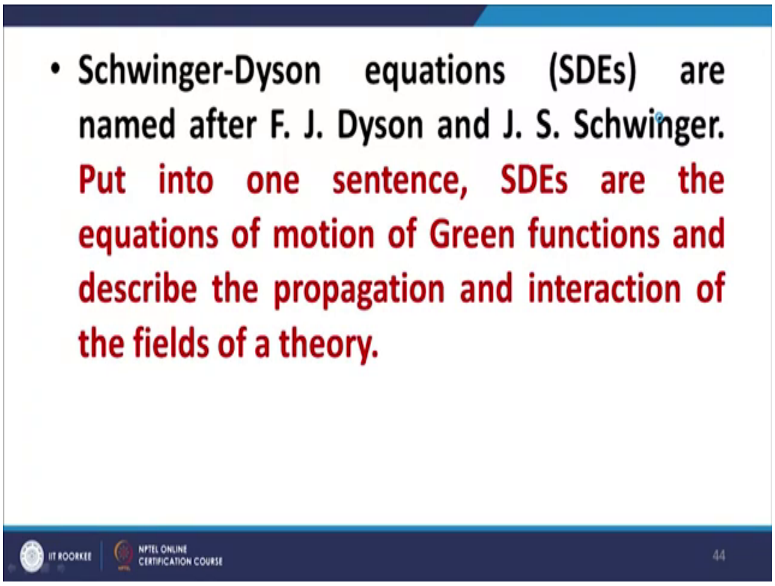

Then, there is an approach or there is a formalism or extension of this formalism which is called the Schwinger-Dyson Equations. The Schwinger-Dyson Equation gives us the equations of motion, the dynamics of the Green functions and that is describe the propagation and interaction of the fields in the theory.

So, they provide a very attractive formalism of how we are here, how the field interactions develop and the field propagates in time.

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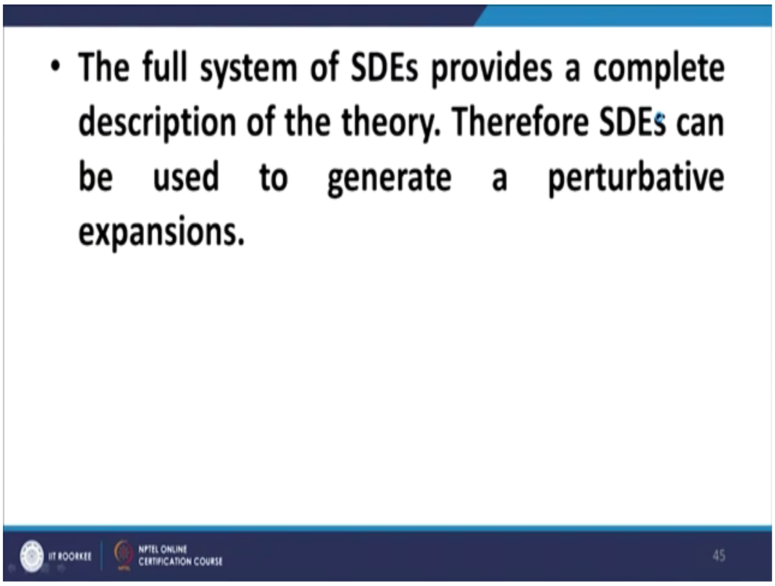

This is the this system of equations are fundamental to any field theory although, they are they are somewhat complicated, but they are fundamental to the theory the one of the reasons in fact, of selection of this zero-dimensional field theory is to enable us to understand the utility of these kind of approaches. So, in the next lecture I propose to take up the Schwinger-Dyson equations.

Thank you.