Path Integral Methods in Physics & Finance
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Lecture – 31
Field Theory Basics

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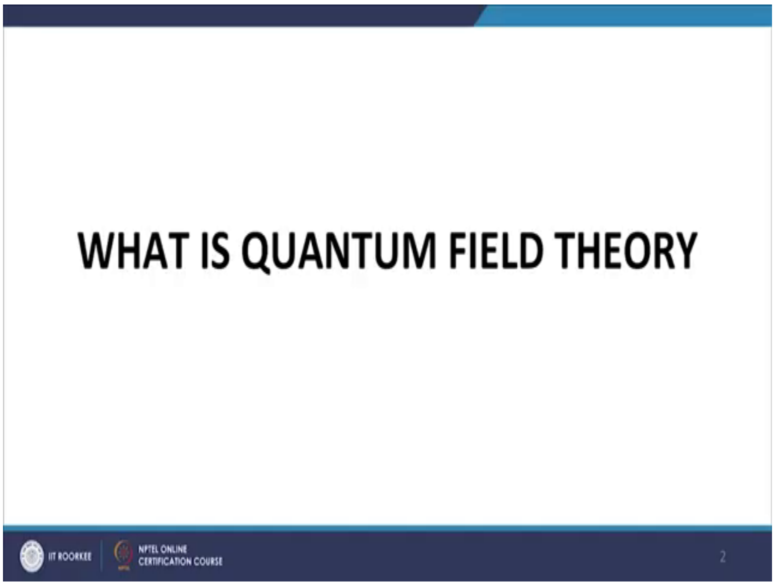

Welcome back. So, in the last lecture I gave a summary of the various short comings that arise when we look at the particle based interpretation of quantum mechanics or interpretation based on these space time as it is. Now, this paved the way for the introduction of additional structure over the space time framework; in the form of a quantum field. I also introduced the concept of the field.

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Basically, the field is a continuous is an allocation or an assignment of values to every point in space time. Represent continued field functions are continuous functions which allocate or assign values to every point in space time. Now, these values may be take the form of a scalar, they may take the form of a vector or even a tensor.

And correspondingly we have a scalar field like temperature a vector field or a tensor field as well, so that is the important thing. In the case of a field we have a two layered structure we have an underlying space time. And over that we have a field which is which allocates a physical quantity to each point on the space time.

And then on that we construct our desired framework comprising of either the Lagrangian or the Hamiltonian as the case may be. Now therefore, the Lagrangian and Hamiltonian which were earlier functions of coordinates and their derivatives; now get elevated and they become functionals of the field variables and their derivatives.

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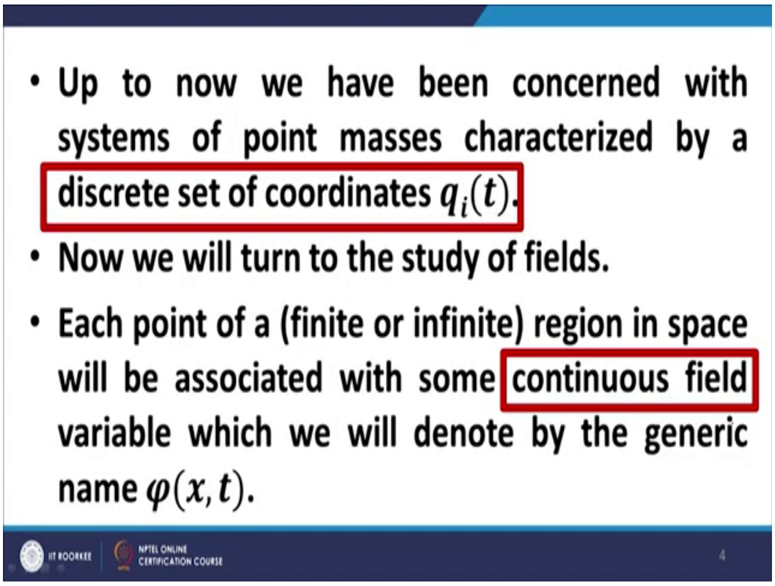

The important point is we were talking about so far we have talked about point masses and those point masses were characterized by discrete set of coordinates. Field functions on the other hand are continuous functions which allocate or assign values to each port in this coordinate space or underlying space or space time.

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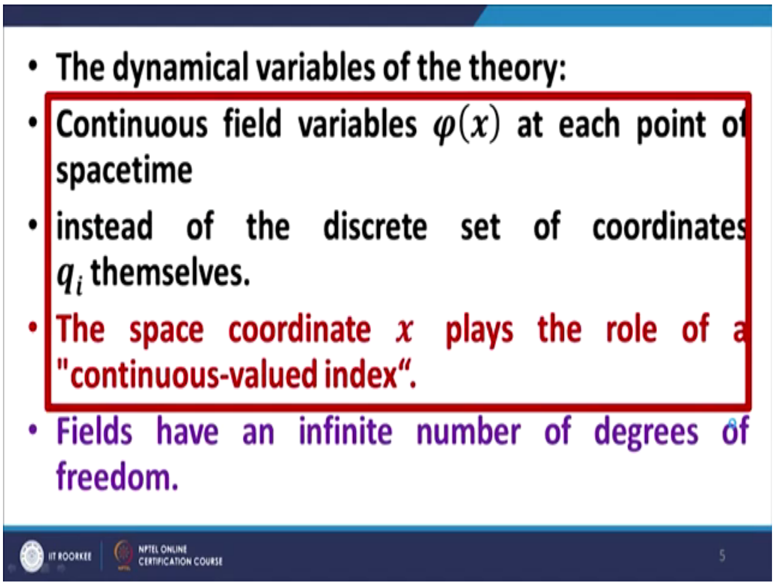

Now, the important implications are that the continuous field variables at each point of exist at each point of space time the allocation is there at each part of space time. And instead of the discrete set of coordinates we now have these the field variables representing the or taking about the role of the continuous the physical entities.

The dynamical variables the dynamical variables now are represented by the field variables rather than the coordinates and then and derivatives. And the coordinates of course, simply serve as the index set, it is a serve to index those field variables.

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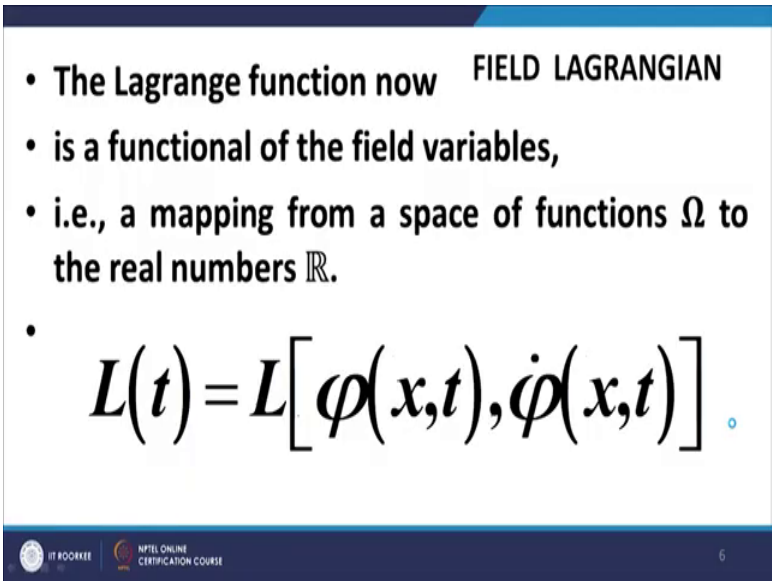

As I mentioned just now the Lagrangian function now becomes our functional of the field variables and it is derivatives. That it is a mapping from the set of functions to the real space or the real line or the set of real numbers.

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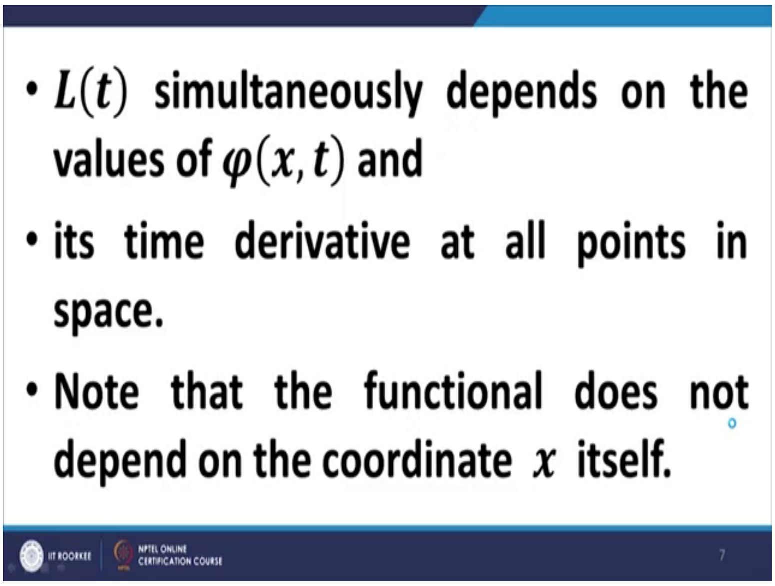

The and the Lagrangian simultaneously depends on the field variable. And the field variables phi x of t phi is itself a function of the space time variables x and t which are indexing phi; please note this which are indexing phi.

And it is time derivative of course, is also there in the Lagrangian also has a functional dependence of the time derivative of phi x t. But phi at the Lagrangian does not explicitly depend on the space time variables x and t.

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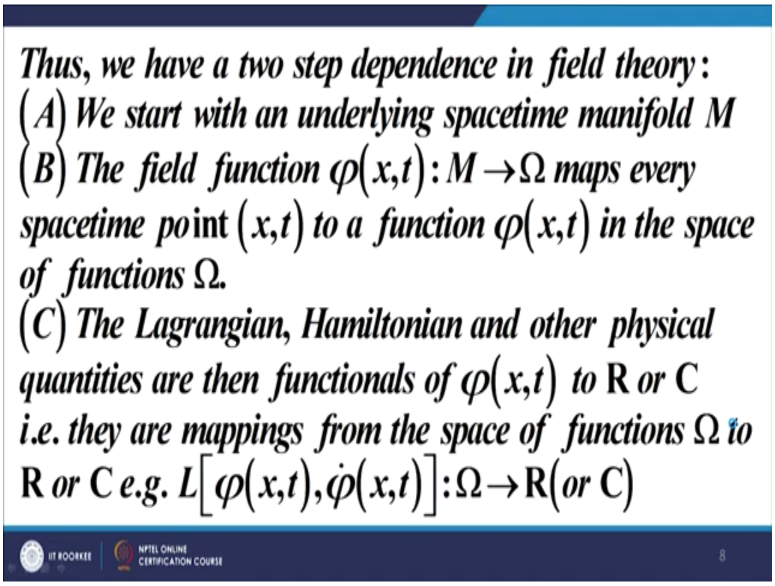

So, we have this two layer structure; we have an underlying space time manifold M. We have the field function that allocates physical quantities to every point on the space time manifold. That is the mapping from the set M to this set of functions omega. And then we have the Lagrangian or the Hamiltonian as the case may be which is a mapping from omega to the set of real numbers.

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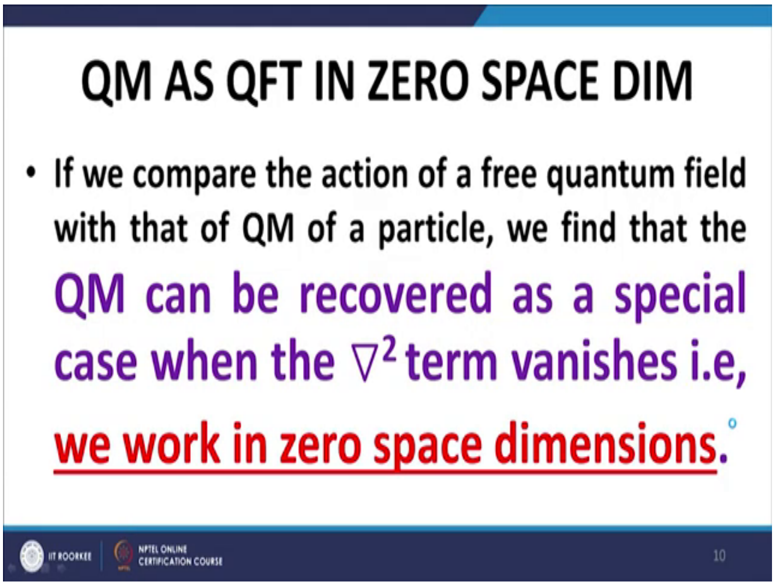

Now, comes the issue of quantization. Now, to understand quantization in this context in the current context we note that we can recover the harmonic oscillator Lagrangian harmonic oscillator action in fact, if we ignore or if we remove the del square term from the field action.

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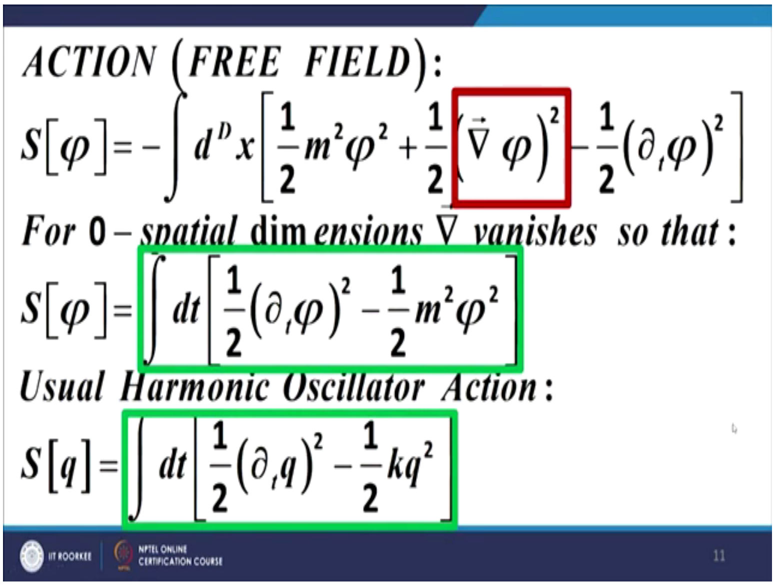

To make it more explicit you look you can look at the slide here, the action of the free field is given in the first equation. And, if you remove the expression of the this expression in the red box you get the expression which is there in the green box. If you by removing the expression on the in the red box.

And if you compare this with the harmonic oscillator there is a literally one to one correspondence between the field variable and the dynamical variable of the Hamilton of the harmonic oscillator. And therefore, because when we quantize the harmonic oscillator we are upgrading these dynamical variables q and p to the status of an operator.

The rational logic is that we upgrade the field variables or quantize with respect to the quantized the system with respect to the field variables now. And that is precisely what is done in canonical quantization of the quantum fields.

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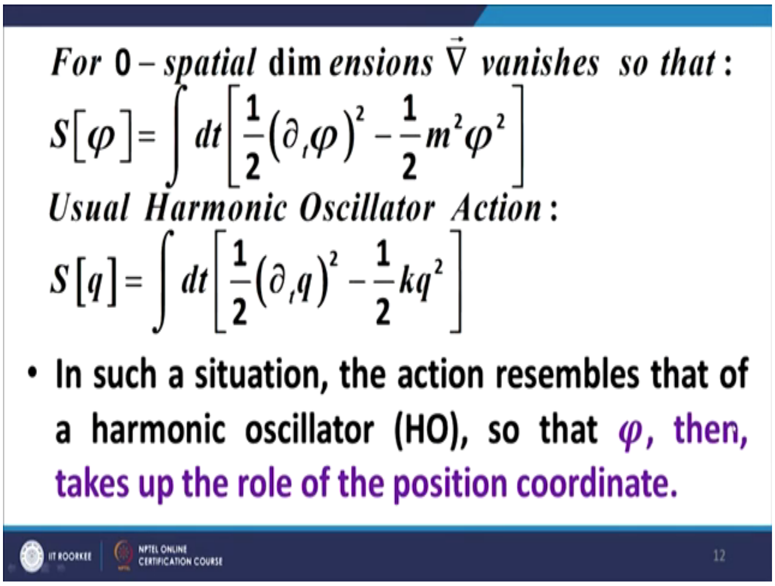

Now, if you have as I mentioned in the 0 spatial dimensions we recover the harmonic oscillator action by removing that term. Because if you have 0 spatial dimensions the derivatives with respect to space will and disappear. And we shall only have derivative with respect to time and then we have this correspondence with the harmonic oscillator.

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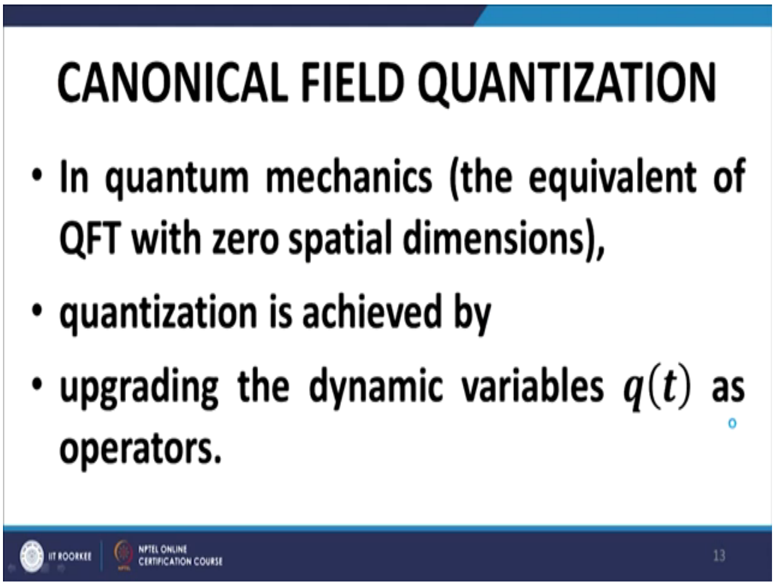

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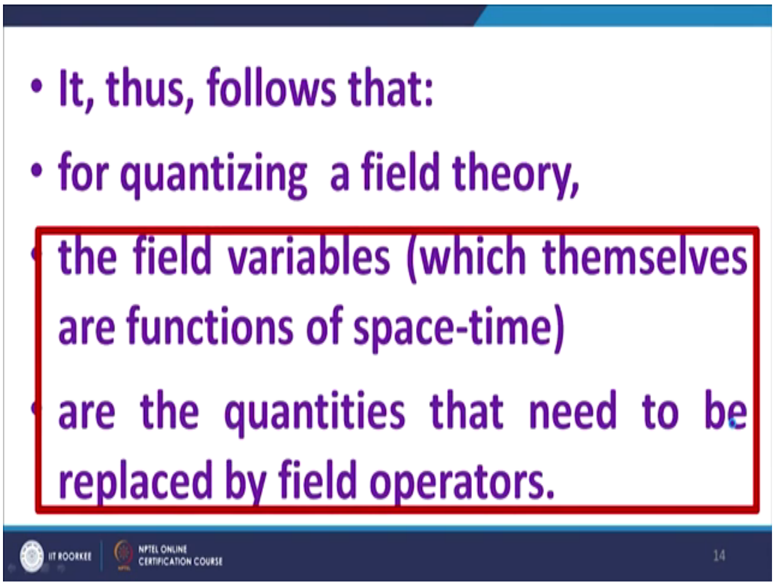

So, that is the relevant part about field quantization. The variables that are to be quantized are not the underlying space time variables, but the variables that form the field variables that is phi and it is a conjugate derivative right. So, then there is another issue here.

The other issue is that because we are going to merge special relativity with quantum mechanics. And special relativity the fund the cornerstone or special relativity is that the time and space variables are need to be placed on equal footing. So, in the context of quantum field theory we need to give equal relevance or equal positioning to the space and time variables.

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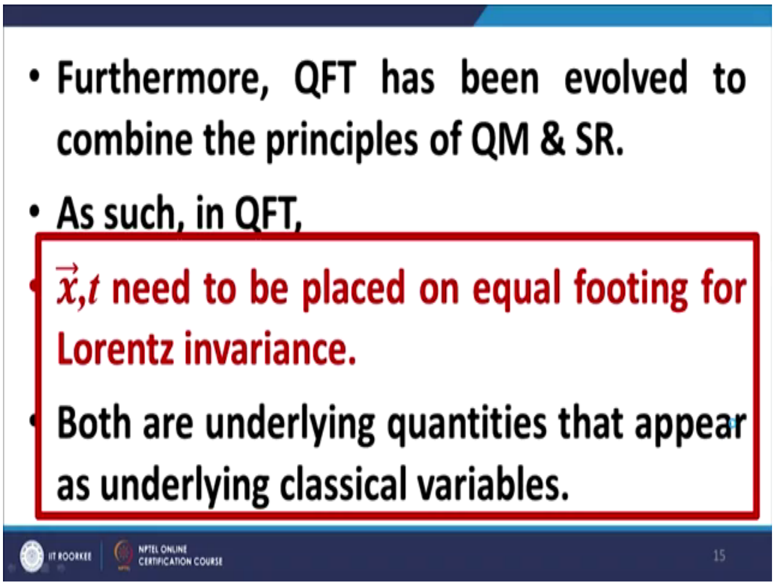

And these are in fact x and t are the underlying quantities that they that represent the classical variables. And the quantization will be done with respect to the field variables which are functions of these underlying where classical variables x and t. Then I started talking about the; zero dimensional field theory.

The logic of starting with the; zero dimensional field theory is to acquaint the viewer with the nuances with the mythology; the formalism in it is simplest form. And then carried over to the more complex; form the scalar field, the direct field and the gauge fields.

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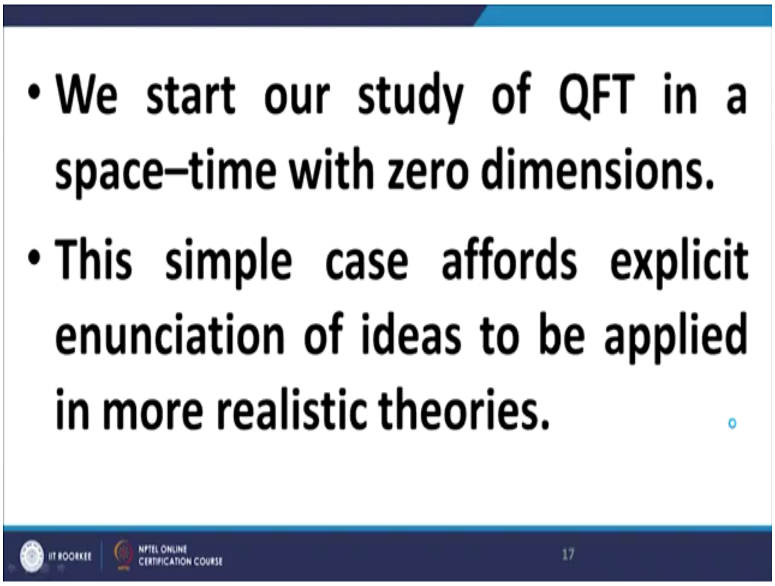

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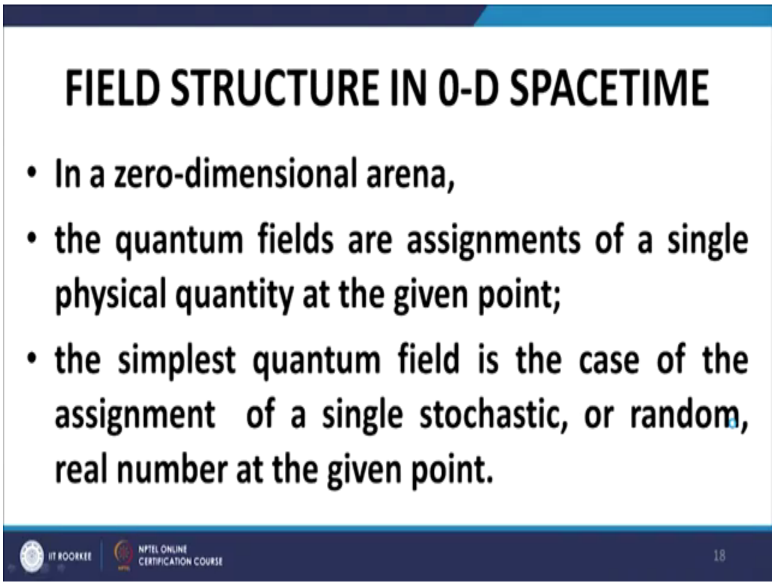

So, in a field structure in zero dimensional space time and that is 0 plus zero dimension; zero dimensional space zero dimensional time. The quantum fields are nothing, but assignments of a single number single physical quantity to a point to a given point. And the simplest form is when the simple the physical quantity being assigned to the point zero dimensional point is and nothing, but the real number. And that is the case that we are going to investigate to start with.

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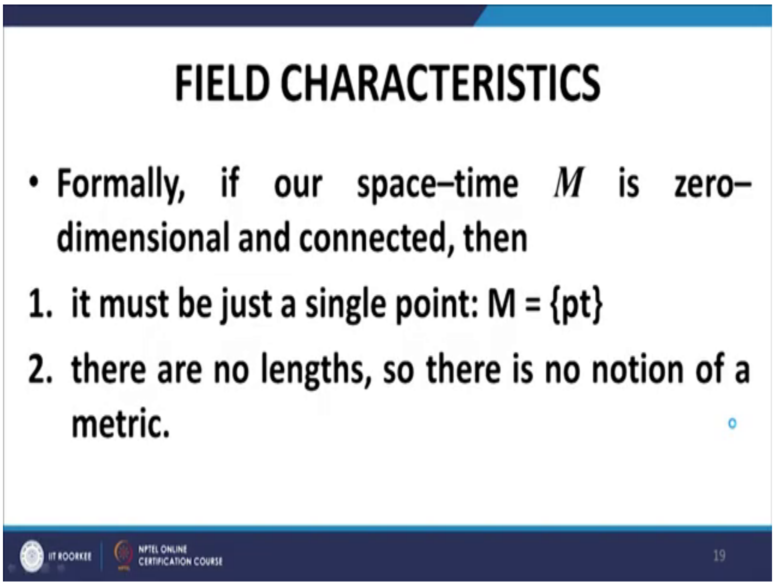

So, formally if we have our space time M and if it is compact and if it is zero dimensional it is nothing, but a point. So, we represent it in the form given here. Furthermore because we our space time is only a point there are no lengths and there are and there is no metric. And because there is no matrix there will be no derivative terms here in the action or elsewhere.

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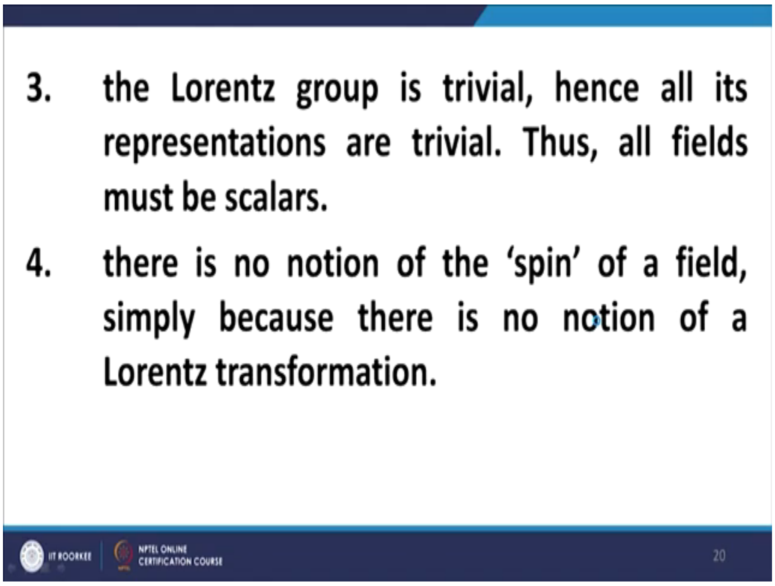

The concept of derivative will not be there because there is no metric as such. So, the issue of derivatives when does not arise. The Lorentz group is trivial and it is representations are also trivial.

And therefore, all fields have to be scalar fields that is an immediate consequence of the trivial representations of the Lorentz group. There is notion of spin in the field; because there is no notion, no concept of Lorentz transformations in this scenario zero dimensional field.

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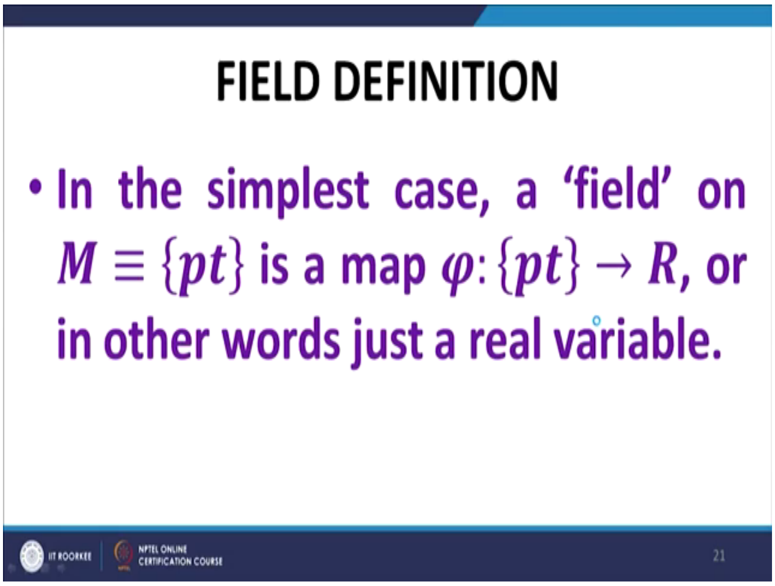

And the field is therefore, defined as a mapping keeping in M in all have what whatever I have said just now. The field may be considered as a mapping from a point to the set of real numbers; mapping phi from the point and to the set of real numbers. Recall that a zero dimensional space underlying squares if it is compact is nothing, but a point.

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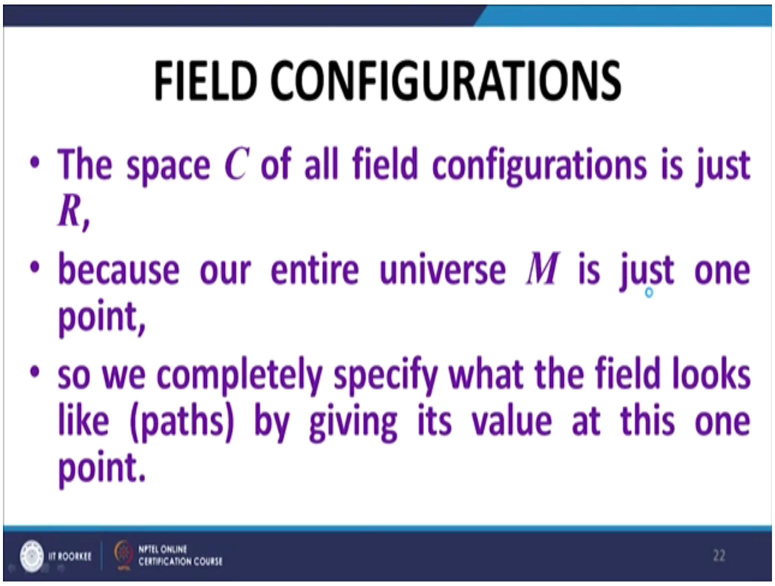

Now, the space of C of configurations is also just the space of real numbers; why is that? You see how will we represent a path in this space? We will represent a path in this space simply by representing a value at the given point because the path is nothing, but the value of the field at that given point. And that value can take any value on the real line.

And therefore, the set of all paths of the or the configuration space C of all compute field configurations is nothing, but the field of real numbers or the set of real numbers. To repeat because we have just the one point in the manifold, so we can describe the field by describing it on that point by describing it is value on that point.

And that value can take any as value on the set of real numbers. Therefore the all the paths any particular path is simply a value at that point at that given point which can be any real number. And therefore, the space of all the paths is simply the real line or the set of real numbers.

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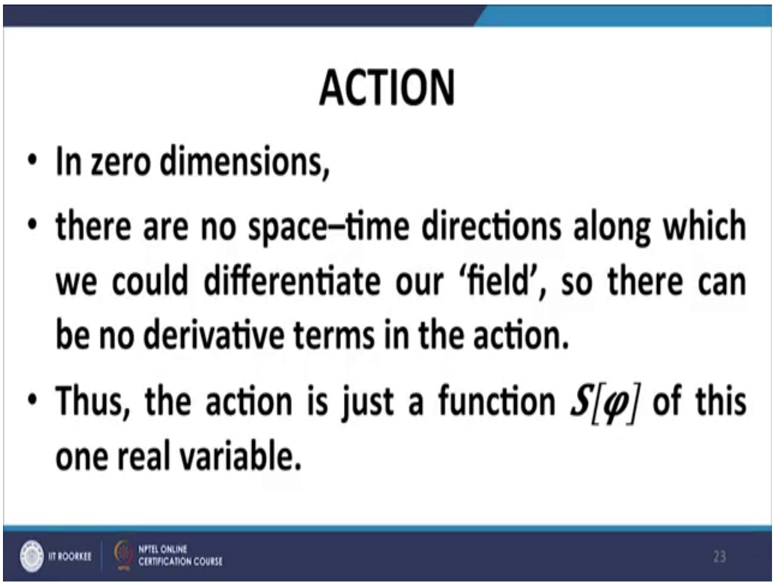

Now, we come to action. In the context of action in zero dimensions there will be no space time directions as I mentioned earlier. And therefore, there will be no derivatives no differentiation. And in fact, there is no matrix so there is no question of any differentiation no derivative. Terms would be present in the action and therefore, the action will just be a function of the field variables and no derivative of the field variables will appear in the action.

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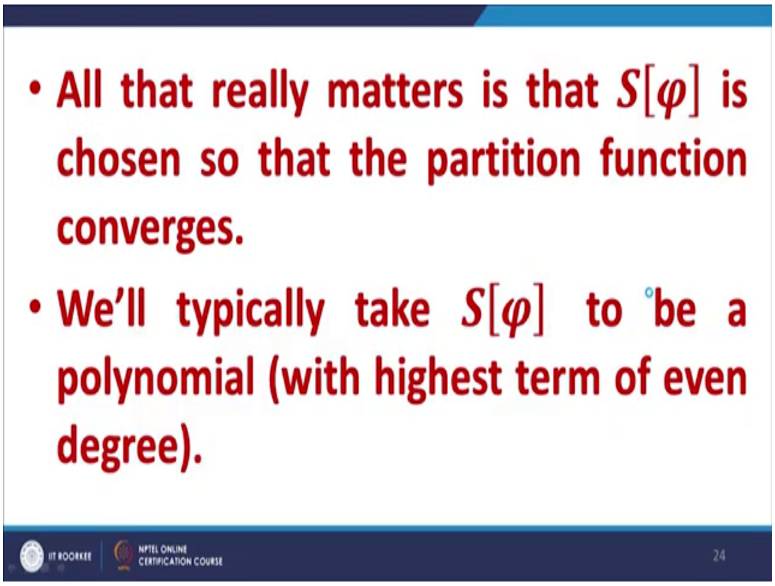

And; however, it is important that the action is so chosen that the partition function a function in that we will be talking about converges. And we shall take the action to be a polynomial with the highest term of even degree. Why this has to be an even degree? We will also discuss that.

So, the for the moment the important thing is that the action will be so chosen that the partition function converges. And number two the action shall be a polynomial in even degrees because otherwise the issue of convergence becomes non trivial. So, we shall come back to it.

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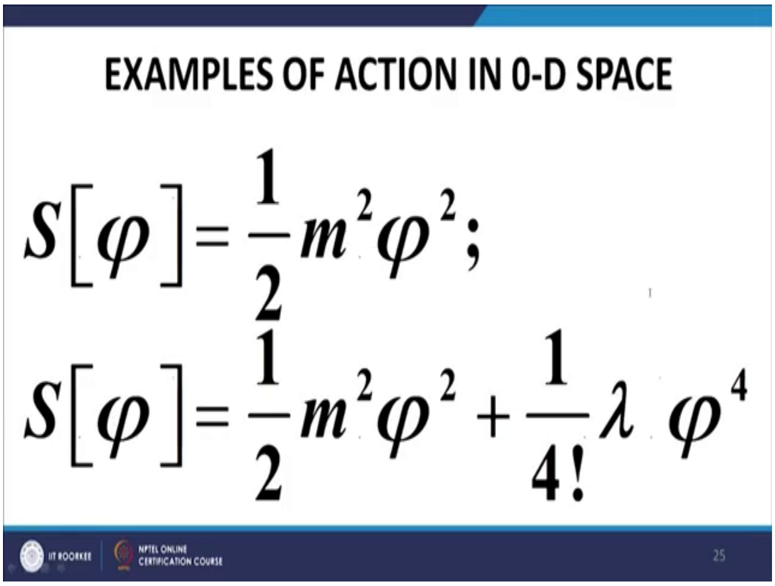

So, the examples of action in zero dimensional space; this is the pre action. The first is the example of the pre action and the second is the example of an action with an interaction term with the coupling constant represented by lambda. So, we shall be talking about them.

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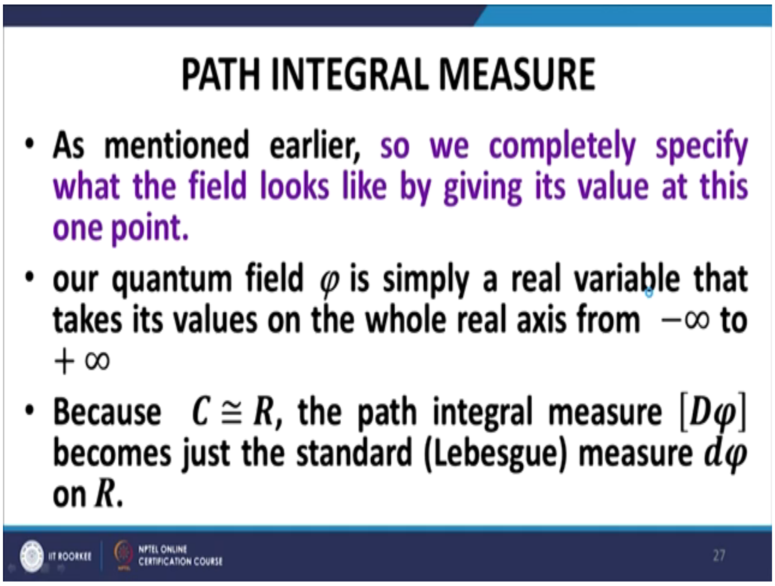

And the path integral measure as I mentioned; as I mentioned the paths are simply the values of the fields at the point under reference and one because therefore, and that value is a real number. And therefore, this entire set of path is simply the set of real numbers.

So, the when we talk about the path integral measure it simply becomes the conventional integral measure that we are accustomed to a standard calculus. That is what we shall be adopting here.

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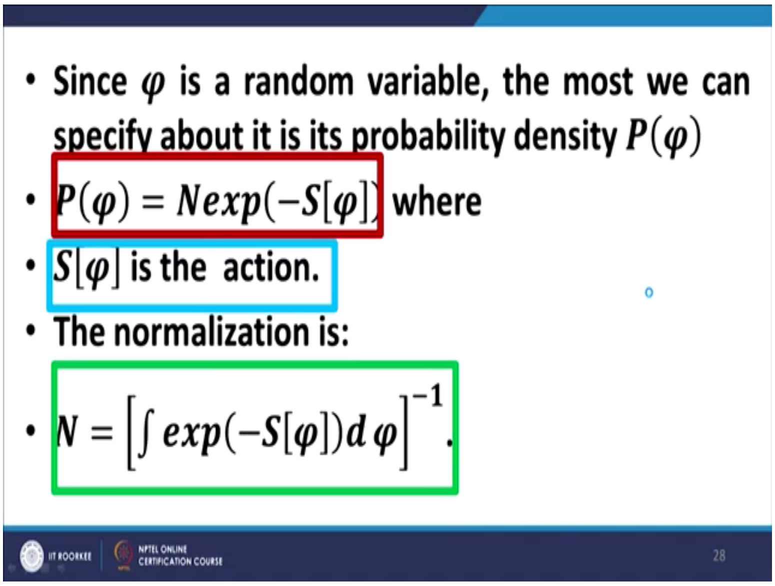

Now, the important thing is that the field is being represented or the field as we have; as we have developed the field formalism as we have developed so far in the context of zero dimensional field. Allocates or assigns a random number; random real number to a point in which point represents the manifold zero dimensional manifold.

Now, because we are talking about a random number the most we can know about the random number are it is probabilities. And, probability density function and the parameters associated with it is probability density functions. So, we define the probability density function of the field variable phi as this expression N exponential minus S phi where, N is a normalization constant and S phi is the action. The normalization is given by the expression in the green box.

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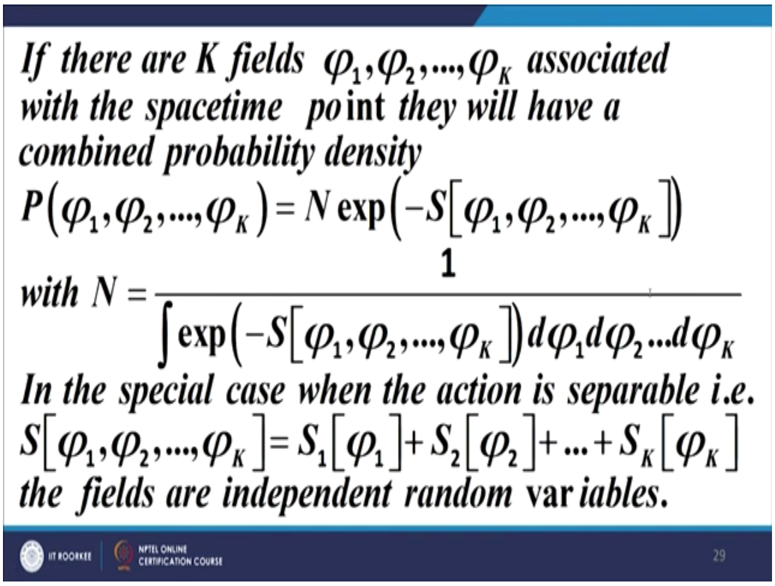

Of course then it is not necessarily that we necessary that we should have only one field at the point under reference. We can have a situation where we are more than one fields acting on the point under reference. We shall be starting with just the one field and we shall be exploring are investigating the case of just the one field acting on the given point.

But it is certainly not necessary that this is the only case possible then. We can have a situation where we have a number of fields acting on the point under a reference. In that case the probability density function will take the form of this expression and the normalization constant will also take the form of this multiple integral.

And the important thing is that if this action is separable if this action is separable this section S phi 1 phi 2 and phi K is separable then of course, the action simplifies. And therefore, the probability also probability density function also simplifies considerably.

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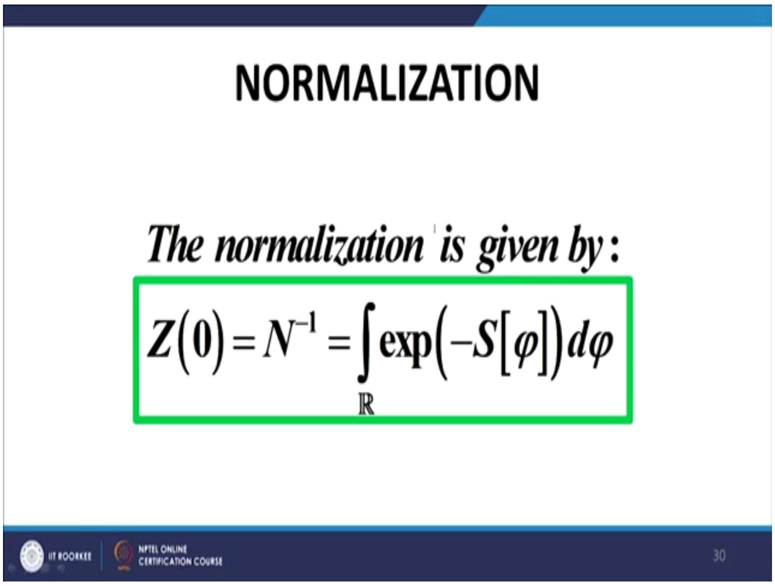

Now, normalization as I mentioned is given by this expression Z 0 this is equal to N inverse. Please note this is N inverse and this is given by this expression integral exponential minus S phi d phi.

As I mentioned please note this that the path integral measure is same as the conventional standard integral measure which is d phi because of the 0 dimensionality of the underlying space.

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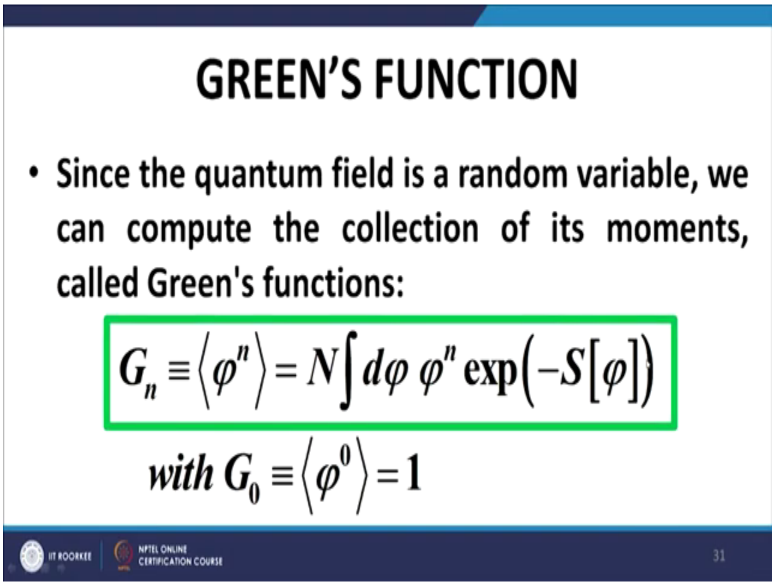

Now, we define the Green functions of this quantum field as the moments of the distribution moments of the probability distribution of phi. So, we define the n point Green function as the nth moment of phi and that is given by the expression that is in the green box.

Obviously, G 0 will be equal to the 0th moment and the 0th moment is nothing, but 1. So, G 0 will always be 1 and the nth moment will be given by this expression. The expectation value of phi to the power n in the given probability distribution, that we have assumed for the field variables.

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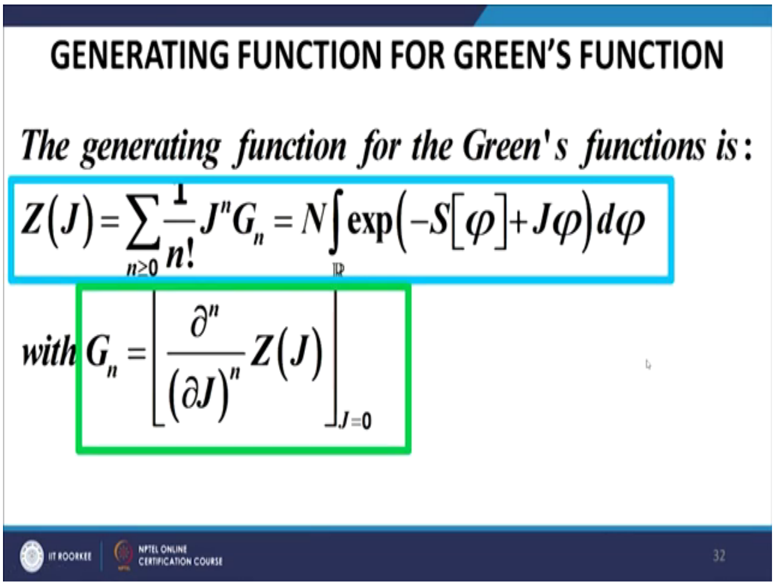

In the generating function; the generating function is exactly the definition is exactly the same that we had and that was discussed earlier. In the context of the statistical description of path integral Z J is given by the expression in the blue box here.

And from this expression we; now please note we have a source term added to this J pi is a source term which is added to the action added to the action in the path integral. And from here we can recover the N point functions N point Green functions by taking the respective.

And derivatives of the generating function and putting J equal to 0 thereafter. First we take the derivatives of Z J with respect to J and then we put J equal to 0 and we recover the corresponding Green functions with the order of the derivative determining the Green function that we have to this to that will recover.

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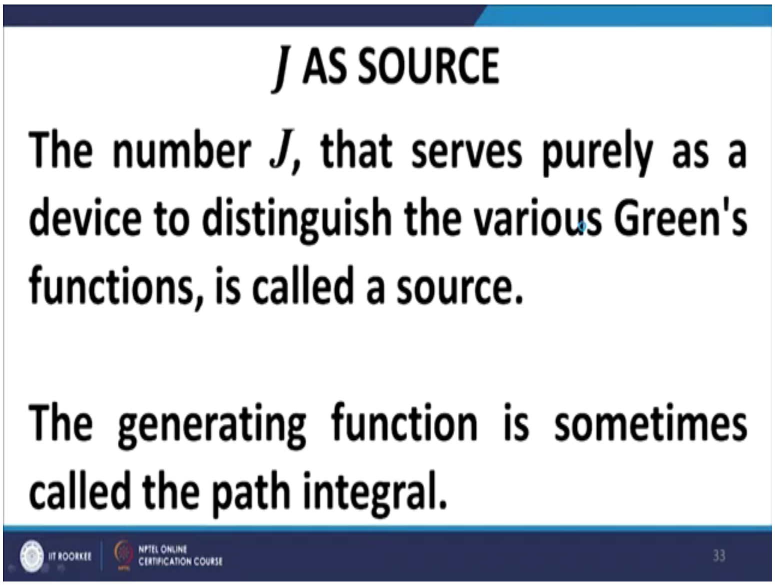

As I mentioned the number J is which is the which is just a mathematical mechanism to recover the Green functions is a source term. And the generating function is occasionally called the path integral.

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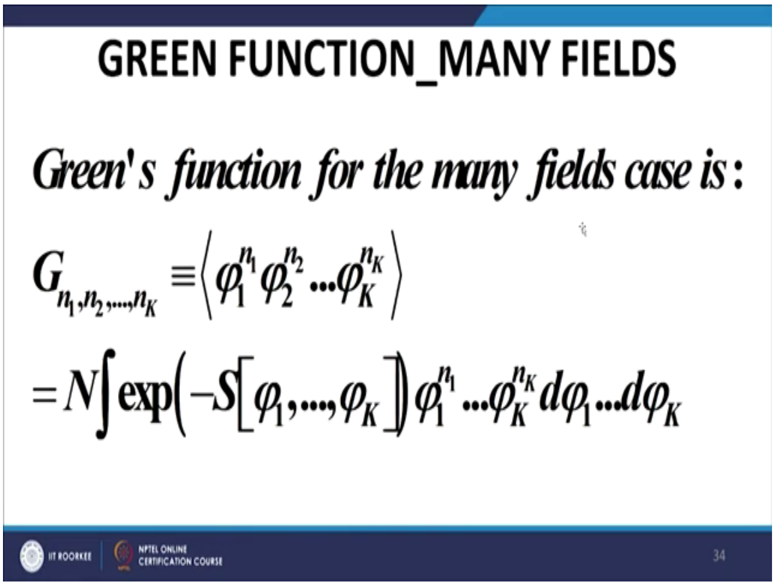

It is and this generating function is also sometimes referred to as the path integral. In the many fields case the expression for the N point function can be generalized in the form that is given in your slide.

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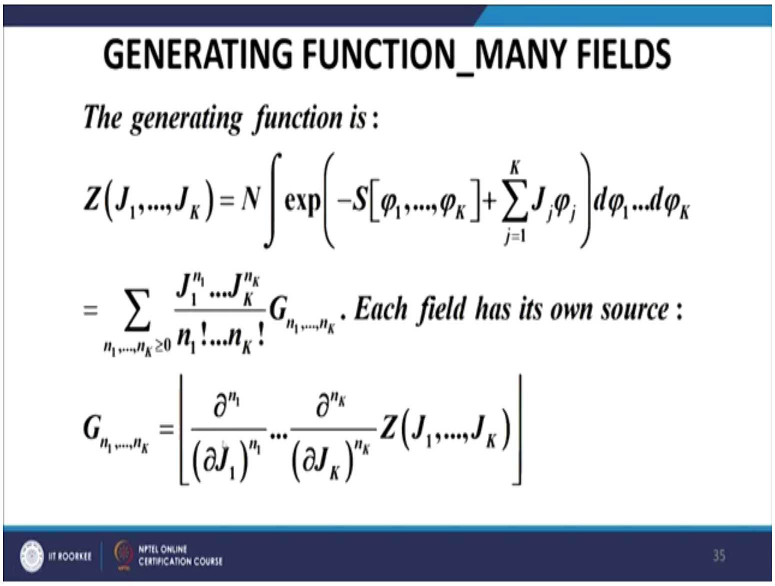

And correspondingly it can be determined the generating function for the many fields case also can be written in this form which is given in your slide. And please note this we have different sources different J as corresponding to different fields. In other words each field carries it is own J; J 1 for phi 1, J 2 for phi 2 and so on. It is interesting to mention here that summation J i phi i can also be written as a scalar product in this expression.

And of course, on the standard process for recovering the various moments or the various Green functions from the generating functional is again the same. You take the derivatives with respect to the with respect to the J as which relate to the phi. So, for which the function the Green function is required to be obtained. The expression is given in the last equation on this slide.

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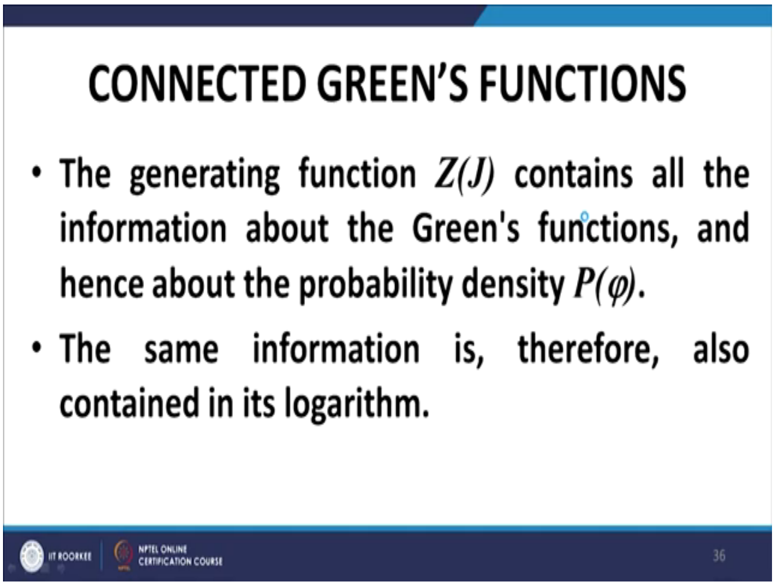

Now, we come to the concept of connected Green functions. Now, the generating function Z J contains all the information about the Green functions. In other words we can recover the Green functions from the generating function as we have seen earlier.

We simply we can simply recover the Green function by taking derivatives first step and then second step setting J equal to 0 after taking the derivatives. So, that being the case we can recover all the Green functions from Z J. And that being the case we can also recover the Green functions from the logarithm of Z J.

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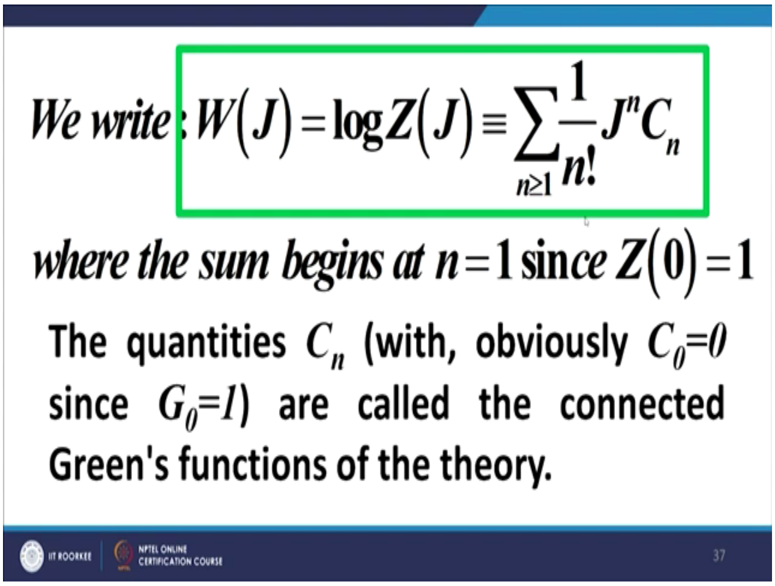

So, the logarithm of Z J is called the generating function for the connected Green functions generating functional or generating function for the connected Green functions is given by log of Z J.

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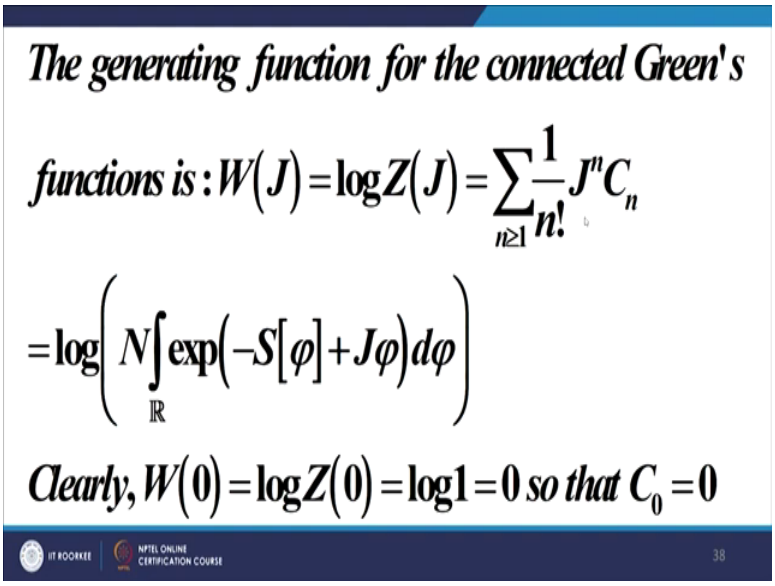

The generating function for the connected Green functions is given by this expression which in terms of the; in terms of the original expressions can be written as log of N integral exponential. In terms of the action it can be written as log N; N is the normalization exponential minus S phi plus J phi d phi.

Because W 0 is equal to 0 therefore, C 0 becomes a 0. And the expression starts from the series starts from N equal to 1.

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$$\text{For } n=1: C_1 = \left[\frac{\partial}{\partial J} W(J)\right]_{J=0} = \left[\frac{\partial}{\partial J} \log Z(J)\right]_{J=0}$$

$$= \left[\frac{1}{Z(J)} Z'(J)\right]_{J=0} = \frac{G_1}{1} = G_1 = \langle \varphi \rangle$$

For n equal to 1; let us see what we get? For n equal to 1 we have C 1 is equal to the first derivative of W J. And then we substitute J equal to 0 first we work out, the first derivative of W J that is given by this expression; del by del J log of Z J that is 1 by Z J Z dash J. And that is nothing, but G 1 upon 1 that is equal to phi the expectation value of phi the first moment of or the 1 point Green function.

So, for n equal to 1; C 1 is equal to G 1 is equal to the expectation value of the field variable or field operator. And this is for; this is for n equal to 1 that is the one point Green function.

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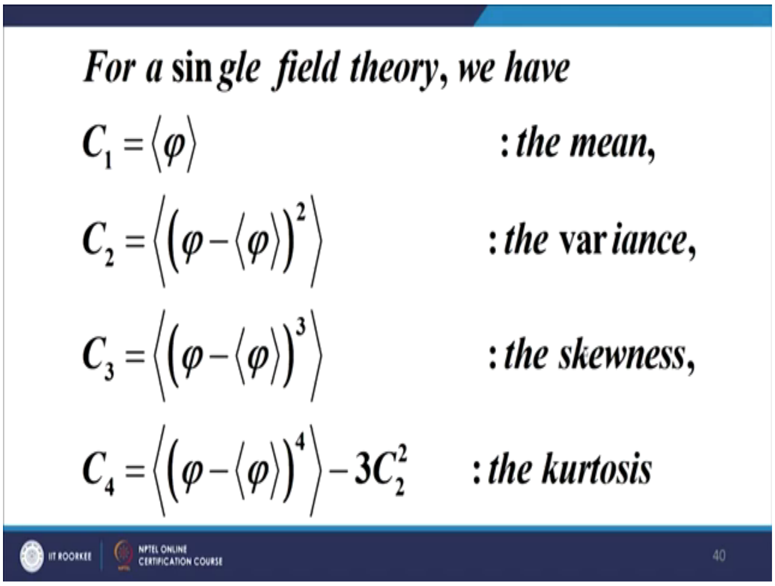

Similarly, we have the 2 point and the 3 point Green 3 point 4 point Green functions. As I mentioned the 1 point Green function gives you the mean of the distribution, the 2 point Green function gives you the variance.

And the 3 point connected Green functions I am sorry connected Green function gives you the mean, the 2 point connected Green function gives you the variance. The 3 point connected Green function gives you the skewness and the 4 point gives you the kurtosis.

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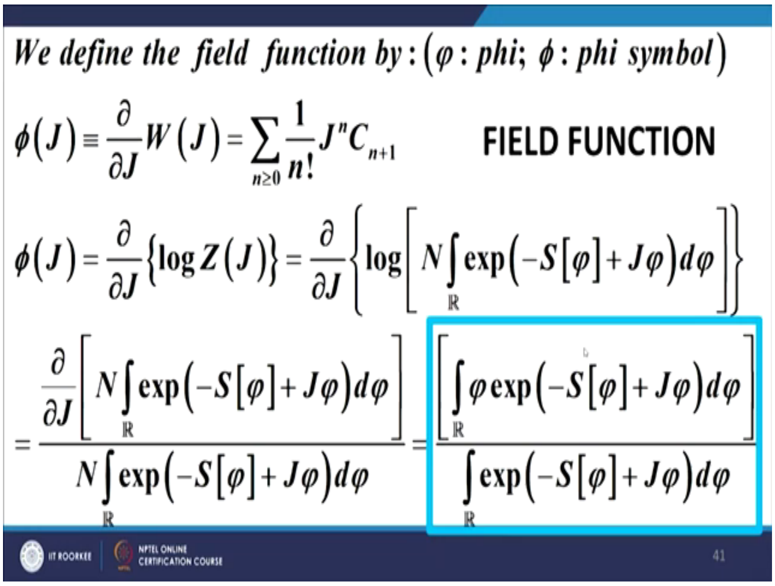

Now, the field function we define the field function as the first derivative of the generating function for the connected Green functions. The first derivative of; the first derivative of the generating functional for the connected Green functions. When you differentiate the expression in the square bracket this del upon del J attaches itself to this J pi here attaches itself to this J phi here and brings back or pulls back a factor of phi. So, the net result is there net result is what we get in the blue box.

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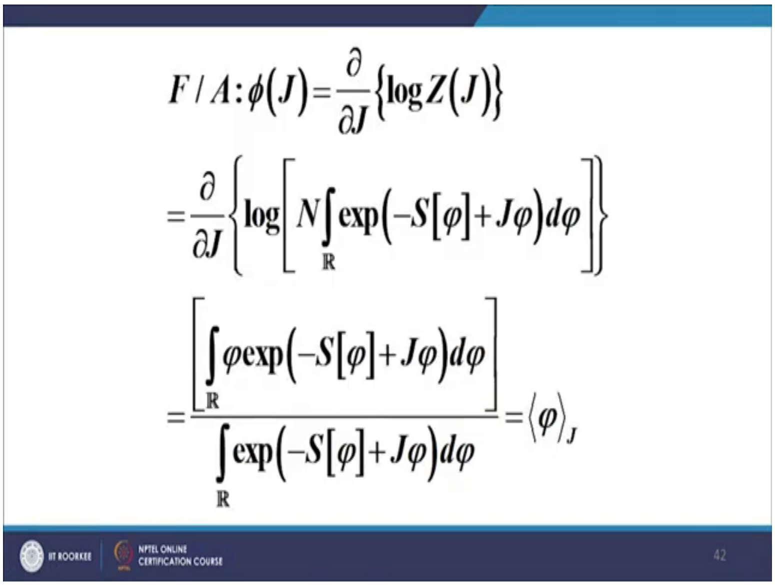

So, this is the expression that we brought forward in the expression that we had. And if you look at this carefully; if you look at this carefully this is nothing, but N into this integral of phi exponential this whole expression. And that is nothing, but that is nothing, but the expectation value of phi in the presence of source J.

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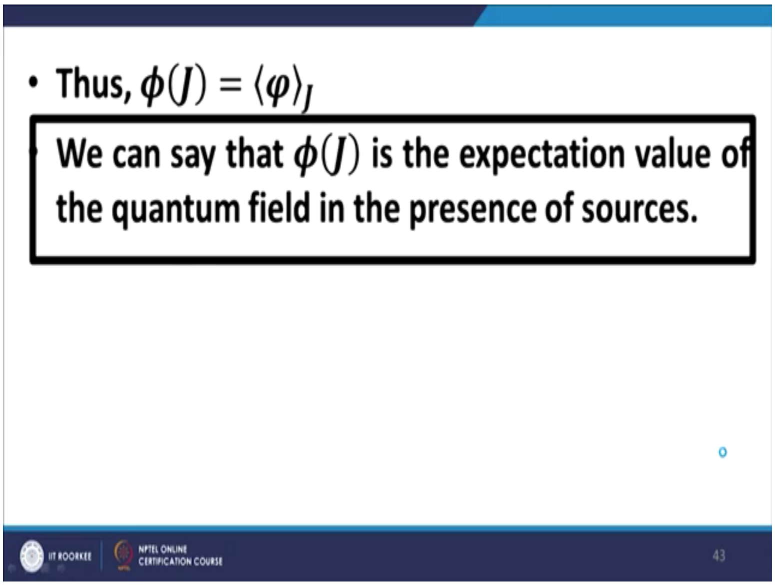

So, in other words what do we conclude? We conclude that our field function field function is nothing, but the expectation of the field variable in the presence of the source term J.

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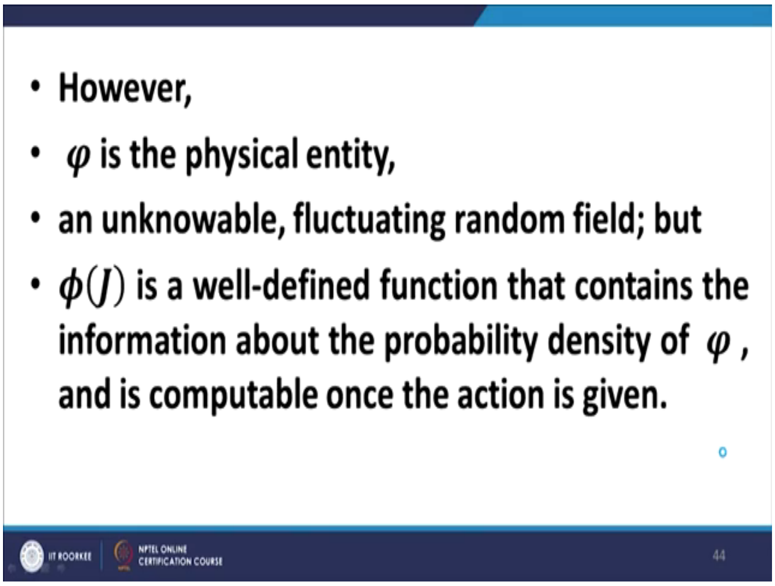

Now, the important thing here is that phi is remain the physical entity this round phi. And the phi symbol I will call this the phi symbol and I will call this the this is the small phi and this is the phi symbol. I shall be using this because both the expressions would be used frequently. So, phi is this expression and phi symbol is this.

So, phi is the physical entity, this is the random entity it is the fluctuating field. It represents the field whereas; the phi symbol represents an expectation value of that particular expression. It represents the expectation value of this physical quantity phi in the presence of a source J.

So, if it is this is also well defined it is well defined function. It is not a random function it is an expectation value it is a number it is a quantity and not a random number. It is the expectation value phi, but in the presence of a source J.

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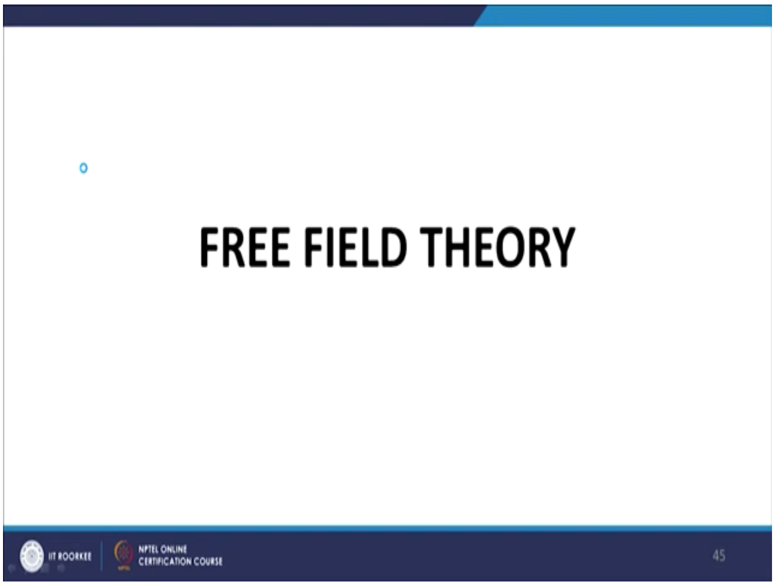

So, now it we use the mechanism that we have developed above so far. And we use it for the free field theory free field theory means there is no interaction term that is just the. And I remember we shall have no derivatives because we are still working in the 0 dimensional field theory.

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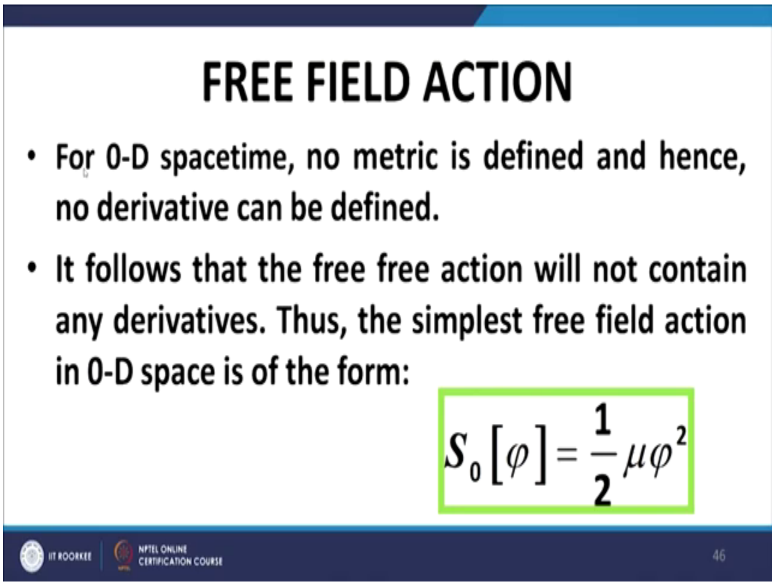

So, our action is very simple our action is straightforward; this is the free field action S 0 phi is equal to 1 by 2 mu phi square.

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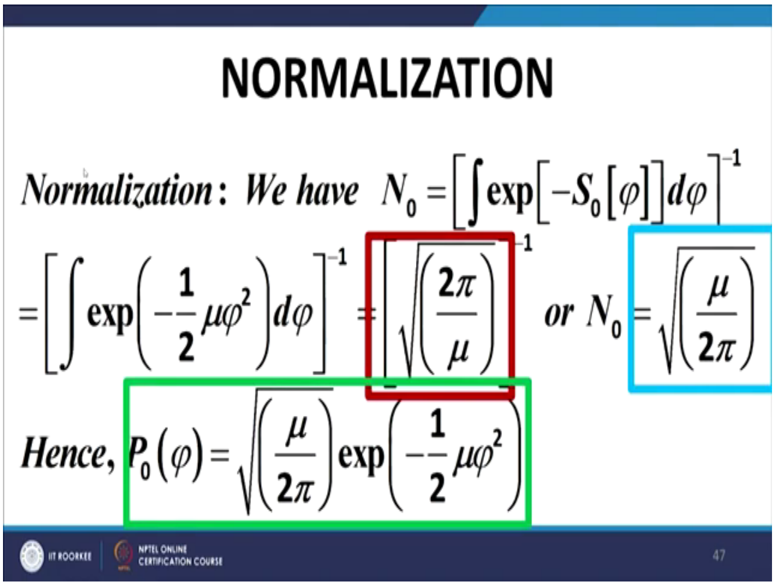

The normalization is quite straightforward it is a Gaussian normalization. It is the integral required for normalization is a simple Gaussian integral. It has a value of under root 2 pi upon mu and therefore, the normalization constant is under root mu upon 2 pi. And therefore, the probability distribution of the field variables becomes under root mu upon 2 pi exponential minus 1 by 2 mu phi squared.

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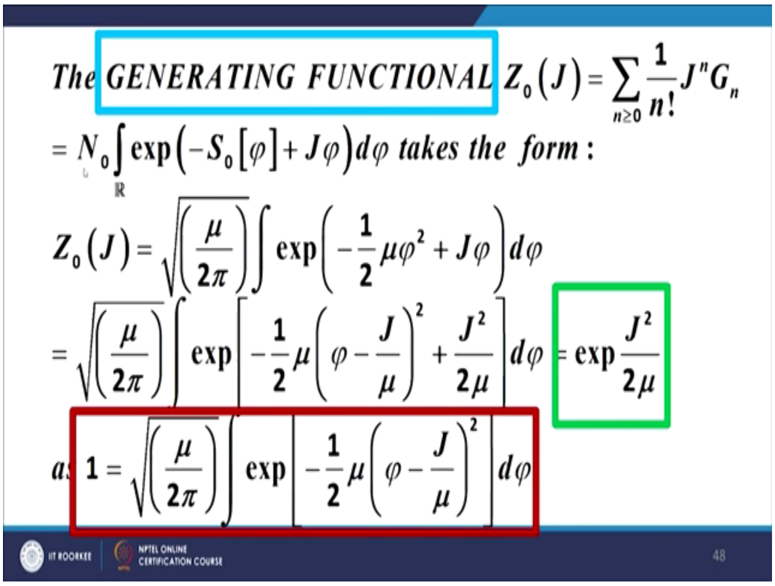

The generating functional Z 0 J; remember I am using the subscript 0 to indicate that it is a free field case we are working in the free field environment. So, using the generating functional Z 0 J is given by and then this expression we now have J the source J coming into the integral path integral. N 0 we have already determined N 0 we have determined as under root mu upon 2 pi.

So, we put in the value of the normalization and we have this integral exponential minus 1 by 2 mu phi square plus J phi d phi. This is again a very straightforward Gaussian integral, we can do it by completing the square. And when we complete the square we get this extra piece J square upon 2 mu which is independent of the integral variable phi.

And therefore, it can be taken outside the integral exponential J square upon 2 mu goes outside the integral and the rest of the term is equal to 1. And therefore, the net result of the integration after integration is equal to exponential J square upon 2 mu. And this whole expression is equal to 1; as you can see here.

This is simply the Gaussian distribution integrated over the entire spectrum of values of the random variable. And please note that this phi minus J mu represents a shift of the origin. So, the condition that is given in the red box continues to hold right. We will continue after the break from here.

Thank you.