Path Integral Methods in Physics & Finance
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Lecture – 30
Quantum Field Theory, Introduction

Welcome back. So, before the break we were talking about the clean paradox and as I mentioned the paradox emanated from a relativistic bombardment of electrons on a potential barrier.

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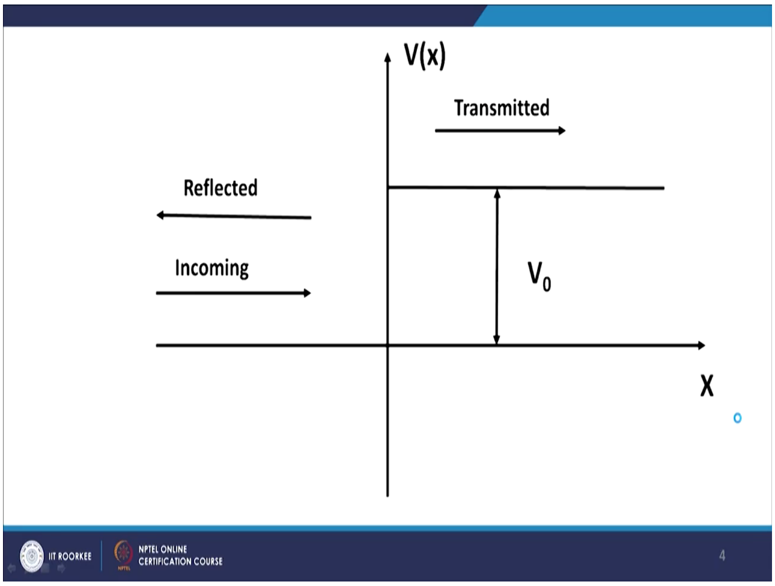

We were given a potential barrier V 0 and an incident beam of electrons was focused on their potential barrier and we examined the various parameters that is the transmission and reflected probabilities.

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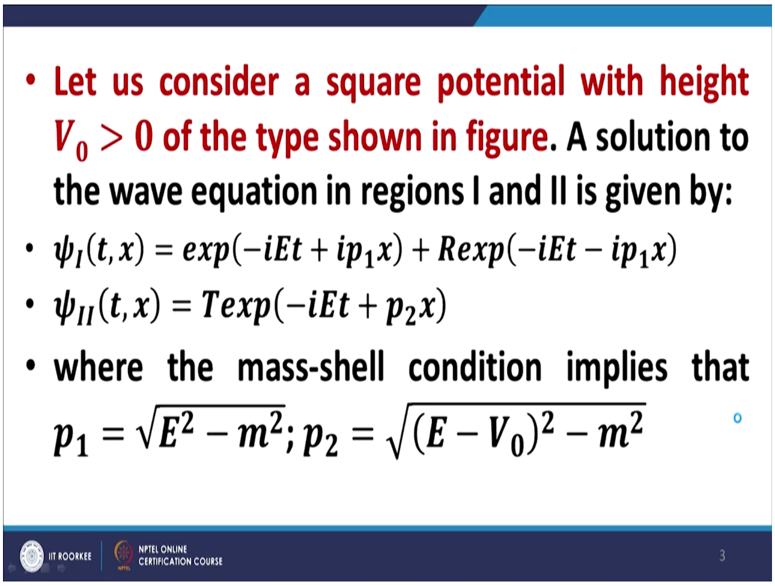

So, the equate the wave equations and the let us look at the two regions that are involved the first region is the region where the incident beam is flowing through or passing through and the second region, region two is the region where we are the transmitted beam flows after interaction with the potential barrier.

So, the wave equation across the two regions is given by the expression that is psi 1 and psi 1 and region 1 and psi 2 and region 2. Now to make physical sense the wave functions psi 1 and psi 2 need to be continuous across the boundary or across the potential wall.

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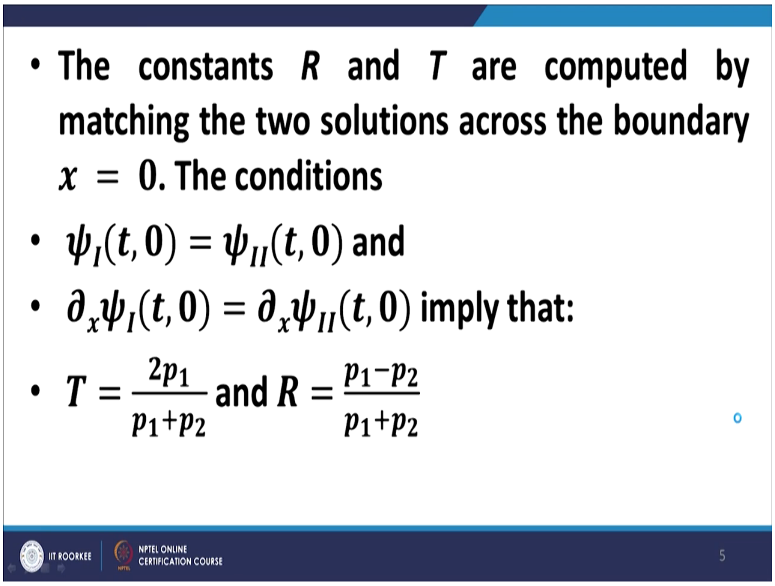

And not only that their first derivatives also need to be continuous, now when we impose these two conditions the continuity of the wave function and the continuity of their first derivatives we obtain expressions for the transmission coefficient and the regression reflection coefficient. I am sorry the transmission coefficient and the reflection coefficient which are given at the bottom of your slide T and R, T represents the transmission coefficient, R represents the reflection coefficient.

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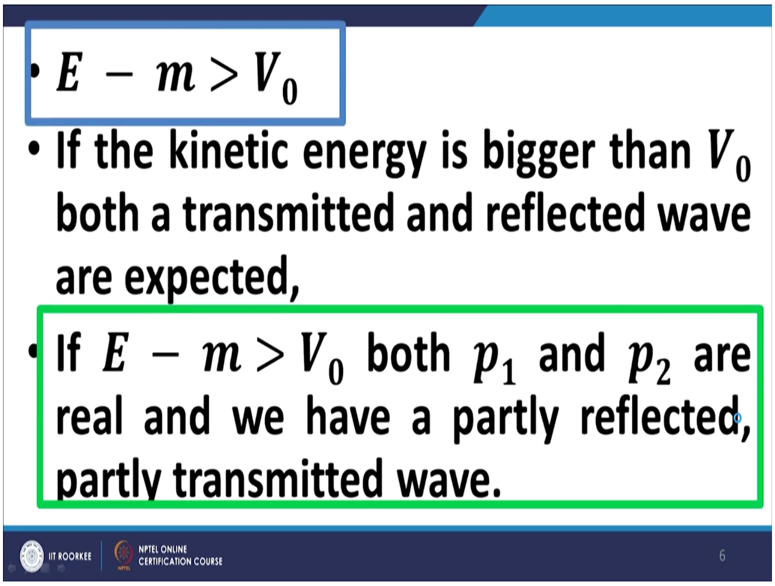

Now, let us examine the behaviour of this transmission and reflection coefficients in various scenarios. Now if the kinetic energy is bigger than V 0 that is the kinetic energy exceeds V 0, V 0 recall is the potential.

So, the kinetic energy of the incident beam of electrons exceeds the potential barrier then we have both a transmitted wave and a reflected wave ah, because both p 1 and p 2 turn out to be real and we have a partly transmitted wave and a partly reflected wave that this is the situation when the kinetic energy of the incident beam is more than the potential barrier.

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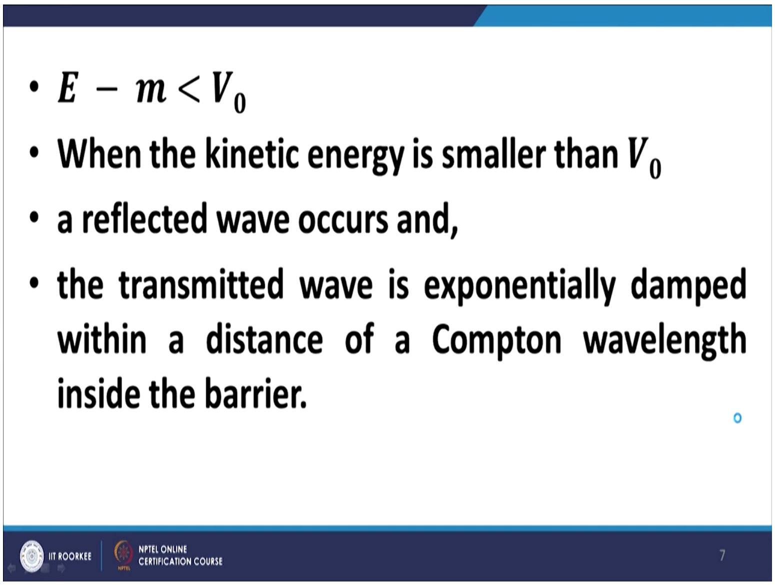

When on the other hand E minus m is less than V 0 and the kinetic energy is smaller than V 0 we get a reflected beam and the transmission and transmitted beam gets exponentially damped. So, we have two situations if the kinetic energy exceeds V 0 we have both transmission and reflection and if we have if the kinetic energy is less than V 0 of course, there is tunnelling and there is exponential damping on the on the transmission side and of course, we have reflection as well, but the interesting part is still to come.

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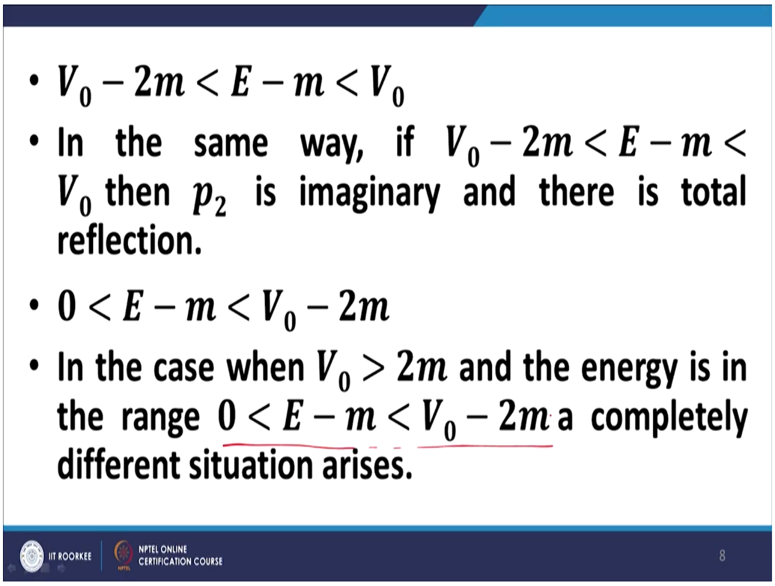

However, the if V 0 minus 2 m is less than E minus m is less than V 0 then we have a similar situation as we had earlier p 2 becomes imaginary and we have a total reflection p 2 becomes imaginary therefore, we have a total reflection, remember t 1 and t 2 are given by these 2 expression.

So, in this case p 2 becomes imaginary and we have total reflection what happens if the kinetic energy is less than 2 m; if the potential bar is greater than 2 m and the energy in the range of this figure. This is the most interesting case when the potential bar or the potential barrier is greater than 2 m and the energy the incident energy is in this range.

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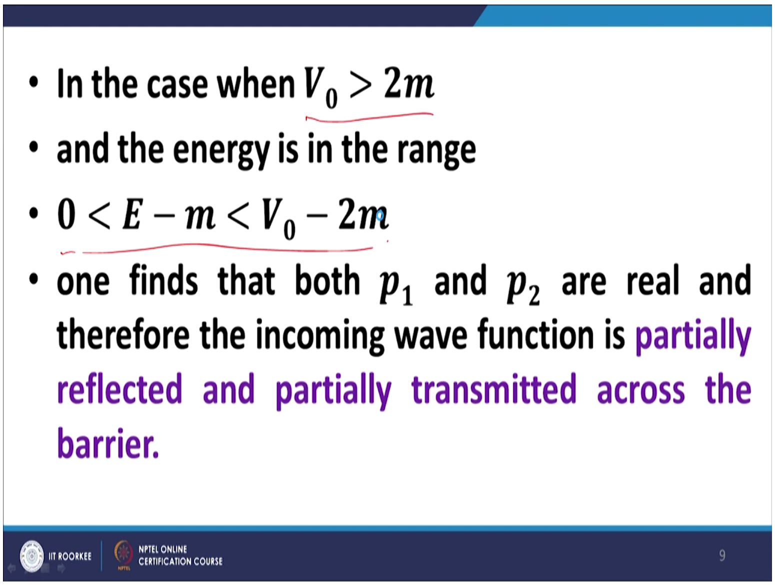

This is the potential barrier V 0 is greater than 2 m and this is the energy of the incident beam and in this case both p 1 and p 2 turn out to be real and therefore, we have transmission as well as reflection partly transmitted and partly reflected, but V 0 what, now let us look at what happens.

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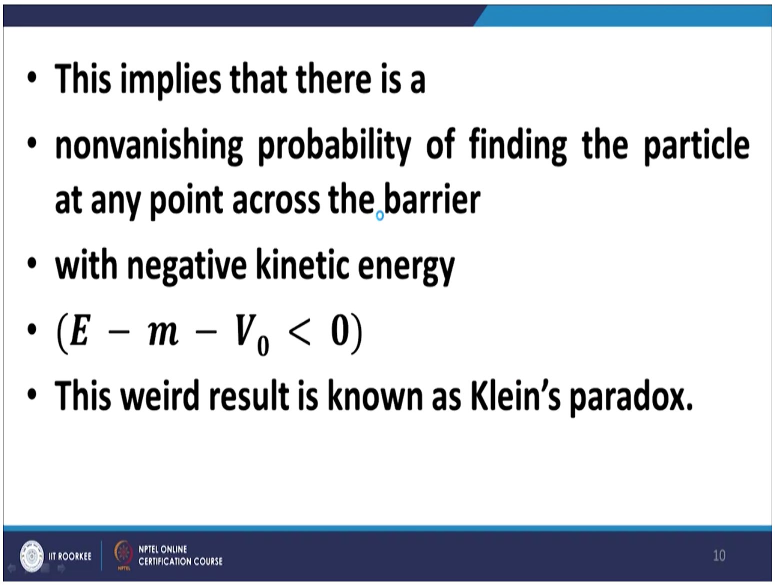

Now, the potential barrier is of 2 m or greater, now if the potential barrier is of 2 m or greater and the incident energy is in this range; that means, what; that means, after interacting with the barrier after interacting with the barrier you have E minus m minus V 0 and this is going to be negative E minus m minus V 0 is going to be negative why because V 0 is greater than 2 m. So, E minus m minus V 0 is going to be there could be situations at least where this expression is negative and not withstanding that we are having a non vanishing probability of finding the electron post or after passing through the potential barrier.

In other words we are likely to find out or we are likely to encounter electrons with negative kinetic energy the probability of finding electrons with negative kinetic energy to the right side of the barrier is not 0. This was a very interesting result; this again needed a radical solution to lend credibility to the underlying quantum theory or quantum philosophy.

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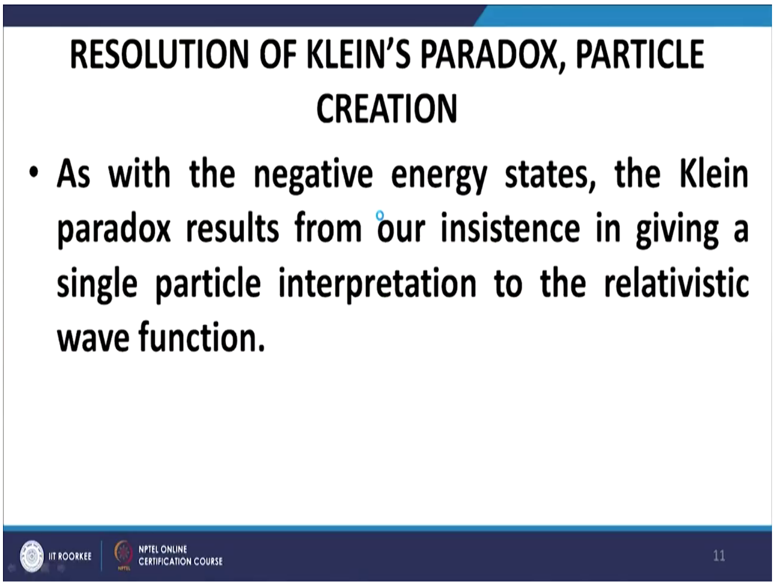

The argument that was propounded to resolve this issue was again to give up the particle interpretation and to look at to provide for the creation of particle pairs at the due to the interaction with the barrier.

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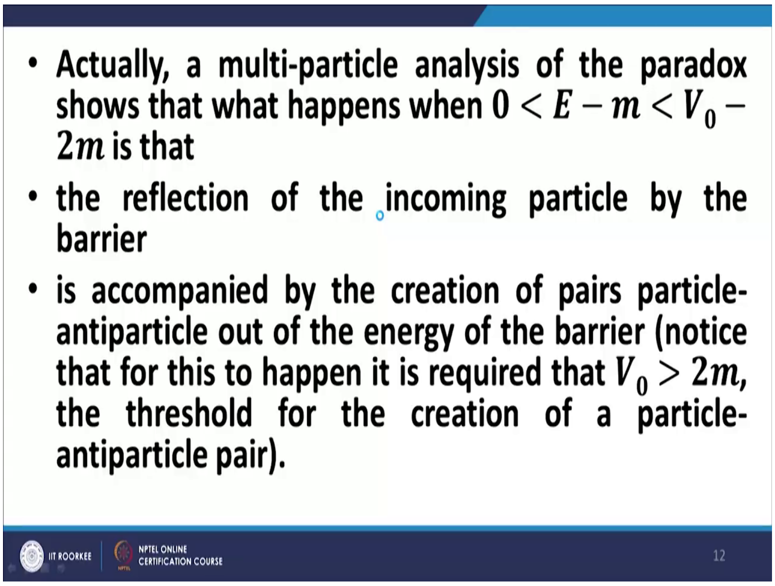

Recall that V 0 is greater than 2 m. So, recall that V 0 is greater than 2 m therefore, when the particle interacts with the potential barrier there is a possibility that due to the interaction a particle antiparticle could be created and as a result of which we could have particles moving across the barrier with negative kinetic energies.

So, that was again proposed resolution of the claim paradox due to proposed on the basis of the fact that the interaction with the potential barrier of 2 m enabled creation of particle antiparticle pairs. And it was this pair creation which accounted for the generation of particles with negative energy across the barrier.

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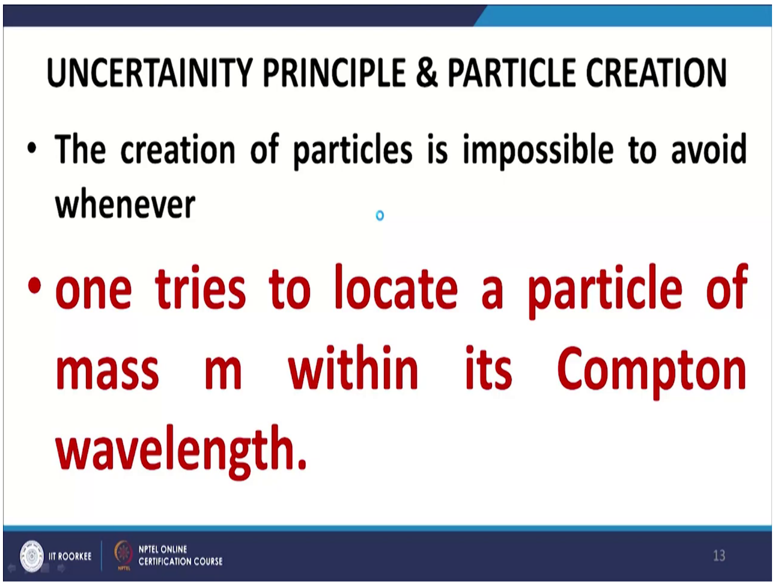

Now, it is see if you have a situation where you try to localize a particle or locate a particle within its Compton wavelength let m be the mass of a particle say an electron and you want to localize the particle or you want to identify the particle, you want to locate the particle within a region of its Compton wavelength, in such a situation Heisenberg's uncertainty principle creates havoc.

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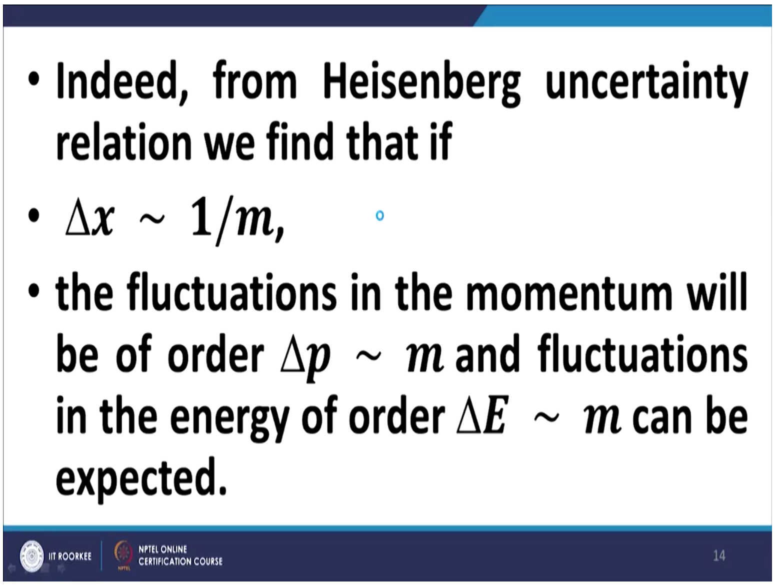

In the sense that if delta x is of the order of 1 upon m if delta x is order of the order of 1 upon m and that is the Compton wavelength then the fluctuations and momentum will be of the order of m and therefore, the fluctuations in energy will be of the order of m and that implies that even Heisenberg's uncertainty principle acknowledges because of the mass energy relationship that if the energy variance is of the order of m or the energy deviation with of the order of m that m could will arise due to the creation or annulation of particles which are involved in the experiment.

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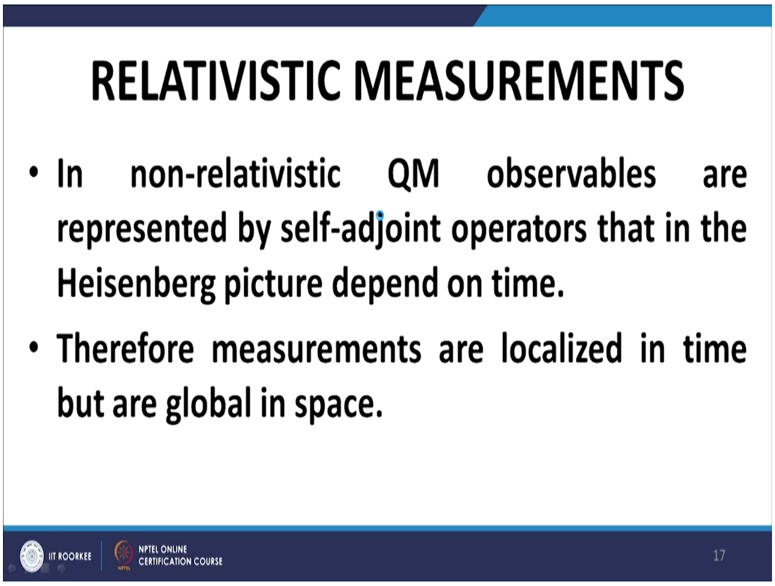

Now, we come to the issue of relativistic measurements now in the case of non-non-relativistic quantum mechanics observables are represented by self-adjoint operators which are localized in time which are functions of time and therefore, they are localized in time therefore, such operators are global in the context of coordinates of positions, but are localized in terms of time.

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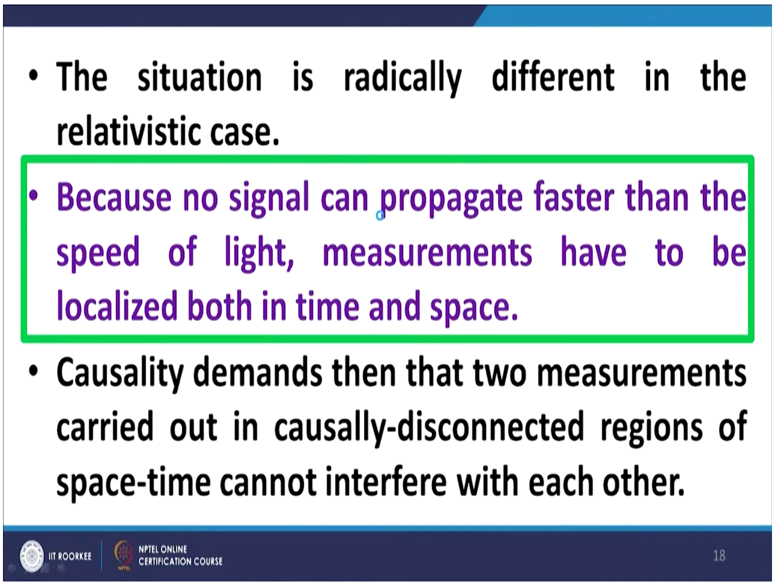

However, what happens in the when we move towards a relativistic framework. In the relativistic framework most as you know the fundamental postulate of special relativistic an upper bound to the external to the speed of propagation to the speed of light and therefore, no signal because no signal can propagate with more than the speed of light therefore, measurements have to be localized now both in terms of time and space. In other words relativity relativistic operators or measurement operators must be localized both in space and time.

And as a coronary to this what happens the causality requirement says that if two points in space term are connected by a space like interval, then in other words if they are causally disconnected then their operators cannot influence each other their measurements cannot influence each other and because their measurements cannot influence each other the corresponding operators must necessarily commute.

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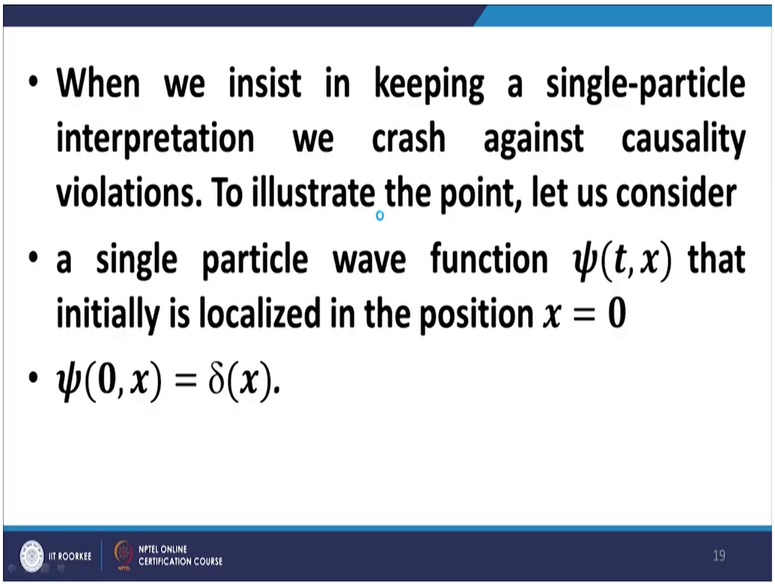

But let us see what happens there if we retain the single particle case. Let us consider a localized particle or a localized wave function as a delta wave localized at the position x equal to 0 we represent it by delta of x the initial wave function at time t equal to 0 is localized at the at the origin at x.

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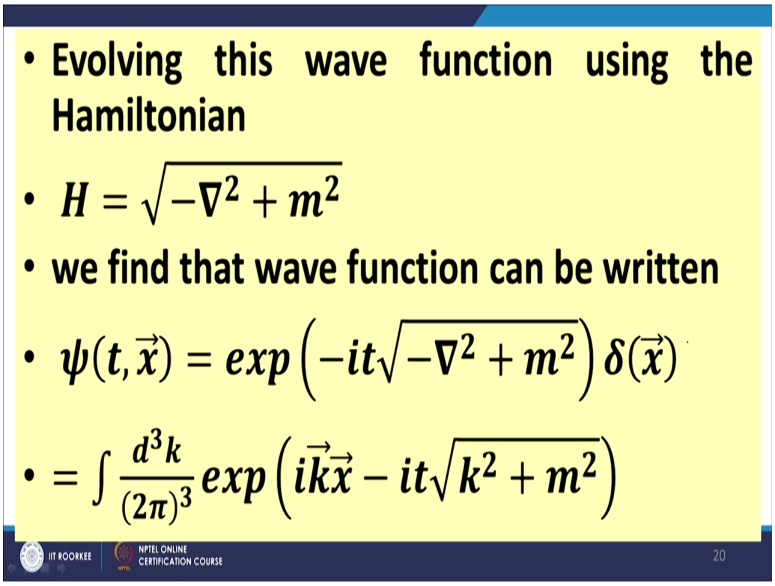

Now, we evolve this wave function using the Hamiltonian under root minus del square plus m square you recall that this is the relativistic Hamiltonian arising from or emanating from the Marshall condition e square is equal to p square plus m square you replace them with the respective operators you get the Hamiltonian and this Hamiltonian for the relativistic Klein Gordon equation.

The wave function solution of the corresponding wave equation can be written as psi of t x is equal to exponential minus it into delta x remember delta x is the initial wave function. Converting it converting delta x the whole expression into Fourier space what we get is the expression that is given right at the bottom of your slide.

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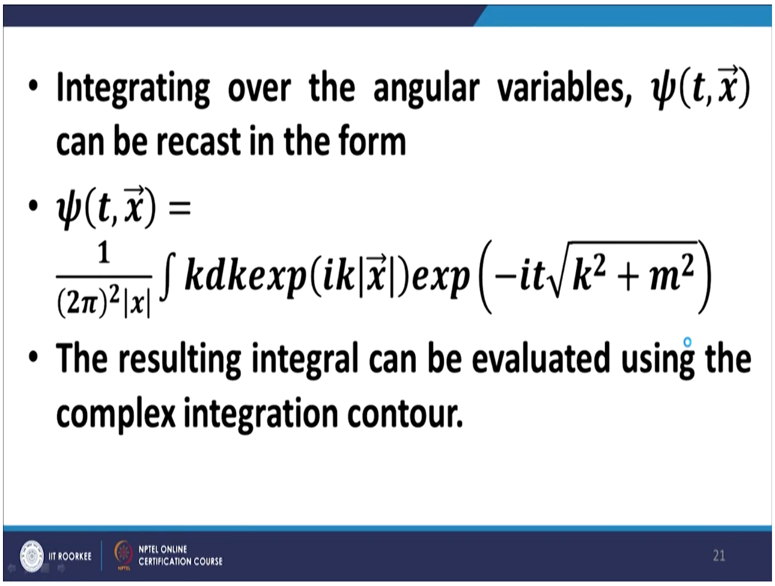

Now this equation can be solved exactly, first of all you integrate over the angular variables and by integrating over the angular variables you get the expression on the slide then you do complex contour integration.

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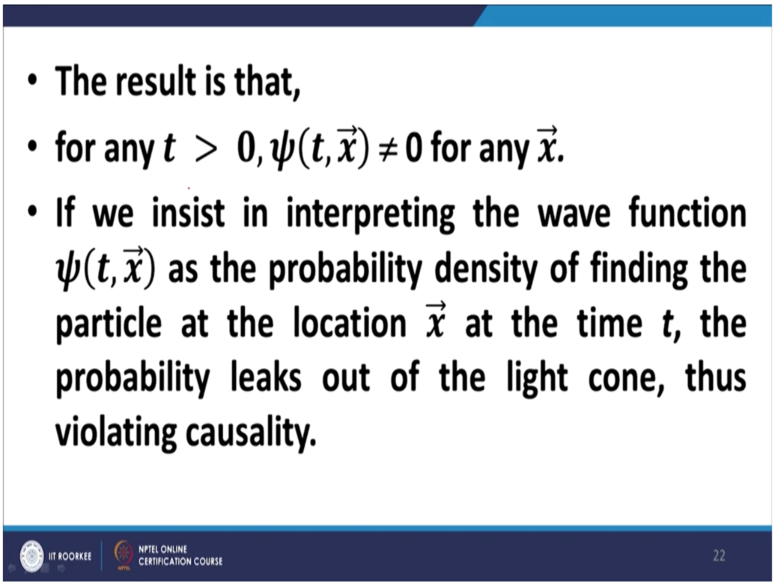

And after doing complex counter integration what you find is that for any t greater than 0 at any t greater than 0 you find psi t x unequal to 0 for any x. Now, this result is very fundamental what does it say?

It says howsoever small t may be so long as it is positive; that means, starting at t equal to 0 even for infinitesimal evolution in terms of time the wave function completely spreads and across all space all coordinates space and the value of the finding the prior particle the probability of finding the particle at any point in space is not 0. So, this again violates causality because causality requires that propagation should be confined at most to the speed of light which in this case is clearly violated.

In other words you can say the probability moves or leaks out of the light cone in by virtue of this result. So, these problems all these problems enabled or are required a radical rethinking of the quantum mechanical framework as it stood at that time. And it was not really possible in keeping in you such blatant inconsistencies to be able to merge the fundamentals of quantum mechanics a particle based quantum mechanics with the special relativity.

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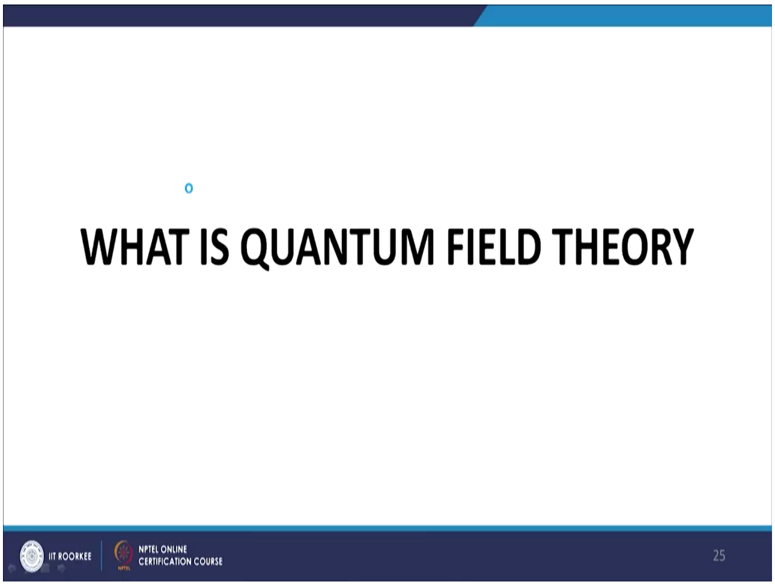

So, the outcome of all this was the origin of the concept of quantum field theory, before we talk about the concept of quantum field theory let us understand what a field is. Now given a certain space given a any space for example, the space in this room you could allocate a particular variable to every point in this space.

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For example, simplest example that comes to mind is the temperature at every point in this 3 dimensional space you could allocate the or you could allocate the temperature in terms of degree centigrade or Fahrenheit or whatever, but that particular temperature will; obviously, depend on the point at which it is being worked out or calculated.

So, naturally every point will have a different temperature, a different value of that expression similarly not now this is the case of a scalar field where you are simply specifying a number and by virtue of specifying that number you are giving a complete description of the physical variable in this case the temperature this is called a scalar field scale the temperature field is a scalar field you could also possibly assign a vector at every point.

In this particular room every part in this particular room you assign a vector that has a magnitude and has a particular direction that would be a vector field for example, electromagnetic fields.

So, this is an example on your slide of what a field represents, every point on this screen is carrying a certain number is allocated a certain number and that is how we define a field, that there is an underlying manifold on this manifold we assign every point on that manifold is assigned a number and then we will we create functions out of those numbers which are assigned to every point on this space.

So, there are really two layers in this case, in the case of quantum mechanics classical mechanics or whatever we just have one layer, we have a space time and on the space time we define our variables of interest like we have the trajectory in configuration space or we have the in this case in the case of quantum system we have our Hilbert space. So, there is just an underlying space and physical variables are defined over that space.

In the case of fields there is an there is a two tyre structure really you have the underlying space and underlying manifold that can be Minkowski space, that can be Euclidean space, that can be whatever space you like multi-dimensional space or whatever space structure you want a torus may be.

But on that space you define a variable at every point on that space and that constitutes our field these will be the operators that this these variables representing or that are defined on every point of our space time represent the operators which will which will later be quantized and so on in canonical quantization.

And then we construct functions of these operators like the Lagrangian, Hamiltonian and so on. So, it is the two tyre structure in contrast to what we have in traditional dynamical systems right.

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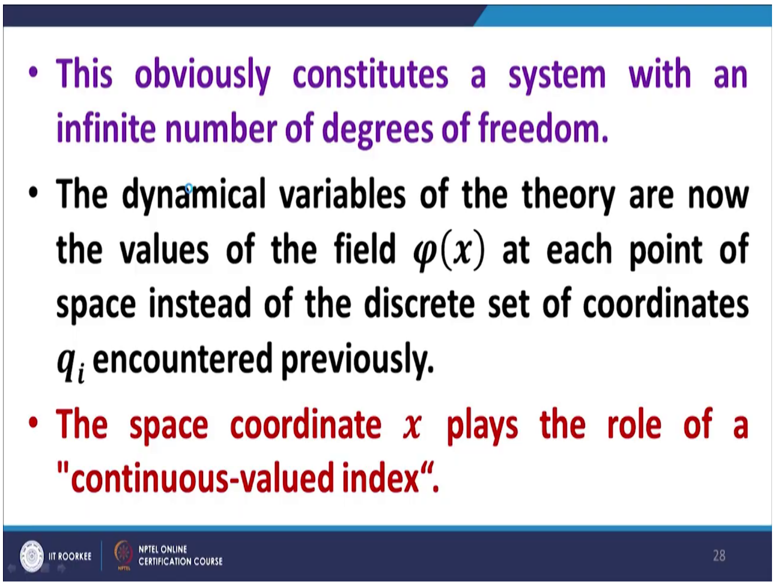

So; obviously, the field as its field constitutes of an infinite number of degrees of freedom and it is quite natural that it will have infinite number of degrees of freedom.

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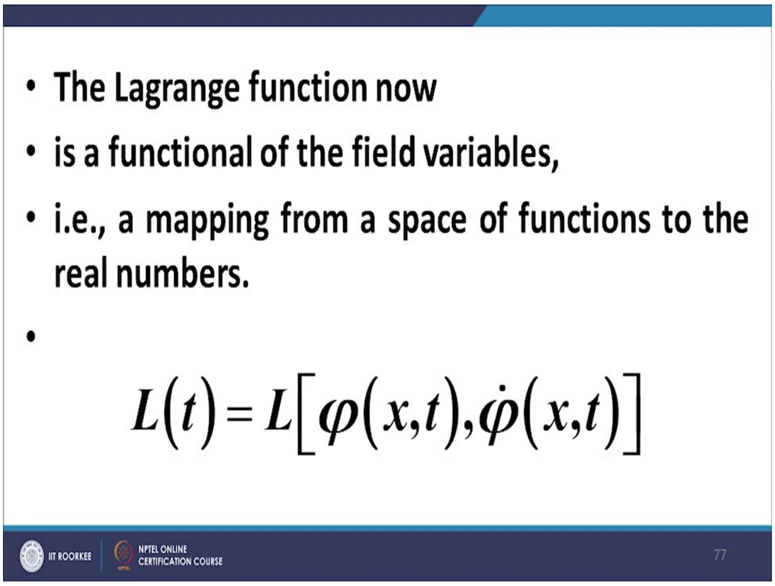

The Lagrangian is a functional of the field where you know here we come in where we come to the second tyre the expression that you see within the square brackets this is the first tyre.

Next are the underlying space are the variables in the underlying space the manifold phi of x t phi dot of x t these are the field variables that are defined on every point of the underlying space. L of phi and phi dot is the function of these field variables these field variables can be quantized what is called normally called the second quantization; however, of course, we shall not be focusing much on second quantization we shall be focusing more on the path integral approach which is an alternative to quantization.

So, or rather another different mode of quantization you can say an alternative to canonical quantization, but nevertheless quantization, but in a different approach so that is what a field is.

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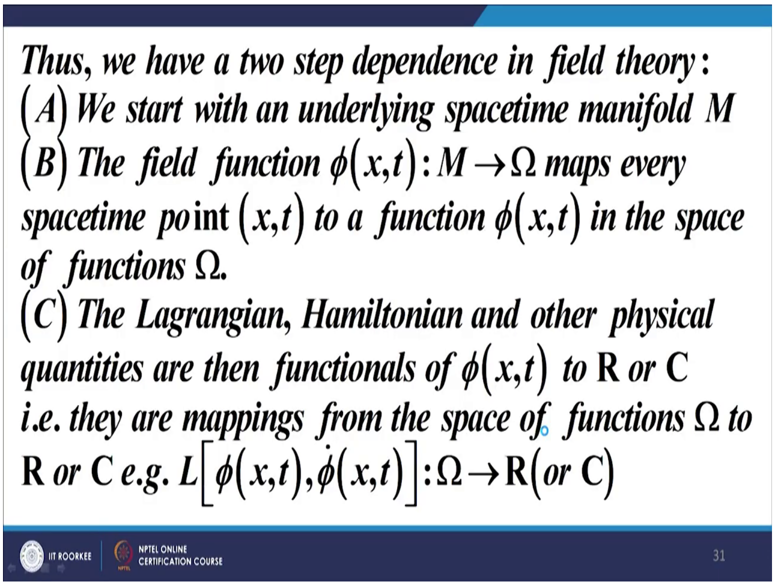

So, as I mentioned we have a two tyre structure here start with an underlying manifold M the field function phi x of t is a mapping from M to omega, it maps every space time point to a function, function space phi x t in the space of functions omega, omega is the space of functions phi x t is the function which is mapping the underlying space to omega and then Lagrangian and all those are functions of phi which are mappings from omega to R or C as the case may be when you are talking about complex numbers.

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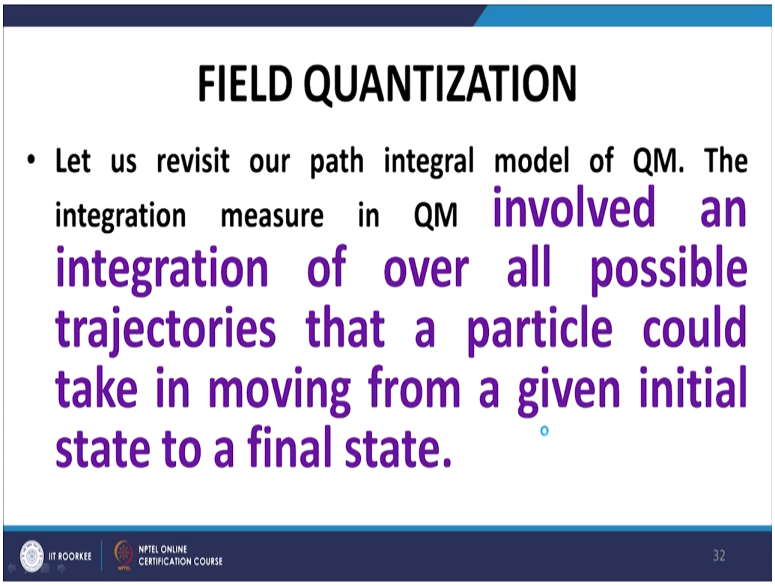

Now, let us try to understand how the quantization went about or the how the second quantization operated.

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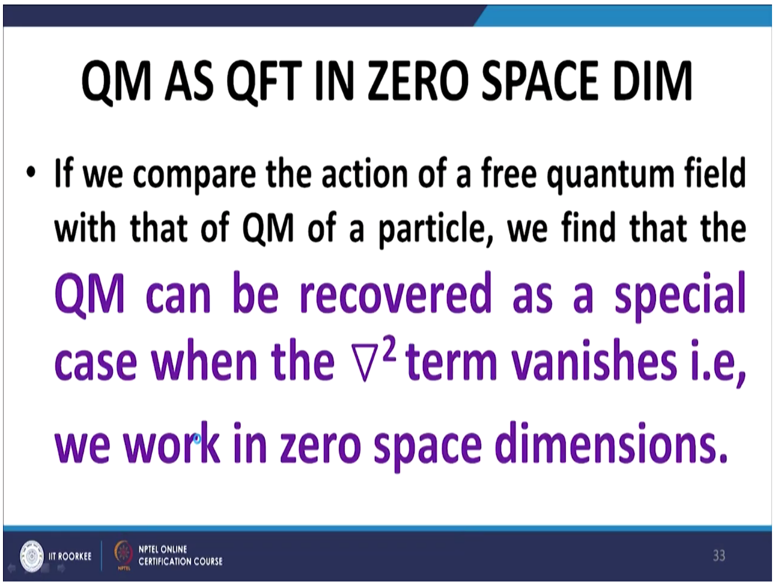

If we look at the look at quantum mechanics what we have done so far, if we look at quantum mechanics as we have done so far we can recover the quantum mechanics as a special case a quantum field theory, how is that let us have a look at it.

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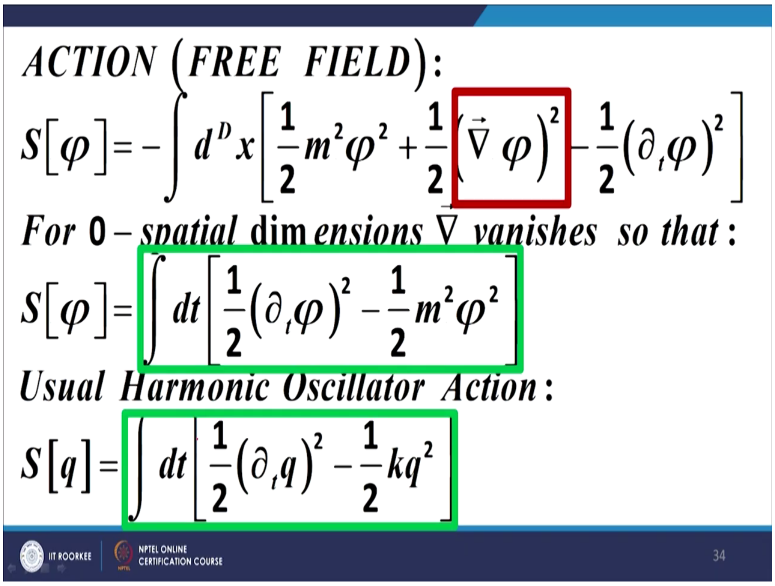

The first expression the expression in the red box or the entire first equation is the equation for the action in the case of a field theory in this case it is the scalar field theory.

So, in this case this is the action for a scalar field theory, if you look at the expression this expression for a 0 with 0 spatial dimensions let me repeat. The first expression the first equation is the expression for the action in the case of; a in the case of a scalar field a regular scalar field that is a Klein Gordon field for a 0 spatial dimension case if you are taking looking at a 0 spatial dimension case then you end up with this particular expression which is there in the green box.

Now, if you compare this expression with the harmonic oscillator action there is literally a one to one correspondence between phi and q if you look carefully between the 2 green boxes the green box representing the first equation and the the first green box representing the 0 spatial dimension field and the other one representing the harmonic oscillator action there is a clear cut one to one correspondence. Now you recall you recall that when we did quantization of the harmonic oscillator we quantized which variable we quantized q.

So, the natural corollary is that if we have to quantize the variables in the context of QFT or quantum field theory we will be quantizing the field variables phi. So, q corresponds to phi and we quantize q when we talked about quantum mechanics and therefore, we will quantize phi when we talk about quantize field theory and our underlying say Minkowski space or other Euclidean space variables will simply serve as indices for the identification of the fields at various points in the space underlying space time.

So, the important thing is as I illustrated quantum mechanics can be recovered as a special case of quantum field theory when we substitute when we take the spatial dimensions to be 0, when we work in just the one space one time dimension 0 spatial dimension 0 plus 1 dimensional space underlying space.

Then we recover the quantum mechanics as a special case of quantum field theory that enables us to move or get a idea a feel of what we should do to quantize our field theory, because the quantized the dynamic variables q we shall now be quantizing their correspondence in quantum fields and that is the field variables phi right.

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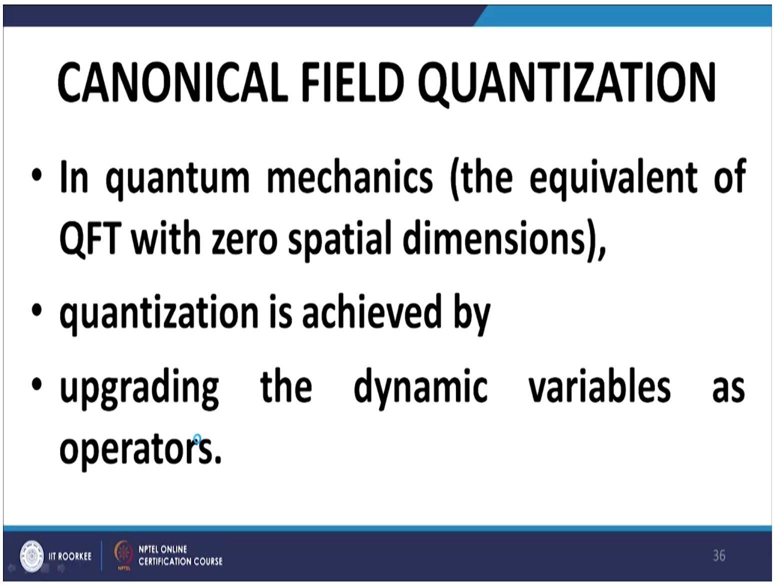

So, in canonical quantization what we do is we upgrade the dynamical variables as operators we upgraded our q and p two operators and so, we will do the same prescription here we will have clear the field variables phi and the corresponding conjugate momenta pi as the dynamic variables of the theory.

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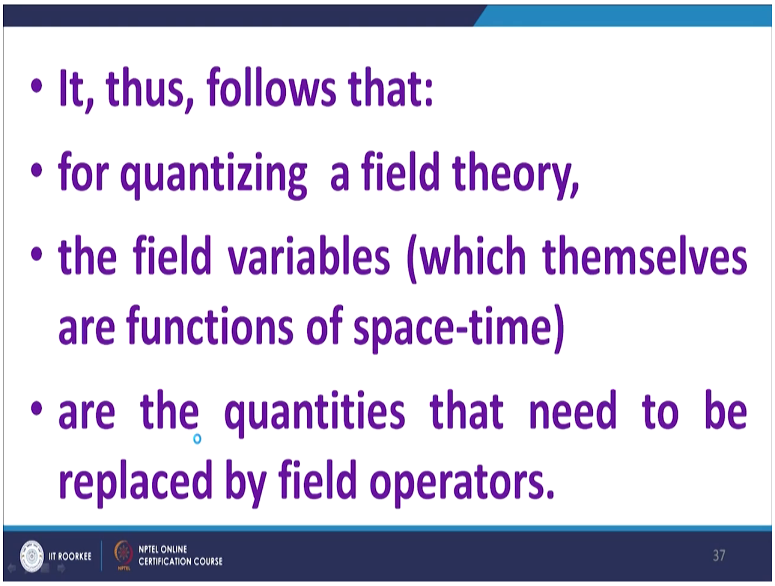

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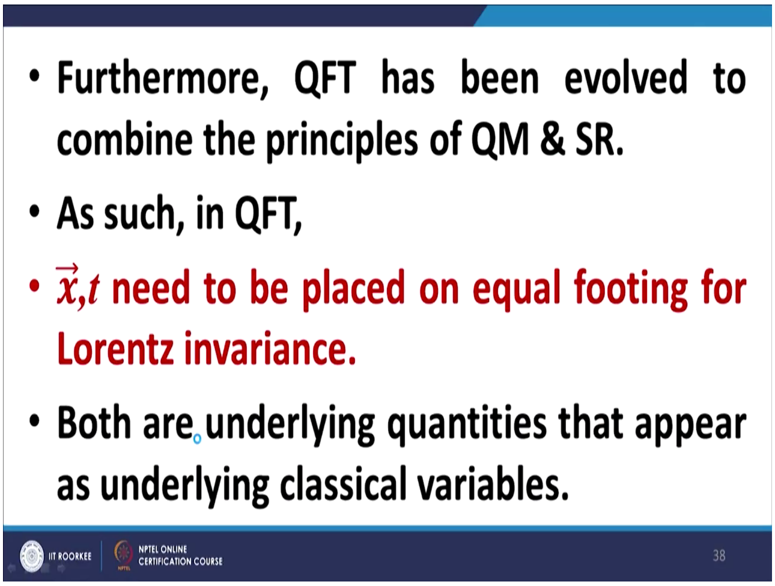

And moreover there is one more point here that is given in red the fundamental feature of special relativity is that space and time need to be placed on equal footing and because QFT is a formalism that has been devised in order to combine together the quantum mechanics and fields and special relativity we shall be using our formalism as far as practicable shall be based on an equal identification or parity between the spatial and time variables.

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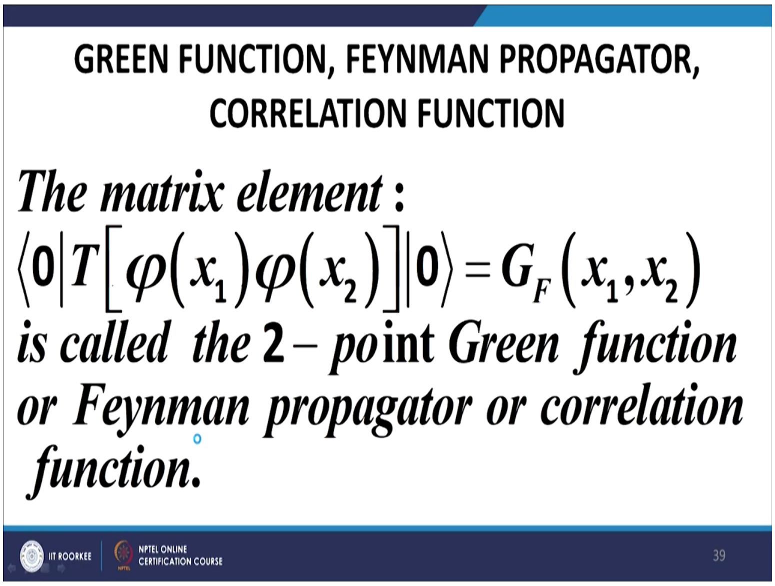

And that is in fact, required for Lorentz invariants as a matter of fact and these the underlying space time of course, will serve as the in the index for identifying the fields at various points. The matrix element which is given here the time ordered operators phi x 1 phi x 2 in the ground state are usually called the 2 - point green functions or the correlation functions or the Feynman propagator you call them. So, many almost synonymous names they are basically the time order product of the respective number of operators in the ground state.

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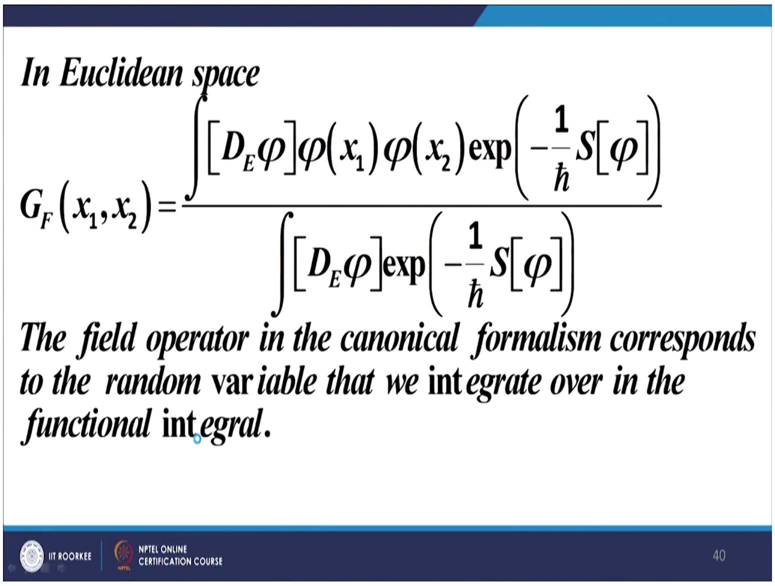

We have similar expressions in the interaction states as well, in Euclidean space this is the expression that we shall of course, we shall be working it out, but this is the space expression for the 2 - point function that we end up with in the.

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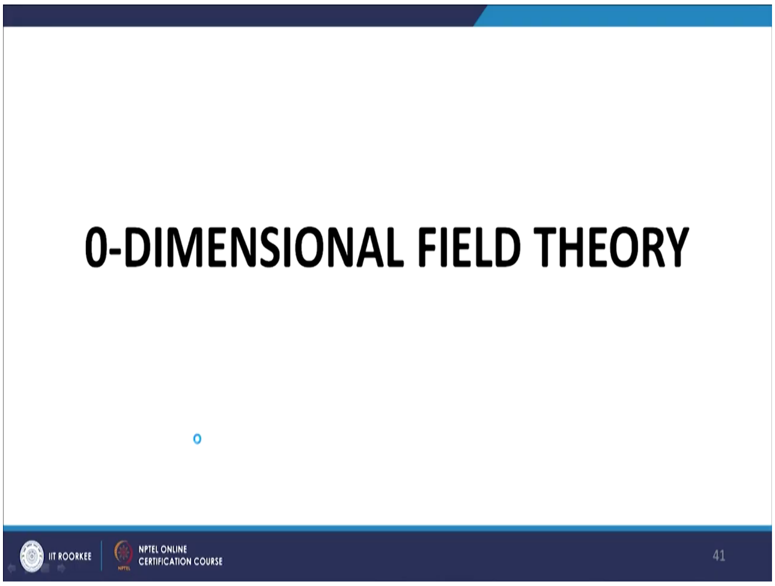

Now we talk about this 0 - dimensional field theory, see my scheme is that I will start with discussing the 0 - dimensional field theory, 0 -dimensional field theory I have selected because it gives you an overview it gives you a feel of the various first of all it gives you a feel of the various terminology that are singular to quantum field theory that is one point.

The second is the methodologies the approaches that will be followed to work out those particular terminology terms or to work out those particular quantities that is the second step that we are in that is relevant. And thirdly another important point is that it without the complications arising from the structure of Minkowski space by having 0 - dimensional field theory you are we are able to simulate the fundamentals and then apply them gradually to the complex Minkowski space.

So, let us start with 0 - dimensional field theory, what is 0 - dimensional field theory? In the case of 0 - dimensional field theory what is the underlying space time, the underlying space time is a point if we look at a 0 - dimensional manifold which is compact then we have only one point in it.

So, the therefore, the quantum field as can be described as the assignment of a number at that given point any real number in the simplest case a real number and the assignment of a random real number at the at our point which corresponds to our space time, we will define the field, we will identify the field so, in other words our field is a mapping from the point to the set of real numbers.

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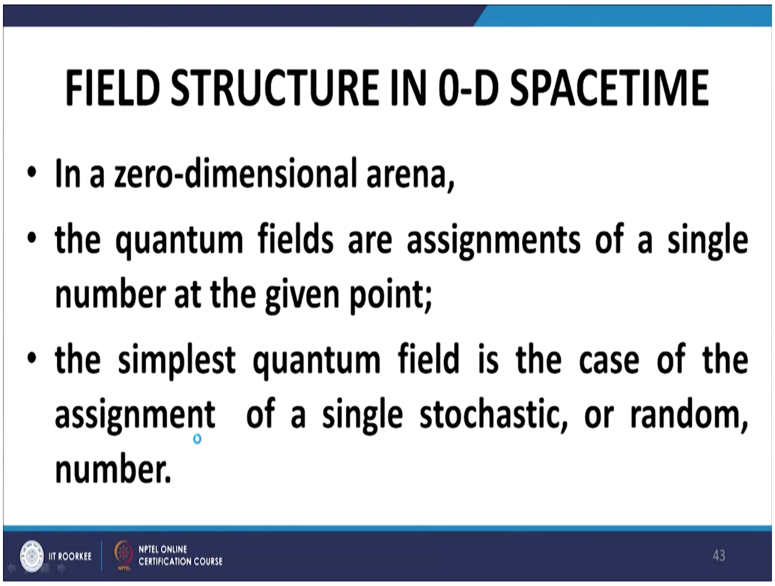

Secondly, because we have got just one point there is no metric there is no definition of length and therefore, the terms that involve the length quantities are irrelevant will not figure in the action and the action will contain only the quadratic term in the variable.

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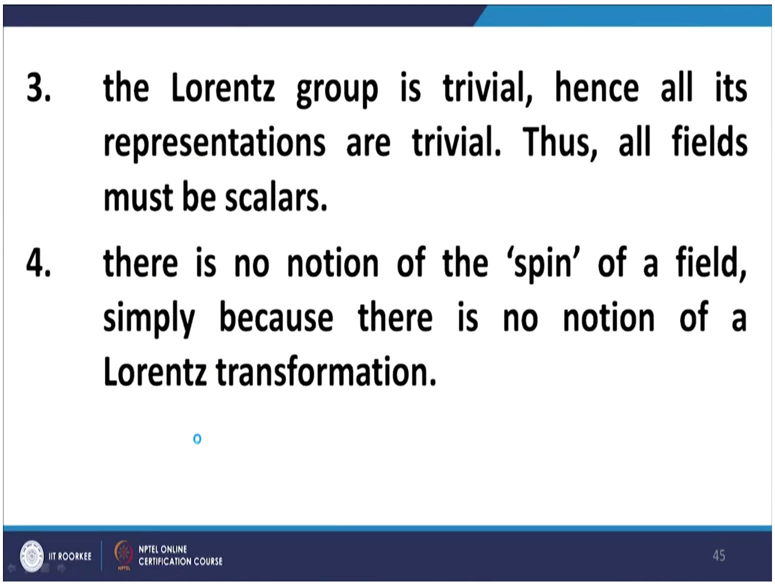

The other point is that the Lorentz group is trivial and all it is representations would be trivial and second the next point because the Lorentz group is trivial there would be no spin involved.

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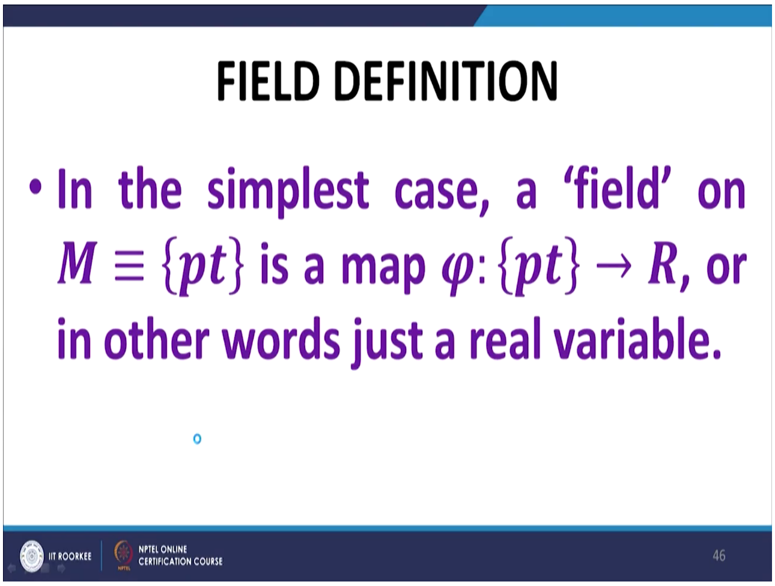

So, these are the features; these are the features that define a 0 - dimensional field theory and in the simplest case as I mentioned the field function is simply allocation or a relation between a point which acts as the underlying manifold and this space of real numbers. We will continue from here.

Thank you.