Path Integral Methods in Physics & Finance
Prof. J. P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee

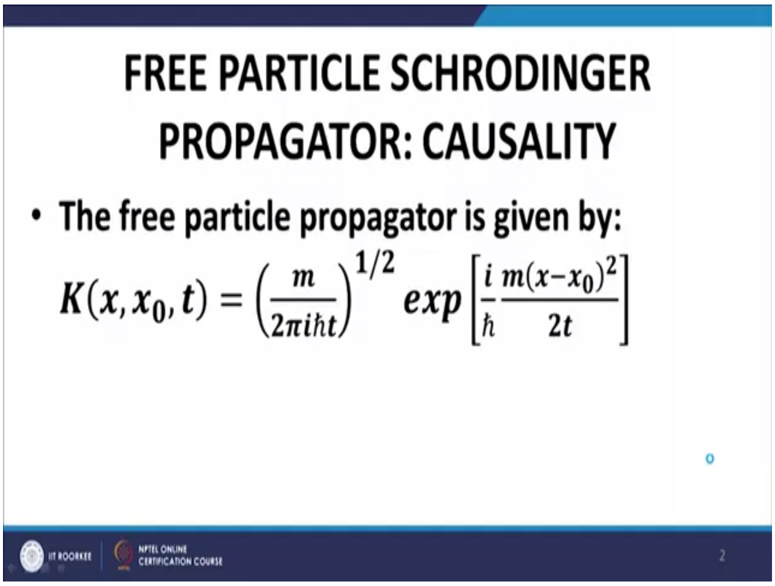
Lecture - 29
Need for Quantum Field Theory

Welcome back. So, in the last lecture, we obtained an expression for the path integral for the relativistic point particle. And thereafter, we also obtained the path integral in terms of the stationary phase approximation which explicitly enabled us to investigate how and why the classical paths become the most prominent paths.

Today we digress a bit, and we examine the problems associated with the particle interpretation of quantum mechanics or the where we look at physical objects as particles in the framework of quantum mechanics where do we end up with controversies with contradictions. And therefore, we have to move towards a more abstract formalism of the quantum field theory.

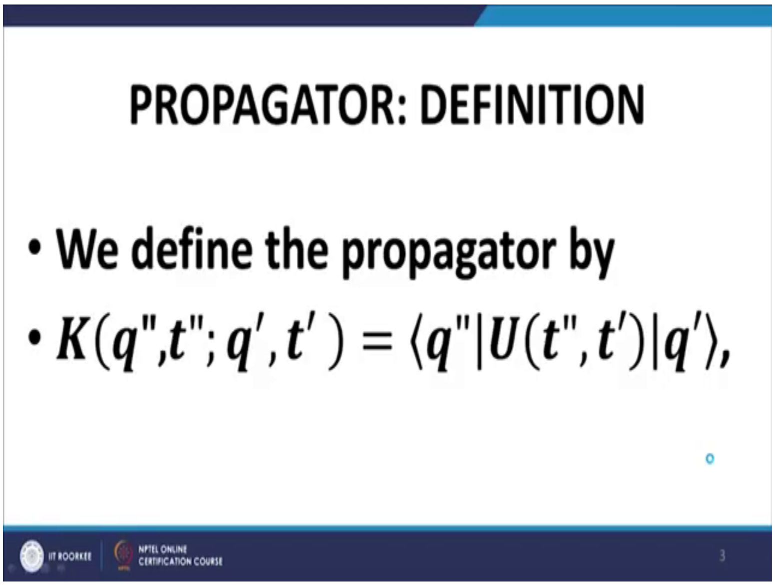
So, my objective today is to elaborate on the impediments that we face in interpreting the particle formulation of quantum mechanics, and thereby pave the way to the field theoretic approach to the investigation or to the examination of physical systems at the quantum level.

(Refer Slide Time: 01:56)



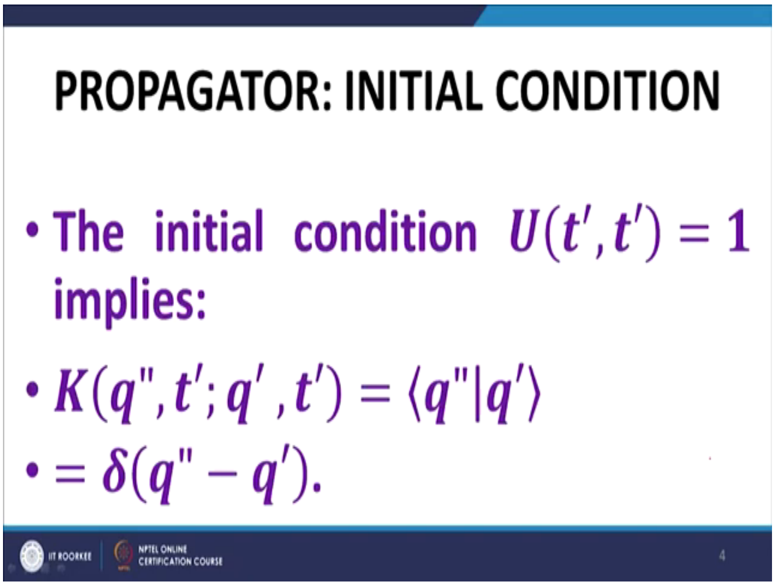
So, let us start. The free particle Schrodinger propagator we are obtained by a couple of methods in fact and this is the expression that we ended up in all cases. So, this expression is what we are going to talk about later on in this discussion. So, just keep that at the back of your mind, this is the free particle propagator that we also worked out in the last lecture on the bases of the stationary phase approximation.

(Refer Slide Time: 02:24)



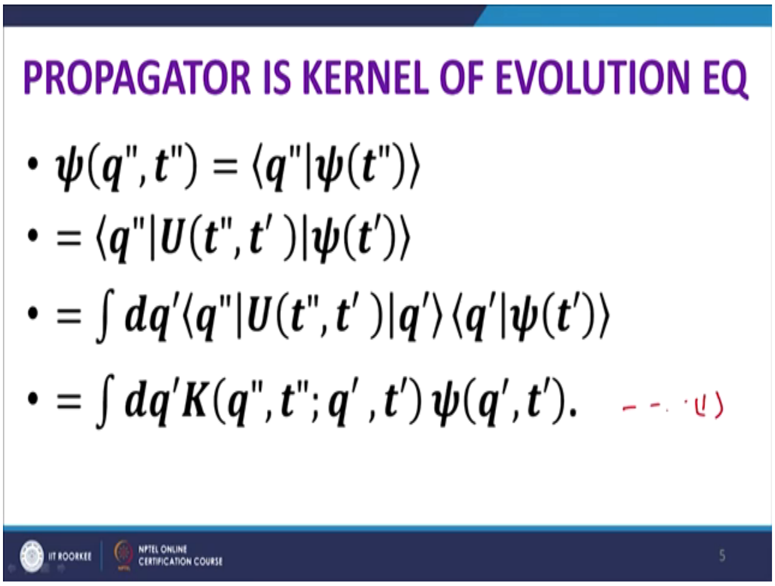
And we defined the propagator, this is a more or more or less revision material. So, we define the propagator or the path integral as the matrix element sandwich between matrix element of the evolution operator, sandwich between the initial and final states.

(Refer Slide Time: 02:47)



The initial condition; the initial condition for the propagator or the path integral can be obtained on the basis of the initial condition for the evolution operator which is given here U t dash comma t dash is equal to 1. And, that gives us the initial condition for the path integral or the propagator as a delta function of q double dash minus q dash.

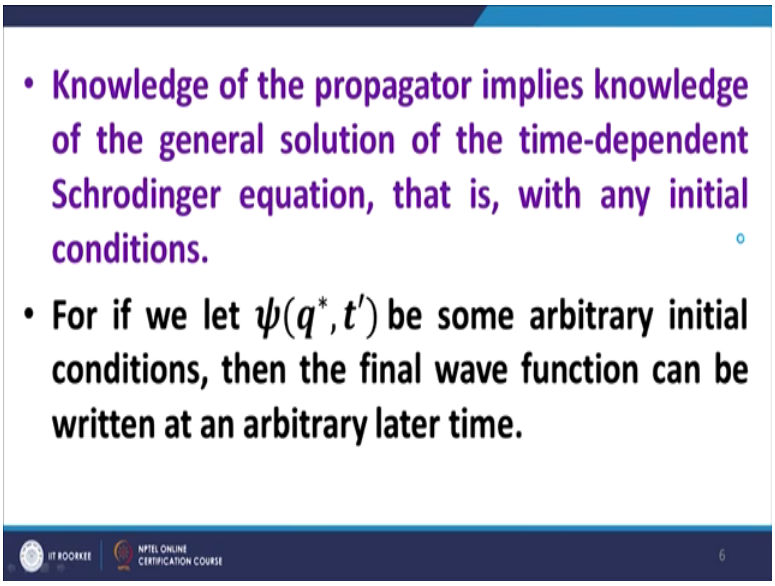
(Refer Slide Time: 03:21)



The propagator as can be seen here turns out to be the kernel of the Schrodinger evolution equation expressed as an integral equation. The path integral, the propagator happens to be the or works up to be the kernel for this particular equation.

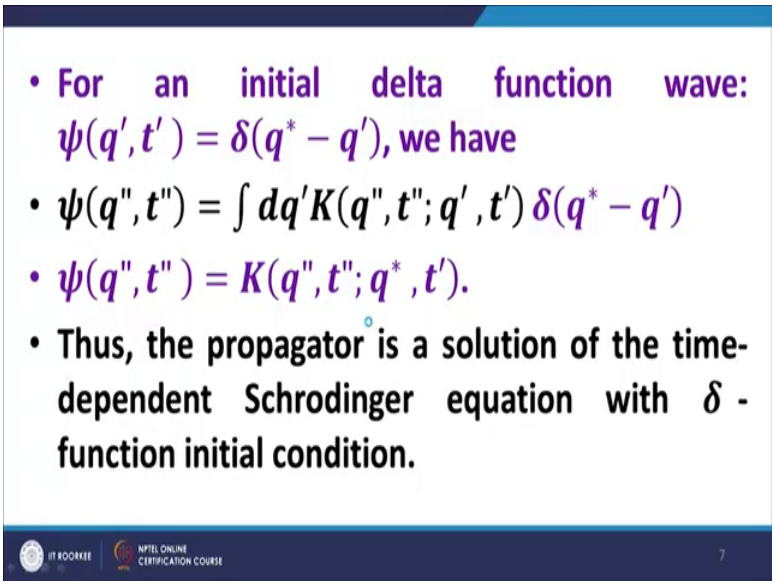
And in fact, if you see this it see this expression carefully the kernel or the propagator transforms or projects the initial wave function or the initial wave function in terms of space and time to the final wave function in terms of the space and time in coordinate space.

(Refer Slide Time: 04:07)



So, therefore, if we know the propagator, we can obtain the general solution of the time dependent Schrodinger equation with any initial condition that we want, at least in principle we can do that. For if this is an initial wave function, this is an initial wave function, then by acting on with the propagator, we can obtain the corresponding wave function at any later point in time.

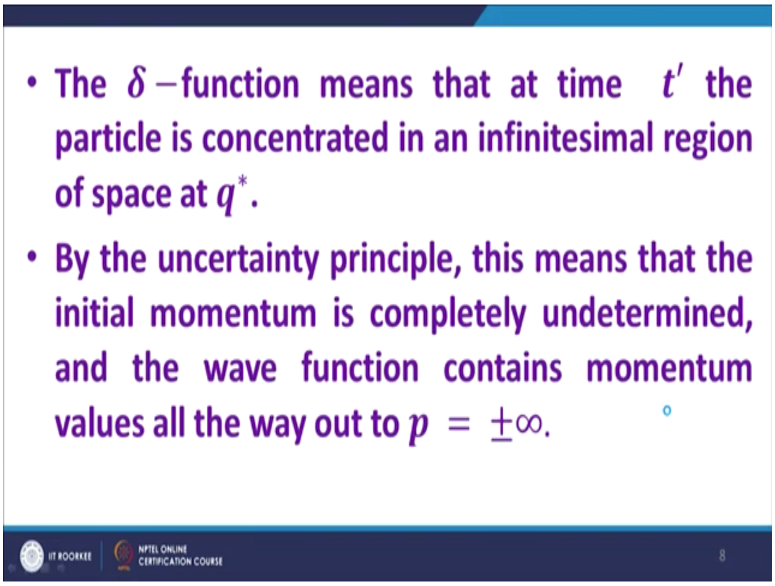
(Refer Slide Time: 04:38)



Now, for an initial delta function; for an initial delta function wave, the situation turns out to be very interesting. As you can see if I use the delta function as the initial wave function, I end up with propagator as the final wave function. So, in other words, the propagator becomes the solution of the time dependent equation with a delta function initial condition.

Just to recap if I use the delta function as the initial wave function and put it in the evolution equation, let us go back, let me this particular equation, let us call it equation 1. If I put it in equation 1, then I end up with the propagator itself as the solution or as the final wave function.

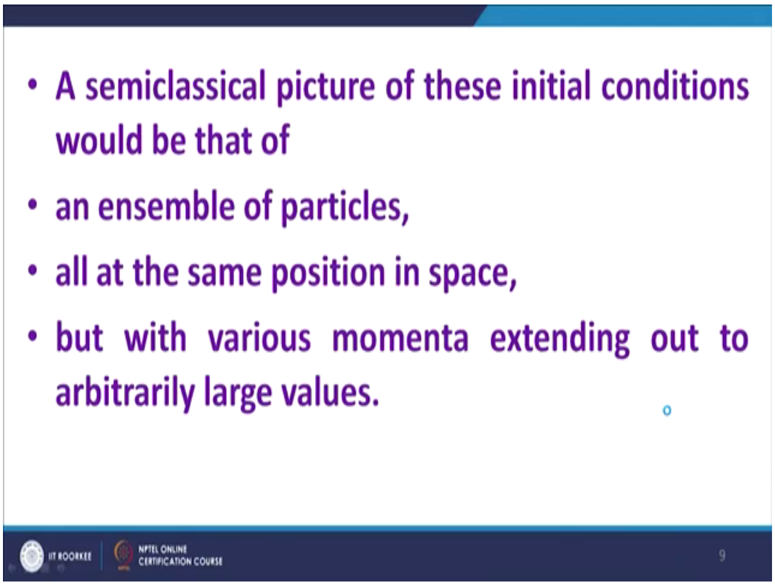
(Refer Slide Time: 05:32)



Now, this means what? This means that the delta function it means what the what is the delta function? Delta function represents an infinite infinitesimal or an impulse over an infinitesimal point time. So, if we use a delta function initial condition delta function wave, that means, the particle is concentrated in an infinitesimal region in configuration space and coordinate space.

If it is so; if it is so, in other words, if the wave is the delta function wave that means it is focused or it is limited or a localized to infinitesimal volume in a position space, then it implies it is implied by the Heisenberg uncertainty principle that the momenta of the wave or the particles constituting the wave would be in determinate, infinite ranging from minus infinity to plus infinity.

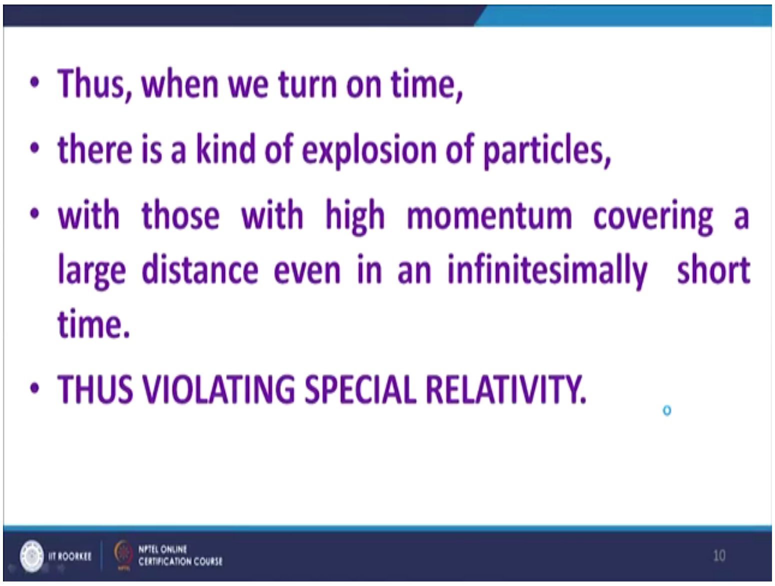
(Refer Slide Time: 06:49)



In a semiclassical picture, this kind of an initial condition would be construed as would be interpreted as an ensemble of particles, localized at a particular position, but having a huge spectrum of continuous spectrum of momenta extending from minus infinity to plus infinity.

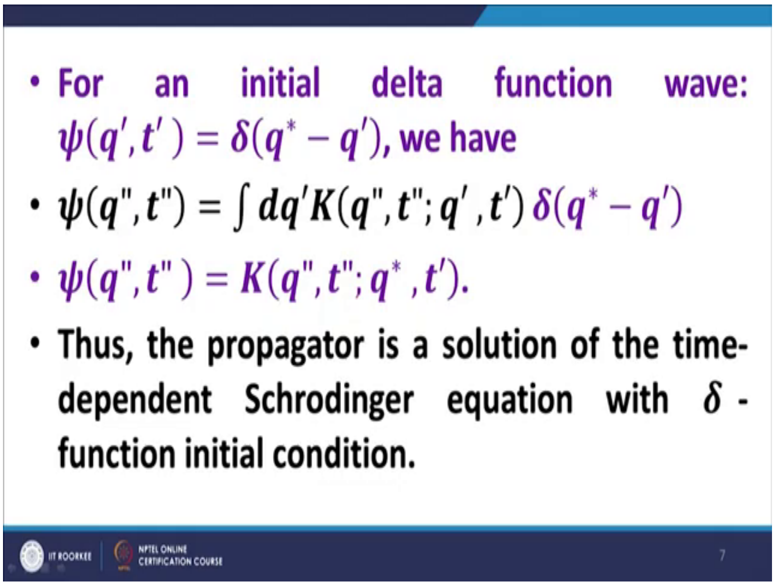
And immediately when we switch on time, what happens that they occurs an explosion of particles and the explosion of particles means that the final wave function occupies all of coordinate space which violates the principle of special relativity so far it relates to the upper bound on the propagation of a light.

(Refer Slide Time: 07:44)



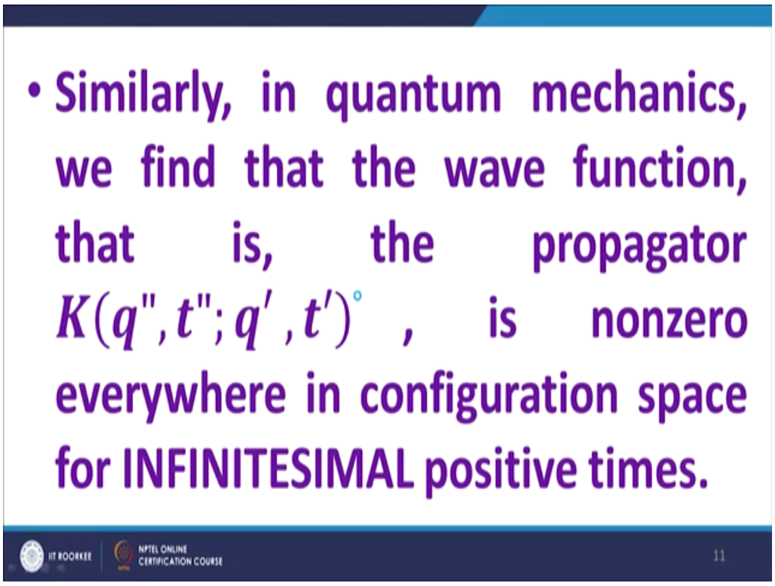
So, here again here we start facing or we start encountering in consistencies between the Schrodinger the framework, and the special relativity in terms of the instantaneous propagation.

(Refer Slide Time: 08:00)



Because remember as usual also see in due course, this the propagator fills up in the entire volume of space when it is acted on by a delta or when it acts on a delta function wave. When this delta function wave as the initial wave is acted on by this propagator, you get the propagator as the final solution, and final time solution. And therefore, the delta function wave at as the initial time becomes wave spontaneously unlocalized and the entire region of coordinate space.

(Refer Slide Time: 08:50)



There was the important thing that I mentioned just now in fact is that the propagator is nonzero everywhere in configuration space in it becomes nonzero everywhere, it blows up and to a nonzero value everywhere in infinitesimal time which violates special relativity.

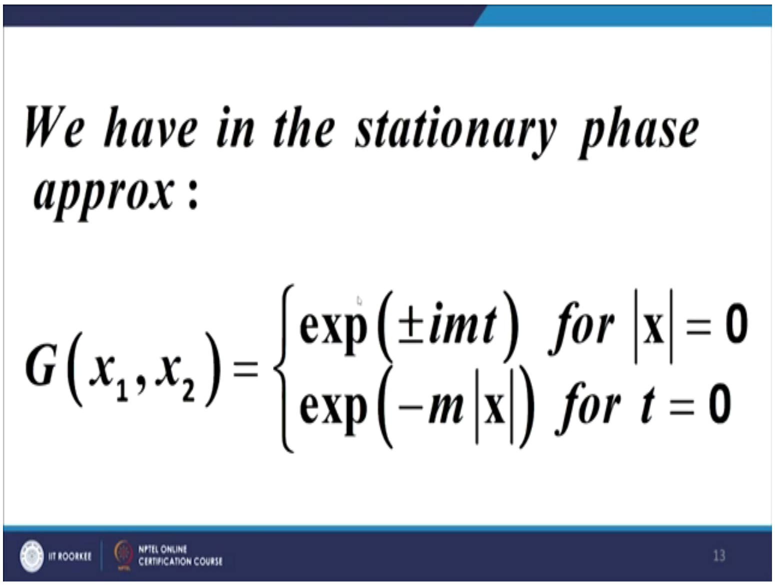
(Refer Slide Time: 09:19)

$$G_M = -\left(\frac{i}{16\pi^2}\right)\int_0^\infty \frac{1}{s^2}\,ds\exp\left(-im^2s + \frac{i\left(|\mathrm{x}|^2 - t^2\right)}{4s}\right)$$

$$= -\left(\frac{i}{16\pi^2}\right)\int_0^\infty \frac{1}{s^2}\,ds\exp\left(-im^2s - \frac{ix_M^2}{4s}\right)$$

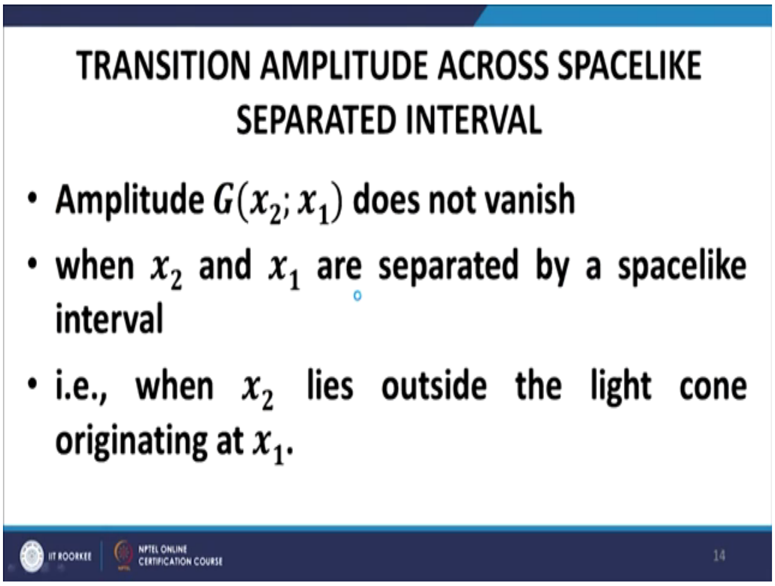
Now, let us look at what happens in the case of the relativistic single particle propagator. You would recall that this is the expression that we got for the relativistic single point propagator. Let us see what happens when we investigate this.

(Refer Slide Time: 09:28)



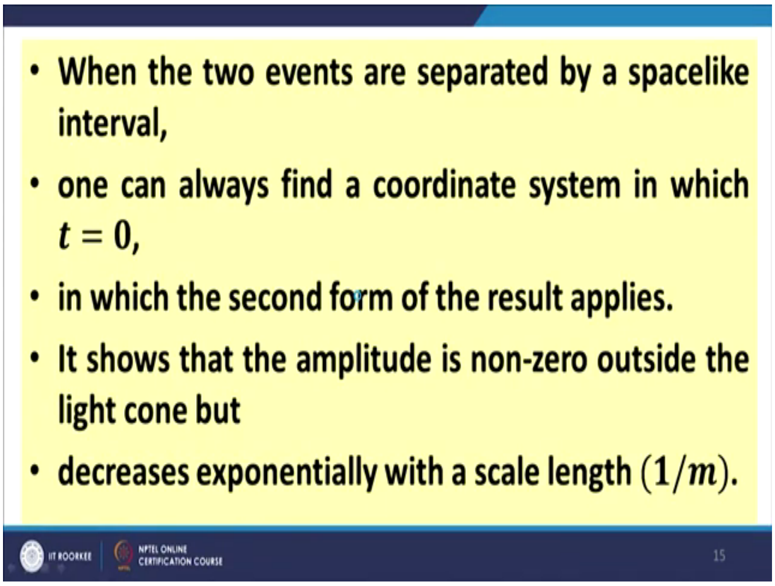
Now, in the saddle point approximation or the stationary phase approximation, this the propagator can be approximated by the expression that is given on your slide. For x equal to 0 that is for zero special separation the propagator can oscillates, and for zero times separation the propagator decays exponentially.

(Refer Slide Time: 10:01)



Now, what is the implication of this? The implication of this is that the propagator does not vanish where if x 2, x 1 has separated by a spacelike interval. Why do I say so? In other words if x 2 and x, if x 2 lies the outside the light cone of x 1, and the light cone that is generated by x 1 x 2 lies outside that light cone in spite of the two points being spacelike separated they the propagator does not have a zero value. In other words there is nonzero probability of propagation between two spacelike separated points.

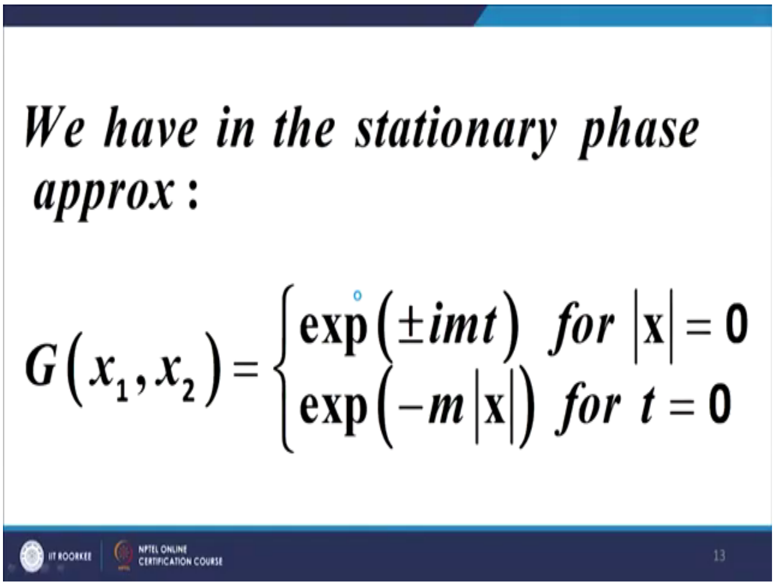
(Refer Slide Time: 10:46)



Why do we say so? Let us look at that see the point is if two points are separated spacelike in a particular Lorentz frame, then you can find another Lorentz frame in which the time in which the two events can be instantaneous.

Let me repeat if you have two points spacelike separated in a particular Lorentz frame, then by a Lorentz frame transformation you can go to another frame in which those two events would be instantaneous in which the time would be zero between the two space time points, two space time events.

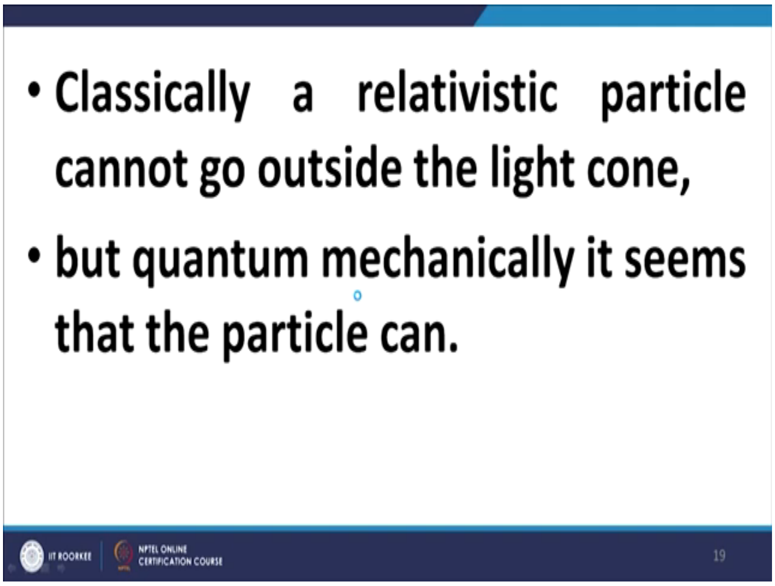
(Refer Slide Time: 11:35)



So, that being the case; that being the case, if you look at this particular term here, this particular approximation, the second approximation applies the second approximation applies and the amplitude does not become 0 for immediately.

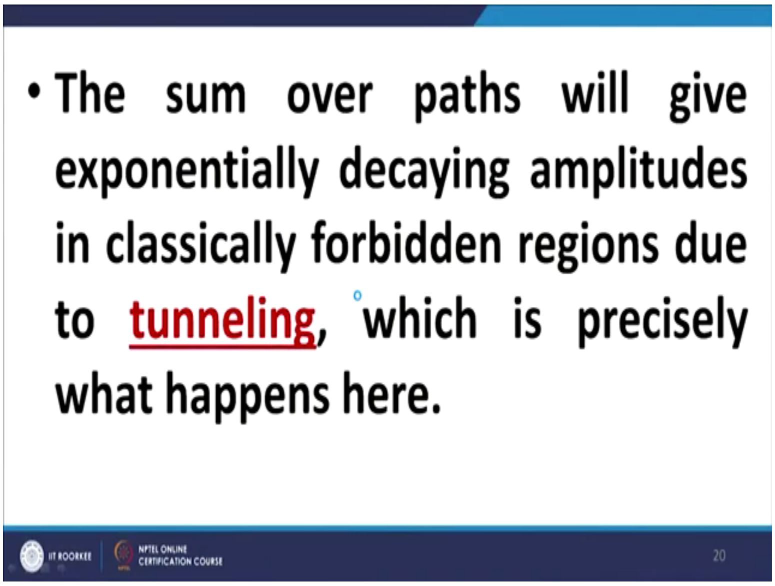
It exponentially decays to 0 on a scale of 1 upon m, so that that is important. This the amplitude or the propagator, the transition amplitude does not immediately forthrightly go to 0 outside the light cone; the it gradually diminishes gradually falls gradually, dams down to 0 at scale of 1 upon m.

(Refer Slide Time: 12:16)

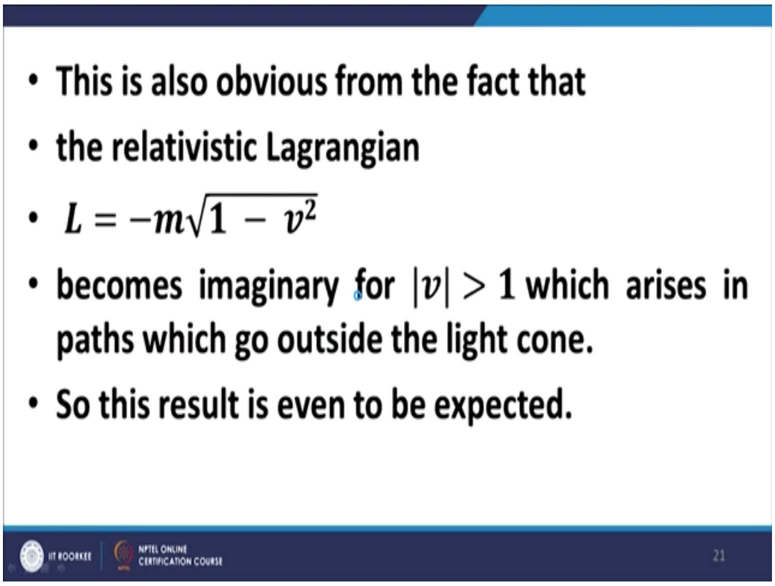


So, the outcome of this discussion is that even if we look at a relativistic single point particle or a point particle, we have a situation where special relativity is violated and therefore, we end up in a problem. So, notwithstanding the fact that Schrodinger equation was in itself non-relativistic and it was expected to violate relativity which it did in fact even in the relativistic framework when we talk about particle localized particles, we have a problem on our hands. Let us say investigate more.

(Refer Slide Time: 12:58)

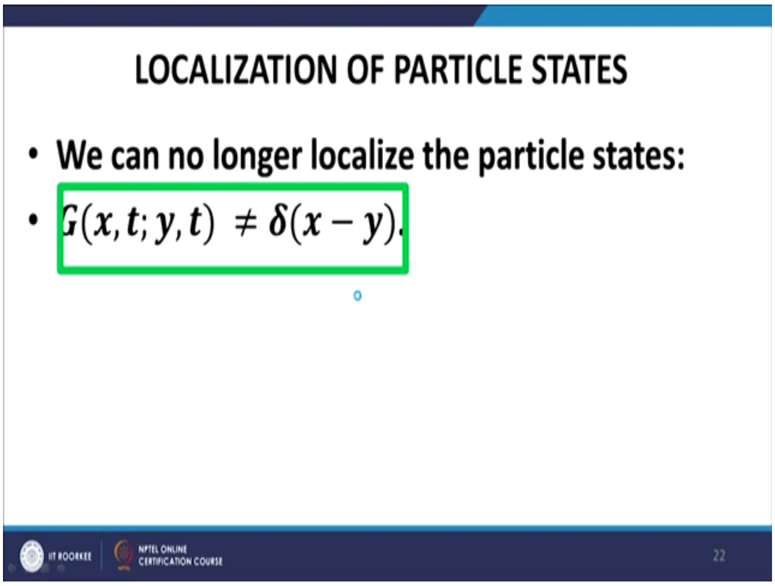


(Refer Slide Time: 13:05)



Now, this process that I elaborated just now is called tunneling. It is very very common, or very very consequence to quantum systems. And in fact, if you can look at the Lagrangian of the single particle, the Lagrangian also becomes imaginary for velocity is greater than the velocity of light, and therefore, this behaviour which was anticipated or which occur in the propagator was more or less unexpected lines.

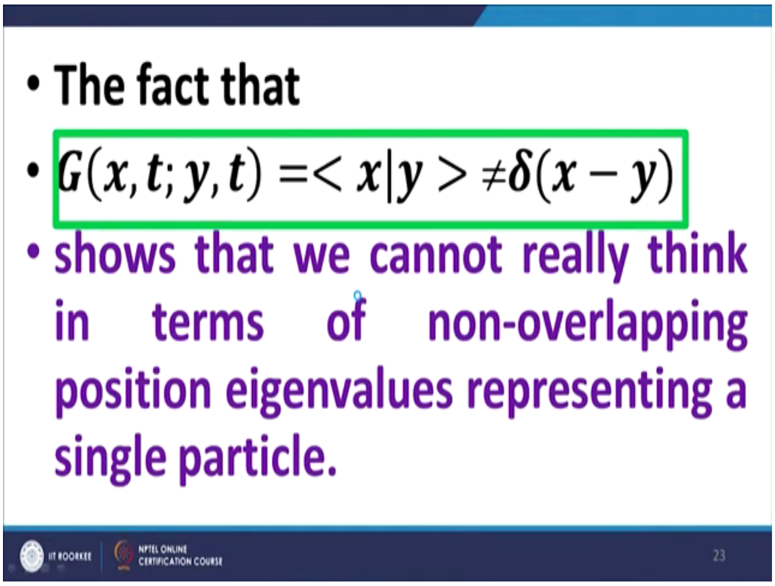
(Refer Slide Time: 13:33)



Now, there is another impact of another inference that can be drawn from the expression that we obtained for the single particle relativistic propagator, and that is that a particle states cannot be localized; particle states cannot be localized. That is clearly seen if you look at this the expression in the green box where the contrary true.

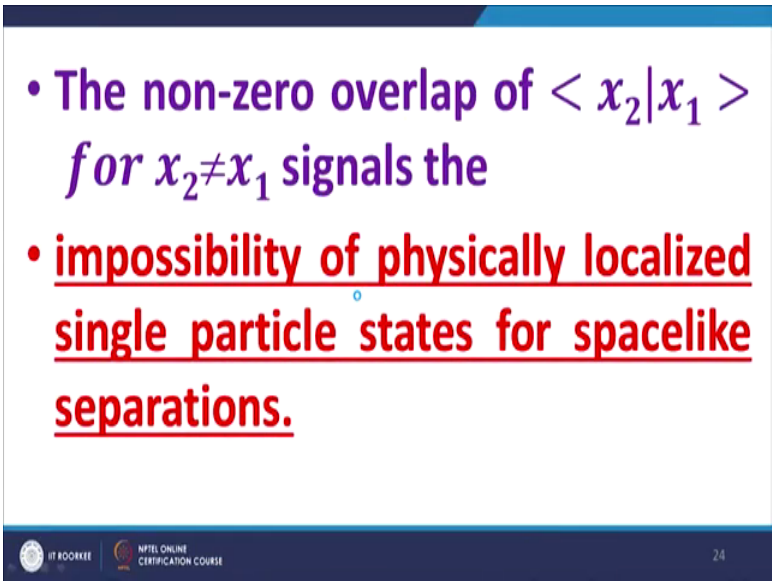
In other words were it were it possible to localize the particle states completely, we should have obtained the delta function for the expression that is on the left hand side of the transition amplitude at a given time, at a fixed time, at a initial point time of two different states, they should be related or they should yield us the delta function, but it does not do so. When we actually work out the transition amplitude the left hand side for the expression that we have derived earlier, we do not get a delta function.

(Refer Slide Time: 14:47)



So, what is the implication of this? This the implication is this that you do not get non-overlapping position eigenvalues. And because you do not get non-overlapping position eigenvalues representing a single particle, the localization fails. And it becomes physically impossible to localize the single particle in the cases particularly in the cases spacelike separations.

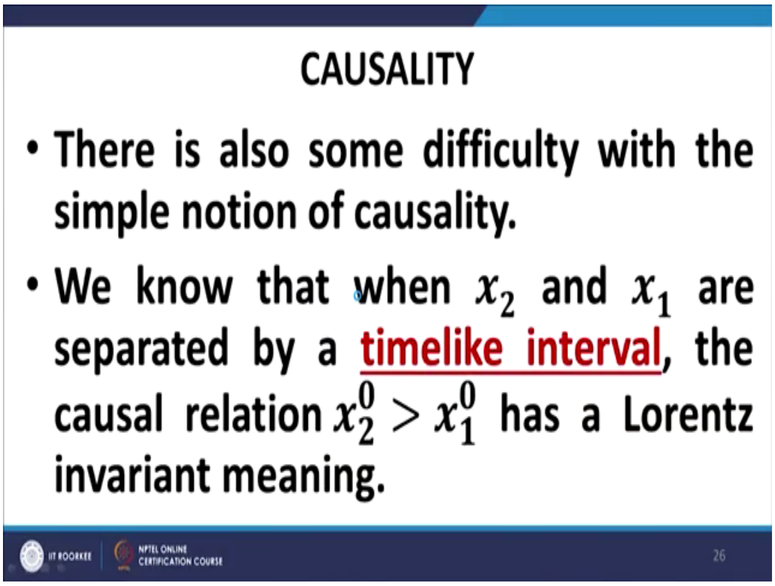
(Refer Slide Time: 15:22)



You cannot attribute specific eigenvalues and eigen states corresponding to specific particles or corresponding to specific measurements on specific particles. So, that problem arises that is another problem which we encounter when we talk about a particle based system or a particle based formalism which includes special relativity. We cannot physically localize single particle states for spacelike separations, so that is another issue that we encounter.

So, in both cases we have problems. And now we look at another issue that again arises from the same expression for the propagator that we obtain for the single particle relativistic a single relativistic particle.

(Refer Slide Time: 16:18)



Now, if x 2 and x 1 are separated by a timelike interval, if x 2 and x 1 are. Separated by a timelike interval, then this expression is Lorentz invariant. In other words, you look at any Lorentz transformation, you would work on any operator or apply any Lorentz transformation on a timelike interval, you will end up with a time like interval. However, so therefore, that order of time likeness was preserved.

If x 2 occurs after x 1 in a particular Lorentz frame, and then it if x 2 occur after the x 1 in terms of a timelike interval, then that order I will always be preserved. But what happens if x 2 and x 1 is separated by a spacelike interval, then we have an issue. Why, because if x 2 and x 1 are separated by a spacelike interval and let us say in that particular frame, let us say there is a frame o in which x 2 occurs after x 1, x 2 occurs after x 1, but x 2 and x 1 are separated by spacelike intervals, then there can be through a Lorentz transformation we can through a

Lorentz transformation transform to another frame o dash in which the time interval between x 2 and x 1 becomes the other way around.

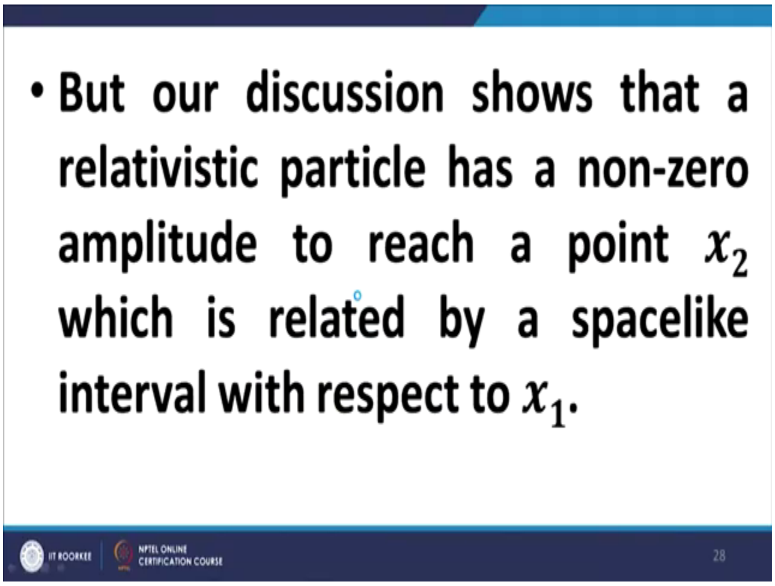
In other words, if in o in the frame o x 2 of follows x 1 in terms of time but the two events are separated spacelike, then in the other frame o dash in the other frame o dash; the time order maybe reversed, where o and o dash are connected by a Lorentz transformation. So, the order of time is not preserved in the cases spacelike intervals. So, that is another important issue another important factor then that is called causality or violation of causality.

So, let me repeat it, because it is very important. If x 2 follows x 1 in a particular frame o and x 2 and x 1 are related through timelike or timelike in a sense, then this statement is invariant or is this statement is Lorentz invariant. In other words, in every Lorentz frame through every Lorentz transformation x 2 will follow x 1 always x 2 will not proceed x 1 provided of course, x 2 and x 1 are connected with through a timelike interval; and number 2 x 2 follow the x 1 in a particular frame. So, x 2 follow the x 1, and x 2 and x 1 being timelike connected through a timelike interval, the two together form a Lorentz invariants statement.

However, in the case where x 2 follows x 1, but x 2 and x 1 connected by spacelike interval, it is not a Lorentz invariant. In other words, through a Lorentz transformation, we can move to a frame from o to o dash in which the time order of occurrence of x 2 and x 1 gets reversed.

In other words, if x 2 follows x 1 in frame o, it may be possible that x 2 precedes x 1 in a frame o dash, where o and o dash are connected by a Lorentz transformation, and x 2 and x 1 are connected by a spacelike interval. So, that is another violation of causality that we encounter in the case of particle based interpretation of physical systems right.

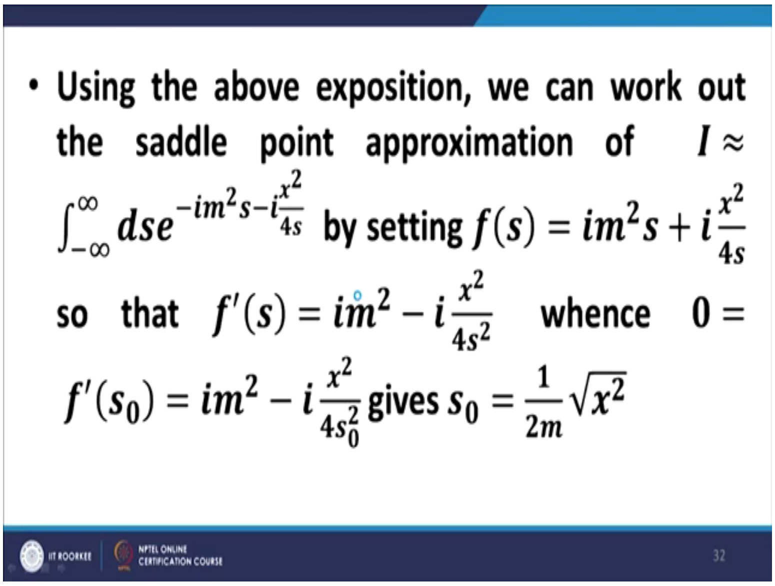
(Refer Slide Time: 20:42)



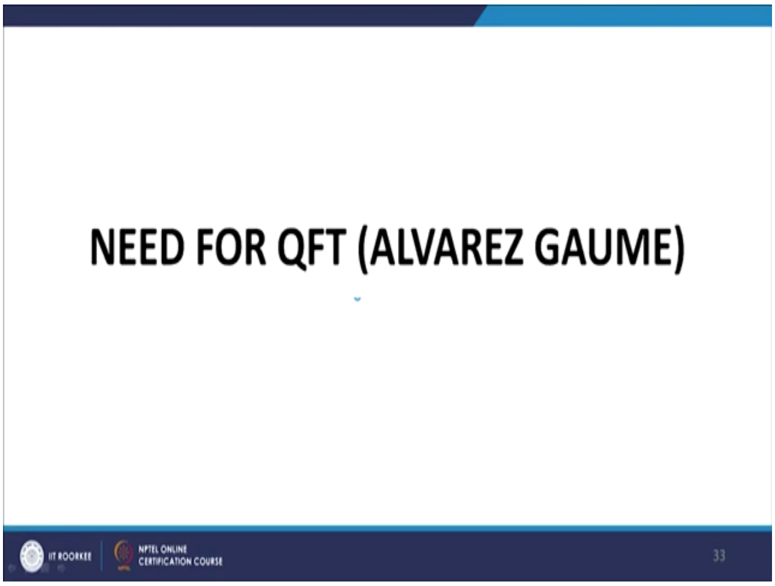
(Refer Slide Time: 20:49)

- Let us say, $f(x)$ is minimum at $x = x_0$. Then we could approximate f(x) by the first few terms of the Taylor expansion $f(x) \approx f(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \cdots$. There is no linear term because $f(x_0)$ is a minimum so that $f'^{(x_0)} = 0$.
- The point is that if f(x) is sufficiently singular close to $f(x_0)$ in the given region, then the contribution to the integral I due to the $f(x)$ on either side of x_0 will be negligible so that
- $I \approx \int_{-\infty}^{\infty} dx e^{-f(x)} = \int_{-\infty}^{\infty} dx e^{-f(x_0) - \frac{1}{2}(x-x_0)^2 f''(x_0)}$
- $= e^{-f(x_0)} \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}(x-x_0)^2 f''(x_0)} = e^{-f(x_0)} \sqrt{\frac{2\pi}{f''(x_0)}}$

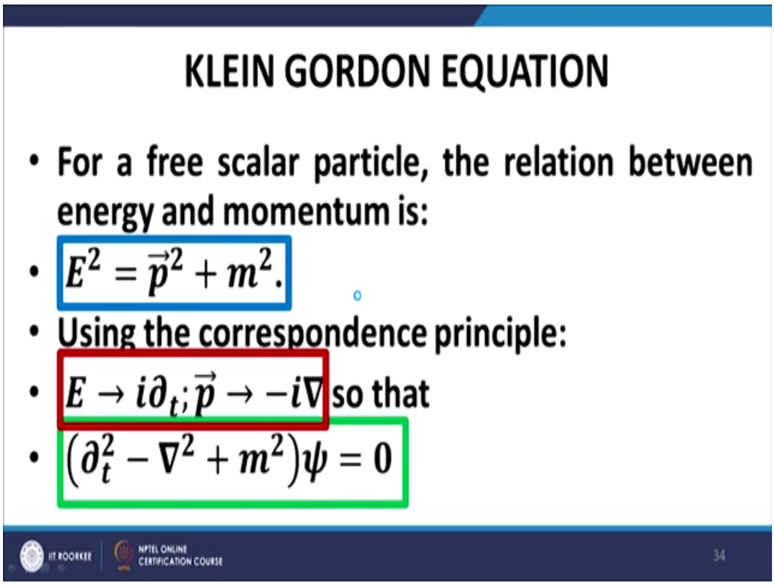
(Refer Slide Time: 20:50)



(Refer Slide Time: 20:50)



(Refer Slide Time: 20:52)

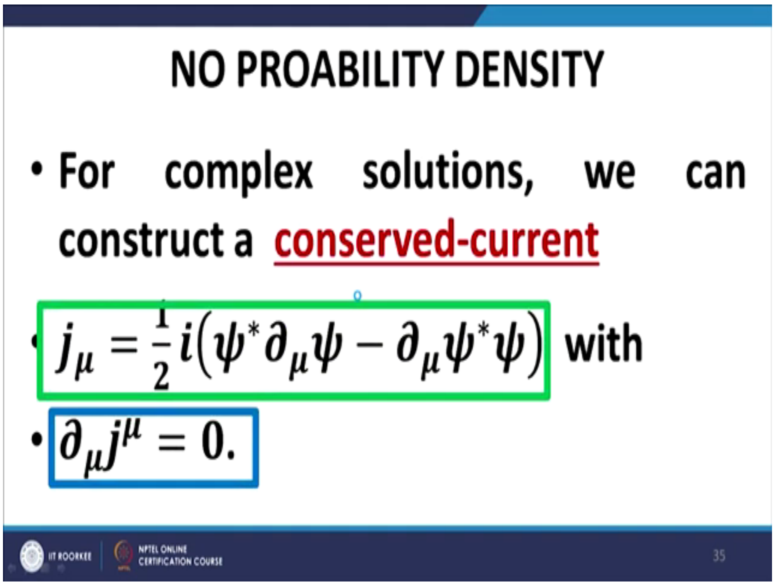


So, continue on this way in we now move towards the relativistic particle based relativistic generalization of the Schrodinger equation. The first attempt of course, was the Klein Gordon equation which is which was the straightforward generalization or straightforward extrapolation of the principles of non-relativistic quantum mechanics.

We have the mass shell condition given in the blue box, what we do is we simply substitute E and p in respect of their quantum mechanical operators that as per the standard practice; standard practice of canonical quantization, we replace E by minus i remember we are putting h bar equal to 1 here.

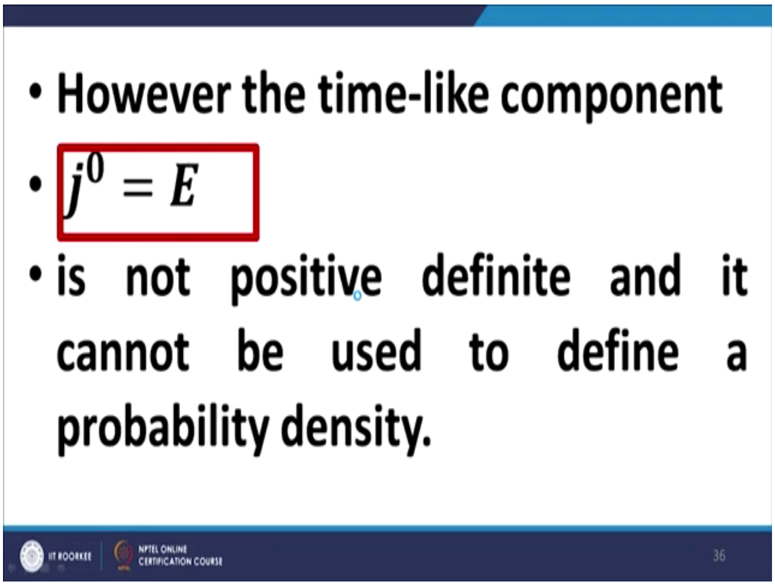
So, E is equal to i del t or the derivative with respect to t, and p as minus i del. So, substituting these values we get the equation that is in the green box and which is refer to as the Klein Gordon equation.

(Refer Slide Time: 22:03)



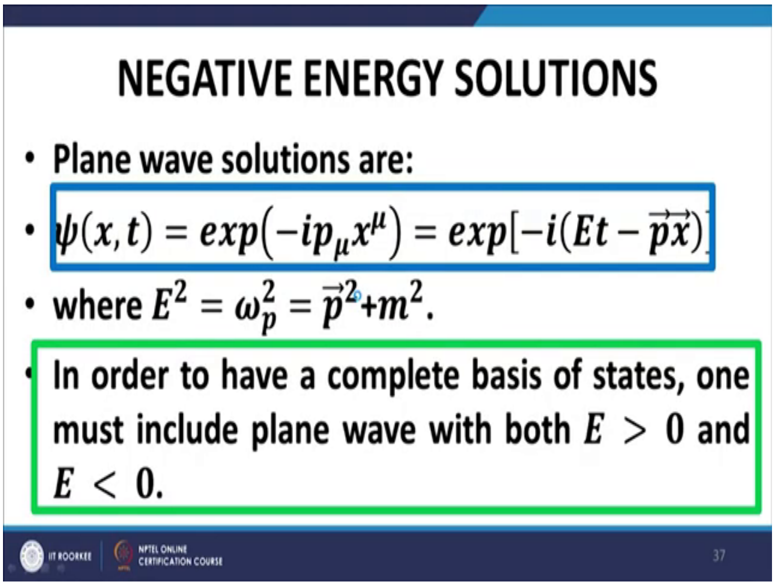
Now, the Klein Gordon equation enables us the good part is; the good part is that the Klein Gordon equation gives us a conserved current in the form of the expression that is given in the green box. It gives you a conserved current, so that part is the positive part and the continuity equation that the conserved current follows is given in the blue box that is again the standard, 4 divergence of the current vanishes.

(Refer Slide Time: 22:39)



However when we look at the time-like component, the time-like component of this conserved current is not positive definite and because it is not positive definite; it cannot be interpret preted as a probability density. And it cannot be used or it cannot be employed to work out a probability density, which is the standard formalism of quantum mechanics.

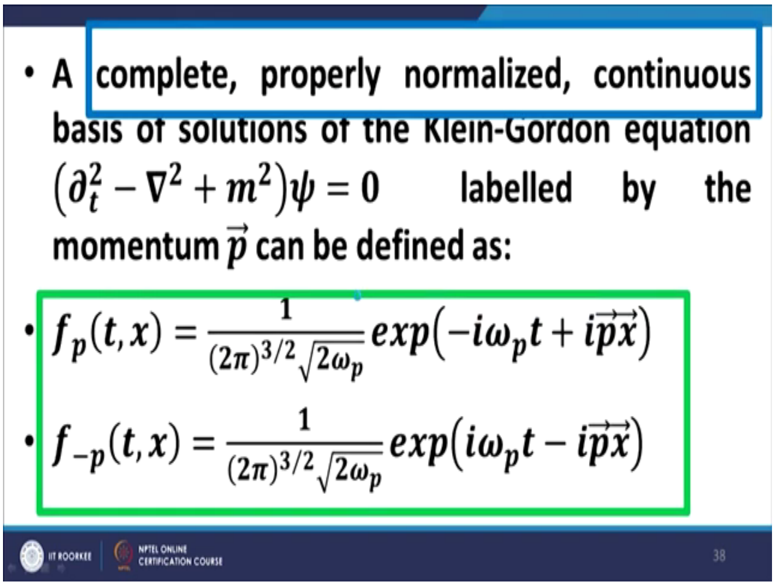
(Refer Slide Time: 23:12)



The second problem with the Klein Gordon equation is also very gigantic, very important. And the plane wave solutions of the Klein Gordon equations are given in the blue box is quite straightforward to solve these equations, to get the plane wave solutions; of course the normalization factor is not here. We have the mass shell condition here omega p square or E square is equal to omega p square is equal to p square plus m square.

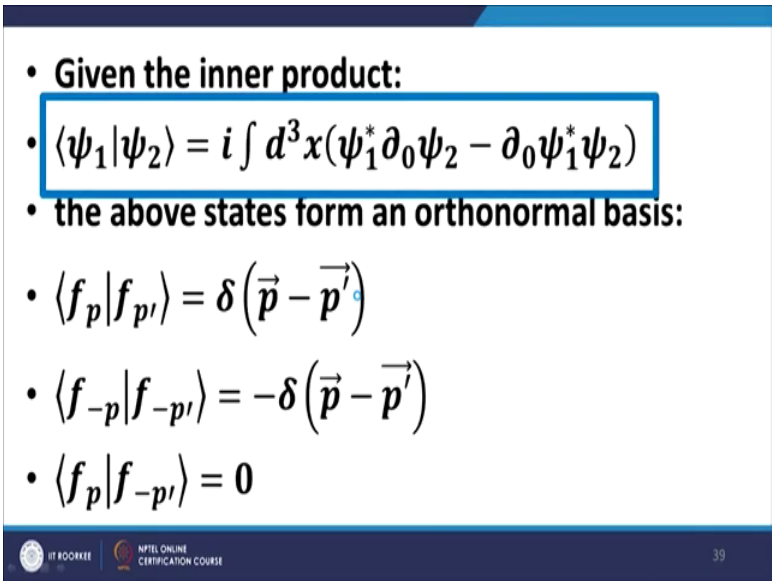
But the important point is if you have want to have a complete basis of states, as you can see from the equation in the blue box; if you want to have a complete basis of states, then you must include plane waves for E greater than 0 and E less than 0; E greater than 0 alone does not provide you with a complete basis, for complete for completion of the basis of states, you also need E less than 0 as well.

(Refer Slide Time: 24:20)



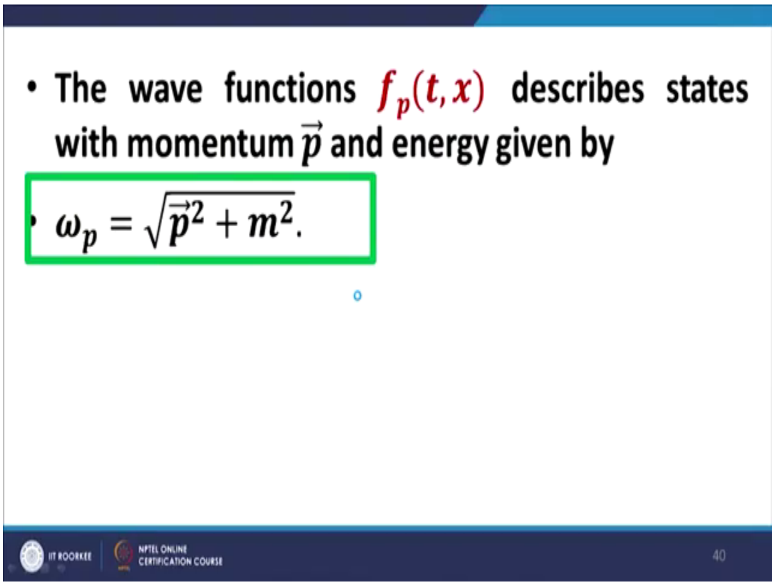
So, the a complete for properly normalized continuous basis of states or solutions of the Klein Gordon equation are given in the green box. The upper set they are labeled by the momentum p, but recall that the momentum p is connected to the energy by the mass shell condition. So, this is the properly normalized complete basis; now the upper states as we I shall mention in the next slide.

(Refer Slide Time: 24:56)



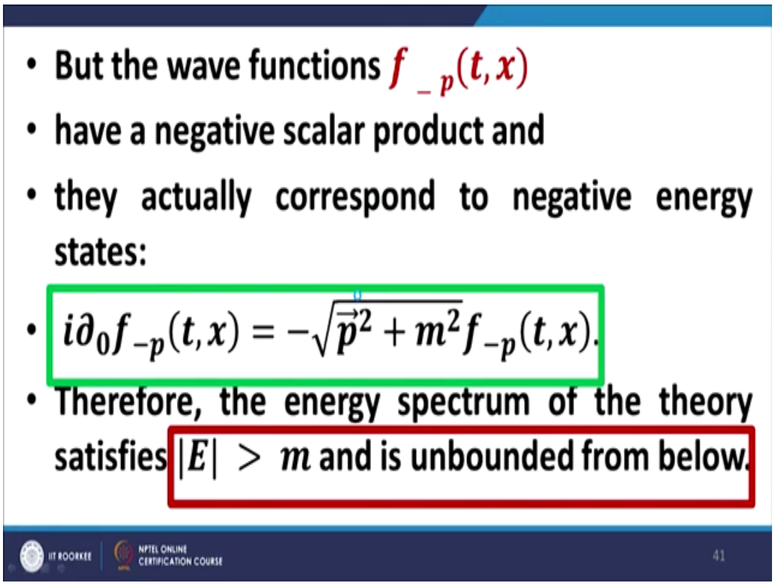
If this is how we defined the inner product, the expression in the blue box; if this is how we define the inner product in our given Hilbert space, then the above states; above states means we are referring to these states, these states above the orthogonality relations given here in the bottom of your slide.

(Refer Slide Time: 25:34)



Now, the important thing is as far as the positive momentum wave functions are concerned, as far as the positive momentum wave momentum functions are concerned there is not much of a problem. They follow the mass shall condition, omega p to under root p square plus m square.

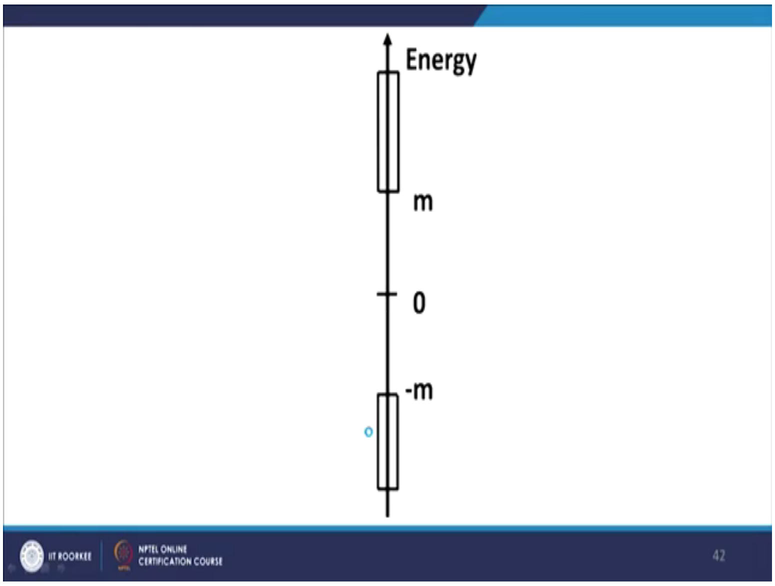
(Refer Slide Time: 25:47)



And furthermore if you operate on them with i d 0, you get a positive eigen value and so there is nothing wrong with that nothing to worry about. But when we look at the negative momentum, states; the states that are given here in the first line f minus p t and x, they not only have a negative scalar product. If you look let us go back, these states if you look at the second equation in the bottom set of equation, they have a negative scalar product.

They not only have that negative scalar product, they have another varying consequence; and the varying consequences if you apply the energy operator to these states, we get a negative energy eigen value. A negative energy eigen value means a worrying feature, how do we explain the existence of negative energies. So, it follows from this that the energy spectrum of the theory; the energy spectrum energy eigen values of the theory that the Klein Gordon equation yields is of the form mode of E is greater than m of this form which is given in your slide.

(Refer Slide Time: 27:04)

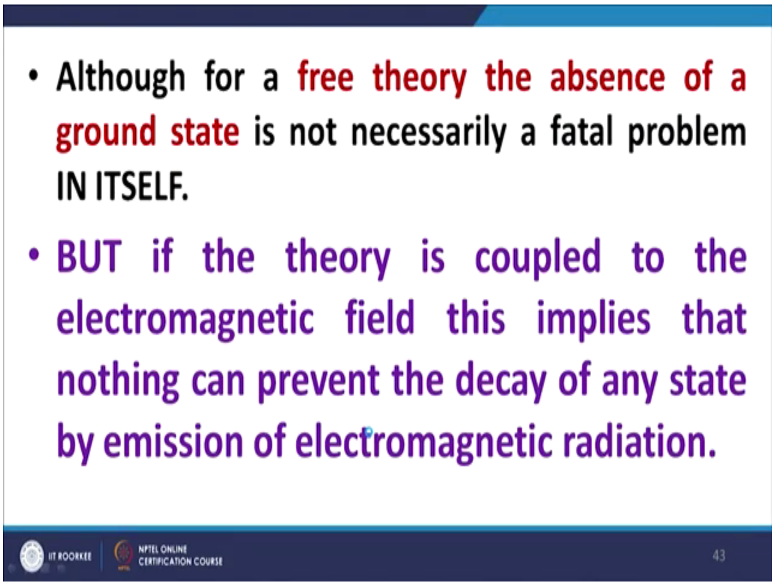


So, we have energy is greater than m and then we have energy is less than minus m, so that is then outcome that is an outcome of the Klein Gordon equation which is again not prime of s i at least not acceptable. Firstly, we have the problem of no positive definite, probability density and secondly, we do not we end up with having negative energy solutions of the Klein Gordon equation which are necessary in fact, to ensure that the basis is complete.

So, these are two problems and this problem implies that if there is an interaction between a scalar particle that is a particle that obvious the Klein Gordon equation, and an electronic electromagnetic field; if there is an interaction between a particle obeying the Klein Gordon equation.

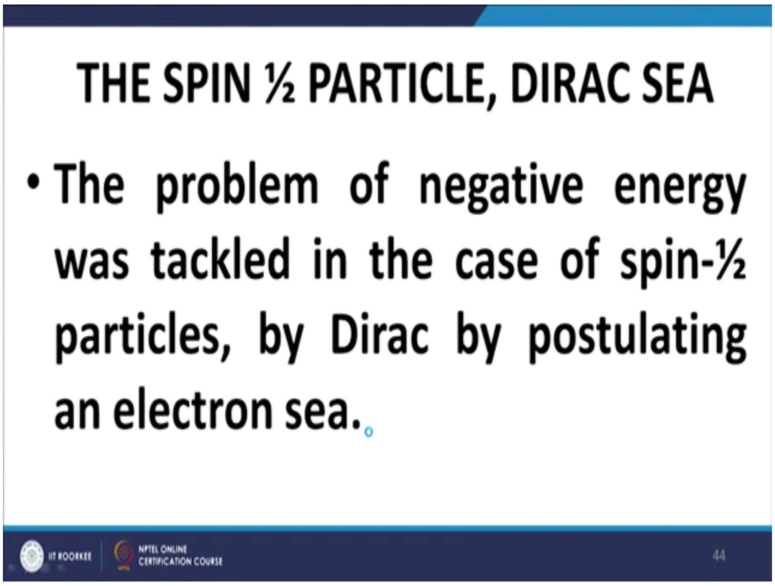
And the electromagnetic field, then works labor maybe the particle can go down to lower, lower and lower states without any lower bound; so that was a serious drawback of the Klein Gordon equation in the form that we have discussed now in the particle formulation.

(Refer Slide Time: 28:24)



Where we talk about the Klein Gordon equation in the particle formulation, single particle formulation.

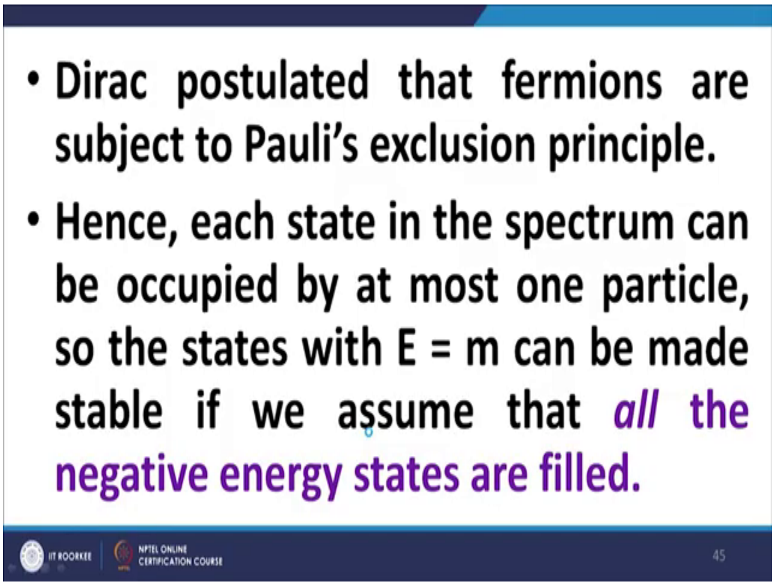
(Refer Slide Time: 28:39)



However, to some extent Dirac was able to counteract to this issue of negative energy states by postulating that at least in the case of spin half particles, because the Pauli's exclusion principle operates and you can have only one electron filling of a state. And therefore; and therefore, the argument propounded by Dirac was that all the negative energy states were initially filled up and they found what is later on a name as the Dirac sea.

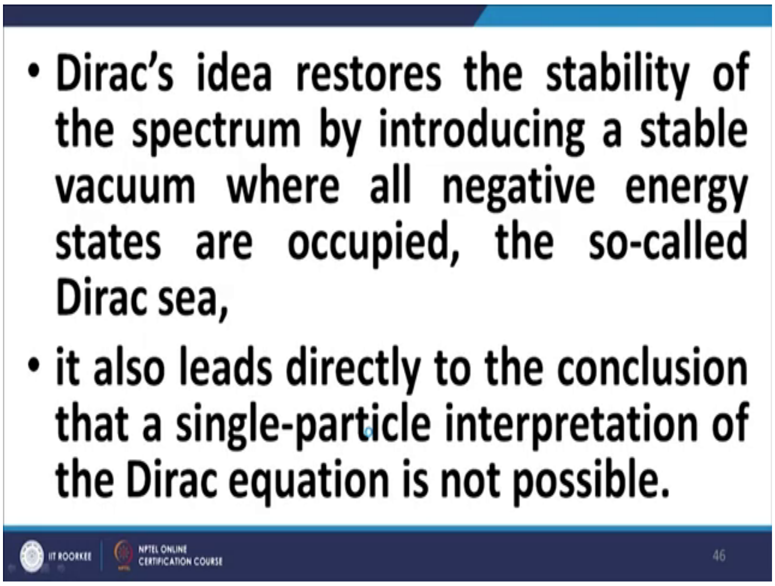
And as such the negative energy states being already filled up, do not pose any problem and consequently that interpretation of or the existence of negative energy is not an issue with the particle proposition or the particle base theory, contemplated here.

(Refer Slide Time: 29:43)



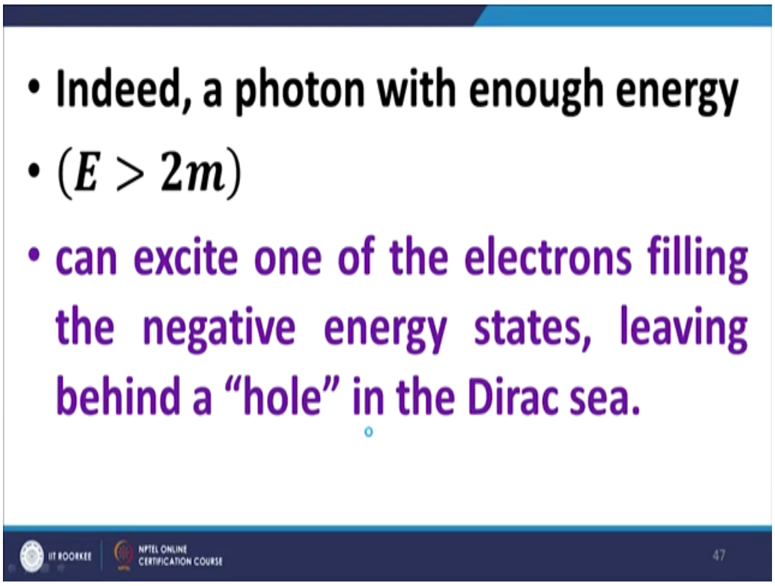
So, that was an argument based on Pauli's exclusion principle. What is it was that every state could contain only one electron and every state was filled up every negative energy state was filled up, and as a result of which the particles could exist or the particles may move only in the positive energy states.

(Refer Slide Time: 30:11)



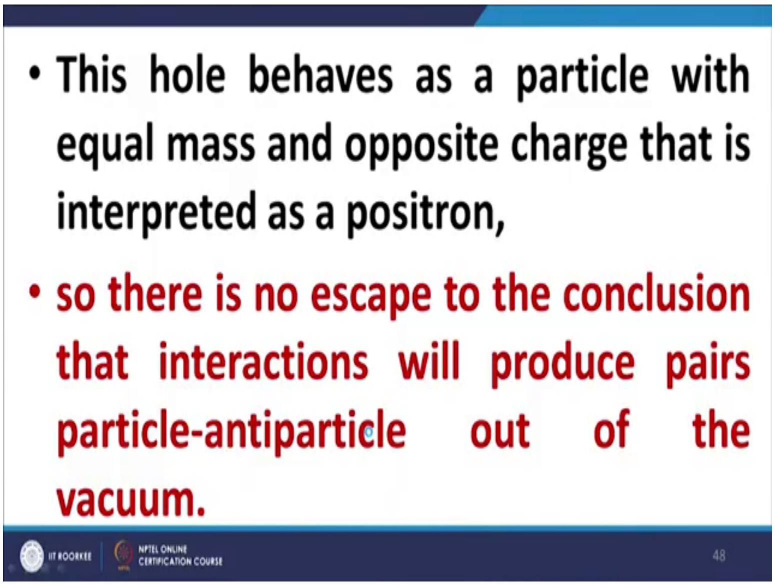
But however, however the issue is that once he say once a Dirac introduced the concept of negative energy states being filled up by electrons already. Then he already is moving towards a single-particle to a multi particle based interpretation, so that is that was the initiation in some sense that everything is not well with the single-particle based interpretation of quantum mechanics.

(Refer Slide Time: 30:52)



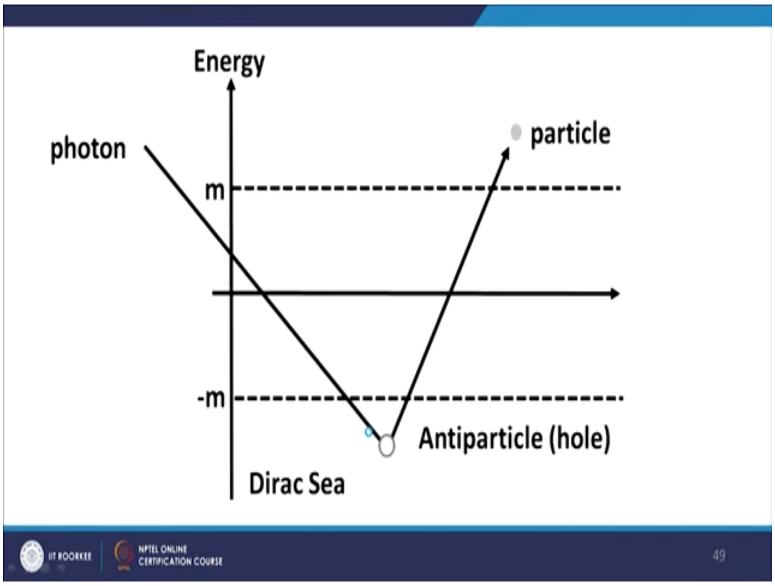
And in fact, if a particle is bombarded on this Dirac sea with energy greater than 2 m; where m is the mass of the electron. Then it is quite, quite practical that the particle could form a electron positron pair and the electron could shootout and the positron could be represent or could represent or could be present as a whole in the Dirac sea.

(Refer Slide Time: 31:18)



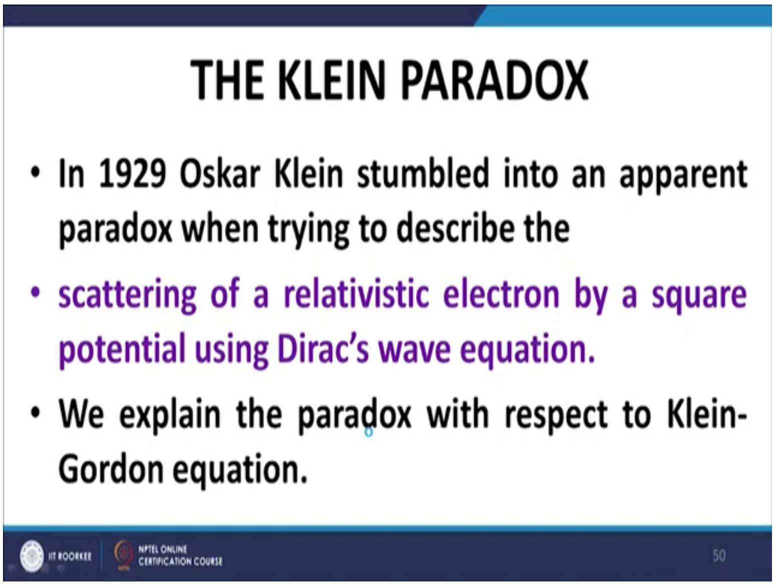
And then this whole which is a positron really could be as the an electron of equal mass and opposite charge, and it was named as the positron.

(Refer Slide Time: 31:33)



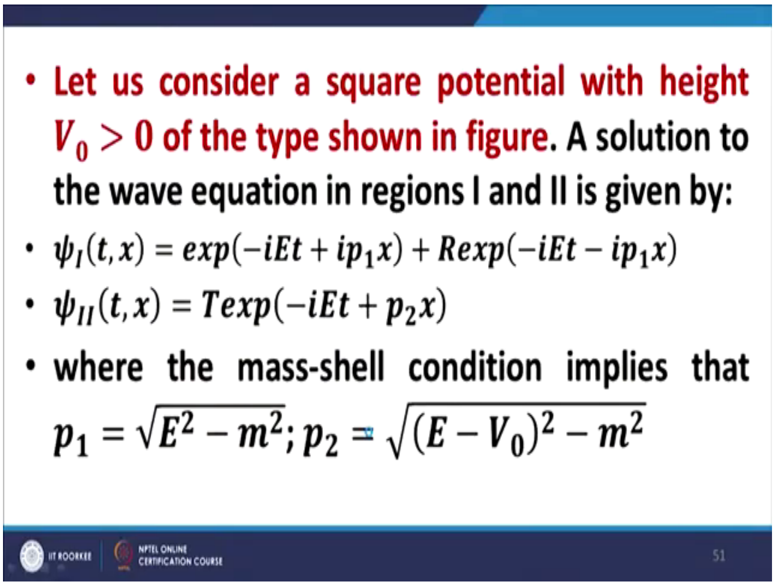
And this is the figure that displays this interaction, a photon, carrying energy greater than 2 m is interacting on the Dirac sea at a particular point where appear is generator consisting of the electron and the antielectron or that is the positron. The electron flies off and the positron remains in the Dirac sea creating or appearing in the form of a whole in the Dirac sea.

(Refer Slide Time: 32:02)

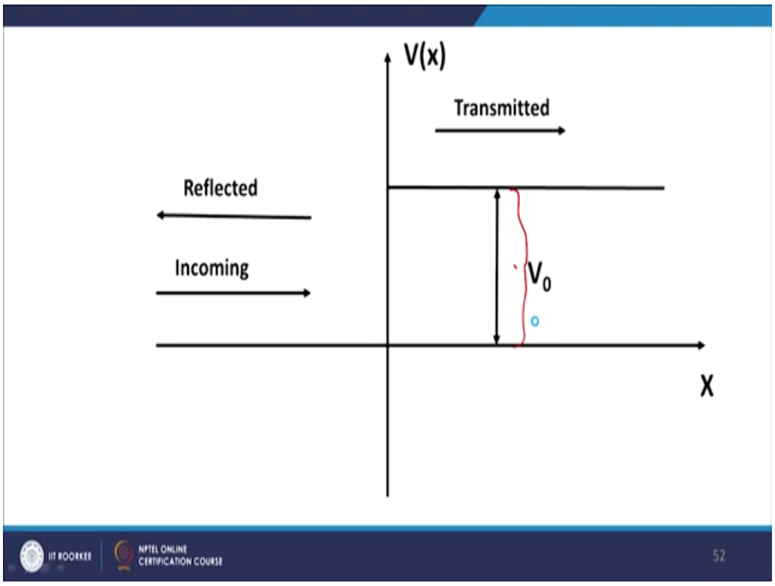


Then we had a more serious problem also that that was the Klein paradox, which again required a multi particle interpretation in order to make some anything sensible out of it. Klein was studying, the scattering of a relativistic electron by a square potential using Dirac wave equation. So, we will I will enunciate elaborate on this issue, but I will use the Klein Gordon equation since that is simpler to consider simpler to discuss.

(Refer Slide Time: 32:46)



(Refer Slide Time: 32:47)



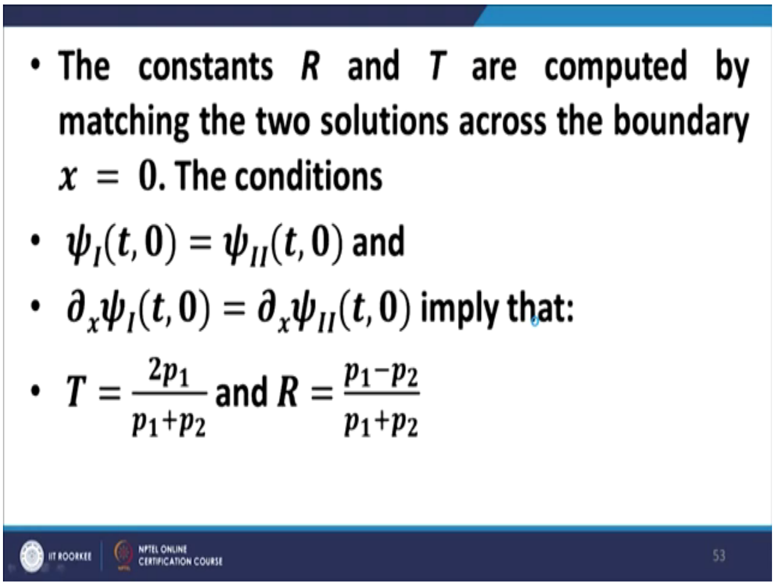
What happens is suppose we have a square potential of height V 0, this is my square potential of height V 0. And the particle is are being bombarded on the square well potential well, particles are moving towards the potential, the potential barrier as it is the height of the potential barrier is V 0. Particles are bombarded towards the positive X direction from a particular source and towards this potential barrier and the impact is studied.

So, what happen; now we arrive at the expressions for the various parameters of the problem, the reflection coefficient and the transmission coefficient by making certain mandatory requirements, by imposing certain mandatory requirements on the wave function that the initial wave function and the wave function post the interaction with the potential barrier have to follow.

First of all is that the wave functions have to be continuous. So, wave functions need to be continuous in both the region; this is a region 1, this is region 2. So, wave functions in both the regions need to be continuous at across the barrier or at the barrier. And the second is the first derivatives of the wave functions also need to be continuous, along the barrier.

So, these were two conditions that need to be imposed on the wave functions to make arrive at sensible solutions. And making use of these two sensible, these two conditions, we arrived at results which are given in the slide.

(Refer Slide Time: 34:46)



We get the transmission coefficient as this expression and the reflection coefficient as this expression. From here, we will continue after the break.

Thank you.