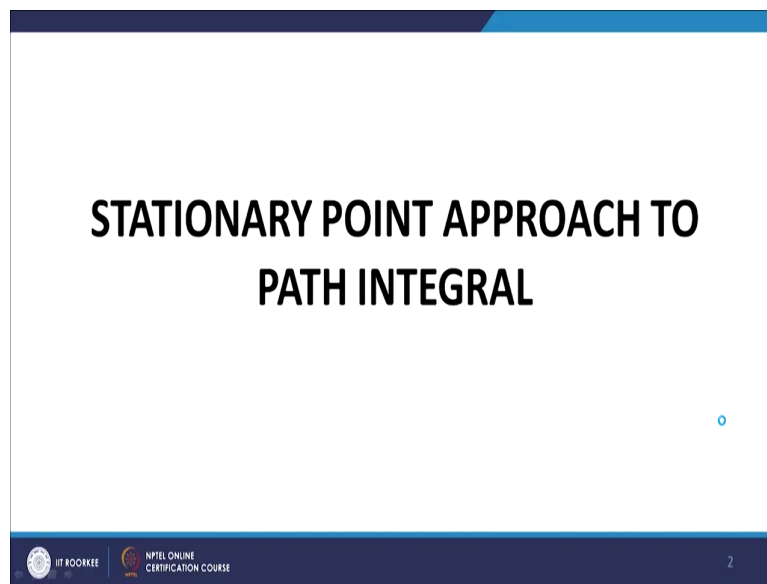


**Path Integral Methods in Physics & Finance**  
**Prof. J. P. Singh**  
**Department of Management Studies**  
**Indian Institute of Technology, Roorkee**

**Lecture - 28**  
**Interpretation of Path Integral**

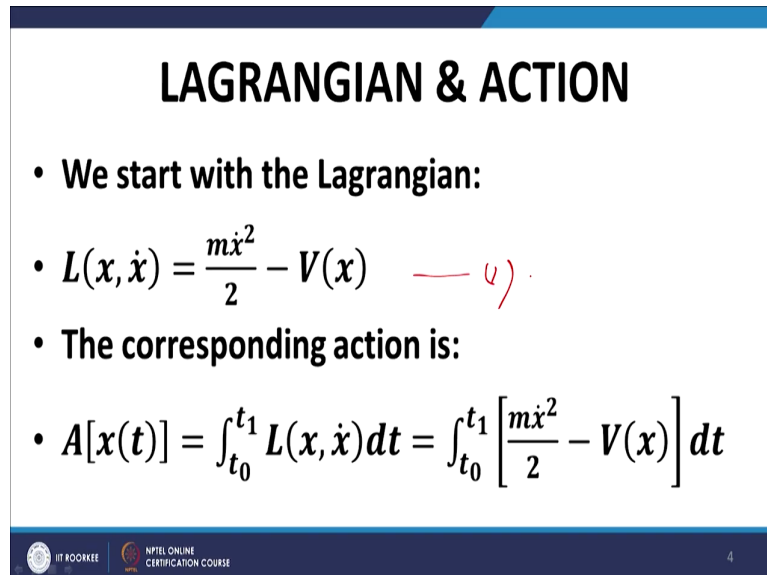
Welcome back. So, before the break I told you that we shall be talking about at different approach, modified approach to working out the Path Integral. Now, this approach is certainly more instructive, more intuitive and more explicit than the conventional approach that we have been following so far. It is the stationary point approach to the path integral.

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Again, it works on the Hamiltonian it starts with the Hamiltonian formalism and then we progress to the same version of the path integral as we had worked out based on the Lagrangian formalism.

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**LAGRANGIAN & ACTION**

- We start with the Lagrangian:
- $L(x, \dot{x}) = \frac{m\dot{x}^2}{2} - V(x)$  —  $\psi$ .
- The corresponding action is:
- $A[x(t)] = \int_{t_0}^{t_1} L(x, \dot{x}) dt = \int_{t_0}^{t_1} \left[ \frac{m\dot{x}^2}{2} - V(x) \right] dt$

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Now, let us first work out the Lagrangian, let us start with a simple physical system. The Lagrangian of the system can be written in the form of equation 1, which is here. The first term is the kinetic energy term; the second term is the potential energy term.

So, and the corresponding action which is a function of every path it is a functional. So, it is a function of every path, it can be written as the integral of the Lagrangian with respect to; the with respect to time over the carried out over the time of evolution of the system.

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## HAMILTON'S PRINCIPLE & PHYSICAL PATHS

- Hamilton's principle asserts that a
- path is physical, that is, a solution of Newton's equations of motion, if and only if
- THE ACTION IS STATIONARY I.E.
- $\frac{\delta A}{\delta x(t)} = 0$
- Thus, first order variations about a physical path cause only second order variations in the action.

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Now, Hamiltonian principle says that paths that are physical paths satisfy a certain requirement and that requirement is the stationarity of the action. The action is stationary with respect to the variation of the paths. The first order variation in the action is stationary, first order variation in the action vanishes.

In other words, we can say that first order variations about a physical path do not cause any first order variation in the action. They may cause second and higher order variations in the actions, please note this point. Now, let us see quantitatively evaluate the variations of the action due to a variation in the path.

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- Now we evaluate the variations of the action along a modified path, obtaining
- $A[x(t) + \delta x(t)] = \int_{t_0}^{t_1} \left\{ \frac{m}{2} [\dot{x}(t) + \delta \dot{x}(t)]^2 - V(x(t) + \delta x(t)) \right\}$

- $= \int_{t_0}^{t_1} \left\{ \begin{aligned} &\frac{m}{2} [\dot{x}(t)^2 + 2\dot{x}(t)\delta \dot{x}(t) + \delta \dot{x}(t)^2] \\ &-V(x(t)) - V'(x(t))\delta x(t) - \frac{1}{2}V''(x(t))\delta x(t)^2 - \dots \end{aligned} \right\} dt$   
 $= T_0 + T_1 + T_2 + \dots,$

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The variations in the action as you can see can be written, it can be expanded the power series in delta x and it therefore, it takes the form which is given in the green box in your slide.



The first term is comprises of the action of the original path  $m \int \dot{x}^2 dt - \int V(x) dt$  we call it  $T_0$ , the second term  $m \int 2\dot{x}\delta \dot{x} dt - \int V'(x)\delta x dt$  that we call it as  $T_1$  and the third the other term  $m \int \delta \dot{x}^2 dt - \int \frac{1}{2}V''(x)\delta x^2 dt$  we call it as the  $T_2$ , the quadratic variation, and the first order variation and the zeroth order variation.



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- where we have expanded the integrand in powers of  $\delta x(t)$  and written the zeroth order, first order, etc terms as  $T_0, T_1$ , etc.
- The zeroth order term is just the action evaluated along the unmodified path  $x(t)$ ,

$$T_0 = \left[ \frac{m}{2} \dot{x}(t)^2 - V(x(t)) \right] dt = A[x(t)].$$

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So, this is what we have done by expanding the integrand in powers of  $\delta x(t)$ ,  $T_0$  is the zeroth order variation,  $T_1$  is the first order variation, and  $T_2$  is the second order variation. The zeroth order variation is just the action evaluated along the unmodified path, when there is no variation in the path. So, this is what the various the green box represents the zeroth order variation  $T_0$ , which is the variation of the unmodified path.

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- The first order contribution to the action,  $T_1$ , has two terms, of which we integrate the first by parts to obtain

$$T_1 = \int_{t_0}^{t_1} [m\dot{x}(t)\delta\dot{x}(t) - V'(x(t))\delta x(t)] dt$$

- $m\dot{x}(t)\delta x(t)|_{t_0}^{t_1} + \int_{t_0}^{t_1} [-m\ddot{x}(t)\delta x(t) - V'(x(t))\delta x(t)] dt.$

Now, the first order contribution to the action or the first order variation if we do an integration by parts and if we ignore the surface terms, what do we get?

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- The boundary term vanishes because of the end point condition, and the rest can be written,

- $T_1 = \int_{t_0}^{t_1} [-m\ddot{x}(t) - V'(x(t))]\delta x(t)dt.$

- We see that  $T_1$  vanishes if the path  $x(t)$  is physical, that is, if it satisfies Newton's laws,

- $m\ddot{x} = -V'(x).$



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We get this expression which is there in the green box here. So, this is the first order variation remember. So, what happens if we substitute, if we take the condition or if we impose the condition  $T_1$  equal to 0, if we impose the condition  $T_1$  equal to 0, because delta x is an arbitrary variation the integrand must vanish and if the integrand or the coefficient of delta x must vanish and that gives us the expression  $m \ddot{x}$  is equal to minus  $V'(x)$  which is nothing, but the Newton's laws.

So, in other words what have we concluded that  $T_1$  vanishes, if this the system that we are considering follows the Newton's laws and conversely even the converse holds the converse holds in the sense that if  $T_1$  is equal to 0, then the system will follow Newton's law.

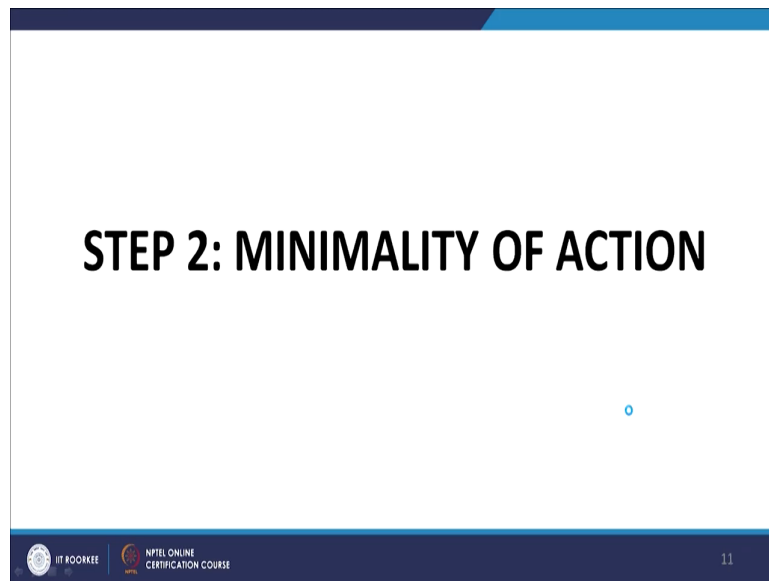
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- **Conversely, if  $T_1 = 0$  for all choices of  $\delta x(t)$ , then the Newton's laws must be satisfied.**
- **This is Hamilton's principle for the type of one dimensional Lagrangian we are considering.**
- **The notation in the present case is**
- $$\frac{\delta A}{\delta x(t)} = -m\ddot{x}(t) - V'(x(t)) = 0$$

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So, this is the Hamilton's principle. It is really the it is commonly known as the principle of least action, but it would be more precise to call it as the principle of stationary action rather than least action right.

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Now, that is that brings us to the question whether under what circumstances the action is actually least and whether stationary action automatically implies that the action of the physical path of the system along the physical path is least, that is what we will investigate now.

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- To see whether the action is minimum or not along a classical path, we must look at the second order variations in the action viz.

$$\int_{t_0}^{t_1} \left[ \frac{m\delta\dot{x}(t)^2}{2} - \frac{1}{2}V''(x(t))\delta x(t)^2 \right] dt = T_2.$$

- Here we are assuming that  $x(t)$  is a classical path so  $T_1 = 0$  and  $T_2$  is the first nonzero correction to the action along the classical path.

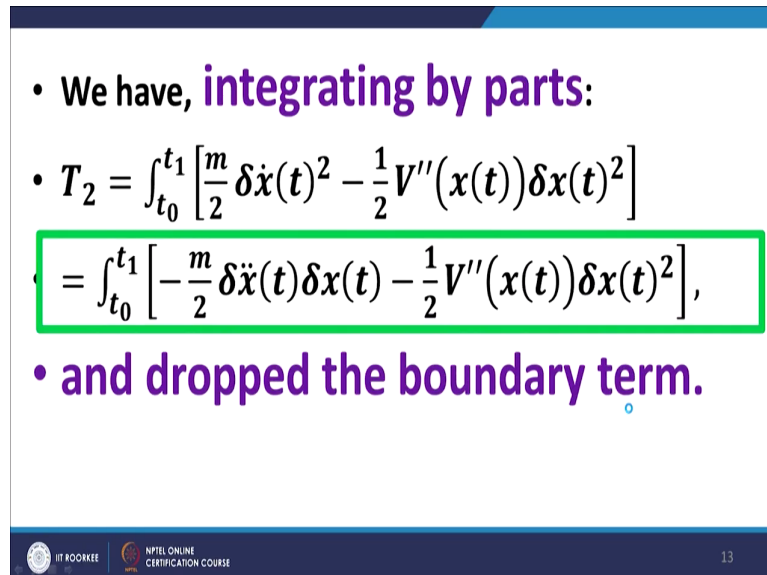
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To see whether the action is minimum along a classical path, let us look at the second order variation in the action. This is the various second order variation  $T_2$ . You can recall it is the, it was there in the earlier slide, the slide at the beginning of our discussion, this one. It has been extracted from this particular equation. The second order terms in  $\delta x$  square.

Now, the delta quadratic term  $\delta x$  were the coefficients of that constitute  $T_2$ , right. So, so let us come back to this. So, this is  $T_2$  this is  $T_2$  the second order variation. Now, please note, because the we are considering a physical path we have automatically assumed that  $T_1$  is equal to 0 and therefore  $T_2$  is now, the first nonzero correction to the action.

Remember  $T_0$  was nothing, but the original path the action along the original path without any variation,  $T_1$  we have taken as 0, because we can consider or investigate physical path therefore,  $T_2$  becomes the first nonzero correction in the action.

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• We have, **integrating by parts:**

•  $T_2 = \int_{t_0}^{t_1} \left[ \frac{m}{2} \delta \dot{x}(t)^2 - \frac{1}{2} V''(x(t)) \delta x(t)^2 \right]$

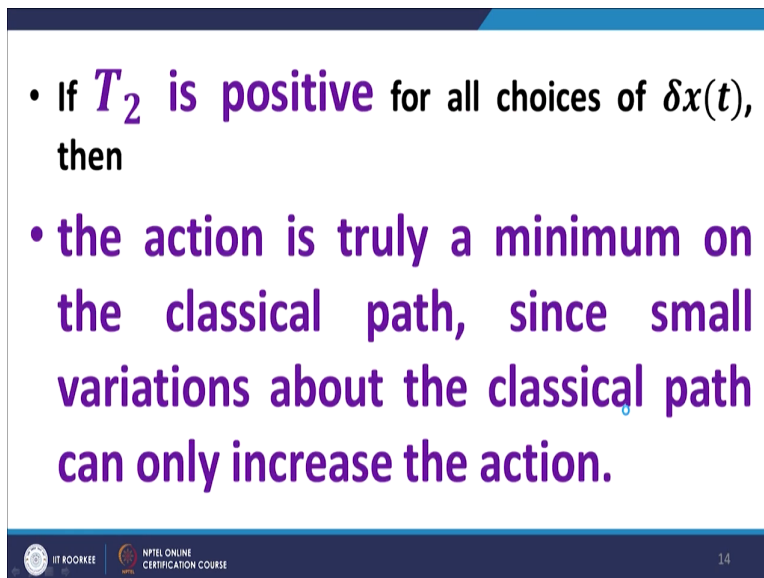
$= \int_{t_0}^{t_1} \left[ -\frac{m}{2} \delta \ddot{x}(t) \delta x(t) - \frac{1}{2} V''(x(t)) \delta x(t)^2 \right],$

• **and dropped the boundary term.**

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If we integrate this expression by parts, we get this result and of course, when we drop the boundary term which is just usually assumed to hold and if that is the case then we get the expression that is within the green bracket.

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- If  $T_2$  is positive for all choices of  $\delta x(t)$ , then
- the action is truly a minimum on the classical path, since small variations about the classical path can only increase the action.

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Now, if  $T_2$  is positive, now, comes the important part. If  $T_2$  is positive for all choices of  $\delta x$ , if  $T_2$  is positive for all choices of  $\delta x$  this implies that our action is indeed minimum at the given path, at the classical path, at the given path, at the physical path.

But, because what happen even a small variation here small variations in the action will increase the action, will not increase will not decrease the action, because the second order derivative or the second order variation is positive. So, if the second order variation is positive it means that the action along the classical path or the physical path is actually a minimum.



If it is not so, then we it is not mandatory, it is not obligatory that the action along the given path that we are considering is actually a minimum. Now, to formulate a test for this minimality of action, one could devise a scheme on this form.



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**To formulate a test for this minimality of action, let:**

- 1.  $f(t), g(t)$  etc be real functions:**
- 2. defined on  $t_0 \leq t \leq t_1$**
- 3. that vanish at the endpoints,  $f(t_0) = f(t_1) = 0$ , etc.**
- 4. Define a scalar product of  $f$  and  $g$  in a Dirac-like notation by:  $\langle f|g \rangle = \int_{t_0}^{t_1} f(t)g(t)dt$ .**


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We define  $f(t)$  and  $g(t)$  as functions of time, real functions of time and in the defined on the interval  $t_0$  to  $t_1$  and these functions  $f(t)$  and  $g(t)$  both vanish at the boundaries of time that is at  $t_0$  and  $t_1$ . Both these functions  $f(t)$  and  $g(t)$  vanish at the time boundaries  $t_0$  and  $t_1$  and otherwise they are defined on the interval  $t_0$  to  $t_1$ .

We also define a scalar product of  $f$  and  $g$  in the conventional way that we have in the case of quantum mechanics, but the important part is here, we do not consider the conjugate, because we are talking about real functions, because  $f(t)$  and  $g(t)$  are both real function so; obviously, the conjugates coincide with the originals.

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- **These functions form a Hilbert space, and the scalar product looks like that of wave functions  $\psi(x)$  in quantum mechanics except the variable of integration is  $t$  instead of  $x$ .**
- **The boundary conditions are like those of a particle in a box.**



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Now, this formalism of the structure enable us to form a Hilbert space and the scalar product is very similar to what we encounter in quantum mechanics. Although, if you look at it the only difference is that here we are having an integration with respect to  $t$ , normally when we have talk about scalar product in Hilbert space we have an integration with respect to  $x$ . The boundary conditions resemble the boundary conditions that are there in a box.

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- Now let us write  $T_2$  in the form:
- $T_2 = \int_{t_0}^{t_1} \left[ \frac{m}{2} \delta \dot{x}(t)^2 - \frac{1}{2} V''(x(t)) \delta x(t)^2 \right] dt$  --- (1)
- $= \int_{t_0}^{t_1} \left[ -\frac{m}{2} \delta \ddot{x}(t) \delta x(t) - \frac{1}{2} V''(x(t)) \delta x(t)^2 \right] dt$  --- (2)
- $= \int_{t_0}^{t_1} \delta x(t) \left[ -\frac{m}{2} \frac{d^2}{dt^2} - \frac{1}{2} V''(x(t)) \right] \delta x(t) dt$  --- (3)

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So, we can now let us go back to our  $T_2$ . The expression for  $T_2$  has extracted from the original equation is given in the first equation here. This first equation through an integration by parts is and ignoring, throwing away the boundary term takes the form that is given in the second equation number 1 let us call it, equation number 2.

So, equation number 2 is obtained from equation number 1 by integration by parts and throwing out the boundary condition. This equation number 2 can be written in the form of equation number 3, it is quite straightforward and only the derivatives or operators are explicitly written here.

Now, we introduce an operator  $B$  which represents the operator that is given in the square bracket in equation 2.

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- $F/A.T_2 = \int_{t_0}^{t_1} \delta x(t) \left[ -\frac{m}{2} \frac{d^2}{dt^2} - \frac{1}{2} V''(x(t)) \right] \delta x(t) dt$
- **Let  $B = \left[ -\frac{m}{2} \frac{d^2}{dt^2} - \frac{1}{2} V''(x(t)) \right]$**
- **Then, the second order variation in the action can be written as:**
- $T_2 = \langle \delta x | B | \delta x \rangle$

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So, in terms of that in terms of the operator B, we can write the expression for T<sub>2</sub> in the form of an expectation value of the sandwich between delta x or in the state delta x. So, in other words the second order variation that is T<sub>2</sub> can be written as the expectation of B in delta x. Now, we come to the conditions now, if T<sub>2</sub> is greater than the, T<sub>2</sub> greater than 0 means what means that the expectation has to be greater than 0.

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- Thus the condition that:
  1.  $T_2 > 0$  for all nonzero variations  $\delta x(t)$  is equivalent to the statement that
  2. B is positive definite, which in turn means that
  3. all of its eigenvalues are positive.

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In other words B has to be positive definite and which means that all its eigenvalues must be positive. So, the requirement that the action is minimal along the given path or along the classical path is that the; is that the  $T_2$  must be greater than 0 or B must be positive definite or the eigenvalues of B must all, must all be positive must all be positive.

In other words action is stationary along a physical path, stationarity may or may not imply minimality, stationary and minimality will occur when the first variate of the action is 0 and the second variation of the action satisfies these conditions  $T_2$  is greater than 0 or the operator B defined as above is positive definite, all its eigenvalues are positive.



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- If, on the other hand,  $B$  has some negative eigenvalues, then there are variations  $\delta x(t)$  (the eigenfunctions of  $B$  corresponding to the negative eigenvalues) which cause the action to decrease about the value along the classical path. In this case the action is not minimum along the classical path.

However, if it has some negative eigenvalues then the variations  $\delta x$  eigen functions of  $B$  corresponding to the negative eigenvalues will cause the action, which cause the action to decrease about the classical path and therefore, in this case the action will not be minimal.

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**STEP 3: HAMILTON'S PRINCIPAL  
FUNCTION**

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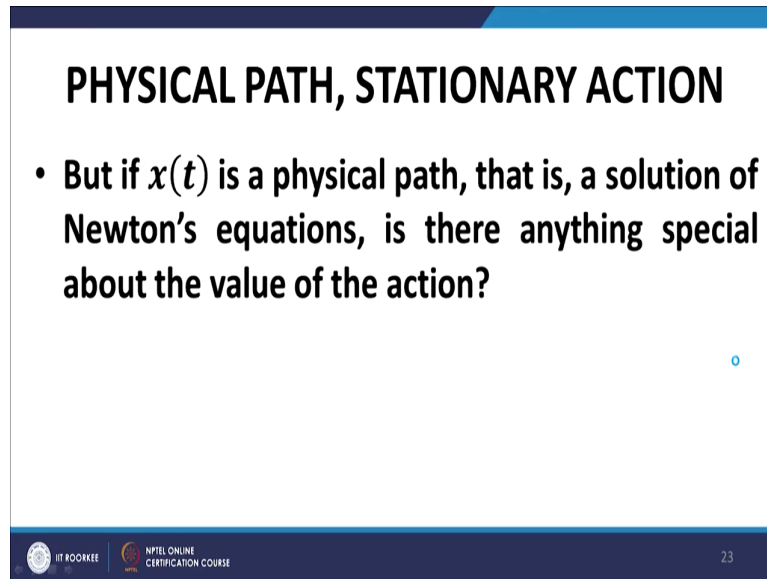
- **The term:**
- $T_0$  is just the action along the original, unmodified path  $x(t)$ .
- It is defined for all paths satisfying the endpoint and endtime conditions, not just physical paths.



Now, we come to Hamilton's principal function. This is the machinery that is required for developing the path integral in this formulation. We have talked about  $T_0$ , it is the action along the original unmodified path and it is,  $T_0$  is defined for all four paths satisfying the end endpoint, end time conditions not necessarily the physical path.



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**PHYSICAL PATH, STATIONARY ACTION**

- But if  $x(t)$  is a physical path, that is, a solution of Newton's equations, is there anything special about the value of the action?



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Physical paths are identified by stationarity of the actions. In other words as we mentioned if a path is to satisfy the, if the path is to satisfy the Newton's laws, then it the path must be must have 0 first variation of the action and vice versa, but what is special; what is special about the action of a path which is a physical path? That is the next question. What is special about the action other than stationarity for a physical path? That is the next question.

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- Hamilton found that the physical path is parameterized by the endpoints and endtimes, as well as by the branch index  $b$ , so we define:
  - $S_b(x_0, t_0, x_1, t_1) = A[x_b(t)]$
  - where  $x_b(t)$  is the  $b$  – th path that satisfies Newton's equations as well as the boundary conditions.

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For this purpose we define a function which is called the Hamiltonians principle function by this expression in the green box and the function please note this, there is an important point here. First of all the subscript  $b$  represents the  $b$ -th path that satisfies the Newton's equation that is a physical path right.

So, that is as far as  $b$  is concerned.  $b$  represents or  $b$  indices or  $b$  identifies a path which is the physical path if there are many such paths, number 1. Number 2 if you look at it carefully, while the action is a function of the path it is a therefore, it is a functional, on the other hand, the principal function the Hamiltonian principle function the left hand side is a function of the endpoints only it is a function of the initial point endpoints, initial point and initial time and the final point and final time.


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- Notice that  $S_b$  is an ordinary function of the endpoints and endtimes, unlike  $A$  which is a functional (something that depends on a function, namely, the path  $x(t)$ ).
- The function  $S_b$  is called Hamilton's Principal Function. ◦

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## PROPERTIES OF PRINCIPAL FUNCTION

- The principal function satisfies:
- $\frac{\partial S}{\partial x_1} = p_1, \frac{\partial S}{\partial t_1} = -H_1,$
- $\frac{\partial S}{\partial x_0} = -p_0, \frac{\partial S}{\partial t_0} = H_0,$





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These properties of the principal function the Hamilton's principal function can be easily establish by considering any simple physical system and working it out. Now, we come to this stationary phase approximation.

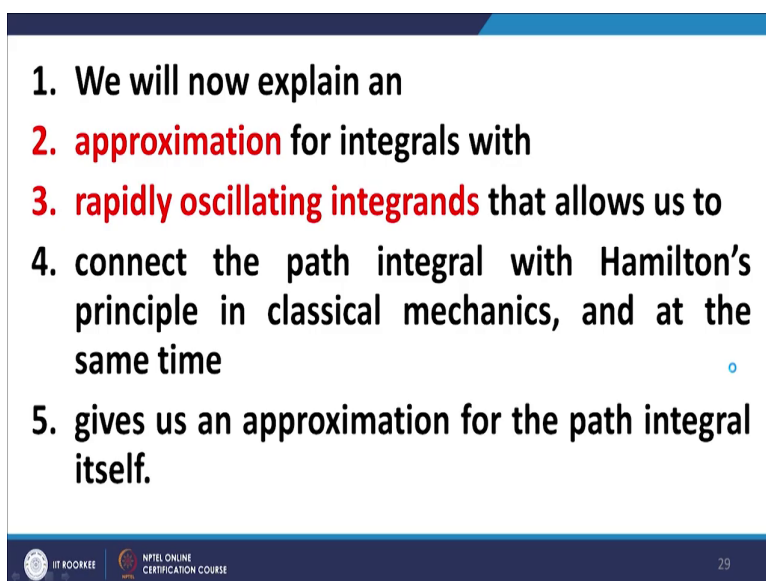
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**STEP 4: STATIONARY PHASE  
APPROXIMATION**

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1. We will now explain an
2. **approximation** for integrals with
3. **rapidly oscillating integrands** that allows us to
4. connect the path integral with Hamilton's principle in classical mechanics, and at the same time
5. gives us an approximation for the path integral itself.

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This is step number 4. I briefly touched on this when we talked about the relativistic path integral this is, this was discussed when we talked about saddle point being involved when working out an approximate solution to the propagator of a point of a point in Minkowski space., but coming back to this.

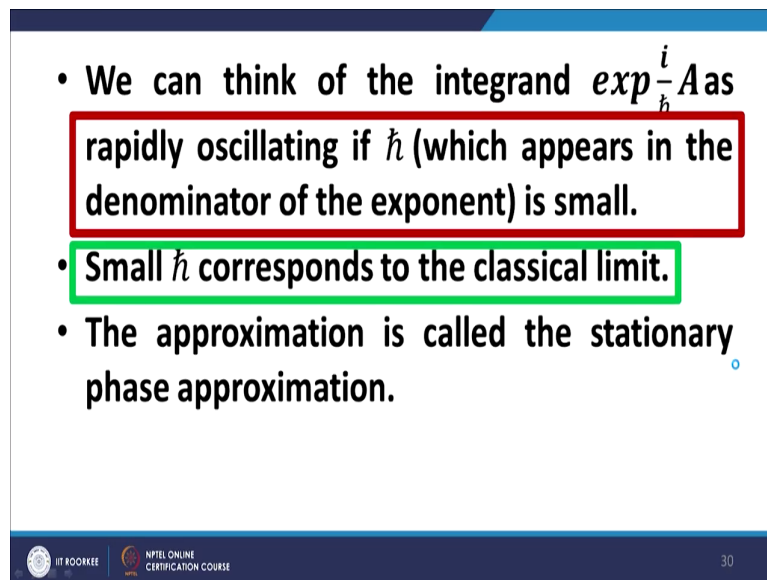
We now, try to approximate integrals which have rapidly oscillating behavior, rapidly oscillating behavior. You see why we are talking about this is let us go back to our path integral expression. It in the integrand involves  $e$  to the weight factor of  $e$  to the power  $i s$ . Now,  $e$  to the power  $i s$  is a oscillate oscillating integral because, it has modulus 1 and as the phase changes naturally the integral oscillates.

Now, if because we have normally we have in this case I have not mentioned the  $\hbar$  factor in the denominator of the exponent, but the precise expression is  $e$  to the power  $i$  upon  $\hbar$

into  $S$  and the  $\hbar$  being. So, very small even marginally variation in the action causes a huge change in the phase and therefore, the oscillations are very rapid and that is why this particular approximation becomes very relevant.

Now so, we are talking about these, this is the backdrop this is the backdrop in which we are talking about these integrals. We are trying to arrive approximations for these kind of integrals which oscillate very rapidly, because of some small factor present in the denominator of the exponent.

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- We can think of the integrand  $\exp\frac{i}{\hbar}A$  as rapidly oscillating if  $\hbar$  (which appears in the denominator of the exponent) is small.
- Small  $\hbar$  corresponds to the classical limit.
- The approximation is called the stationary phase approximation.

So, with now this is a integral I have shown here. This is rapidly oscillating if  $\hbar$  which appears in the exponent is small. The smaller it is the lesser variation in  $\hbar$  is going to cause more rapid oscillations. Now, the approximation, how we introduce the approximation?

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- We begin with an integral of the form:
- $\int dx \exp\left[\frac{i}{\kappa} \varphi(x)\right]$
- Where  $\kappa$  is a parameter. We wish to examine the behavior of this integral when  $\kappa$  is small.

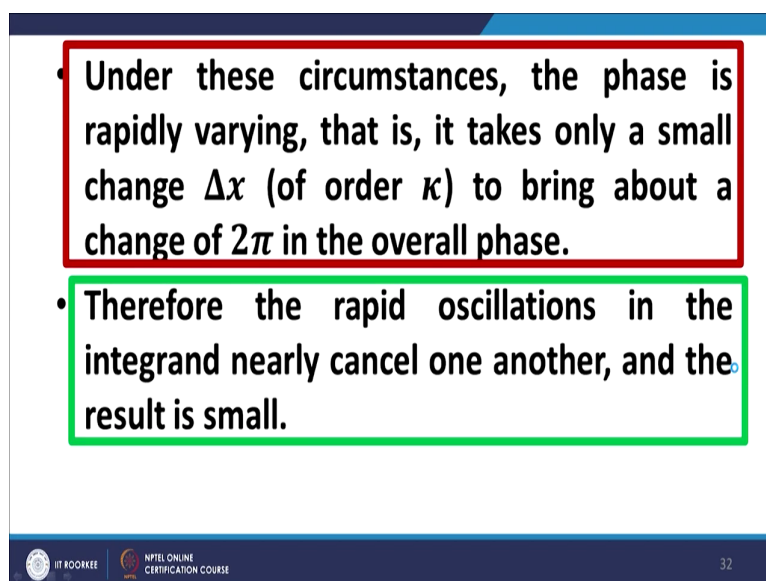


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Let us see, we start with an integral of this form integral d x exponential 1 upon kappa phi of x, where kappa is a parameter.



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- Under these circumstances, the phase is rapidly varying, that is, it takes only a small change  $\Delta x$  (of order  $\kappa$ ) to bring about a change of  $2\pi$  in the overall phase.
- Therefore the rapid oscillations in the integrand nearly cancel one another, and the result is small.

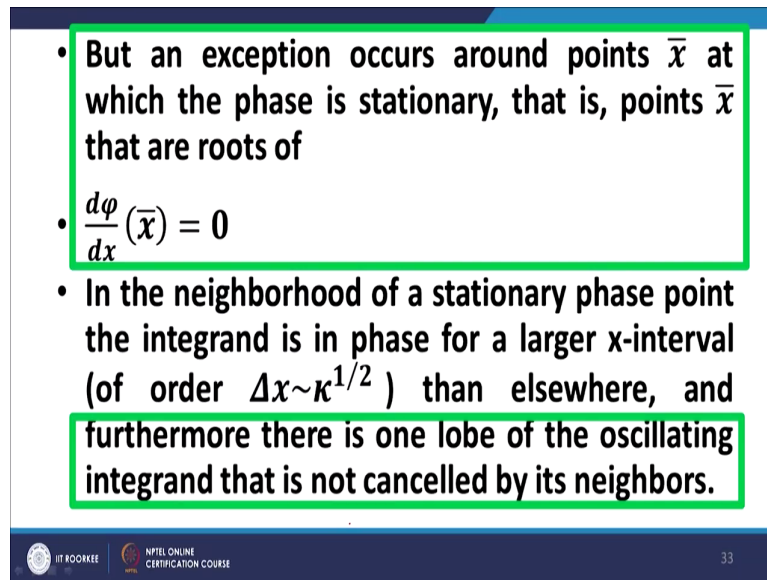
Now, the phase is rapidly varying, because we have got a factor in the denominator. A small change in the numerator will cause a significant change in the phase, a small  $\Delta x$  change in the value of the phase which is of the order of  $\kappa$  will cause a change of  $2\pi$  in the overall phase.

So, that is why the oscillations are extremely rapid and extremely sensitive to changes in the phase, but what happens the oscillations may be rapid and being extremely sensitive to the phase, what happens is even small changes in phase result in larger deviations and these deviations on the average they tend to cancel out.

So, rapid oscillations of the integrand tend to cancel out on the average, because they are in some sense they are not coherent, they are non-structured and even a small change in its for example, the action, because of the very small factor  $\hbar$  here, causes

the significant change in phase and these deviations being non coherent, they tend to cancel each other out.

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A slide with a dark blue header and footer. The main content area is white with a green border. It contains three bullet points. The first bullet point is highlighted in green. The second bullet point is a mathematical equation. The third bullet point is also highlighted in green. The footer contains logos for IIT Roorkee and NPTEL Online Certification Course, and the number 33.

- But an exception occurs around points  $\bar{x}$  at which the phase is stationary, that is, points  $\bar{x}$  that are roots of
- $\frac{d\phi}{dx}(\bar{x}) = 0$
- In the neighborhood of a stationary phase point the integrand is in phase for a larger x-interval (of order  $\Delta x \sim \kappa^{1/2}$ ) than elsewhere, and furthermore there is one lobe of the oscillating integrand that is not cancelled by its neighbors.



But; however, at a particular point for example, which is identified by this  $d\phi/dx$  in our case  $d\phi/dx$  by  $d\phi/dx$  of  $\bar{x}$  at  $\bar{x}$  is that point at which this expression becomes 0, this derivative become 0. At this point what will happen is; if you look at this point, this point it is just stationary, it is a stationary point.

Therefore, if there is a small variation in the in S at this point or in phi at this point the, because it is a stationary point the variation in phase is going to be very small, the deviation is going to be very small and as a result the coherence is maintained and, because the coherence is maintained the total effect blows up and here therefore, this manifests itself as the classical path in a sense.

In other words in the neighborhood of a stationary phase point the integrand is in phase for a larger interval, because the normally we have the bucket shaped curves you know and the bottom is flatter and because the bottom is flatter. If you have a slight deviation to either side it does not affect the value of the function much and that is therefore, coherence is sustained. Structurally there is one lobe of the oscillating integrand which is not cancelled by the neighbors. So, that gives the net result which of coherence of sustenance.

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- Therefore we obtain a good approximation if we expand the phase about the stationary phase point,
- $\varphi(x) = \varphi(\bar{x}) + y\varphi'(\bar{x}) + \frac{y^2}{2}\varphi''(\bar{x}) + \dots$  where  $y = x - \bar{x}$  and where the linear correction term on the right hand side vanishes because of the stationarity requirement.
- Retaining terms through quadratic order and substituting this into
- $\int dx \exp\left[\frac{i}{\kappa}\varphi(x)\right]$
- we obtain a Gaussian integral with purely imaginary exponent.



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Now, we expand this expression  $\varphi(x)$ , the phase  $\varphi(x)$ , we expand this expression  $\varphi(x)$  around this stationary phase approximation, around the point at which the phase is stationary as a Taylor series.

We expand it phase  $\varphi(x)$  as a Taylor series around the stationary point. Of course, at the stationary point you will see that  $\varphi'(x)$  automatically vanishes. So, this term will go and

we are left with  $\phi(x)$  and then the quadratic term and so on. For we retain terms up to the quadratic order, we assume that they are sufficiently representative and therefore, now what we are left with therefore, is  $\phi(x)$  as the first order term  $\phi(\bar{x})$ , the second the zeroth order term  $\phi(\bar{x})$ , the first order term goes, because of the stationarity requirement and we have the second order term.

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- Hence, setting:
- $\phi(x) = \phi(\bar{x}) + \frac{y^2}{2} \phi''(\bar{x})$  in
- $\int dx \exp\left[\frac{i}{\kappa} \phi(x)\right]$  we get
- $\int dx \exp\left[\frac{i}{\kappa} \phi(x)\right] =$   
 $\left[ \exp\left(i\nu \frac{\pi}{4}\right) \right] \sqrt{\frac{2\pi\kappa}{\phi''(\bar{x})}} \exp\left[\frac{i}{\kappa} \phi(\bar{x})\right]$
- Where  $\nu = \text{sgn} \phi''(\bar{x})$



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So, when we this is what we have simplify by simplification, when we substitute it in the exponent of the exponential we get a Gaussian integral and if this Gaussian integral can be carried out conveniently and we get a result here, but the important thing is this Gaussian integral is purely imaginary.

That is an varying factor and therefore, this first factor, the pre factor appears and this pre factor captures the pure imaginary character of the Gaussian integral where nu is the sign of i double dash at the stationary point. Now, there may be more than one stationary points.

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- Finally, we must acknowledge the possibility that there might be more than one stationary phase point. Indexing these roots by a branch index  $b$ , the final answer is then a sum over branches:

$$\int dx \exp\left[\frac{i}{\kappa} \varphi(x)\right] = \sum_b \left[ \exp\left(iv_b \frac{\pi}{4}\right) \sqrt{\frac{2\pi\kappa}{\varphi''(\bar{x}_b)}} \exp\left[\frac{i}{\kappa} \varphi(\bar{x}_b)\right] \right]$$



- This is the stationary phase approximation for one-dimensional integrals.

So, in the event that there are more than one stationary points the whole expression has to be summed over all the stationary points, which are indexed by the index b and therefore, the final answer becomes this expression which is given in the green box in your in this slide and for multidimensional integrals the expression can be extended. The expression that we have here can be extended and we get the expression that is given in the green box on this slide here.

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- For multidimensional integrals, the above generalize as:

$$\int d^n x e^{i\varphi(x)/\kappa}$$
$$= \sum_b \left[ \exp\left(i\nu_b \frac{\pi}{4}\right) \right] (2\pi\kappa)^{n/2}$$
$$\left| \det \frac{\partial^2 \varphi(\bar{x}_b)}{\partial x_k \partial x_l} \right|^{-1/2} \left[ \exp\left(i\varphi(\bar{x}_b) \frac{1}{\kappa}\right) \right]$$

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Please note this pre factor in variably makes an appearance and this pre factor contains nu b and nu b is nothing, but the sign of what? Sign of the phi double dash bar at phi double dash at the stationary point.

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**STEP 5: FREE PARTICLE PATH  
INTEGRAL IN THE STATIONARY  
PHASE APPROXIMATION**

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Now, we come to the final step here we construct the path integral in the stationary phase approximation that is the final step we, because we are having a free particle so there is no potential function here.

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- Let us evaluate the free particle path integral in the stationary phase approximation in one dimension. The Lagrangian is
- $L = \frac{m\dot{x}^2}{2}$ ,
- Hamilton's principal function is:
- $S = \int L dt = \frac{m\dot{x}^2 t}{2}$



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The Lagrangian is given by the kinetic energy alone  $m \dot{x}^2 / 2$  the Hamiltonian principal function as the action which is  $m \dot{x}^2 t / 2$ . So, these two are straightforward.



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- But it is customary to express  $S$  as a function of the endpoints and endtimes, not the velocities, so we invoke the equation of the classical path,
- $x = x_0 + \dot{x}_0 t$ ,
- which we solve for  $\dot{x}_0$ ,
- $\dot{x}_0 = \frac{x-x_0}{t}$ .



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Now, we can express the action in terms of the end points and end times, as I mentioned, because we are here talking about the Hamilton's principal function which is usually expressed as a function of the end points and end times.

So, we express  $S$  as the function of endpoints and end times we recall that  $x$  is equal to  $x_0 + \dot{x}_0 t$  constant velocity case, because there is no potential here free particle, no potential, no force therefore,  $x$  is equal to  $x_0 + \dot{x}_0 t$  and please note  $\dot{x}_0$  is also equal to  $\frac{x-x_0}{t}$  because the velocity remains unchanged and  $x$  is equal to  $x_0 + \dot{x}_0 t$  this is the distance traversed by the particle displacement of the particle.

Simplifying this we get this expression here  $\dot{x} = \dot{x}_0$  is equal to  $\frac{x - x_0}{t}$ , we substitute this in the expression for  $S = \int L dt$  we get  $S = \int \frac{m}{2} \dot{x}^2 dt$ .

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- But since  $\dot{x} = \dot{x}_0$  (the velocity is constant along the path), we get:
- $S = \int L dt = \frac{m\dot{x}_0^2 t}{2}$
- $S(x, x_0, t) = \frac{m(x - x_0)^2}{2t}$ ,
- Furthermore, from  $x = x_0 + \dot{x}_0 t$ , we can see that the classical path connecting the endpoints and endtimes is unique (that is, given  $(x, x_0, t)$ , the initial velocity  $\dot{x}_0$  is unique). Therefore there is only one term in the stationary phase formula.

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Now, this is the classical path constituting the endpoints. Now, the important thing is the classical path connecting the endpoints and end times is unique. In this particular case it is pretty clear, it is quite straightforward that we have only one path the straight line path between the point  $x$  and  $x_0$  and therefore, the initial velocity is also, initial velocity  $\dot{x}_0$  is unique, because we have only one path here.

So, the initial velocity is also unique and therefore, we have only one term in the stationary phase formula to repeat, because there is only one physical path, there is only one physical path, there is only one initial velocity and correspondingly there is only one expression in the

stationary phase formula. The generating function  $S$  gives us the following relations as for the properties that I had mentioned earlier.

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- It is instructive to check the generating function relations for the Hamilton's principal function viz.
- $\frac{\partial S}{\partial x_1} = p_1, \frac{\partial S}{\partial t_1} = -H_1, \frac{\partial S}{\partial x_0} = -p_0, \frac{\partial S}{\partial t_0} = -H_0,$
- for the free particle. From the eq  $S(x, x_0, t) = \frac{m(x-x_0)^2}{2t}$ , we find
- $\frac{\partial S}{\partial x} = m \frac{x-x_0}{t} = p, \quad \frac{\partial S}{\partial x_0} = -m \frac{x-x_0}{t} = -p_0,$
- $\frac{\partial S}{\partial t} = -\frac{m}{2} \left( \frac{x-x_0}{t} \right)^2 = -H.$



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These are the relations that can be verified from simple differentiation of the Hamilton principal function.



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- In the bottom set of eqs we have set  $t_0 = 0$  and  $t_1 = t$ . Note that  $p_0 = p$ . The generating function relations are verified.
- Finally, we need the number of negative eigenvalues  $\mu$  of the operator  $B$  that appears in the second variation of the action.

Now, we want to find out more about the eigenvalues, the behavior of the eigenvalues whether the eigenvalues are negative. If there are any negative eigen values or the eigenvalues are all positive these are the things that we need to examine.

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- Since  $V = 0$  we have:
- $B = -\frac{m}{2} \frac{d^2}{d\tau^2}$  and
- $\langle \delta x | B | \delta x \rangle$
- $= -\frac{m}{2} \int_0^t \delta x(\tau) \delta \ddot{x}(\tau) d\tau = \frac{m}{2} \int_0^t [\delta \dot{x}(\tau)]^2 d\tau \geq 0,$
- where we have integrated by parts and discarded boundary terms.

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Now, since  $V$  is equal to 0 we have  $B$  is equal to minus  $m$  by 2  $d$  square upon  $d$  tau square and this expression if I substitute the value of  $B$  in the integral in the integral form we get this expression for  $t^2$  for  $t^2$ . And you can clearly see here this expression is positive definite and that clearly shows that the action for this particular system is minimum at the stationary point.

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- The eigenvalue problem for  $B$  is
- $-\frac{m}{2} \frac{d^2 \xi_n(\tau)}{d\tau^2} = \beta_n \xi_n(\tau),$  — (4)
- where  $\xi_n(\tau)$  are the eigenfunctions, satisfying  $\xi_n(0) = \xi_n(t) = 0,$  and where  $\beta_n$  are the eigenvalues. — (2)

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Now the eigenvalue problem for  $B$  if you can see here, it can be written in this form and this if you solve this expression with this the boundary conditions which are given, let us call this equation 1 and let us call this boundary conditions 2.

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- $-\frac{m}{2} \frac{d^2 \xi_n(\tau)}{d\tau^2} = \beta_n \xi_n(\tau),$
- This has the same mathematical form of a quantum mechanical particle in a box, so the (unnormalized) eigenfunctions are
- $\xi_n(\tau) = \sin\left(\frac{n\pi\tau}{t}\right), n = 1, 2, \dots,$  and the eigenvalues are
- $\beta_n = \frac{m n^2 \pi^2}{2 t^2}.$  — (4)
- These eigenvalues are all positive, as claimed.

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We can work out the eigenvalues and the eigen functions explicitly. We get the eigen functions as sin. In fact, this problem is similar to the problem of a particle in a box, quantum mechanical particle in a box and when you simplify or work out the eigen functions we get the expression in equation 3 and the expression for the eigenvalues is in equation 4. Clearly all the eigenvalues are positive and therefore the our physical system has minimum at minimum of action at the point of stationary action.

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- We may now gather together all the pieces of the Van Vleck formula for the free particle. We find,
- $K(x, x_0, t) = \left(\frac{m}{2\pi i \hbar t}\right)^{1/2} \exp\left[\frac{i m(x-x_0)^2}{\hbar 2t}\right]$
- Which is the case expression as obtained earlier.



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And collecting all these expressions together we get the expression for the path integral as the quantity that is given on your slide. And which is precisely the same expression that we had when we worked it out as for the Feynman's time slicing approach.



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- **The Van Vleck formula amounts to:**
- **approximating the path integral by including all classical paths connecting the given endpoints at the given endtimes; as well as**
- **the second order contributions coming from fluctuations about such paths.**

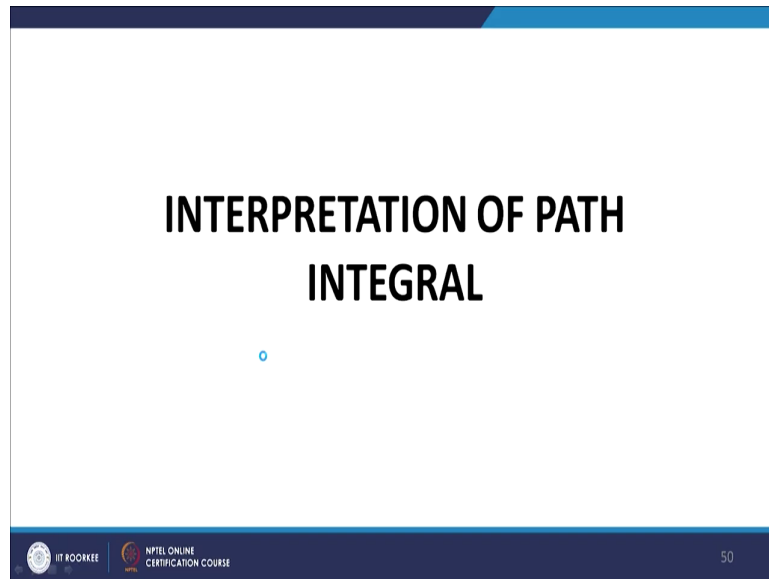


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So, this approach is certainly a different, certainly more informative and certainly more instructive in the sense that it goes through the route of the action principle directly and brings to you the same expression for the path integral. So, of course, the approximation here is introduced by the stationary phase approximation which.



In fact, this stationary phase approximation manifests itself very coherently, very significantly in the context of path integrals when we talk about the, when we talk about the quantum mechanical effects manifesting themselves in the path integral as a classical path in the form of a classical path. How do classic quantum mechanical paths coalesce among themselves to give you the classical path? So, that is a very interesting issue, this is brought it here.

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**INTERPRETATION OF PATH  
INTEGRAL**



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### THE ROTATING PHASOR

1. The phase of each path at an instant  $t$  is  $\frac{1}{\hbar} S_i(t)$  where  $S_i = \int L_i dt$ .
2. The phasor of each path is  $\exp\left(\frac{i}{\hbar} S_i\right)$  where  $S_i = \int L_i dt$ .
3. It can be expressed in the complex plane as a unit length vector with angle  $\phi$  relative to the real positive (horizontal) axis.
4. As time evolves this phasor rotates at the rate  $L/\hbar$ .

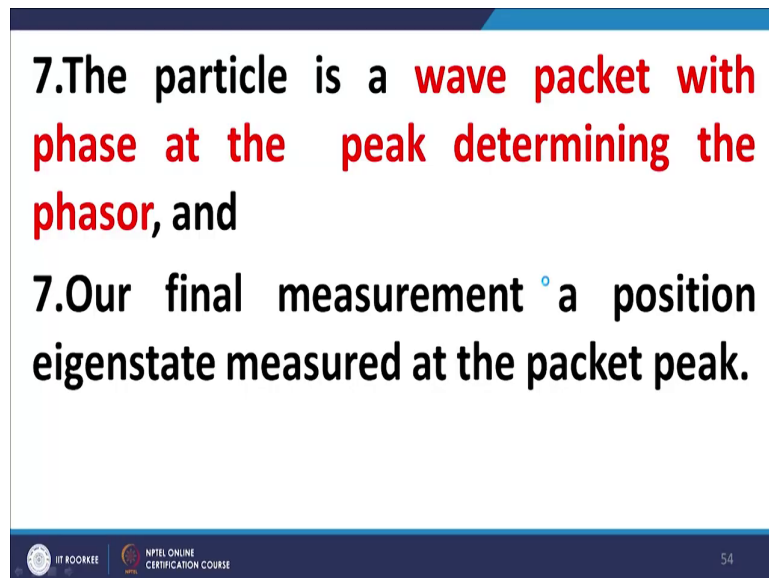
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Now the machinery that we have discussed here can be elaborated more by considering the case of a, or by considering explicitly, or figuratively or pictorially the case of path integral. Let us look at this.

5. The total phase  $\phi = \int \frac{L}{\hbar} dt$ .

6. So we can picture the phasor as a unit length vector rotating like a hand on a clock in a 2D complex plane (though it is a counterclockwise rotation).

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**7. The particle is a wave packet with phase at the peak determining the phasor, and**

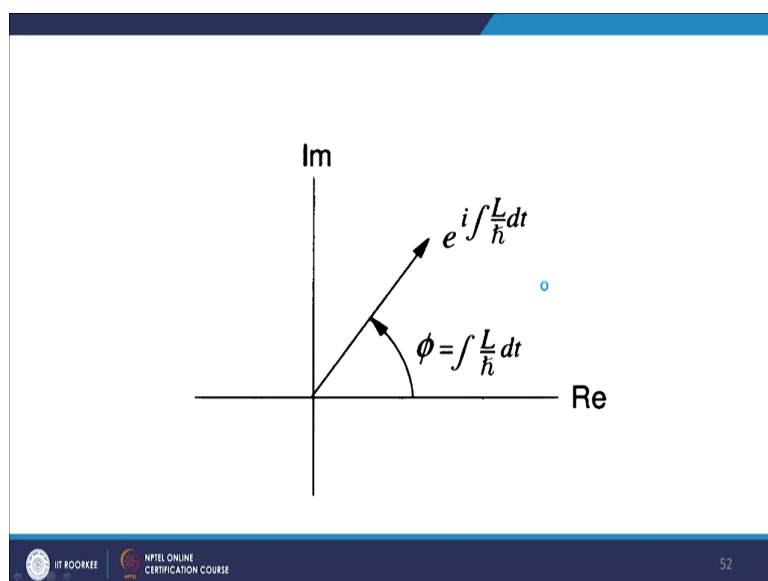
**7. Our final measurement is a position eigenstate measured at the packet peak.**

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The phase of each path at an instant is given by  $\frac{1}{\hbar} S$ , we know that as it is the action, and the action is given by  $L dt$ . And let us call this expression exponential  $i \frac{1}{\hbar} S$  as the phase. Let us call this expression exponential  $i \frac{1}{\hbar} S$  as the phasor.

Phasor not the phase the phasor. The  $\frac{1}{\hbar} S$  is the phase and exponential  $i \frac{1}{\hbar} S$  is the phasor, let us call it the phasor. So, clearly its a vector of unit length which rotates on the complex plane making an angle  $\phi$ , where this  $\phi$  is represented by this phase relative to the real axis. As the way progresses or as time elapses this phasor tends to rotate on this complex plane in the counterclockwise direction mind you.

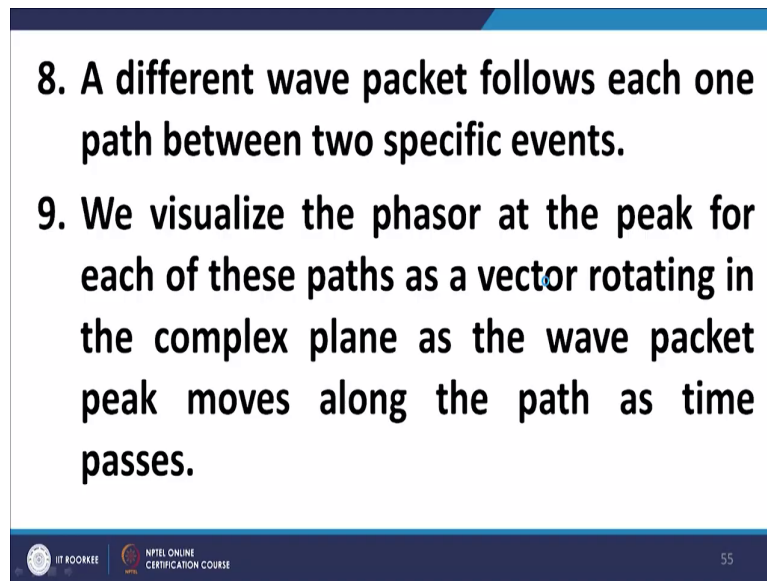
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As this is given in this diagram, the phasor is its a unit vector and it is rotating in the counterclockwise direction as the phases, as the wave is progressing, as the time is elapsing. The total phase between any two points in time is given by this expression. So, this is the shape or the pictureative representation of what is happening in the context of the weighing factor. We will now talk about how that weighing factor manifests itself in the actual dynamics of the quantum wave.



Now the wave, as you know particles or objects in quantum mechanics are deemed to travel as waves or wave packets, and its the peak of the wave packet and that represents the or that determines the phase. And the final measurement that is also determined by the peak of the wave packet. And a different wave factor, a different wave packet follows each of two different paths between two events.

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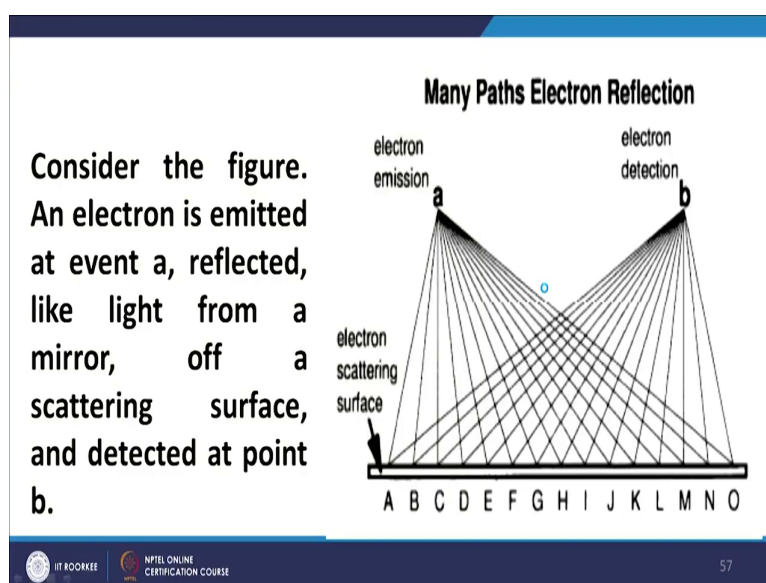
**8. A different wave packet follows each one path between two specific events.**

**9. We visualize the phasor at the peak for each of these paths as a vector rotating in the complex plane as the wave packet peak moves along the path as time passes.**

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So there are different wave packets following different path between two events. So, we can visualize the phasor at the peak of each path, being a vector rotating in the counterclockwise direction as the wave packet progresses along a particular path. That phasor which is tagged to that particular wave packet also rotates in a counterclockwise direction. So, every different path is traversed by a different wave packet and therefore, has a different phasor are tagged to it.

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So, as an example let us consider this simple example. You have an electron which is emitted at the source a, it is reflected say by a magnetic, by a negative charge and mirror and it is identified or detected at the electron detector at the point b. So, for example, we have taken 15 different paths A B C D upto O.

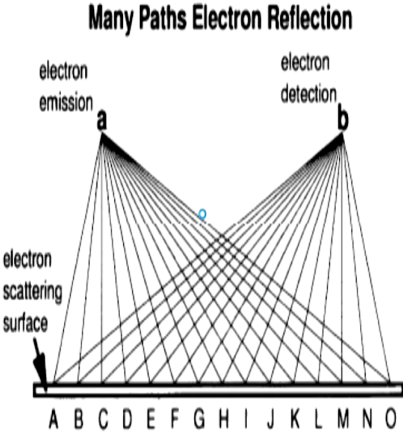
Path H is assumed to be the classical path, because it has a angle of incidence equal to angle of reflection, the other half some of the many paths. Of course, there will be infinitely many paths from the source event to the detector event. Let us to explain the to figuratively explain the dynamics we assume 15 paths around its.



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**Many Paths Electron Reflection**

We look at a representative 15 different paths for the electron, out of the infinite number in the many paths approach, and label them with letters A to O.





The diagram illustrates the many paths approach to electron reflection. It shows an electron emission point 'a' and an electron detection point 'b'. A horizontal line represents the electron scattering surface, with points labeled A through O. Multiple lines represent different paths from 'a' to 'b' via the surface. Labels 'electron emission', 'electron detection', and 'electron scattering surface' are present. The slide footer includes the IIT Roorkee logo, 'NPTEL ONLINE CERTIFICATION COURSE', and the number '58'.

Now each of these path is having its own phasor please note this and as this the electron progresses as a wave along this path that the phasor also rotates on the complex plane or is willing to rotate along the complex plane in the counterclockwise direction. We assume that each path, because the space time points are the same for all the paths initial and final points are the same for all the paths.

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## PATH DESCRIPTION

- Each path takes the same time  $T$ .
- Let path  $H$  is the classical path, having equal angles of incidence and reflection.
- Since it is the shortest, particle speed for that path is lowest.

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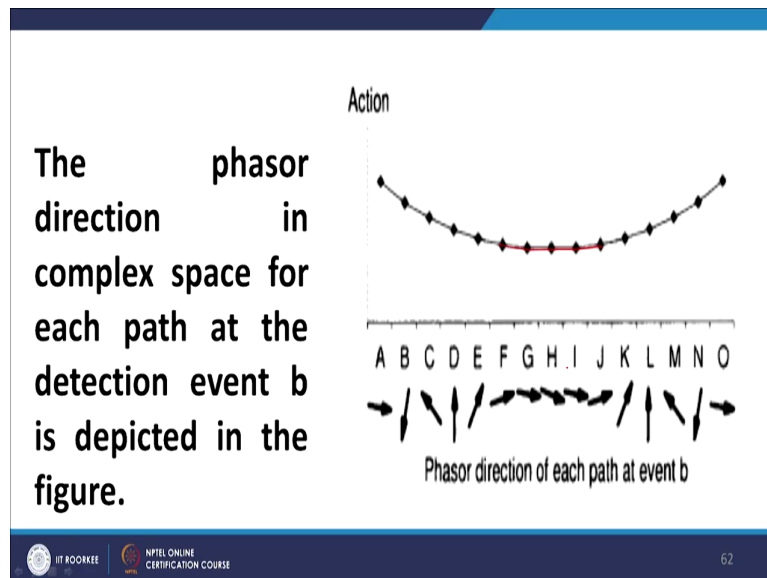
That means, each path takes the same time  $T$  and path  $H$  as I mentioned as is the classical path. It is the shortest path and therefore, it has the minimum kinetic energy and minimum speed.

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## LAGRANGIAN & ACTION

- The Lagrangian here is simply the kinetic energy, and this is constant, though different, for each path.
- Since speed is least for the classical path H, it has the smallest Lagrangian, and thus the least action.
- The other paths do not obey the usual classical laws, such as least action, equal angles of incidence and reflection, etc.

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So, Lagrangian is the kinetic energy. Now look at it what is happening here, now this is the important part if you look at the. Please note the direction of the phasors at the point at which they intersect or they reach the destination point or the detection point b. Please I repeat, please note the direction of the phasors at the point at which they reach or they touch or they are detected by the event b. Now clearly there is a flat bed in the action diagram corresponding to the various paths.

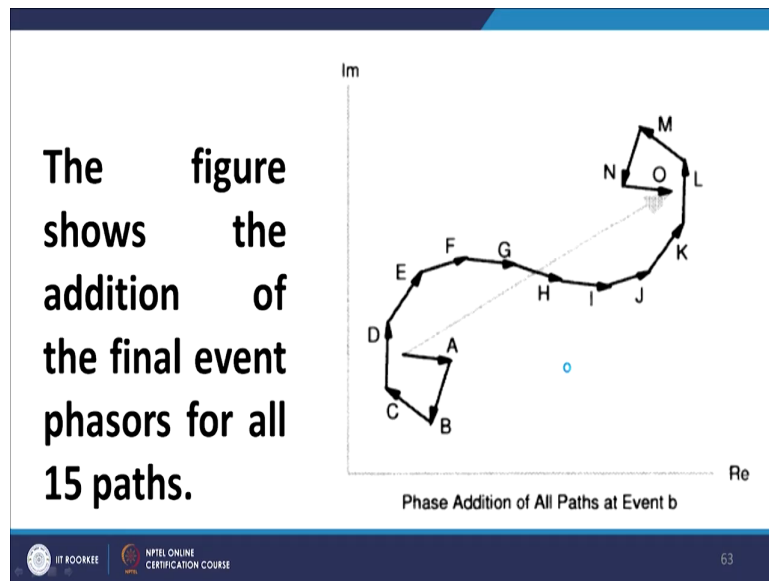
In other words the action is stationary along this bed, let us say F G H I on the J. And you can see here that corresponding to these points F G H I J the phasors are also more or less aligned in the same direction. And as a result of which these phasors being aligned in the same direction, they produce coherence.

And due to this coherence we observe them as the classical path. And if you look at the other phasors that correspond to higher actions on either side of the stationary action, you find that some of them are, most of them are moving in random directions and therefore, they tend to cancel each other on the average.

They do not form any kind of a coherence beam, a coherence structure. The coherence structure is only formed by the set of or the various phasors that correspond to the stationary action or are close to the stationary action. Because as I have been emphasizing a small deviation from the stationary action, because the action is stationary there does not disturb the action significantly.

And as a result of it because the action is not disturbed significantly the path tend to be coherent and we observe them as the classical path. So, that is what it is.

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Thank you so much.