Path Integral Methods in Physics & Finance

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Lecture - 28
Interpretation of Path Integral

Welcome back. So, before the break I told you that we shall be talking about at different approach, modified approach to working out the Path Integral. Now, this approach is certainly more instructive, more intuitive and more explicit than the conventional approach that we have been following so far. It is the stationary point approach to the path integral.

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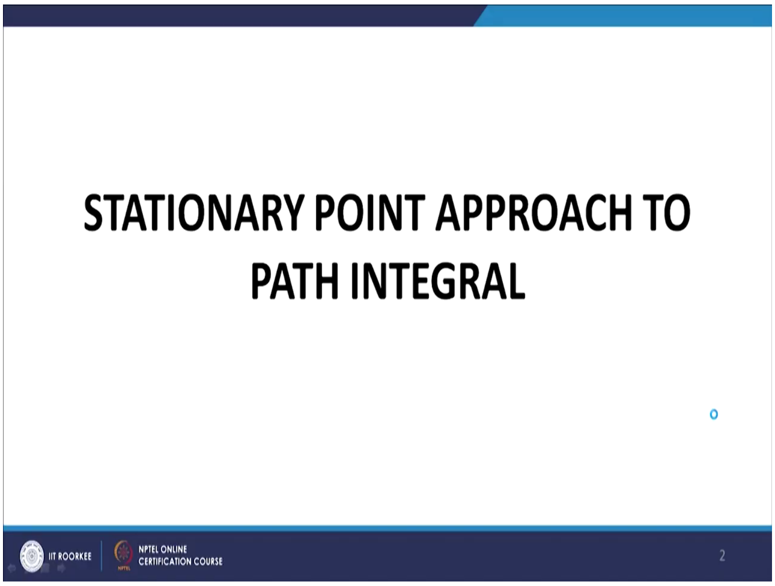

Again, it works on the Hamiltonian it starts with the Hamiltonian formalism and then we progress to the same version of the path integral as we had worked out based on the Lagrangian formalism.

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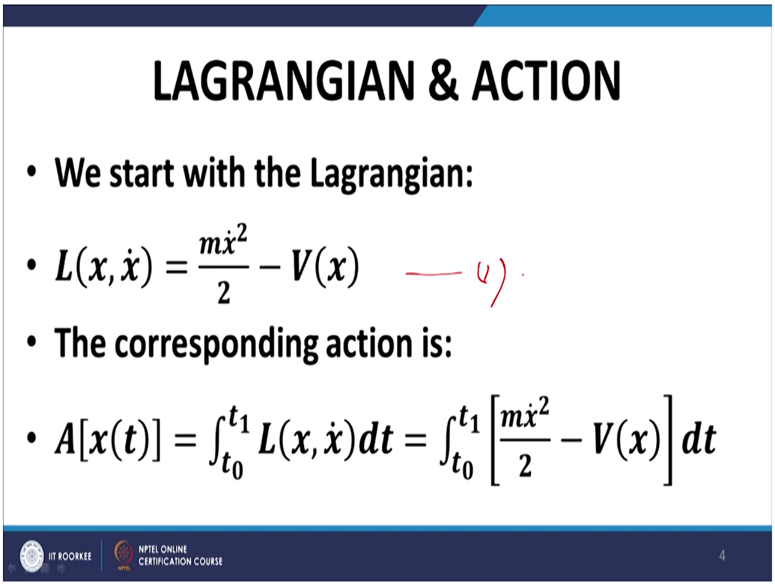

Now, let us first work out the Lagrangian, let us start with a simple physical system. The Lagrangian of the system can be written in the form of equation 1, which is here. The first term is the kinetic energy term; the second term is the potential energy term.

So, and the corresponding action which is a function of every path it is a functional. So, it is a function of every path, it can be written as the integral of the Lagrangian with respect to; the with respect to time over the carried out over the time of evolution of the system.

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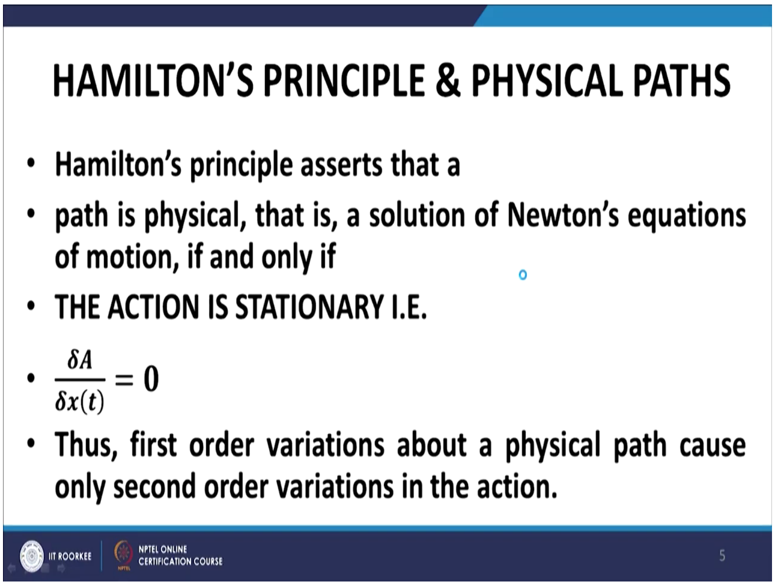

Now, Hamiltonian principle says that paths that are physical paths satisfy a certain requirement and that requirement is the stationarity of the action. The action is stationary with respect to the variation of the paths. The first order variation in the action is stationary, first order variation in the action vanishes.

In other words, we can say that first order variations about a physical path do not cause any first order variation in the action. They may cause second and higher order variations in the actions, please note this point. Now, let us see quantitatively evaluate the variations of the action due to a variation in the path.

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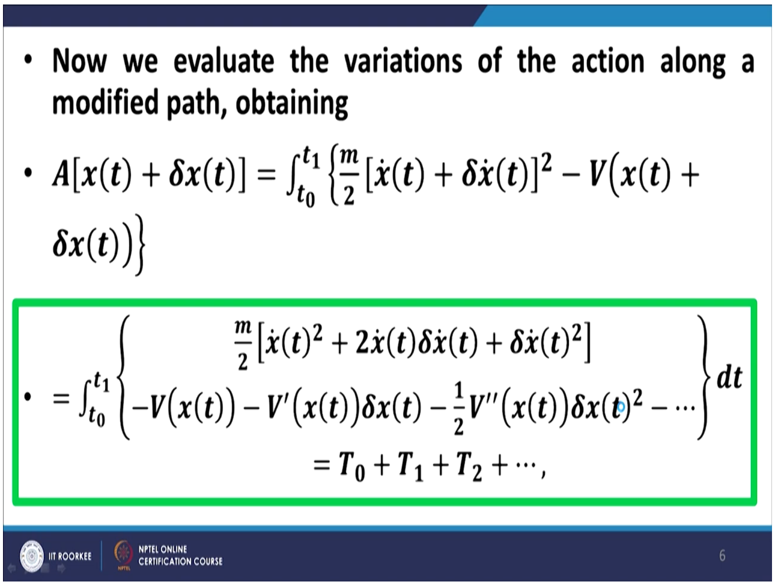

The variations in the action as you can see can be written, it can be expanded the power series in delta x and it therefore, it takes the form which is given in the green box in your slide.

The first term is comprises of the action of the original path m by 2 x dot t square minus V x t we call it t 0, the second term m by 2 into 2 x dot t into delta x dot t minus V dash x delta x that we call it as t 1 and the third the other term m by 2 delta x dot t square minus 1 by 2 V double dash x t delta x t square we call it as the t 2, the quadratic variation, and the first order variation and the zeroth order variation.

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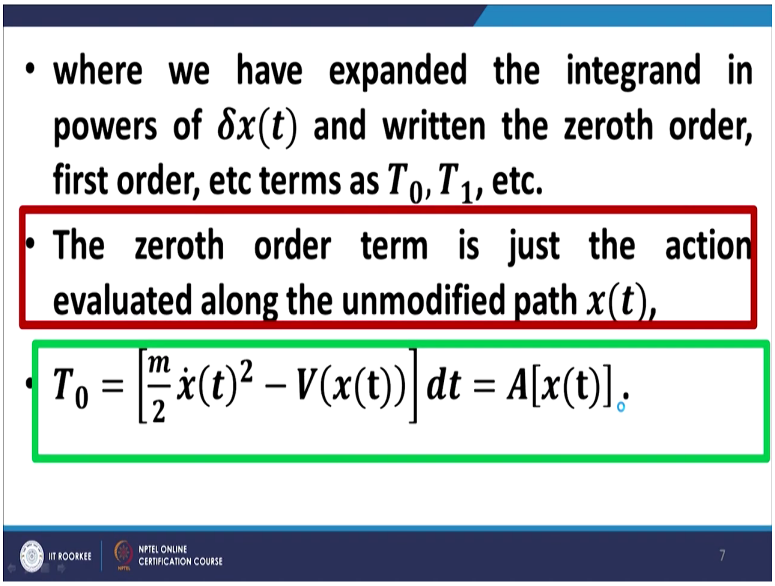

So, this is what we have done by expanding the integrand in powers of delta x t, T 0 is the zeroth order variation, T 1 is the first order variation, and T 2 is the second order variation. The zeroth order variation is just the action evaluated along the unmodified path, when there is no variation in the path. So, this is what the various the green box represents the zeroth order variation T 0, which is the variation of the unmodified path.

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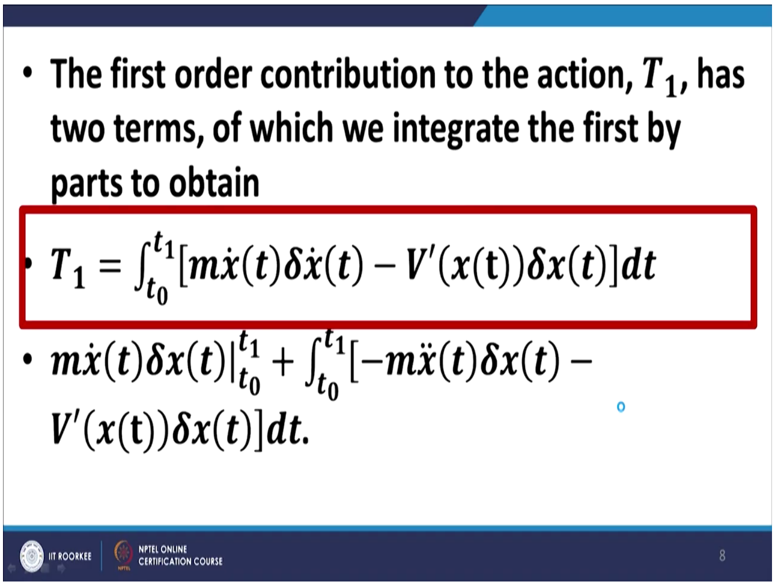

Now, the first order contribution to the action or the first order variation if we do an integration by parts and if we ignore the surface terms, what do we get?

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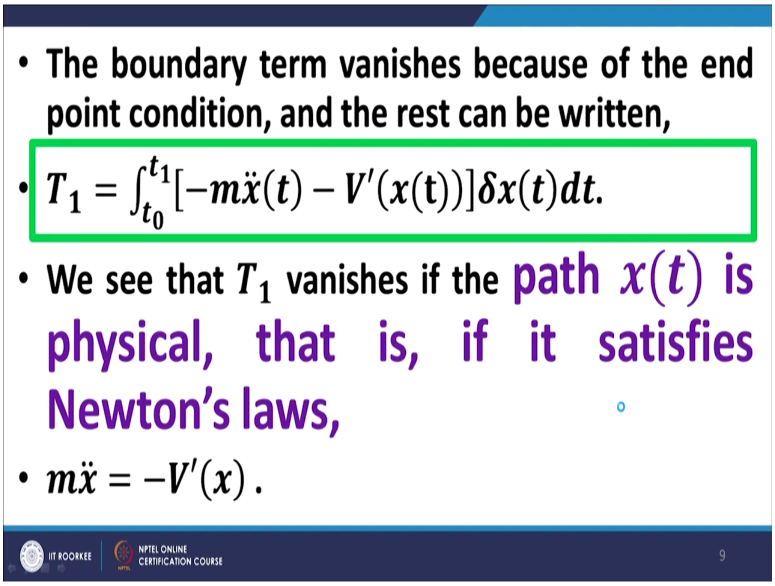

We get this expression which is there in the green box here. So, this is the first order variation remember. So, what happens if we substitute, if we take the condition or if we impose the condition T 1 equal to 0, if we impose the condition T 1 equal to 0, because delta x is an arbitrary variation the integrand must vanish and if the integrand or the coefficient of delta x must vanish and that gives us the expression m x double dot is equal to minus V x t which is nothing, but the Newton's laws.

So, in other words what have we concluded that T 1 vanishes, if this the system that we are considering follows the Newton's laws and conversely even the converse holds the converse holds in the sense that if T 1 is equal to 0, then the system will follow Newton's law.

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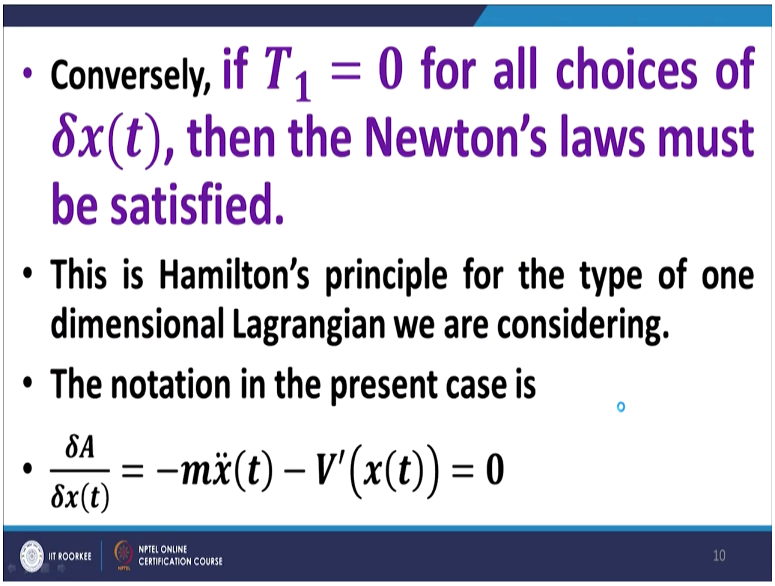

So, this is the Hamilton's principle. It is really the it is commonly known as the principle of least action, but it would be more precise to call it as the principle of stationary action rather than least action right.

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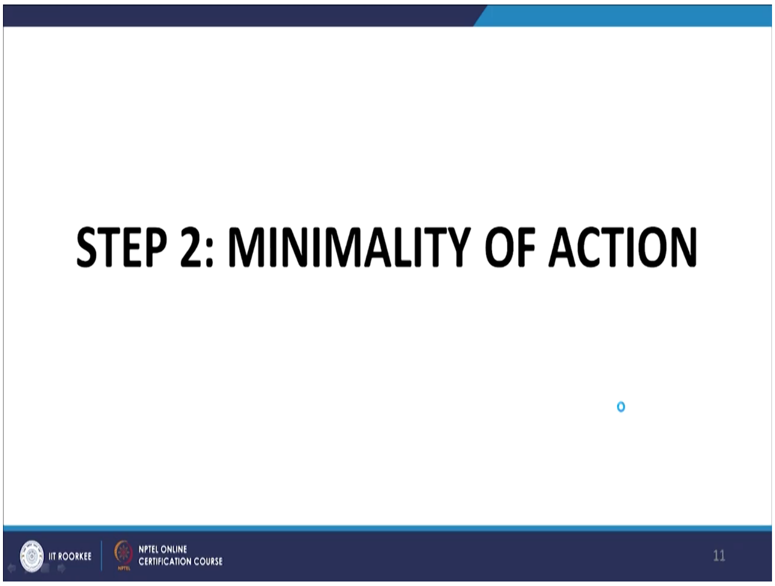

Now, that is that brings us to the question whether under what circumstances the action is actually least and whether stationary action automatically implies that the action of the physical path of the system along the physical path is least, that is what we will investigate now.

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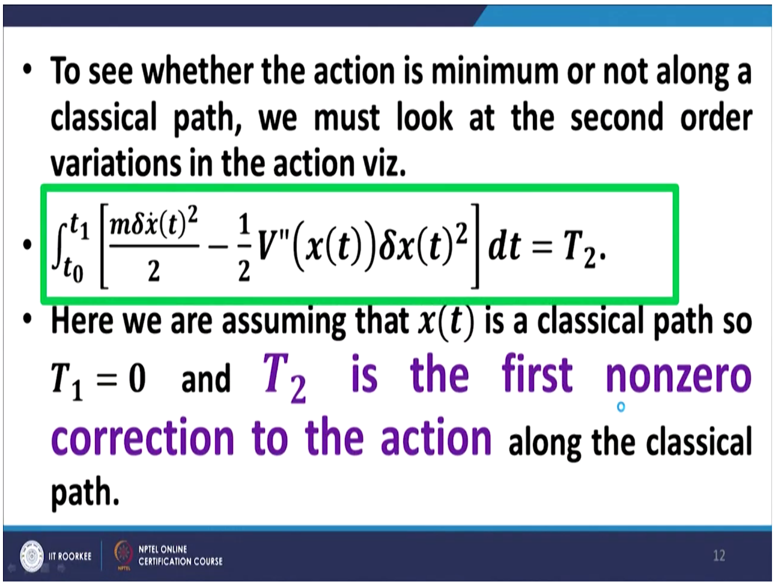

To see whether the action is minimum along a classical path, let us look at the second order variation in the action. This is the various second order variation T 2. You can you recall it is the, it was there in the earlier slide, the slide at the beginning of our discussion, this one. It has been extracted from this particular equation. The second order terms in delta x square.

Now, the delta quadratic term delta x were the coefficients of that constitute t 2, right. So, so let us come back to this. So, this is t 2 this is t 2 the second order variation. Now, please note, because the we are considering a physical path we have automatically assumed that T 1 is equal to 0 and therefore T 2 is now, the first nonzero correction to the action.

Remember T 0 was nothing, but the original path the action along the original path without any variation, T 1 we have taken as 0, because we can considering or investigating physical path therefore, T 2 becomes the first nonzero correction in the action.

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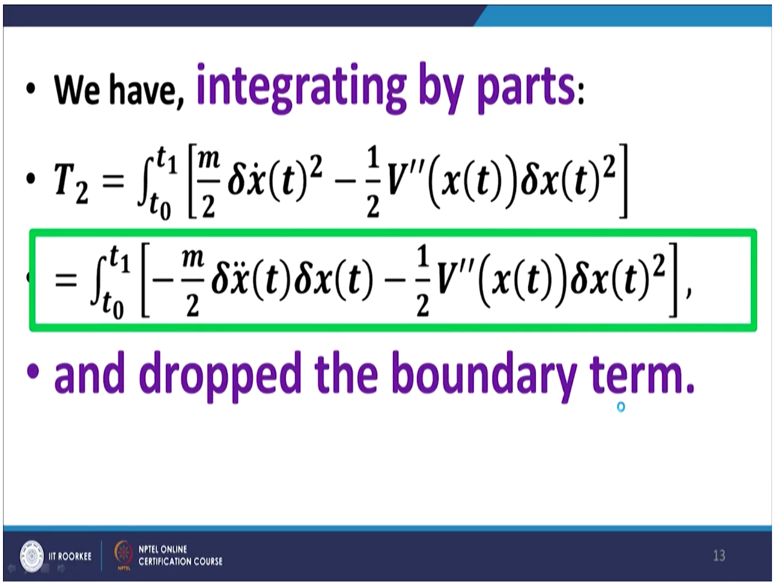

If we integrate this expression by parts, we get this result and of course, when we drop the boundary term which is just usually assumed to hold and if that is the case then we get the expression that is within the green bracket.

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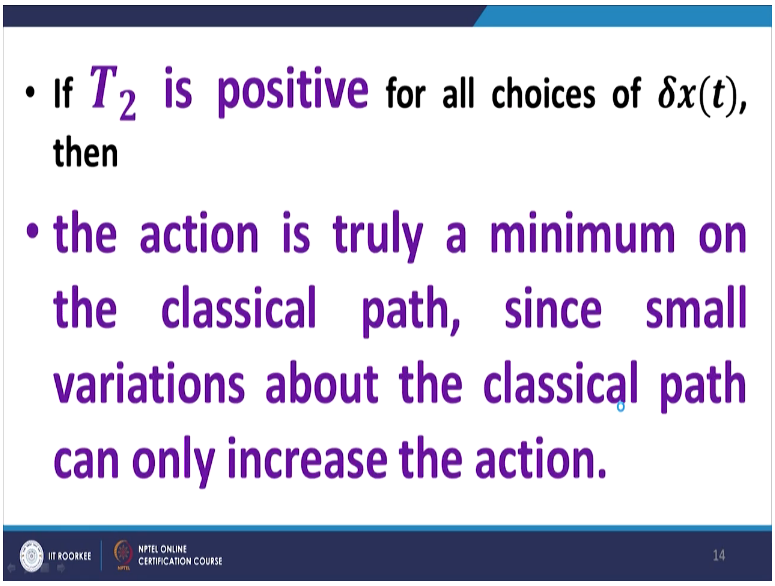

Now, if T 2 is positive, now, comes the important part. If T 2 is positive for all choices of delta x, if T 2 is positive for all choices of delta x this implies that our action is indeed minimum at the given path, at the classical path, at the given path, at the physical path.

But, because what happen even a small variation here small variations in the action will increase the action, will not increase will not decrease the action, because the second order derivative or the second order variation is positive. So, if the second order variation is positive it means that the action along the classical path or the physical path is actually a minimum.

If it is not so, then we it is not mandatory, it is not obligatory that the action along the given path that we are considering is actually a minimum. Now, to formulate a test for this minimality of action, one could devise a scheme on this form.

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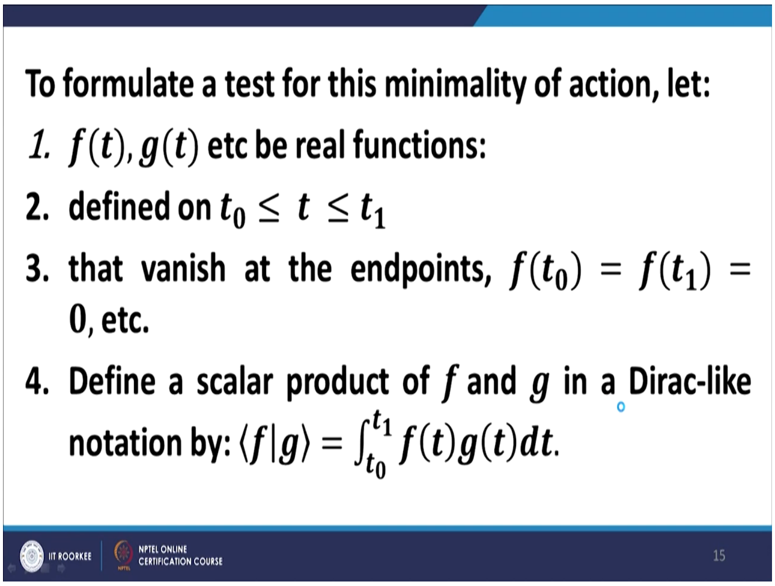

We define f t and g t as functions of time, real functions of time and in the defined on the interval t 0 to t 1 and these functions f t and g t both vanish at the boundaries of time that is at t 0 and t 1. Both these functions f t and g t vanish at the time boundaries t 0 and t 1 and otherwise they are defined on the interval t 0 to t 1.

We also define a scalar product of f and g in the conventional way that we have in the case of quantum mechanics, but the important part is here, we do not consider the conjugate, because we are talking about real functions, because f t and g t are both real function so; obviously, the conjugates coincide with the originals.

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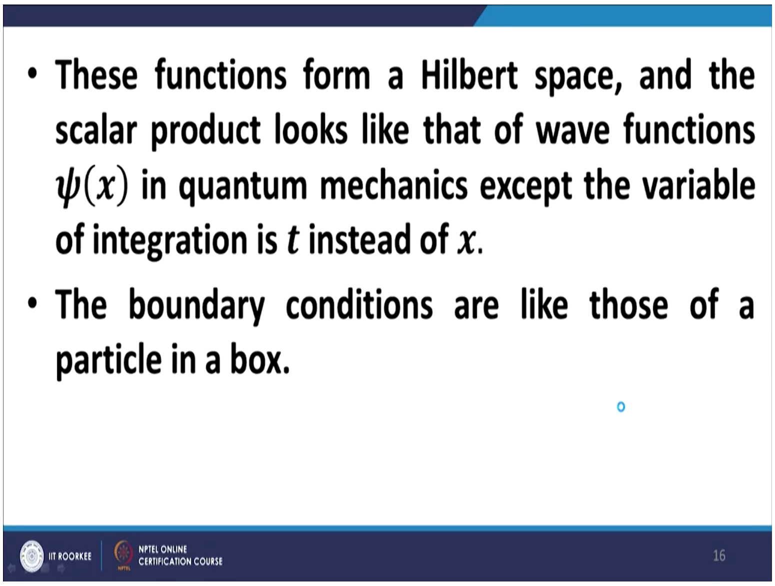

Now, this formalism of the structure enable us to form a Hilbert space and the scalar product is very similar to what we encounter in quantum mechanics. Although, if you look at it the only difference is that here we are having an integration with respect to t, normally when we have talk about scalar product in Hilbert space we have an integration with respect to x. The boundary conditions resemble the boundary conditions that are there in a box.

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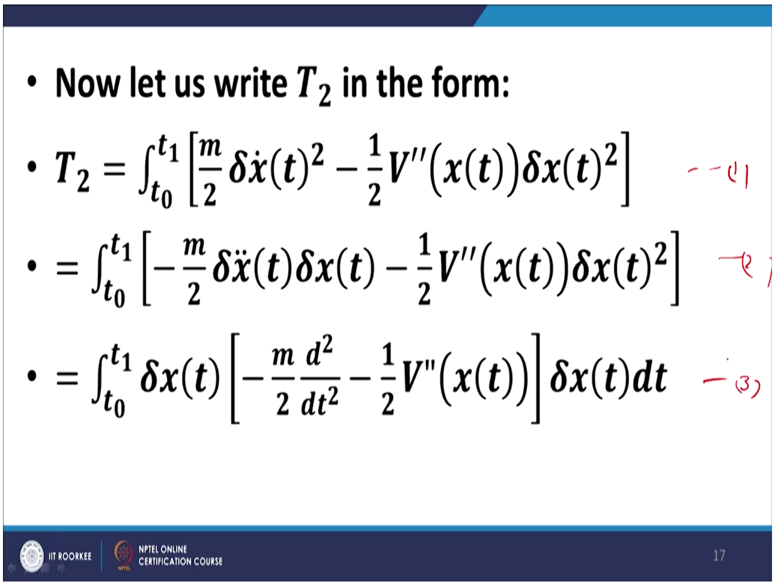

So, we can now let us go back to our T 2. The expression for T 2 has extracted from the original equation is given in the first equation here. This first equation through an integration by parts is and ignoring, throwing away the boundary term takes the form that is given in the second equation number 1 let us call it, equation number 2.

So, equation number 2 is obtained from equation number 1 by integration by parts and throwing out the boundary condition. This equation number 2 can be written in the form of equation number 3, it is quite straightforward and only the derivatives or operators are explicitly written here.

Now, we introduce an operator B which represents the operator that is given in the square bracket in equation 2.

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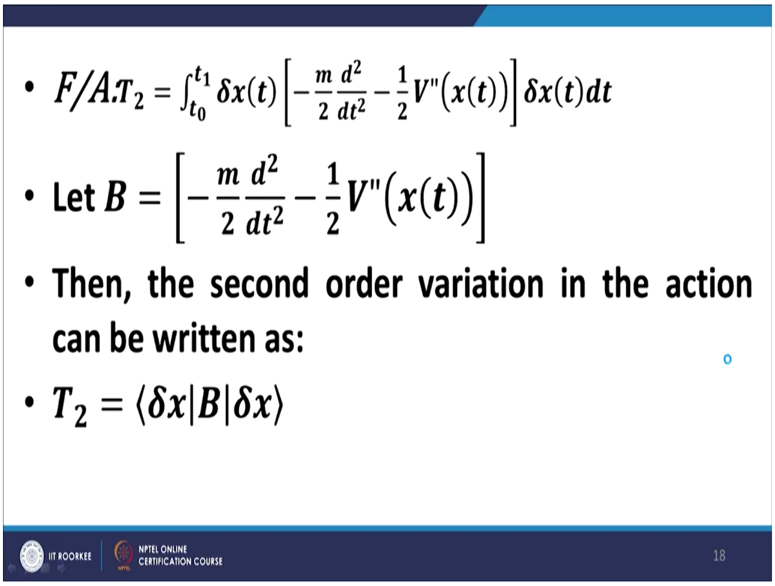

So, in terms of that in terms of the operator B, we can write the expression for T 2 in the form of an expectation value of the sandwich between delta x or in the state delta x. So, in other words the second order variation that is T 2 can be written as the expectation of B in delta x. Now, we come to the conditions now, if T 2 is greater than the, T 2 greater than 0 means what means that the expectation has to be greater than 0.

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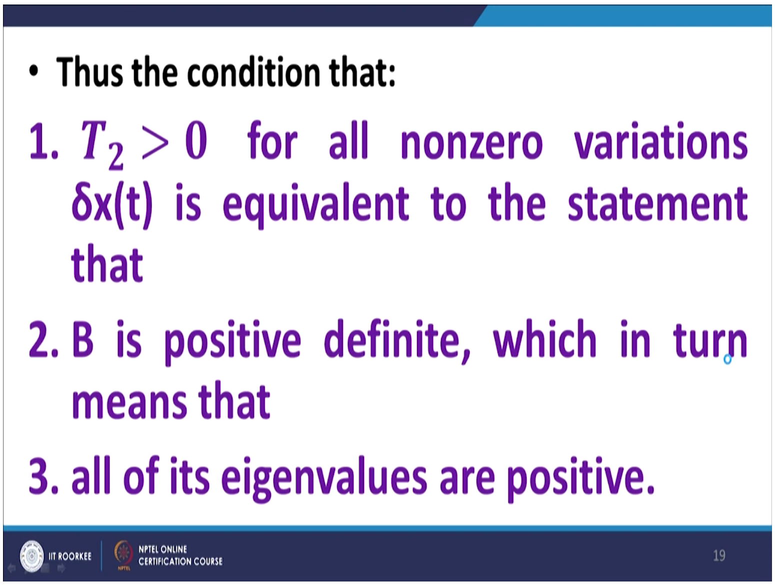

In other words B has to positive definite and which means that all its eigenvalues must be positive. So, the requirement that the action is minimal along the given path or along the classical path is that the; is that the T 2 must be greater than 0 or B must be positive definite or the eigenvalues of B must all, must all be positive must all be positive.

In other words action is stationary along a physical path, stationarity may or may not imply minimality, stationary and minimality will occur when the first variate of the action is 0 and the second variation of the action satisfies these conditions T 2 is greater than 0 or the operator B defined as above is positive definite, all its eigenvalues are positive.

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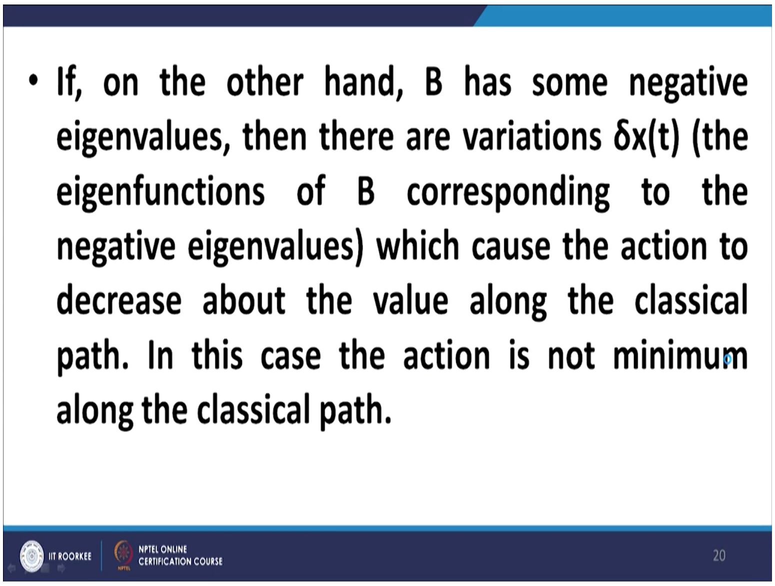

However, if it has some negative eigenvalues then the variations delta x eigen functions of B corresponding to the negative eigenvalues will cause the action, which cause the action to decrease about the classical path and therefore, in this case the action will not be minimal.

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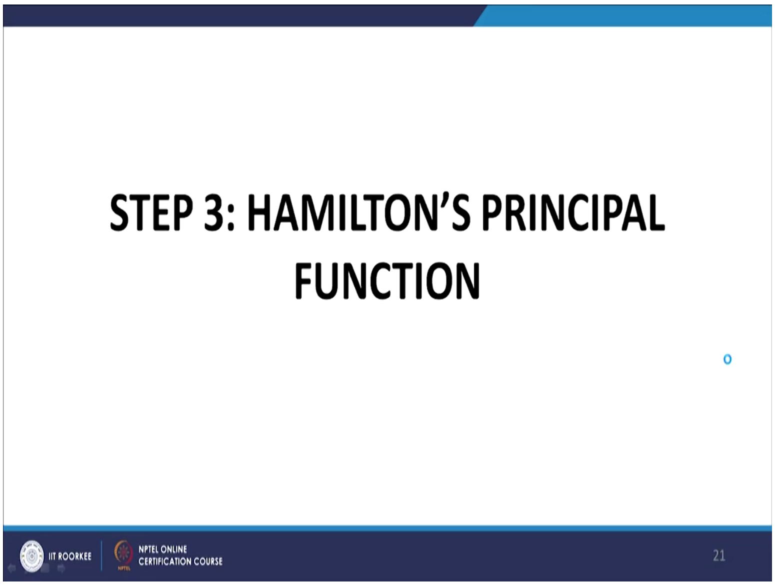

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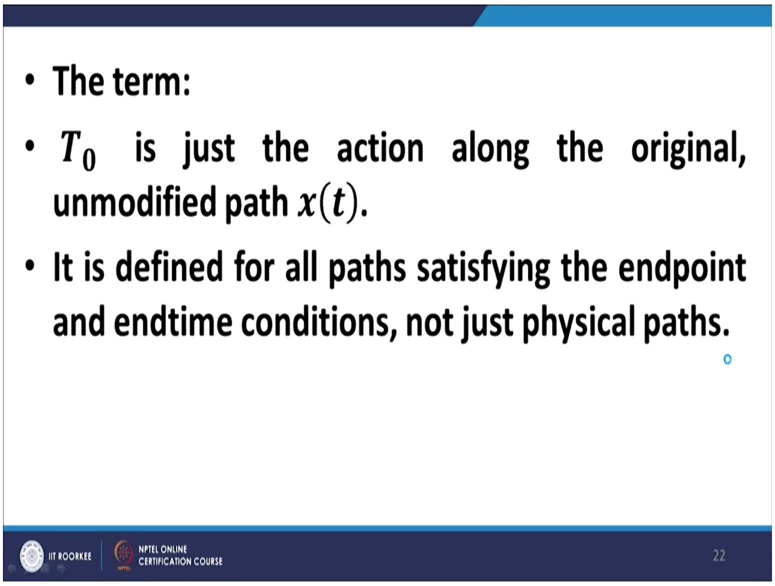

Now, we come to Hamilton's principal function. This is the machinery that is required for developing the path integral in this formulation. We have talked about T 0, it is the action along the original unmodified path and it is, T 0 is defined for all four paths satisfying the end endpoint, end time conditions not necessarily the physical path.

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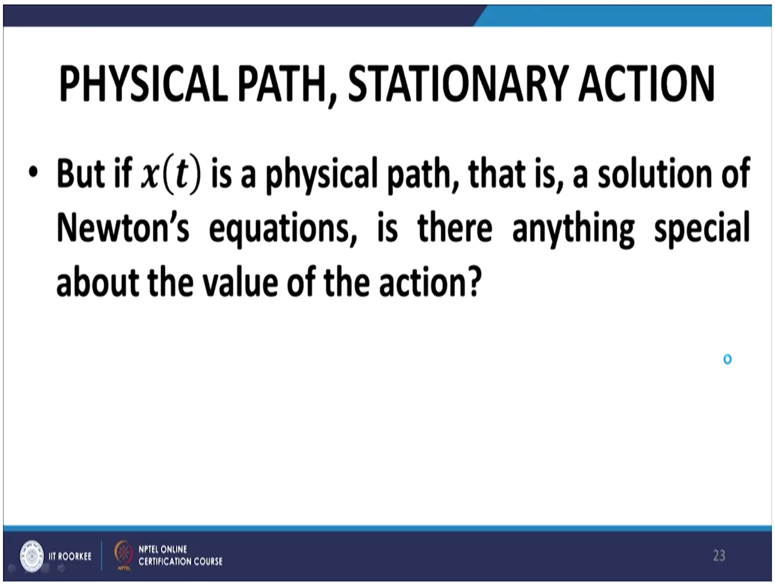

Physical paths are identified by stationarity of the actions. In other words as we mentioned if a path is to satisfy the, if the path is to satisfy the Newton's laws, then it the path must be must have 0 first variation of the action and vice versa, but what is special; what is special about the action of a path which is a physical path? That is the next question. What is special about the action other than stationarity for a physical path? That is the next question.

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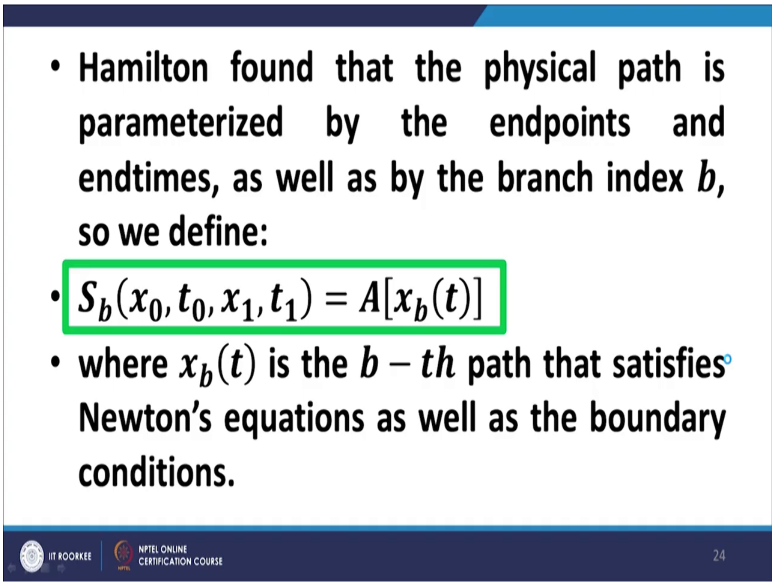

For this purpose we define a function which is called the Hamiltonians principle function by this expression in the green box and the function please note this, there is an important point here. First of all the subscript b represents the b-th path that satisfies the Newton's equation that is a physical path right.

So, that is as far as b is concerned. b represents or b indices or b identifies a path which is the physical path if there are many such paths, number 1. Number 2 if you look at it carefully, while the action is a function of the path it is a therefore, it is a functional, on the other hand, the principal function the Hamiltonian principle function the left hand side is a function of the endpoints only it is a function of the initial point endpoints, initial point and initial time and the final point and final time.

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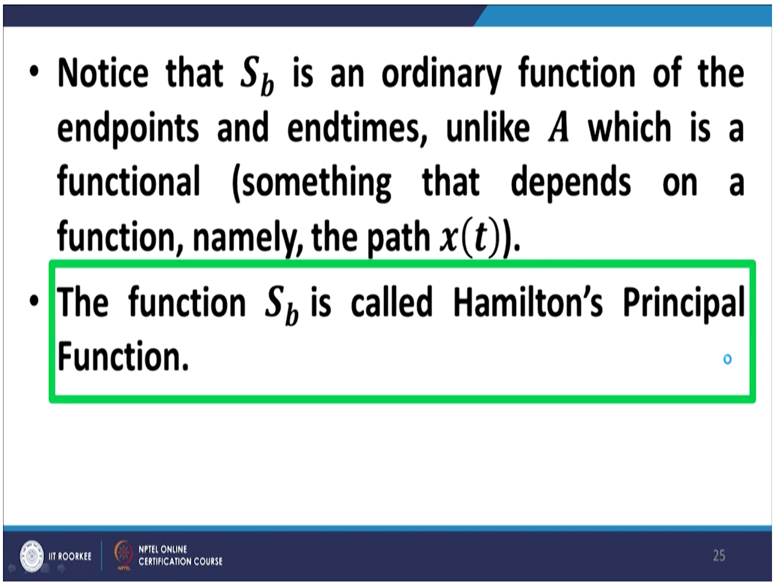

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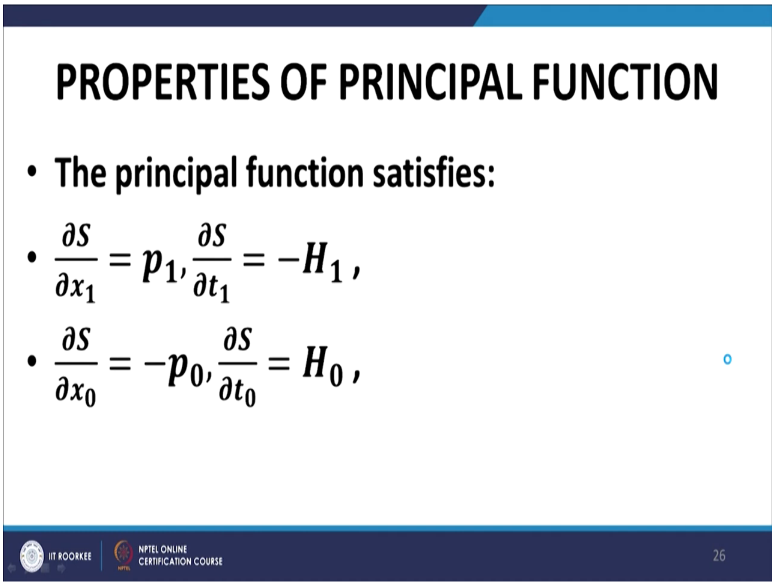

These properties of the principal function the Hamilton's principal function can be easily establish by considering any simple physical system and working it out. Now, we come to this stationary phase approximation.

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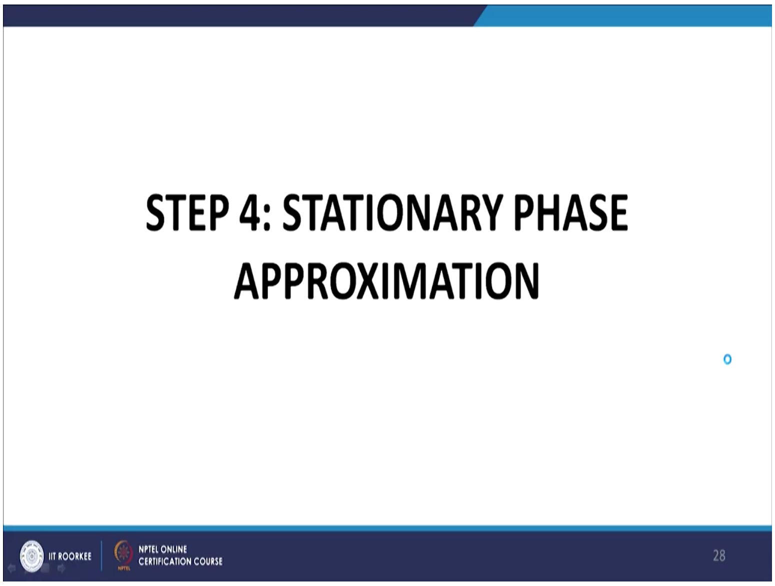

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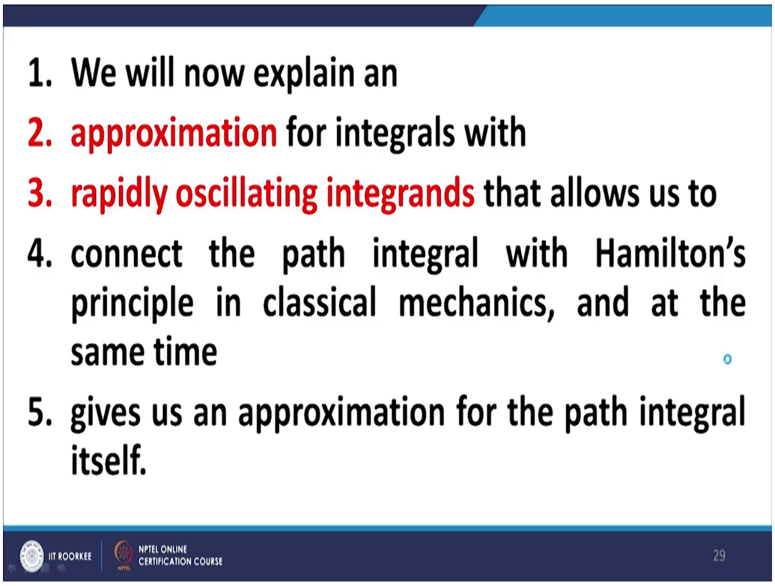

This is step number 4. I briefly touched on this when we talked about the relativistic path integral this is, this was discussed when we talked about saddle point being involved when working out an approximate solution to the propagator of a point of a point in Minkowski space., but coming back to this.

We now, try to approximate integrals which have rapidly oscillating behavior, rapidly oscillating behavior. You see why we are talking about this is let us go back to our path integral expression. It in the integrand involves e to the weight factor of e to the power i s. Now, e to the power i s is a oscillate oscillating integral because, it has modulus 1 and as the phase changes naturally the integral oscillates.

Now, if because we have normally we have in this case I have not mentioned the h bar factor in the denominator of the exponent, but the precise expression is e to the power i upon h bar into S and the h bar being. So, very small even marginally variation in the action causes a huge change in the phase and therefore, the oscillations are very rapid and that is why this particular approximation becomes very relevant.

Now so, we are talking about these, this is the backdrop this is the backdrop in which we are talking about these integrals. We are trying to arrive approximations for these kind of integrals which oscillate very rapidly, because of some small factor present in the denominator of the exponent.

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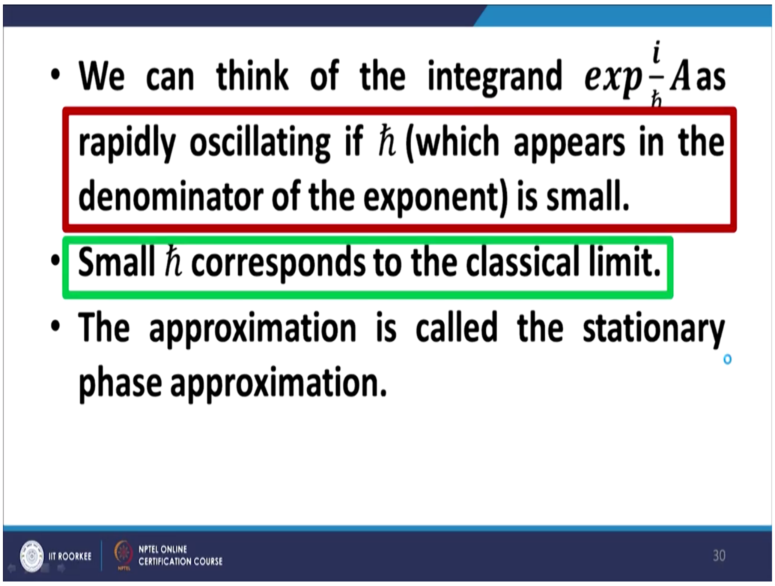

So, with now this is a integral I have shown here. This is rapidly oscillating if h bar which appears in the exponent is small. The smaller it is the lesser variation in h is going to cause more rapid oscillations. Now, the approximation, how we introduce the approximation?

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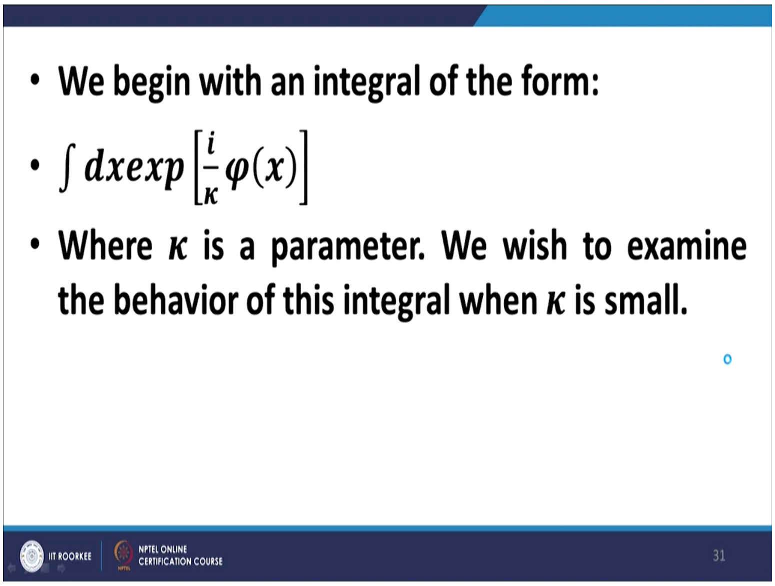

Let us see, we start with an integral of this form integral d x exponential 1 upon kappa phi of x, where kappa is a parameter.

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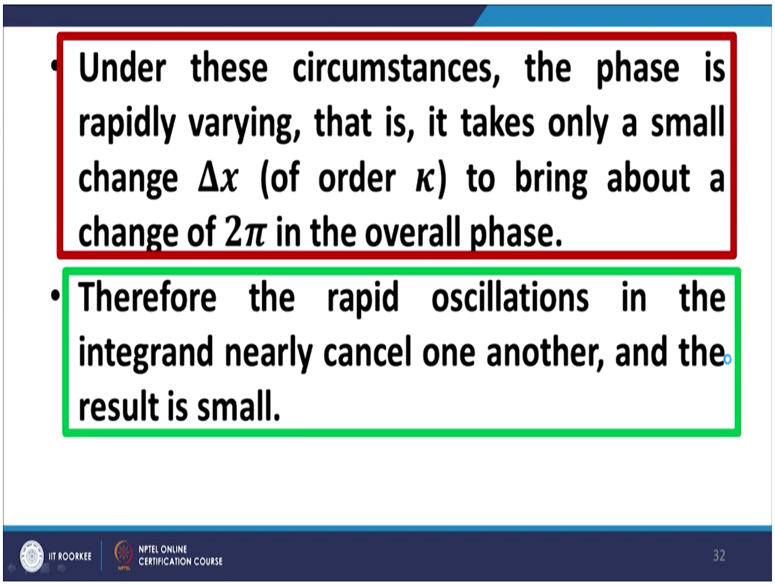

Now, the phases is rapidly varying, because we have got a factor in the denominator. A small change in the numerator will cause the significant change in the phase, a small delta x change in the value of the value of the phase which is of the order of kappa will cause the a change of 2 pi in the overall phase.

So, that is why the oscillations are extremely rapid and extremely sensitive to changes in the phase, but what happens the oscillations may rapid and being extremely sensitive to the phase, what happens is even small changes in phase result in larger deviations and these deviations on the average they tend to cancel out.

So, rapid oscillations of the integrand tend to cancel out on the average, because they are in some sense they are not stuck they are not coherent, they are non structured and even a small change in its for example, the three action, because of the very small factor h bar here, causes the significant change in phase and these deviations being non coherent, they tend to cancel each other out.

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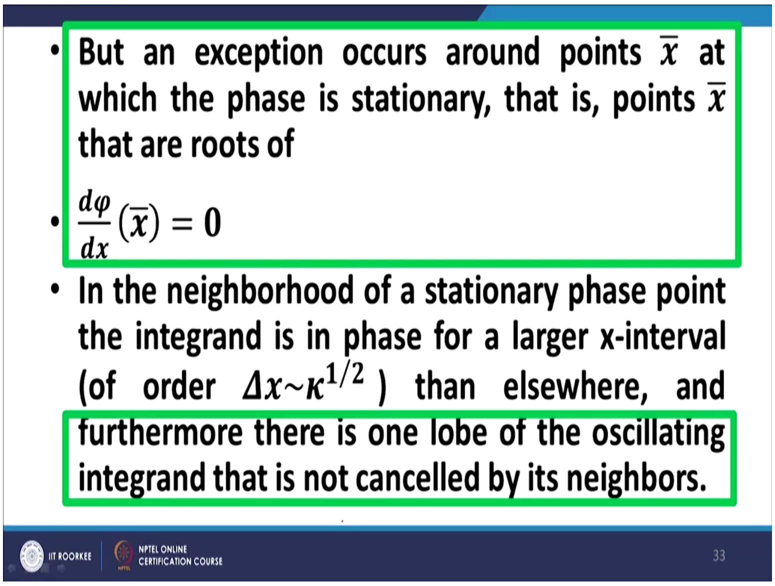

But; however, at a particular point for example, which is identified by this d x d phi in our case phi d phi by d x of x bar at x bar x bar is that point at which this expression becomes 0, this derivative become 0. At this point what will happen is; if you look at this point, this point it is just stationary, it is a stationary point.

Therefore, if there is a small variation in the in S at this point or in phi at this point the, because it is a stationary point the variation is phase is going to be very small, the deviation is going to be very small and as a result the coherence is maintained and, because the coherence is maintained the total effect blows up and here therefore, this manifests itself as the classical path in a sense.

In other words in the neighborhood of a stationary phase point the integrand is in phase for a larger interval, because the normally we have the bucket shaped curves you know and the bottom is flatter and because the bottom is flatter. If you have a slight deviation to either side it does not affect the value of the function much and that is therefore, coherence is sustained. Structurally there is one lobe of the oscillating integrand which is not cancelled by the neighbors. So, that gives the net result which of coherence of sustenance.

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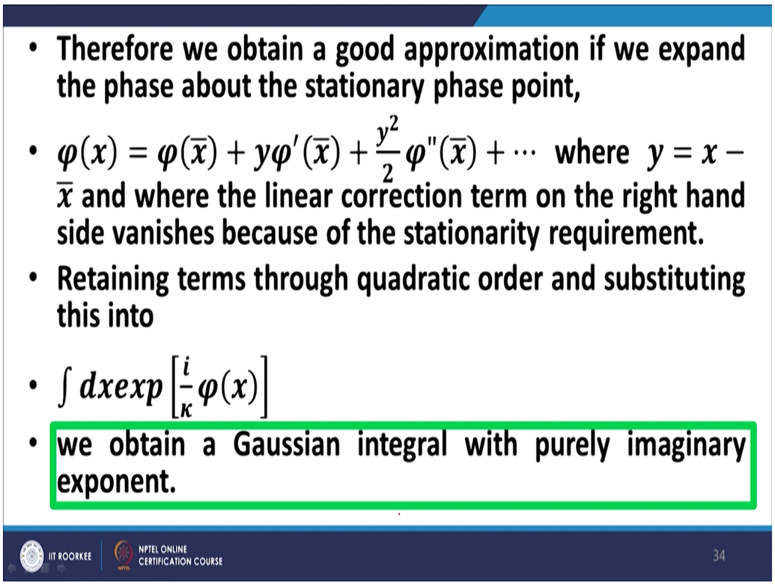

Now, we expand this expression phi x, the phase phi x, we expand this expression phi x around this stationary phase approximation, around the point at which the phase is stationary as a Taylor series.

We expand it phase phi x as a Taylor series around the stationary point. Of course, at the stationary point you will see that phi dash x automatically vanishes. So, this term well go and we are left with phi x and then the quadratic term and so on. For we retain terms up to the quadratic order, we assume that they are sufficiently representative and therefore, now what we are left with therefore, is phi x as the first order term phi x bar, the second the zeroth order term phi x bar, the first order term goes, because of the stationarity requirement and we have the second order term.

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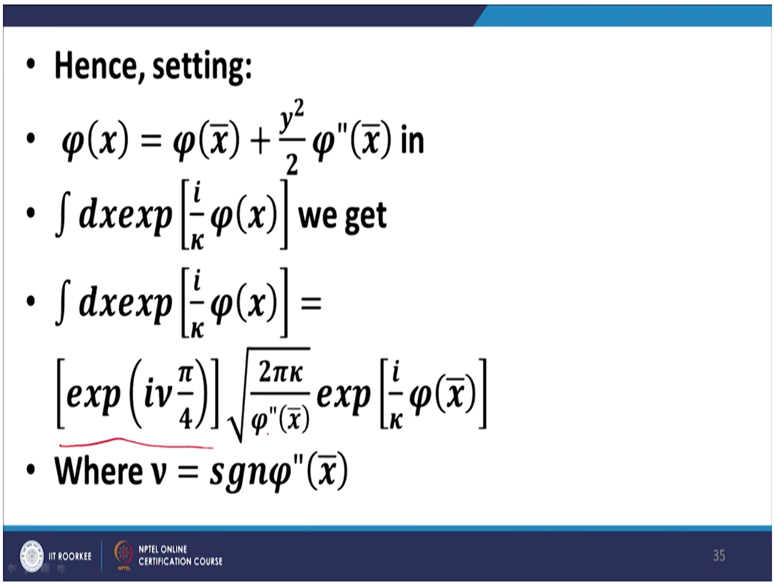

So, when we this is what we have simplify by simplification, when we substitute it in the exponent of the exponential we get a Gaussian integral and if this Gaussian integral can be carried out conveniently and we get a result here, but the important thing is this Gaussian integral is purely imaginary.

That is an varying factor and therefore, this first factor, the pre factor appears and this pre factor captures the pure imaginary character of the Gaussian integral where nu is the sign of i double dash at the stationary point. Now, there may be more than one stationary points.

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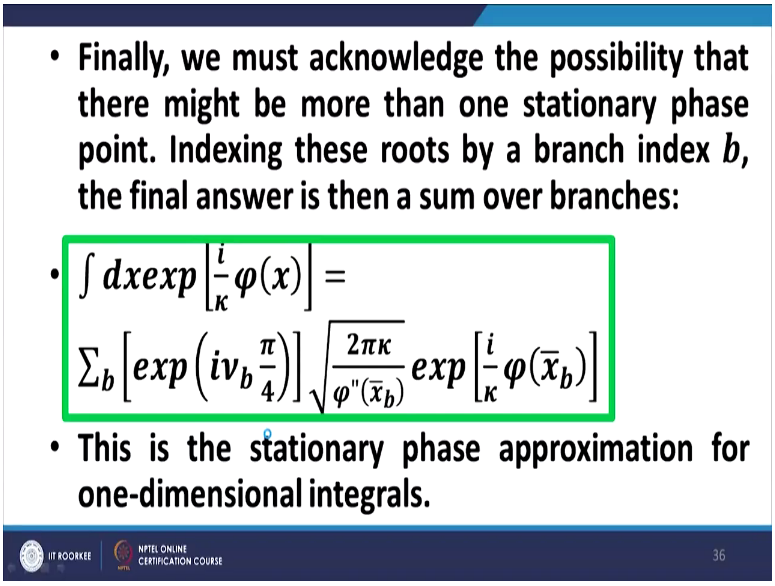

So, in the event that there are more than one stationary points the whole expression has to be summed over all the stationary points, which are indexed by the index b and therefore, the final answer becomes this expression which is given in the green box in your in this slide and for multidimensional integrals the expression can be extended. The expression that we have here can be extended and we get the expression that is given in the green box on this slide here.

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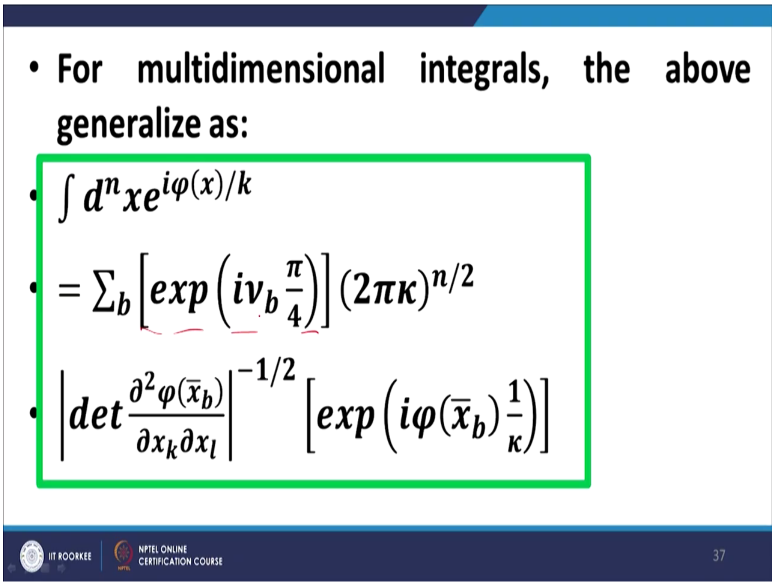

Please note this pre factor in variably makes an appearance and this pre factor contains nu b and nu b is nothing, but the sign of what? Sign of the phi double dash bar at phi double dash at the stationary point.

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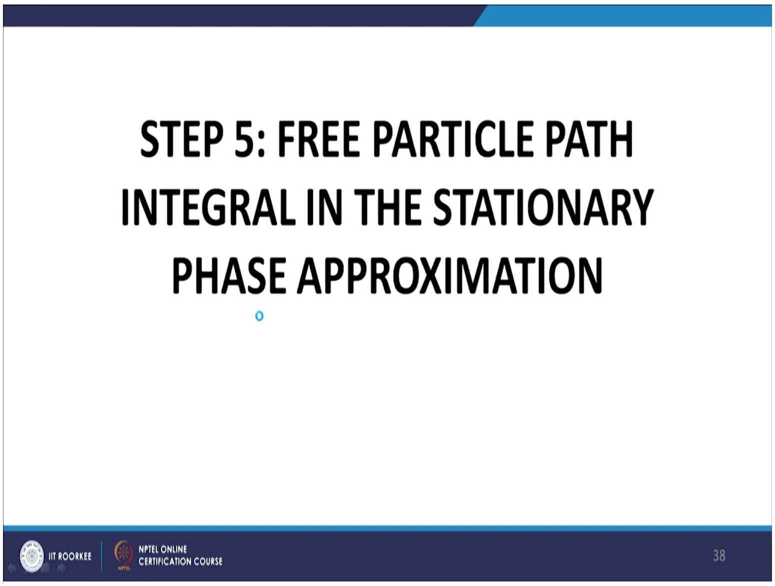

Now, we come to the final step here we construct the path integral in the stationary phase approximation that is the final step we, because we are having a free particle so there is no potential function here.

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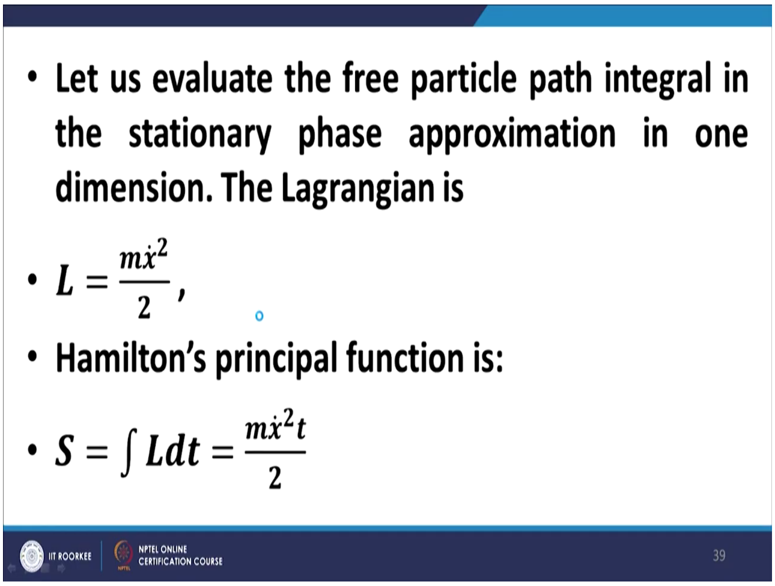

The Lagrangian is given by the kinetic energy alone m x dot square upon 2 the Hamiltonian principal function as the action which is m x double dot square upon 2 into t. So, these two are straightforward.

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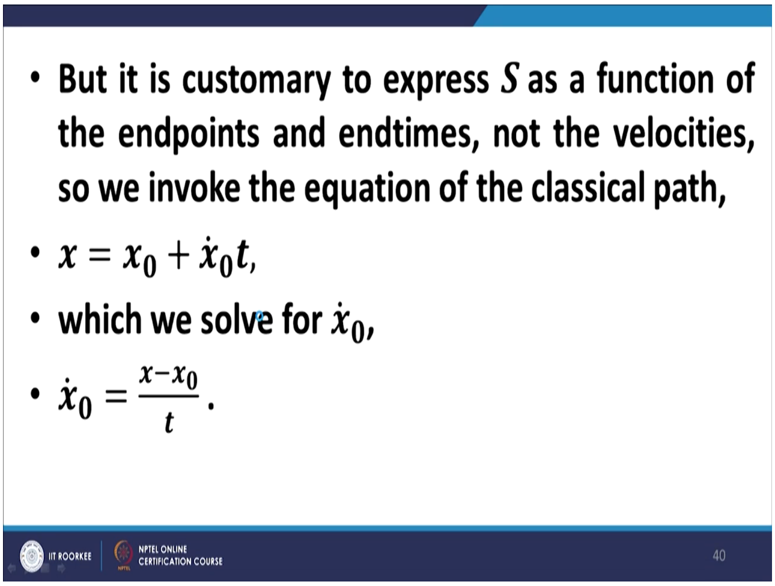

Now, we can express the action in terms of the end points and end times, as I mentioned, because we are here talking about the Hamilton’s principal function which is usually expressed as a function of the end points and end times.

So, we express S as the function of endpoints and end times we recall that x is equal to x naught plus x dot t constant velocity case, because there is no potential here free particle, no potential, no force therefore, x is equal to x dot plus x is equal to x naught plus x dot into t and please note x dot is also equal to x 0 dot x dot is equal to x 0 dot, because the velocity remains unchanged and x is equal to x 0 plus x 0 dot into t this is the distance traversed by the particle displacement of the particle.

Simplifying this we get this expression here x 0 dot is equal to x minus x 0 upon t, we substitute this in the expression for S m x dot square t we get m x minus x dot square upon 2 t.

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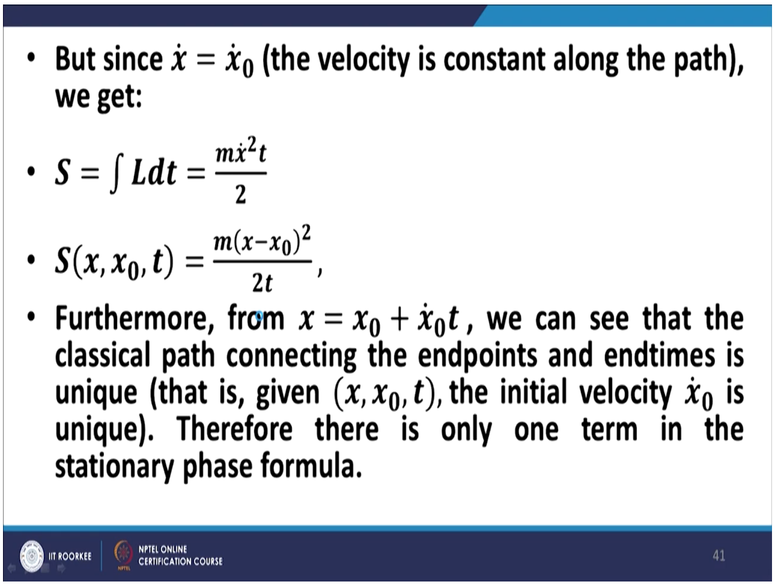

Now, this is the classical path constituting the endpoints. Now, the important thing is the classical path connecting the endpoints and end times is unique. In this particular case it is pretty clear, it is quite straightforward that we have only one path the straight line path between the point x and x dot I am sorry x and x 0 and therefore, the initial velocity is also, initial velocity x dot 0 is unique, because we have only one path here.

So, the initial velocity is also unique and therefore, we have only one term in the stationary phase formula to repeat, because there is only one physical path, there is only one physical path, there is only one initial velocity and correspondingly there is only one expression in the stationary phase formula. The generating function S gives us the following relations as for the properties that I had mentioned earlier.

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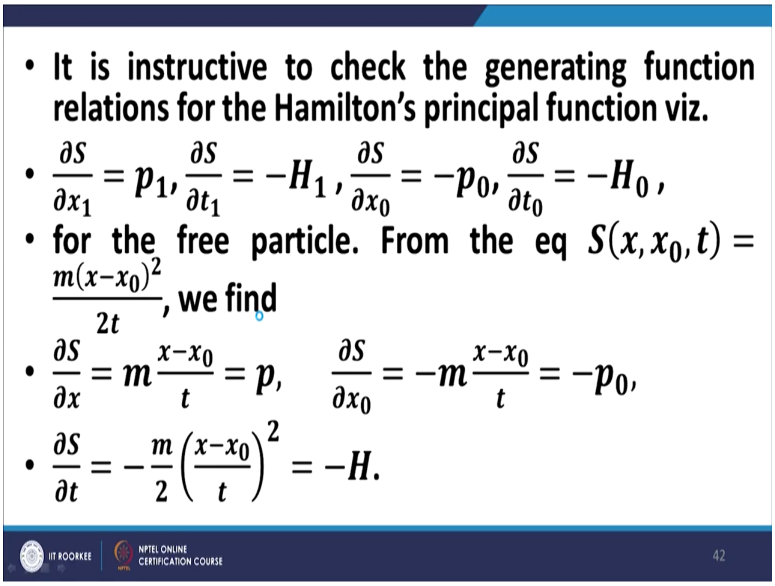

These are the relations that can be verified from simple differentiation of the Hamilton principal function.

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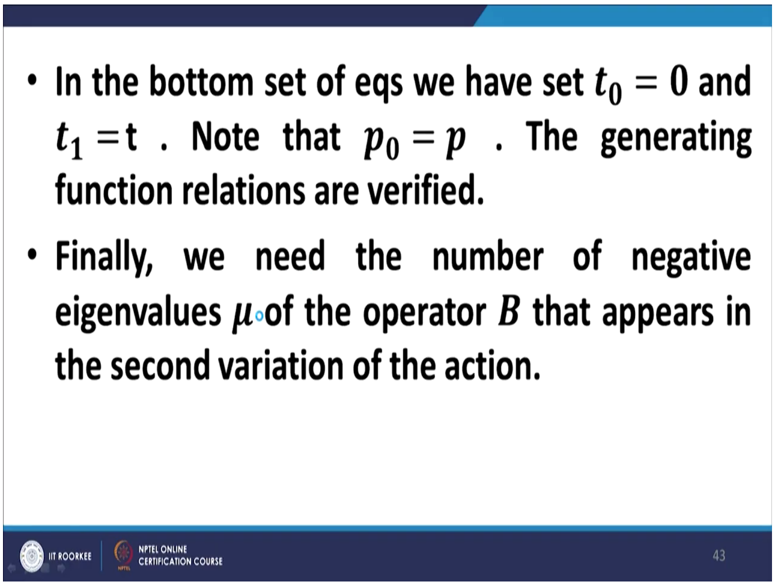

Now, we want to find out more about the eigenvalues, the behavior of the eigenvalues whether the eigenvalues are negative. If there are any negative eigen values or the eigenvalues are all positive these are the things that we need to examine.

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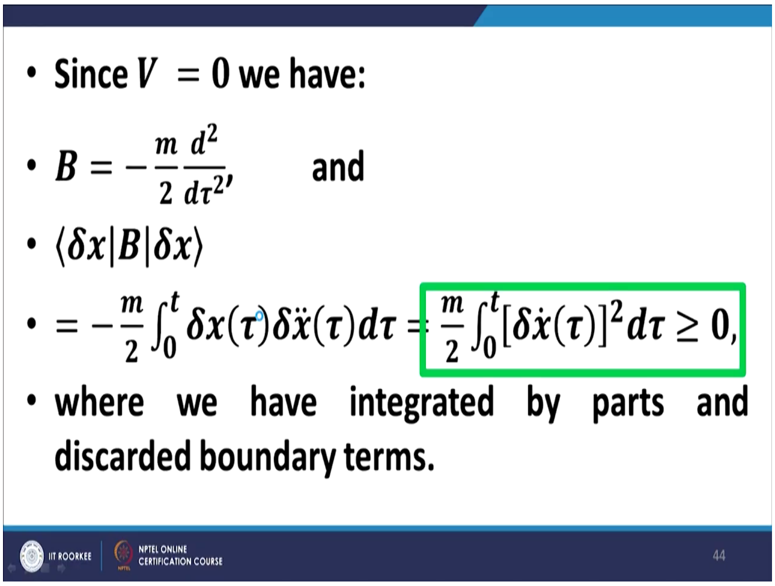

Now, since V is equal to 0 we have B is equal to minus m by 2 d square upon d tau square and this expression if I substitute the value of B in the integral in the integral form we get this expression for t 2 for t 2. And you can clearly see here this expression is positive definite and that clearly shows that the action for this particular system is minimum at the stationary point.

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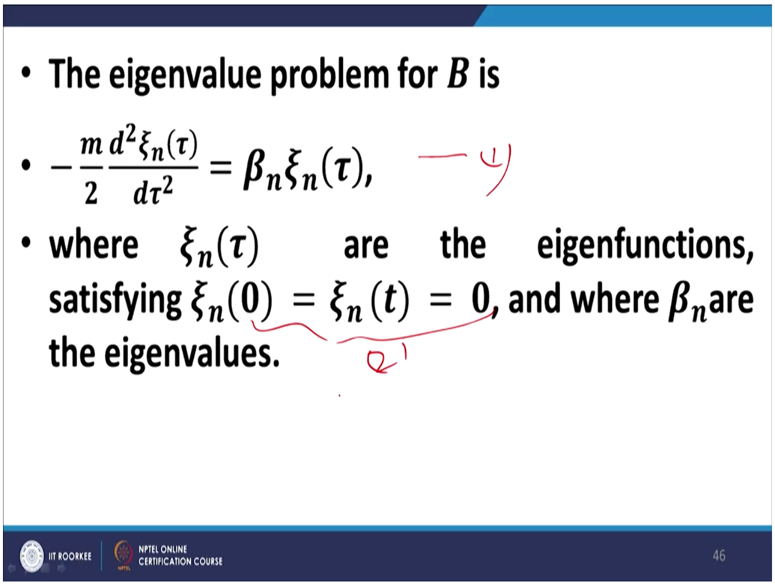

Now the eigenvalue problem for B if you can see here, it can be written in this form and this if you solve this expression with this the boundary conditions which are given, let us call this equation 1 and let us call this boundary conditions 2.

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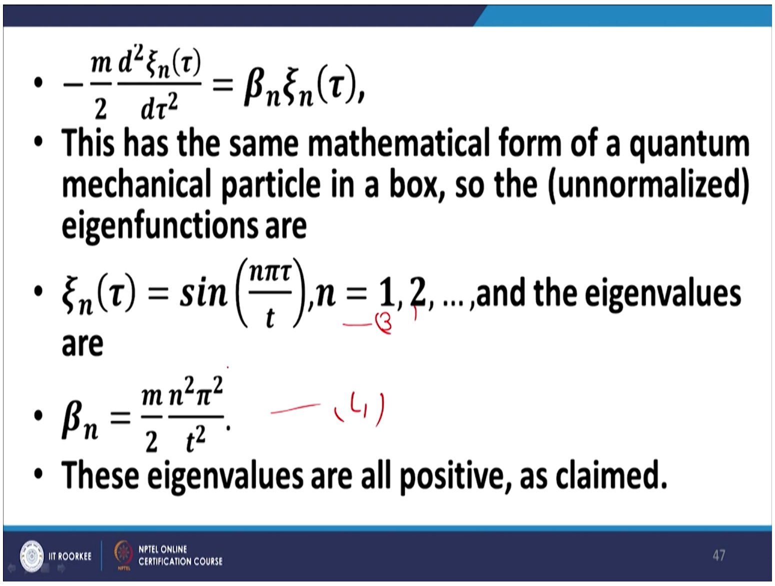

We can work out the eigenvalues and the eigen functions explicitly. We get the eigen functions as sin. In fact, this problem is similar to the problem of a particle in a box, quantum mechanical particle in a box and when you simplify or work out the eigen functions we get the expression in equation 3 and the expression for the eigenvalues is in equation 4. Clearly all the eigenvalues are positive and therefore the our physical system has minimum at minimum of action at the point of stationary action.

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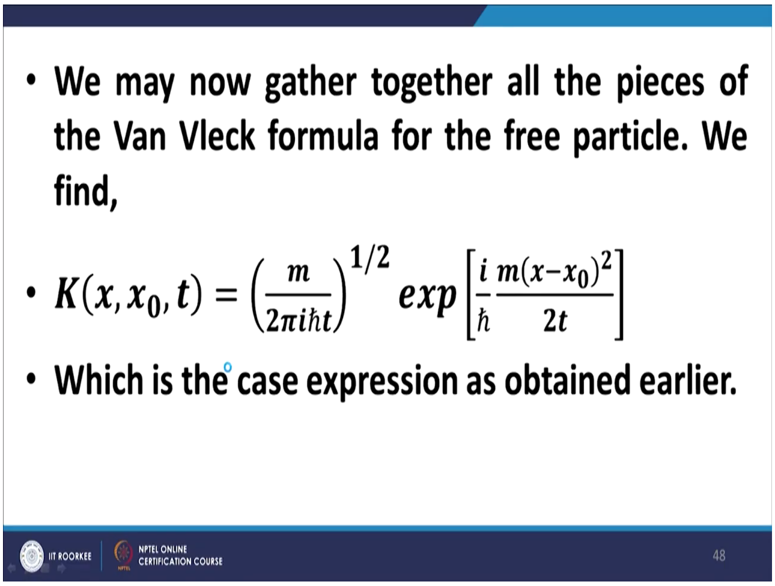

And collecting all these expressions together we get the expression for the path integral as the quantity that is given on your slide. And which is precisely the same expression that we had when we worked it out as for the Feynman's time slicing approach.

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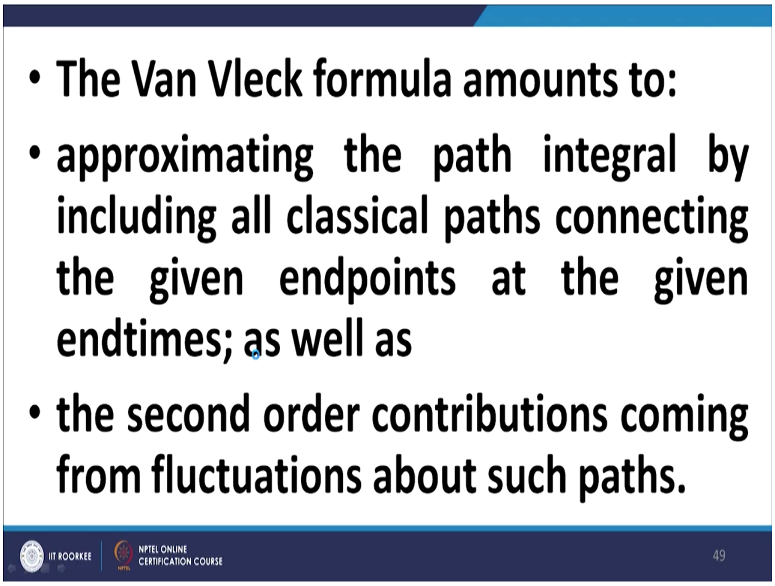

So, this approach is certainly a different, certainly more informative and certainly more instructive in the sense that it goes through the route of the action principle directly and brings to you the same expression for the path integral. So, of course, the approximation here is introduced by the stationary phase approximation which.

In fact, this stationary phase approximation manifests itself very coherently, very significantly in the context of path integrals when we talk about the, when we talk about the quantum mechanical effects manifesting themselves in the path integral as a classical path in the form of a classical path. How do classic quantum mechanical paths coalesce among themselves to give you the classical path? So, that is a very interesting issue, this is brought it here.

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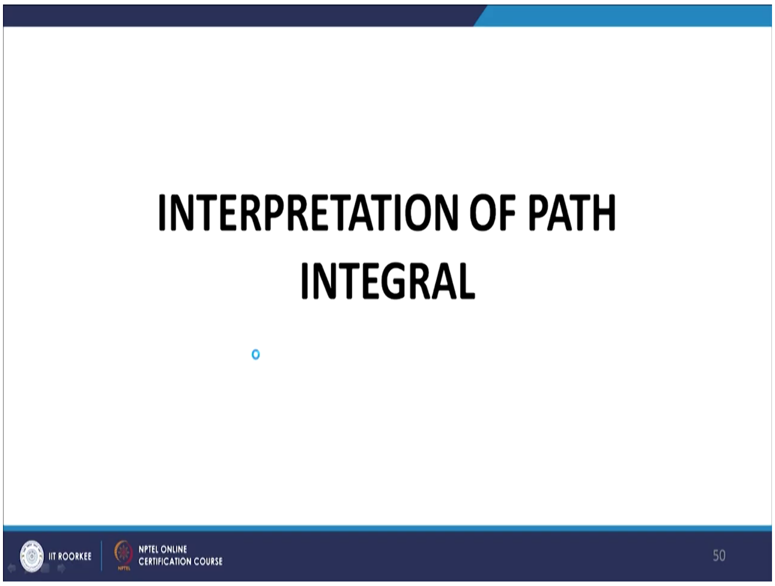

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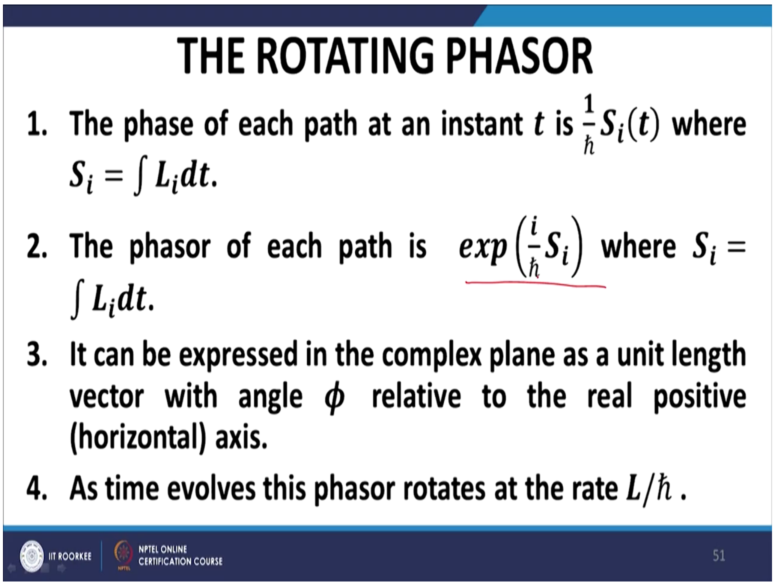

Now the machinery that we have discussed here can be elaborated more by considering the case of a, or by considering explicitly, or figuratively or pictorially the case of path integral. Let us look at this.

5. The total phase $\phi = \int \frac{L}{\hbar} dt$.

6. So we can picture the phasor as a unit length vector rotating like a hand on a clock in a 2D complex plane (though it is a counterclockwise rotation).

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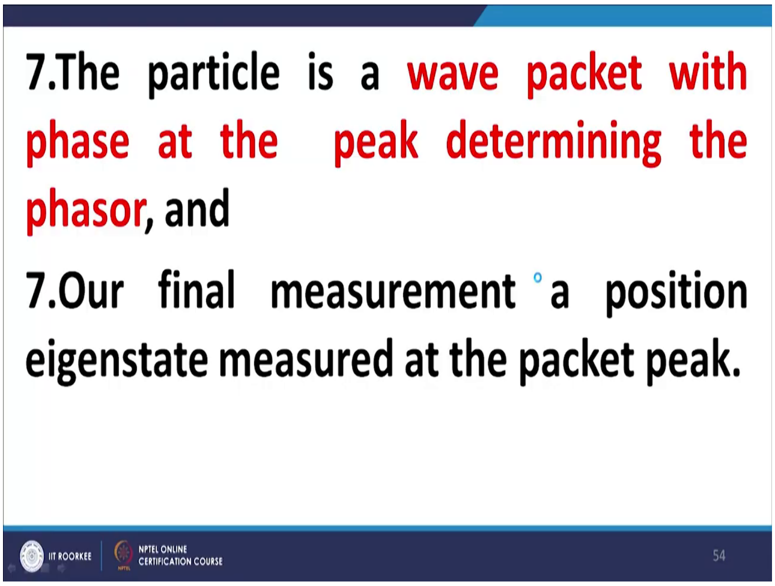

The phase of each path at an instant is given by 1 by h h bar S, we know that as it is the action, and the action is given by L i d t. And let us call this expression exponential i by h bar S as the phase. Let us call this expression exponential i by h bar S as the phasor.

Phasor not the phase the phasor. The 1 upon h bar into S is the phase and exponential i upon h bar S is the phasor, let us call it the phasor. So, clearly its a vector of unit length which rotates on the complex plane making an angle phi, where this phi is represented by this phase relative to the real axis. As the way progresses or as time elapses this phasor tends to rotate on this complex plane in the counterclockwise direction mind you.

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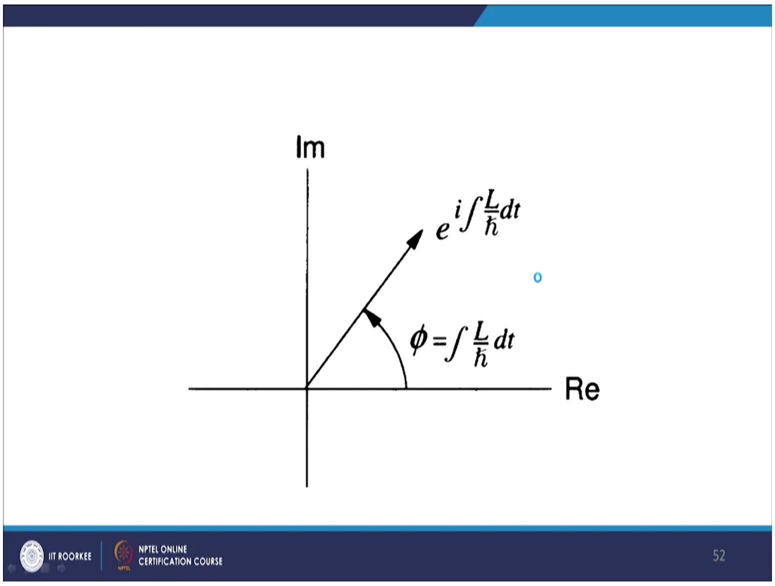

As this is given in this diagram, the phasor is its a unit vector and it is rotating in the counterclockwise direction as the phases, as the wave is progressing, as the time is elapsing. The total phase between any two points in time is given by this expression. So, this is the shape or the pictureative representation of what is happening in the context of the weighing factor. We will now talk about how that weighing factor manifests itself in the actual dynamics of the quantum wave.

Now the wave, as you know particles or objects in quantum mechanics are deemed to travel as waves or wave packets, and its the peak of the wave packet and that represents the or that determines the phase. And the final measurement that is also determined by the peak of the wave packet. And a different wave factor, a different wave packet follows each of two different paths between two events.

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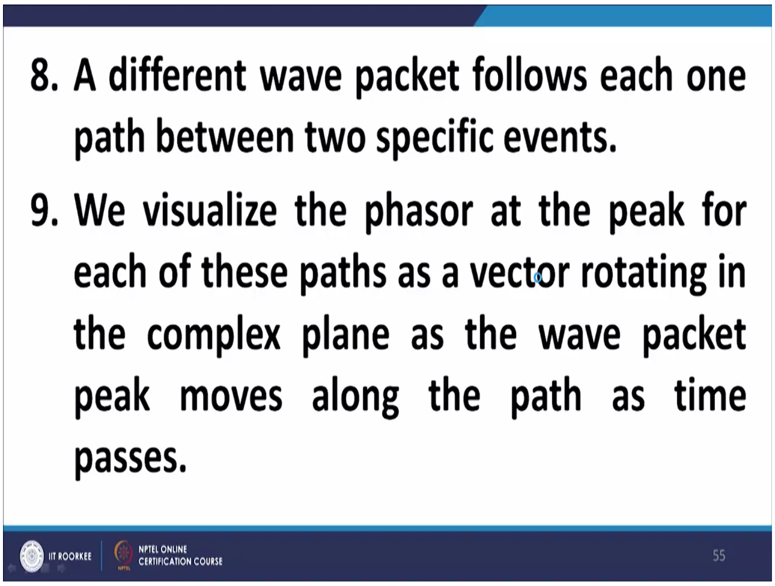

So there are different wave packets following different path between two events. So, we can visualize the phasor at the peak of each path, being a vector rotating in the counterclockwise direction as the wave packet progresses along a particular path. That phasor which is tagged to that particular wave packet also rotates in a counterclockwise direction. So, every different path is traversed by a different wave packet and therefore, has a different phasor are tagged to it.

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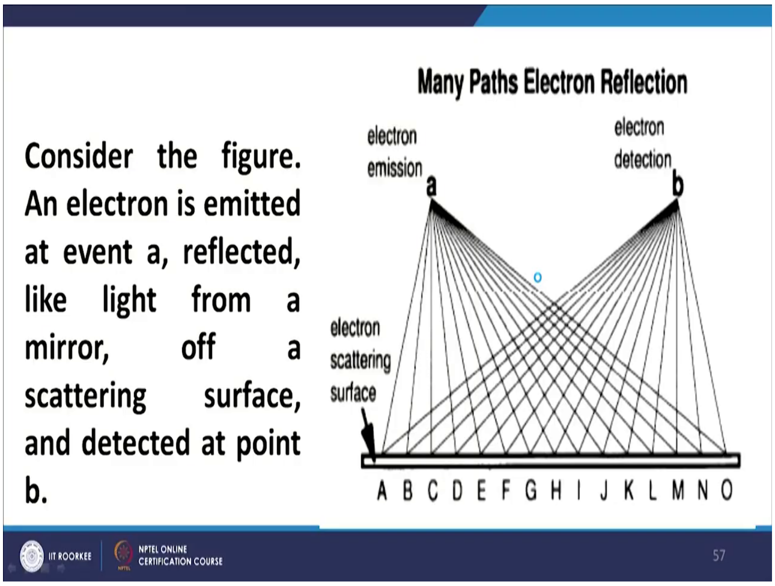

So, as an example let us consider this simple example. You have an electron which is emitted at the source a, it is reflected say by a magnetic, by a negative charge and mirror and it is identified or detected at the electron detector at the point b. So, for example, we have taken 15 different paths A B C D upto O.

Path H is assumed to be the classical path, because it has a angle of incidence equal to angle of reflection, the other half some of the many paths. Of course, there will be infinitely many paths from the source event to the detector event. Let us to explain the to figuratively explain the dynamics we assume 15 paths around its.

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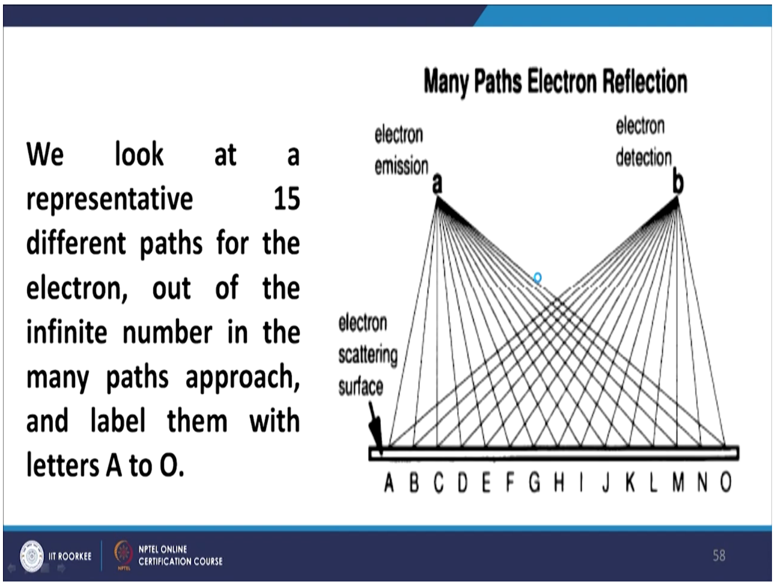

Now each of these path is having its own phasor please note this and as this the electron progresses as a wave along this path that the phasor also rotates on the complex plane or is willing to rotate along the complex plane in the counterclockwise direction. We assume that each path, because the space time points are the same for all the paths initial and final points are the same for all the paths.

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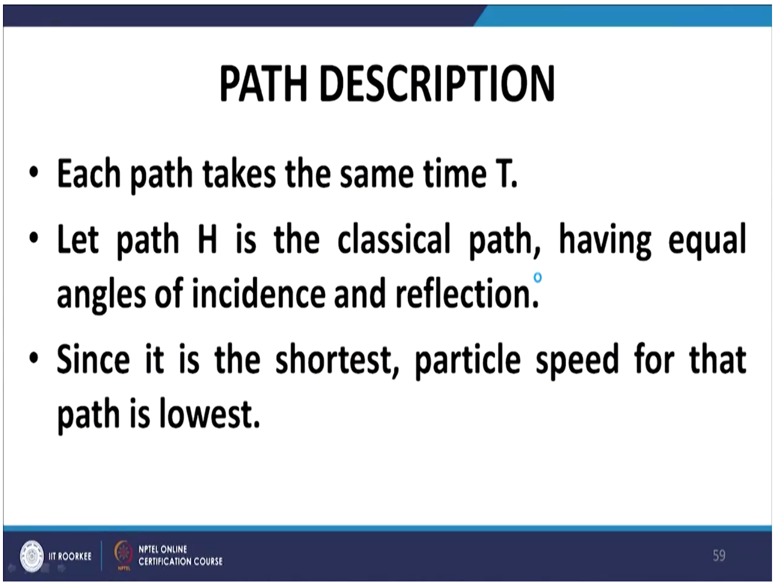

That means, each path takes the same time T and path H as I mentioned as is the classical path. It is the shortest path and therefore, it has the minimum kinetic energy and minimum speed.

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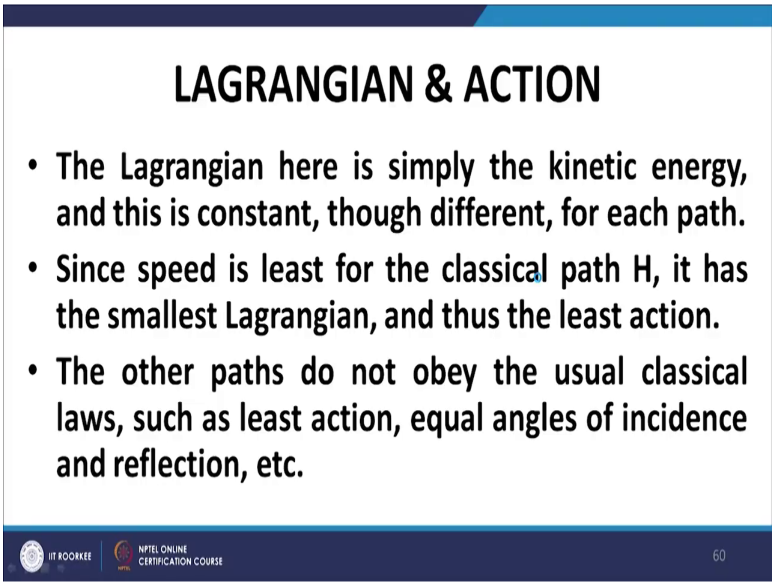

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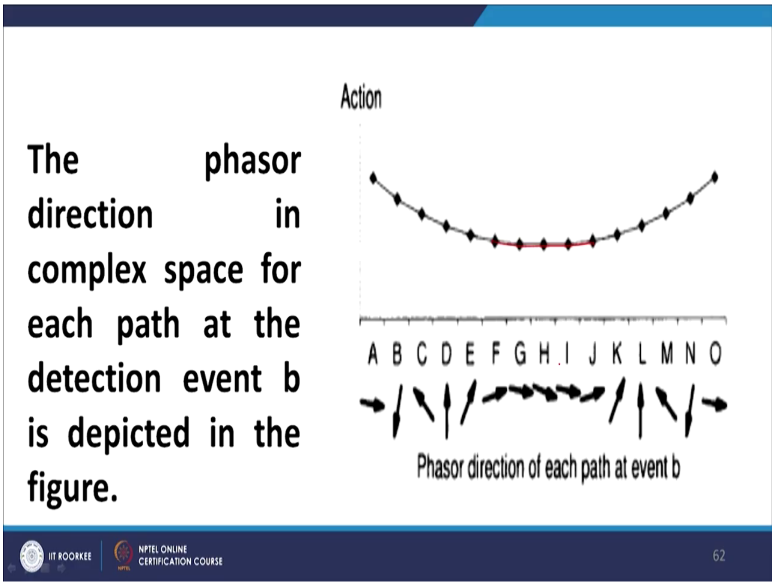

So, Lagrangian is the kinetic energy. Now look at it what is happening here, now this is the important part if you look at the. Please note the direction of the phasors at the point at which they intersect or they reach the destination point or the detection point b. Please I repeat, please note the direction of the phasors at the point at which they reach or they touch or they are detected by the event b. Now clearly there is a flat bed in the action diagram corresponding to the various paths.

In other words the action is stationary along this bed, let us say F G H I on the J. And you can see here that corresponding to these points F G H I J the phasors are also more or less aligned in the same direction. And as a result of which these phasors being aligned in the same direction, they produce coherence.

And due to this coherence we observe them as the classical path. And if you look at the other phasors that correspond to higher actions on either side of the stationary action, you find that some of them are, most of them are moving in random directions and therefore, they tend to anal each other on the average.

They do not form any kind of a coherence beam, a coherence structure. The coherence structure is only formed by the set of or the various phasors that correspond to the stationary action or are close to the stationary action. Because as I have been emphasizing a small deviation from the stationary action, because the action is stationary there does not disturb the action significantly.

And as a result of it because the action is not disturbed significantly the past tend to be coherent and we observe them as the classical path. So, that is what it is.

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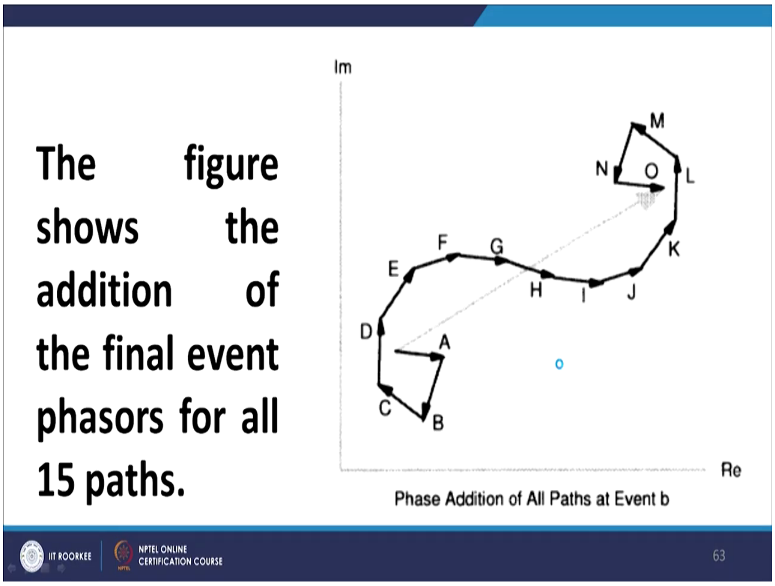

Thank you so much.