Path Integral Methods in Physics & Finance
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Lecture – 26
Harmonic Oscillator 2-Point Problem

Right, let us continue from where we left off. What we will do now is, we shall apply the formalism that we had developed in the previous lecture. Let us just apply that to a specific example, let us just look at working out the ground state expectation values for the harmonic oscillator.

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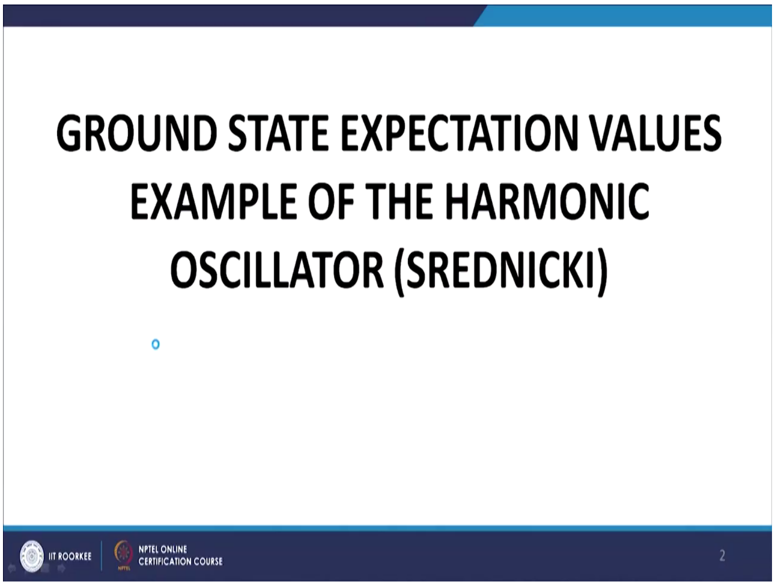

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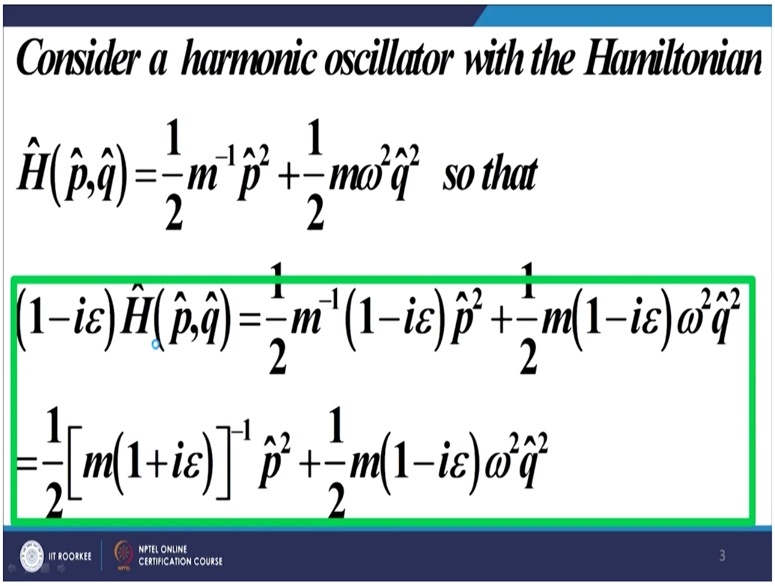

The Hamiltonian for the harmonic oscillator can be written in the form of the first expression on in the first equation on your slide. And, when we introduced the damping parameter or the epsilon parameter in order to manage the Hamiltonian or manage the evolution appropriately to recover the ground state at a later point in time.

We write the Hamiltonian as H into 1 minus i epsilon, and that simplifies to the last equation on your slide ah, which now gives you the Hamiltonian with which you shall proceed. Just to recall the methodology, we first modify the Hamiltonian by introducing this infinitesimal or damping parameter and then take the limit t tending to infinity. And finally, recover the ground state by multiplying by arbitrary function.

So, that is our process that we are going to follow, the formalism that we are going to follow. Let us quick get about the first step we have already done. We have got the Hamiltonian; the new Hamiltonian; the transformed Hamiltonian.

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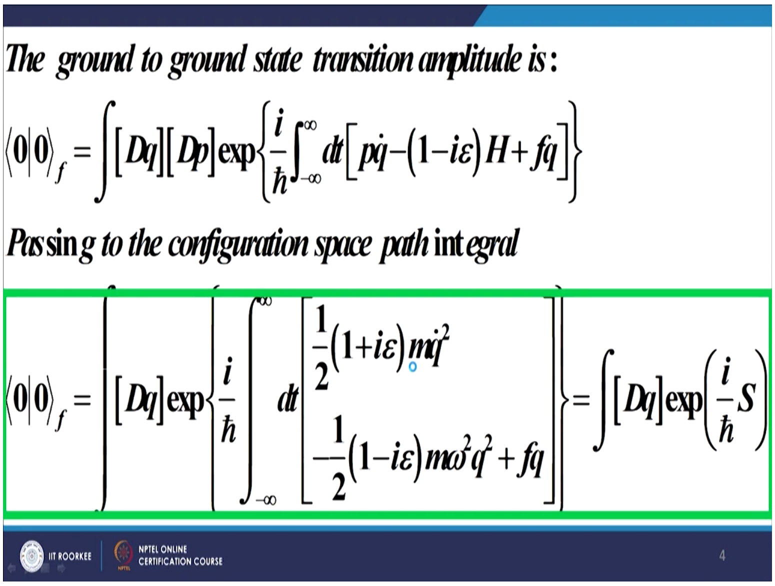

Now, the ground state transition amplitude is given by this expression as usual; however, the Hamiltonian; instead of the original Hamiltonian is now the modified Hamiltonian, other than that there is no change.

And by carrying out the p integrals you can introduce a normalization factor and but the normalization factor for the moment we need not consider and we have focused on the configuration space path integral, which is given by this expression. Now, please note the path integration is with respect to the coordinate elements D q. For further simplification, we said m equal to 1 and h bar equal to 1 and introduce Fourier transform variables.

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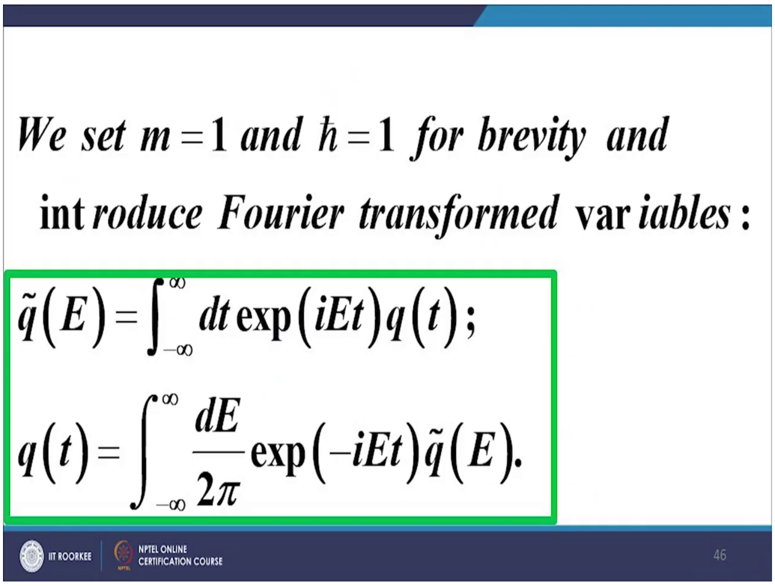

We are now moving from the coordinate space to the energy space.

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$$Hence, L_{\varepsilon,f} = \frac{1}{2}(1+i\varepsilon)m\dot{q}^2 - \frac{1}{2}(1-i\varepsilon)m\omega^2 q^2 + fq$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dE}{2\pi}\frac{dE'}{2\pi}\left\{\exp\left[-i(E+E')t\right]\right\} \times \left\{\begin{array}{l}\left[-(1+i\varepsilon)EE' - (1-i\varepsilon)\omega^2\right] \times \\ \tilde{q}(E)\tilde{q}(E') + \\ \tilde{f}(E)\tilde{q}(E') + \tilde{f}(E')\tilde{q}(E)\end{array}\right\}$$

In terms of the Fourier transform variables the Lagrangian can be written in this form. The Lagrangian in the original variables, just to recall is given by the equation the first equation; the kinetic energy minus potential energy terms, it clearly see that.

Of course there is a extra source term over there. The extra source term is there, otherwise it is kinetic energy minus potential energy. So, this is the Lagrangian and the subscript epsilon represents that the Hamiltonian is the modified Hamiltonian. And the subscript f represents that we are having a classical source term in the Hamiltonian, in the Lagrangian and the Hamiltonian as well.

So, in terms of that Fourier transformed variables we can write this Lagrangian in this form. We are simply substituting the values from the expression from the values given in this green box. And these represents the Fourier transforms of q t and the inverse Fourier transform. We are simply using this and writing the Lagrangian in the Fourier space, right.

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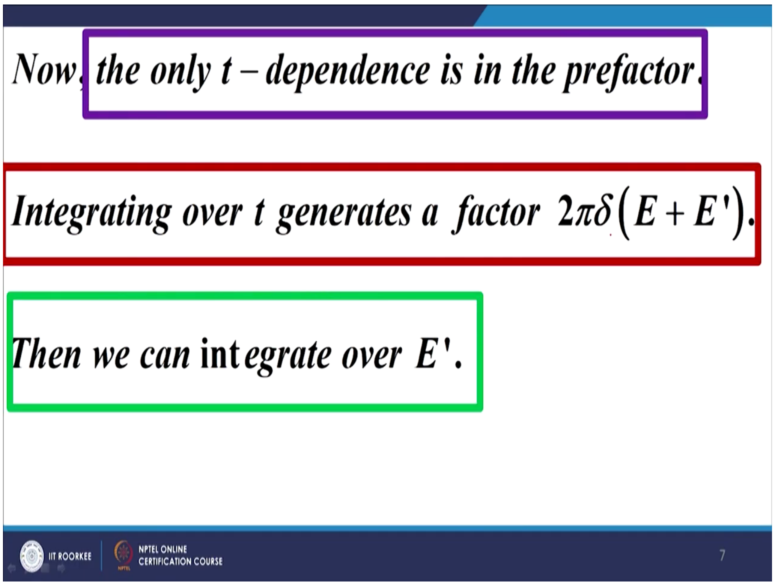

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$$S=\int_{-\infty}^{\infty} dt\left[\frac{1}{2}(1+i\varepsilon)m\dot{q}^{2}-\frac{1}{2}(1-i\varepsilon)m\omega^{2}q^{2}+fq\right]$$

$$=\frac{1}{2}\int_{-\infty}^{\infty} dt\left[\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{dE'}{2\pi}e^{-i(E+E')t}\left\{\begin{matrix}\left[-(1+i\varepsilon)EE'-(1-i\varepsilon)\omega^{2}\right]\times\\ \tilde{q}(E)\tilde{q}(E')\\ +\tilde{f}(E)\tilde{q}(E')+\tilde{f}(E')\tilde{q}(E)\end{matrix}\right\}\right]$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\left[\begin{matrix}\left[(1+i\varepsilon)E^{2}-(1-i\varepsilon)\omega^{2}\right]\tilde{q}(E)\tilde{q}(-E)\\ +\tilde{f}(E)\tilde{q}(-E)+\tilde{f}(-E)\tilde{q}(E)\end{matrix}\right]$$

Now, now if you look at this carefully, let us go back, the first thing is that the t integration is confined to the pre factor only, this rest of the terms do not contain any t. The t integration is with respect to the pre factor. Now, if I carry out the t integration, I get a delta function. Precisely, I get 2 pi delta E plus E dash that is what I get when I integrate the integrate over t this expression, right.

When I substitute that to delta function here, one of the 2 pi goes and I get delta E plus E dash here. And I integrate over E dash, what would the net result would be to substitute E E dash by minus c, everywhere in the remaining integral integrand. And that is precisely what we get here after the simplifications.

So, just to recap, what are the steps that we have done? The first step we have done is; we note that the t factor appears only in this quantity and therefore, we carry out the t integration.

When you carry out the t integration, the net result is a delta function; 2 pi delta E plus E dash. When you now carry out the delta integration with respect to E dash, the net result is that, everywhere where you have an E dash you replace it by minus E.

And, after you do all these three things, you get the result that is shown in the green box here.

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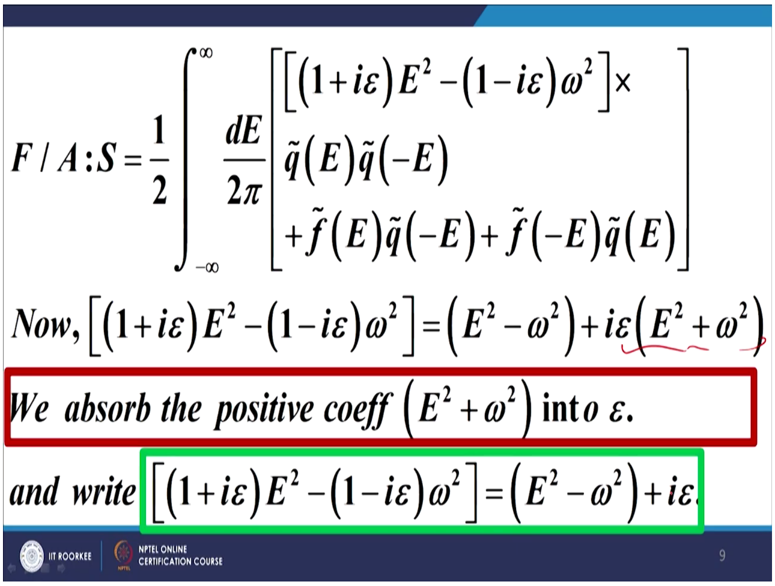

Now, this is what we have, now this is the action; please note that this is the action. We have the Lagrangian action here, action this is this was the Lagrangian. So, integration of the Lagrangian over the, with respect to the time gave us the action. So, this was the action and then we after processing this steps we get this expression for the action, right. Now, look at this expression; this expression 1 plus i epsilon E square minus 1 minus i epsilon omega square. I can write this in the form as E square minus omega square plus i epsilon E square plus omega square.

The next step that I do is, that I absorb this E square plus omega square, which is undoubtedly positive. Remember, epsilon has to be positive and that is the important condition that epsilon has to fulfill. So, and of course, it has to be real. So, because this expression epsilon E square plus omega square is positive, I can incorporate this and express this epsilon as this whole term.

In other words, I can rescale epsilon to be epsilon into E square plus omega square, because E square plus omega square is positive and it will not disturb the sign of epsilon. In other words, I can simply absorb this factor within the epsilon. And therefore, I can write this the whole expression as E square minus omega square plus i epsilon. I can write this in this form.

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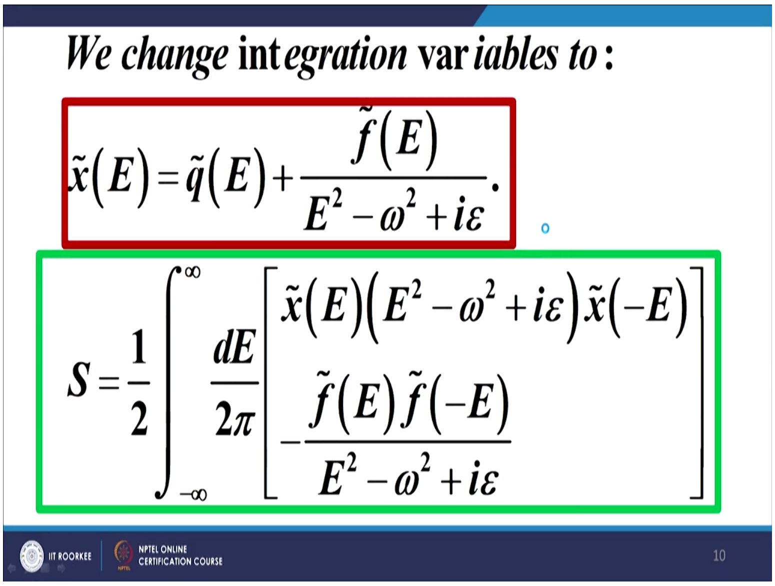

Now, we I introduce a change in integration variables, that is the next step. So, what change in variables? I introduced the variables x tilde as q tilde E plus f tilde E upon E square minus omega square plus i epsilon. After doing the manipulation, after re rescaling or after absorbing that factor into epsilon, I introduce a transformation of variables.

Please note, in this case that this is simply a shift of origin and therefore, dE of this expression will be equal to dE of the this expression, the integration element would not be affected by this transformation. Anyway, let us get back to this, on making this substitution, what we get is this expression in the green box that we have on the slide.

Now; so, the path integral over q is equal to the path integral over x, as I mentioned, because this is only a shift by a constant and we move to the next expression.

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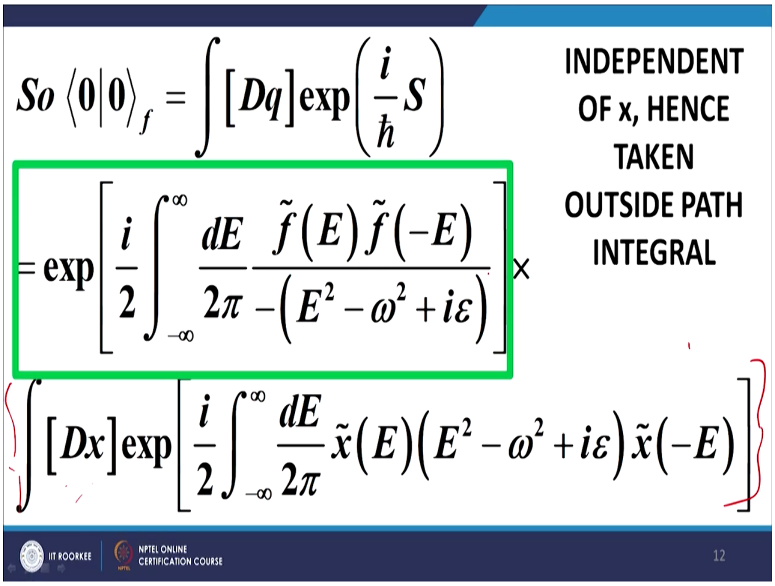

And now, even if you look at this expression, you know this is what we have here, and this is what we have now as the expression. Now, please note this, again yes, this one.

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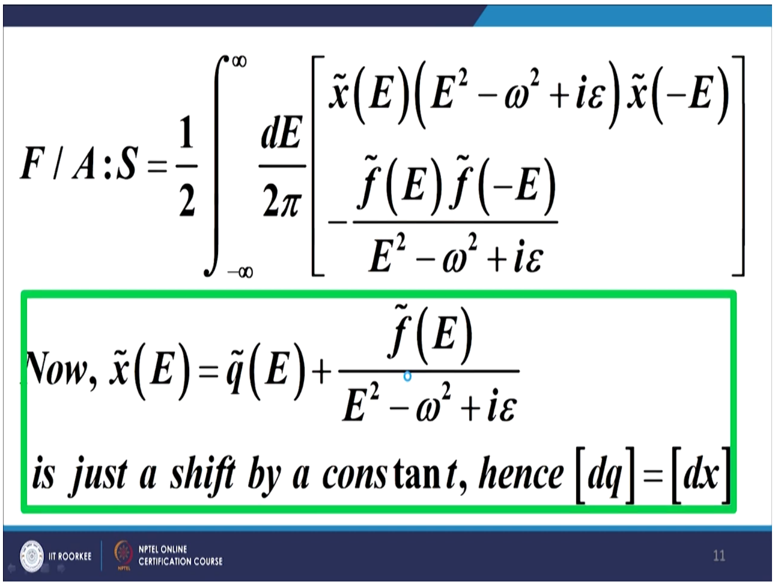

Whenever, I make the substitution, I get this expression. Now, please note this, this expression the second part is; does not contain any x; does not contain any x. So, when I do the path integration E to the power i s, this exhibition can be taken outside the path integral, because the path integral is with respect to x and this expression does not contain any x. This part of the action does not contain any x and it can be taken as a pre factor to the path integral. That is precisely what we have done here. Yes.

So, this is the action and this is the transition amplitude; this is given in terms of this action. Now, when I substitute this expression for the action, and I can take this outside the path integral, because it does not contain any x. The integration is with respect to x, please note this, the path integration is with respect to x.

So, this is important, this part becomes a pre factor to the path integral. The part in the green box becomes a pre factor to the path integral.

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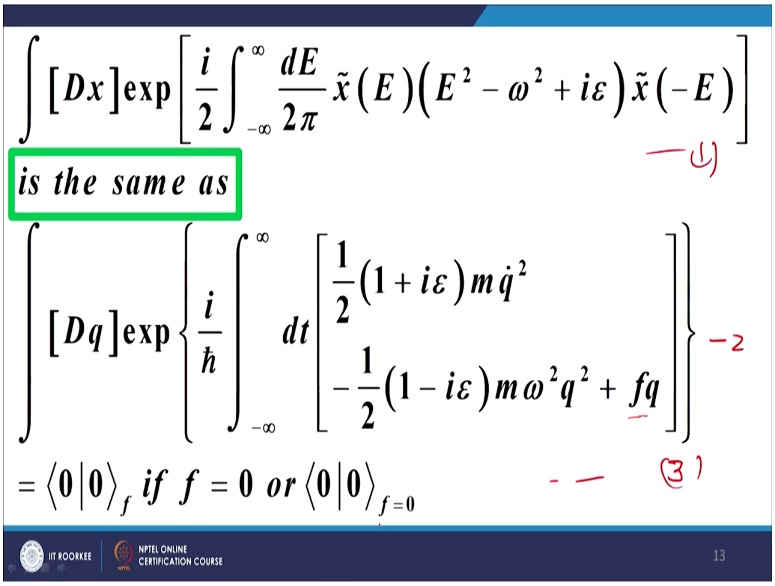

Now; this part, let us look at this path integral, if we rework to our original variables, it is clearly obvious; and that this is equivalent to the second expression, this expression. And this is nothing, but this is nothing, but the transition amplitude of the ground to ground state with the presence of f.

If you look at this expression, the second expression, this expression is nothing, but the transition amplitude transition amplitude for ground to ground state transition with of course, the factor f with the factor f.

In other words, what happens if I put the factor f equal to 0? I recovered the transition amplitude of the ground state to ground state. This expression with the factor f equal to 0 gives me the transition amplitude of the ground state to the ground state.

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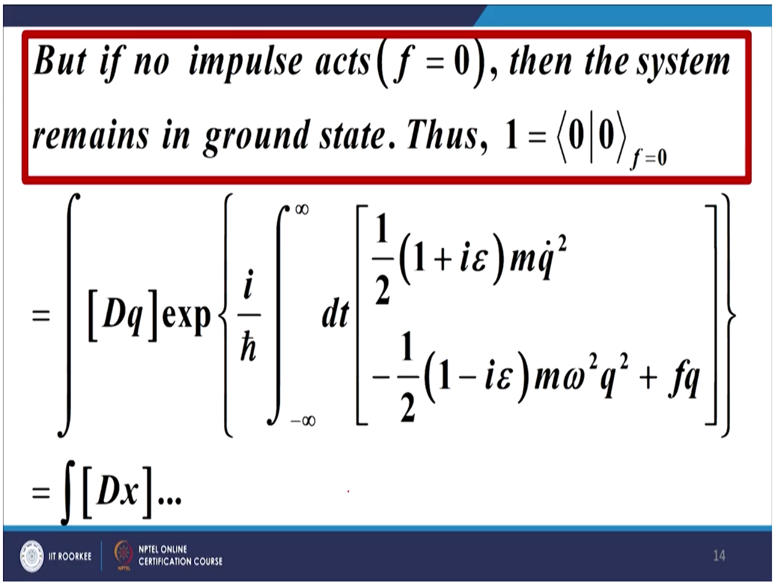

But; let us see, now, if the impulse if there is no impulse then the system would invariably stay in the ground state. If the system is undisturbed absolutely, the physical system it is the quantum system is undisturbed by any external force, then the system would stay in it is ground state.

And; that means, what? That means, that ground to ground state amplitude in the absence of f must be equal to 1. Ground to ground state amplitude in the absence of f must be equal to 1 and; that means, what? That means, this expression which is nothing, but this expression should be equal to 1, if f is equal to 0.

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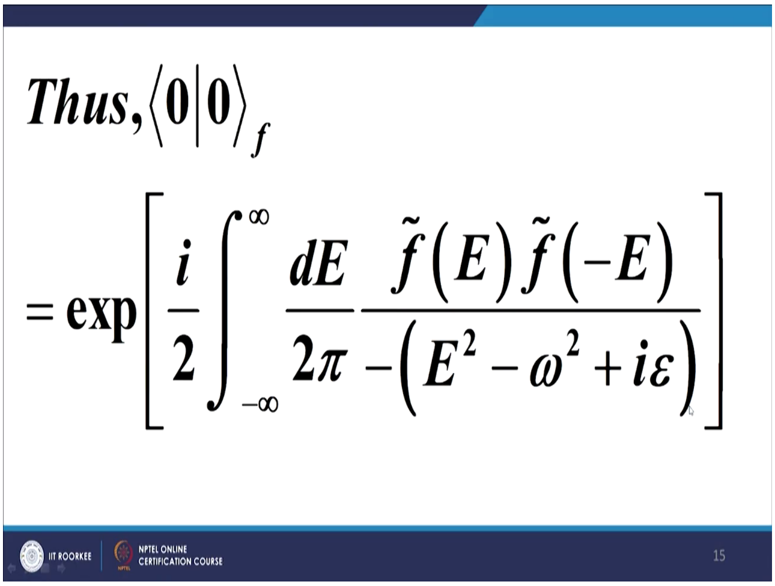

In other words, let us in other words, what do we get? We get that now, look at this, you see actually we need to go one slide back. This part, we are not touching. This part exists. It is still there. As far as this part is concerned, we have shown that this part is, this part this. This is the path integral part the, let us call it the path integral part.

The path integral part, when you look at the path integral part; the path integral part is equal to this expression with f equal to 0, right. That is one thing. The second thing is the path integral part, with this expression with the f equal to 0 is nothing, but 0 the transition from 0 to 0 with f equal to 0, this is number 2.

And number 3 is that if f is equal to 0, then the transition amplitude 0 to 0 must be 1. Why? Because an undisturbed ground state would continue to remain in the undisturbed ground state.

So, the net result of these three arguments is; that in this scenario that f is equal to 0; 0 to 0 transition is equal to 1; 0 to 0 transition is equal to this and this is equal to this. 0 to 0 transition is equal to equation 0 to 0 transition f equal to 0 is equal to 1 and 0 to 0 transition if f is equal to 0 is equal to equation 2 with f equal to 0, of course.

And this equation 2, with f equal to 0 is nothing, but equation 1. In other words, combining these three results what we get is; that 0 to 0 with f equal to 0 is equal to 1 is equal to equation number 1. In other words, our path integral in the given problem, given harmonic oscillator problem in the is equal to; is equal to 1.

This whole expression is equal to 1. This part, this whole path integral Dx exponential i by 2 integral dE upon 2 pi so and so. This whole expression is equal to 1. And therefore, the transition amplitude in the presence of f is given by this expression. In that green box, that is the most important part. So, the transition amplitude in the presence of f is given by this expression and that is the net result.

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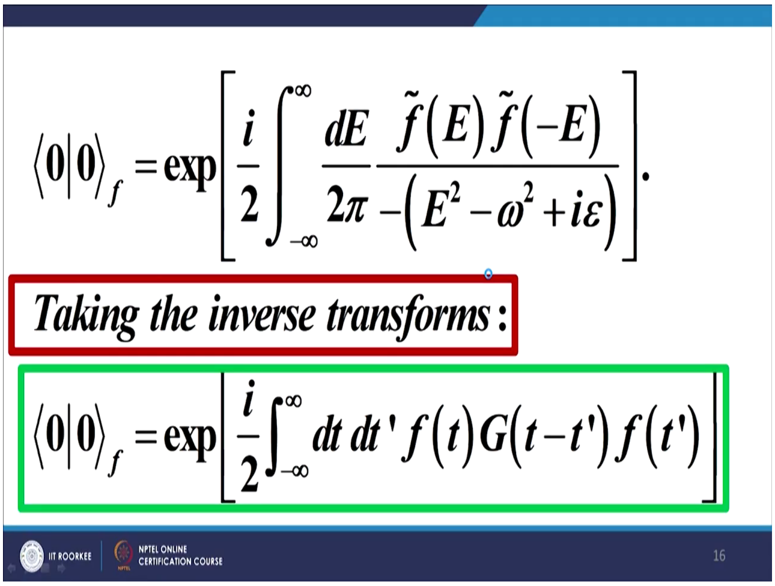

The path integral part has been shown to be equal to 1. Now, by taking the inverse Fourier transform, I can go back, I can revert to my original variables. Please note, these are these energy space, I can go back to my original variable and in original variables I can write this in the form of this expression. In the green box, where this is a green function g t minus t dash is a green function, which has the expression given in this green box. G t minus t dash is equal to this expression.

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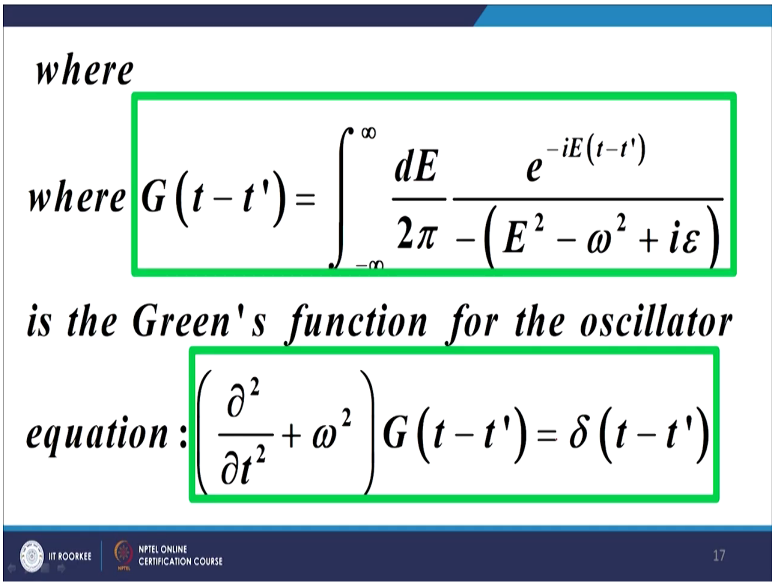

And this is the; this is the green function of the harmonic oscillator equation, which is given in the second green box; the bottom green box. As it can be shown, that the green function in the upper green box satisfies the harmonic oscillator equation of the bottom green box.

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$$G(t-t') = \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-(E^2-\omega^2+i\varepsilon)}$$ can be evaluated explictly as a contour integral in the complex E – plane. We get

$$G(t-t') = \frac{i}{2\omega}\exp\left(-i\omega\left|t-t'\right|\right)$$

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Now, this green function can be evaluated explicitly by using contour integration. If you look at it carefully, there are two poles here, I can, the two poles are plus minus omega minus i epsilon.

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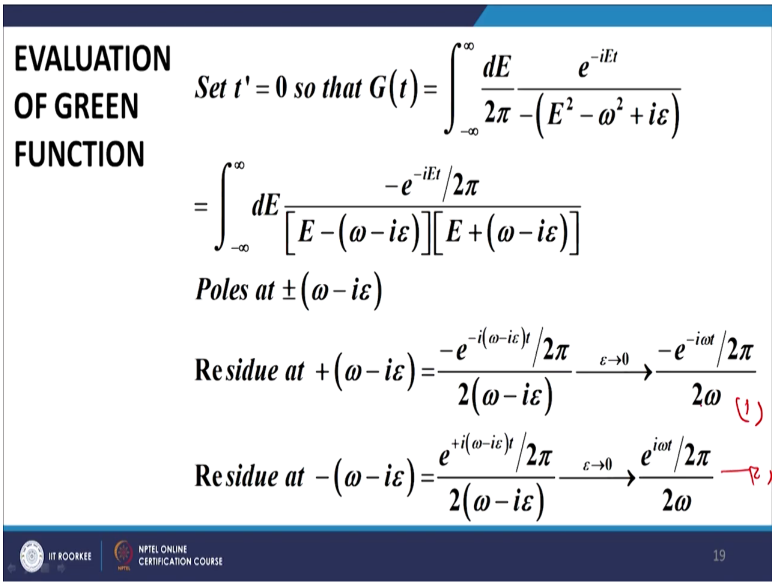

Of course, we ignore r orders of omega. So, I can write this as E minus omega minus i epsilon, E plus omega minus i epsilon, and this shows that the and the poles are at plus minus omega minus i epsilon. The residue at the respective poles is given by this expression 1 and this expression 2, where at the first pole the residue is this and this second pole the residue is this.

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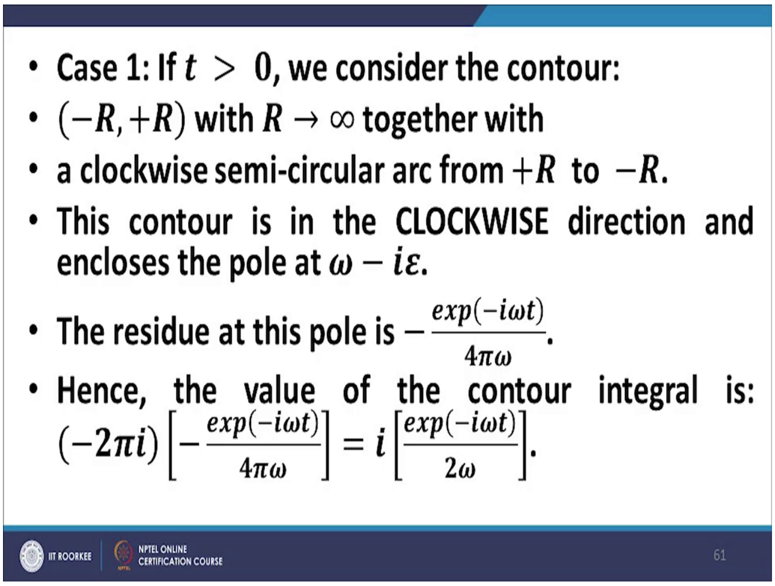

Now, we can carry out the contour integration for the t greater than 0, we I will just draw the contour for t greater than 0, the pole is at omega minus i epsilon. So, it will be somewhere here, and what we do is, as yes, the pole would be somewhere in the fourth quadrant.

And therefore, we complete the contour by at, what we are required is to evaluate the line integral; from minus infinity to infinity. We complete a contour by a clockwise semicircle. In the lower half plane clockwise semicircle, in the lower half plane in the clockwise direction.

As I mentioned, this will include the pole at omega minus i epsilon, and the residue at that pole is given by a, we have seen earlier; the residue at that pole is given by minus E, the exponential minus i omega t upon 4 pi omega.

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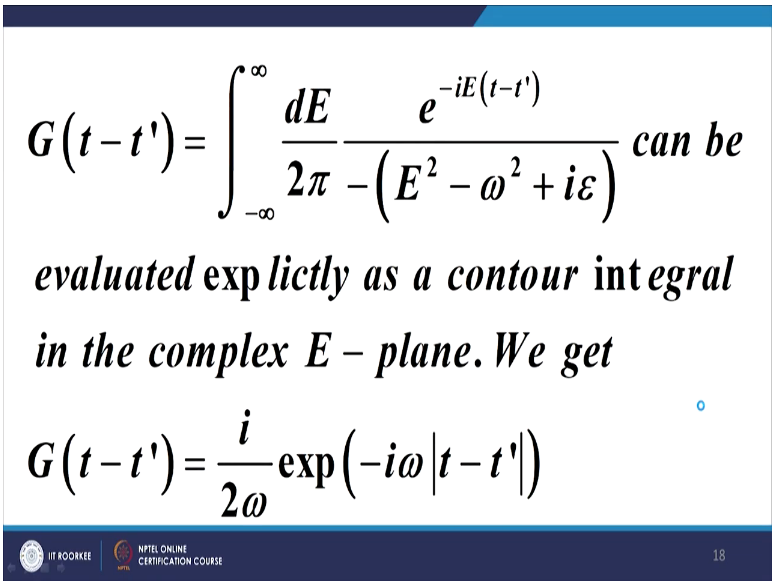

So, that is the value of the contour integral and that is the value of the contour integral when, and therefore, no that is the value of the residue. And therefore, the value of the contour integral, because it is in clockwise direction, the value of the contour integral will be minus 2 pi i minus. This whole thing if you simplify this, you get this expression. This expression, i exponential minus i omega t upon 2 omega. This is the value of the contour integral.

Now, we want to know the value of this line integral. And, the integral along the semicircle as R tends to infinity or as the, say radius of the semicircle tends to infinity will tend to 0 0, because t is greater than 0 and the integrand is minus i omega t.

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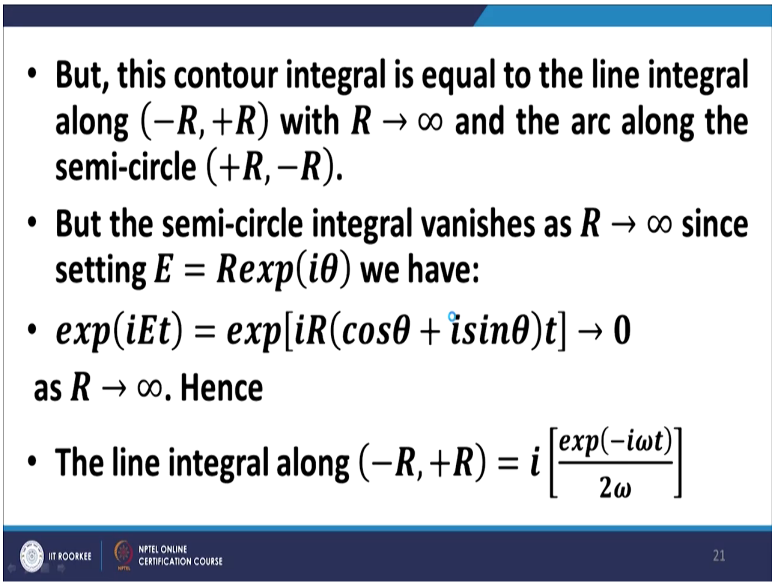

If I substitute for example, in the next slide I have worked it out. If I substitute E is equal to R exponential i theta, then exponential i E t become this expression and R as R tends to infinity. Now, this expression; exponential i R into i sin theta, this is a real term, and real term with minus coefficient minus R sin theta with t, t is positive and therefore, as R tends to infinity this expression goes to 0.

And therefore, this integration along the semicircle goes to 0 and we are left with the line integral being equal to this expression. And similarly, for the t less than 0 case. We can work it out and we end up with we end up with this expression as the final expression for the green function.

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Similarly, if $t < 0$, we can add an arc at infinity in the upper-half plane. This produces a closed contour that encircles the pole at $E = -(\omega - i\epsilon)$ in a counterclockwise direction. The residue of this pole is $\{-\exp[i(\omega - i\epsilon)t]/2\pi\}/[-2(\omega - i\epsilon)] \rightarrow \exp(i\omega t)/4\pi\omega$ as $\epsilon \rightarrow 0$. By the residue theorem, the value of the integral is $+2\pi i$ times this residue. Combining these two cases, we have

$$G(t) = \frac{i}{2\omega}\exp\left(-i\omega|t|\right)$$

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$$G(t-t') \text{ satisfies } \left(\frac{\partial^2}{\partial t^2}+\omega^2\right)G(t-t')=\delta(t-t').$$

$$\text{We set } t'=0. \text{ We have } \frac{d}{dt}|t| = \text{sign } t;\ \frac{d}{dt}\text{sign } t = 2\delta(t)$$

$$\frac{d}{dt}G(t)=\frac{d}{dt}\left(\frac{i}{2\omega}e^{-i\omega|t|}\right)=\frac{1}{2}e^{-i\omega|t|}\text{sign}(t)$$

$$\frac{d^2}{dt^2}G(t)=\frac{d}{dt}\left[\frac{1}{2}e^{-i\omega|t|}\text{sign}(t)\right]=-\frac{i\omega}{2}e^{-i\omega|t|}\left[\text{sign}(t)\right]^2+e^{-i\omega|t|}\delta(t)$$

$$=-\frac{i\omega}{2}e^{-i\omega|t|}+\delta(t)=-\omega^2G(t)+\delta(t)$$

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Now; here, I have tried to show that the green function that we have worked out, satisfies the oscillatory equation. You can go through that, pretty straight forward working.

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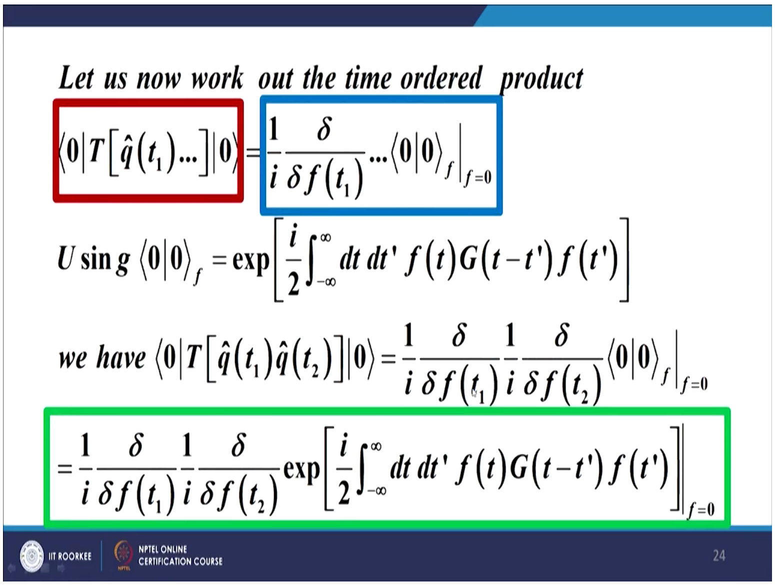

Now, we work out the time ordered products. Let us work out the time ordered products of the given using the given expression for the green function ah. When we work out the time ordered products we take functional derivatives. As I mentioned, we take functional derivatives with respect to f t 1, f t 2 and so on. Of the transition amplitude with the incorporating, where in the source term inhomogeneous source term f.

So, that is precisely, what we have here. This expression is the expression for d G ground state to ground state transition, with f. And, when we work out the when we work out the functional derivatives with respect to t 1 and t 2 and so on. We work out, we get this expression ah.

This is what we have for the amplitude, zero ground state to ground state transition amplitude, and we are taking functional derivatives.

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$$\langle 0|T\left[\hat{q}(t_1)\hat{q}(t_2)\right]|0\rangle =$$

$$\frac{1}{i}\frac{\delta}{\delta f(t_1)}\frac{1}{i}\frac{\delta}{\delta f(t_2)}\times \exp\left[\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt'\, f(t)G(t-t')f(t')\right]\Bigg|_{f=0}$$

$$=\left(\frac{1}{i}\right)^2\frac{\delta}{\delta f(t_1)}\left[\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt' \boxed{\frac{\delta f(t)}{\delta f(t_2)}} G(t-t')f(t')\right]\times$$

$$\exp\left[\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt'\, f(t)G(t-t')f(t')\right]\Bigg|_{f=0}$$

Now, this functional derivative, let us start with one, this functional derivates gets attached to one of these f's, and when it gets attached to one of these, it pulls up a delta function. And when the delta function is integrated over, we get G t t 2, t minus t 2, we get g. We get a delta function here, and that is in the next step. Here it is, you get that in delta function here.

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$$\langle 0|T\left[\hat{q}(t_1)\hat{q}(t_2)\right]|0\rangle$$

$$= \left(\frac{1}{i}\right)^2 \frac{\delta}{\delta f(t_1)} \left\{ \begin{array}{l} \left[\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt' \left[\frac{\delta f(t)}{\delta f(t_2)}\right] G(t-t') f(t')\right] + \\ \left[\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt'\, f(t) G(t-t') \left[\frac{\delta f(t')}{\delta f(t_2)}\right]\right] \end{array} \right\} \times$$

$$\exp\left[\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt'\, f(t) G(t-t') f(t')\right]\Bigg|_{f=0}$$

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And when the delta function is integrated over, yes, this is the delta function.

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$$=\left(\frac{1}{i}\right)^2 \frac{\delta}{\delta f(t_1)}\left[\begin{array}{l}\frac{i}{2}\int_{-\infty}^{\infty} dt dt'\, \delta(t-t_2)G(t-t')f(t') \\ +\frac{i}{2}\int_{-\infty}^{\infty} dt dt'\, f(t)\delta(t'-t_2)G(t-t')\end{array}\right]\langle 0|0\rangle_f \Bigg|_{f=0}$$

$$=\left(\frac{1}{i}\right)^2 \frac{\delta}{\delta f(t_1)}\left[i\int_{-\infty}^{\infty} dt\, G(t_2-t)f(t)\right]\langle 0|0\rangle_f \Bigg|_{f=0}$$

When this delta function is integrated over, we get this expression. And similarly, when this the second for a functional derivative f t t 1 acts, it pulls out another delta function.

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$$\langle 0|T\left[\hat{q}(t_1)\hat{q}(t_2)\right]|0\rangle$$

$$=\left(\frac{1}{i}\right)^2\frac{\delta}{\delta f(t_1)}\left[i\int_{-\infty}^{\infty}dt\,G(t_2-t)f(t)\right]\langle 0|0\rangle_f\Big|_{f=0}$$

$$=\left(\frac{1}{i}\right)^2\left[i\int_{-\infty}^{\infty}dt\,\delta(t-t_1)G(t_2-t)\right]\langle 0|0\rangle_f\Big|_{f=0}$$

$$=\left[\frac{1}{i}G(t_2-t_1)+(term\ \ with\ f's)\right]\langle 0|0\rangle_f\Big|_{f=0}=\boxed{\frac{1}{i}G(t_2-t_1)}$$

When it pulls out another delta function and again when that delta function is integrated over, what we end up with is G t 2 minus t 1. Of course, various other pre factors get adjusted accordingly.

And, this is the final term that we have when you put f is equal to 0, all the other terms vanish. And, this is the expression that we have for the 2 point function, time ordered product q t 1, q t 2, ground state expectation value of this time ordered product.

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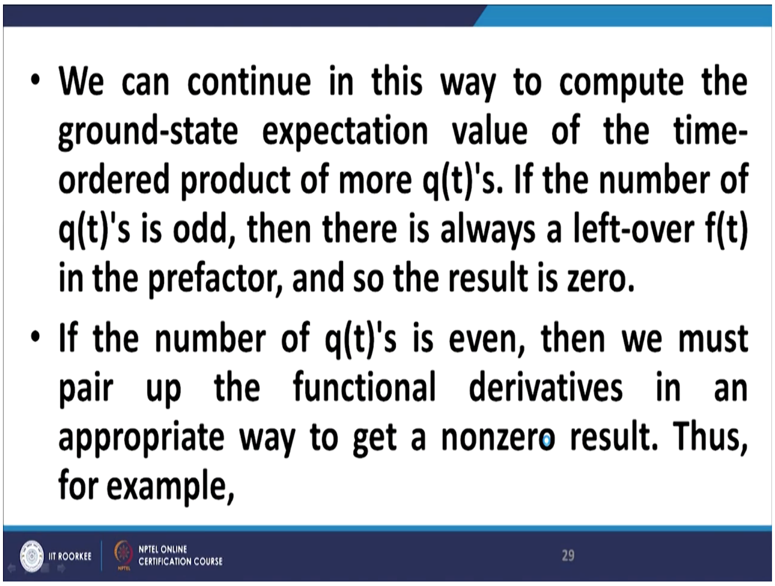

We can continue of course, in this way, right. So, this is what this was an example of the harmonic oscillator. And, the objective of taking this as an example was 2 fold. Firstly, it gives gave you a feel of the manner in which the various steps or the formalism involving functional derivatives was implemented, how it was implemented in a specific example in a specific case.

And the second thing was, that the harmonic oscillators form the essential building blocks of quantum fields. So, we are going to encounter this again and again, when we move from here now to the quantum fields case.

But, before we leave that, as brief discussion on the relativistic single particle path integral. This relativistic single particle path integral has certain nuances attached to it; certain differences which identify itself from the Feynman path integral; non relativistic Feynman path integral, that we have been doing so far.

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- **The standard action for a relativistic particle in the flat metric $\eta_{ab} = diag(+1,-1,-1,-1)$ is given by:**

$$S = -m\int_{t_1}^{t_2} dt\sqrt{1-v^2} = -m\int_{x_1}^{x_2}\sqrt{-\eta_{ab}dx^a dx^b}$$

$$= -m\int_{\lambda_1}^{\lambda_2} d\lambda\sqrt{-\eta_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}}$$

The action in this case, in this problem, we are working in the metric a diagonal plus 1 minus 1 minus 1 minus 1; that means, signature minus 2. This is the metric that I will, flat metric, the Minkowski metric that I would be following in most cases; however, largely I would be focusing on this particular metric throughout the course.

In this metric, the action, the relativistic action can be represented in the form given in this equation where lambda is only a parameter.

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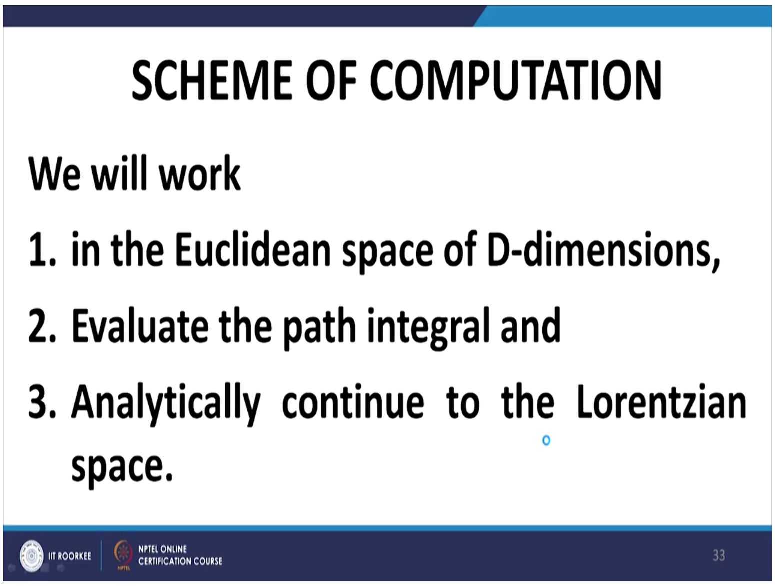

Now, how we will go about completing this path integral? We shift the problem to Euclidean space; we work out the integral in Euclidean space and, then we analytically continue back into the Lorentzian space, with this particular metric. So, that is how I propose to proceed about it.

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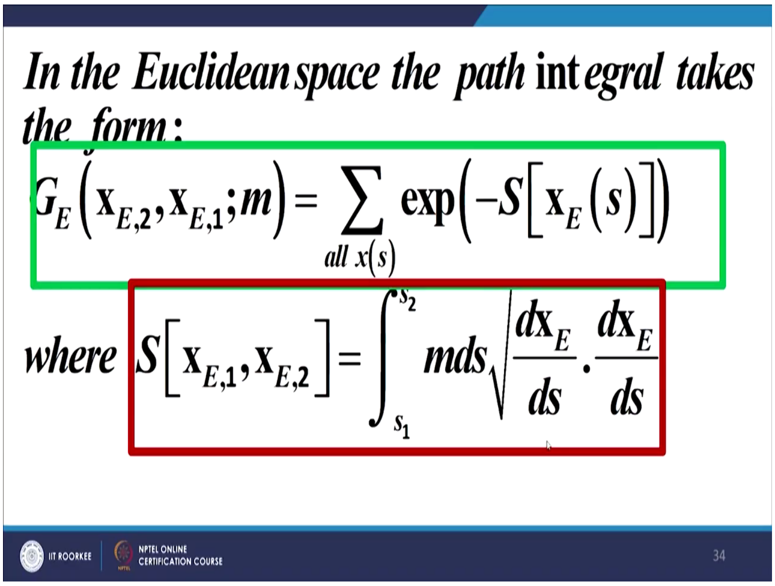

In the Euclidean space, we can write the path integral you know, this suffix E with X is represent, that these are coordinates in the Euclidean space. So, the Euclidean space, the path integral is we will take this form, where and now the action is a function of Euclidean space coordinates and the action is given by this particular integral.

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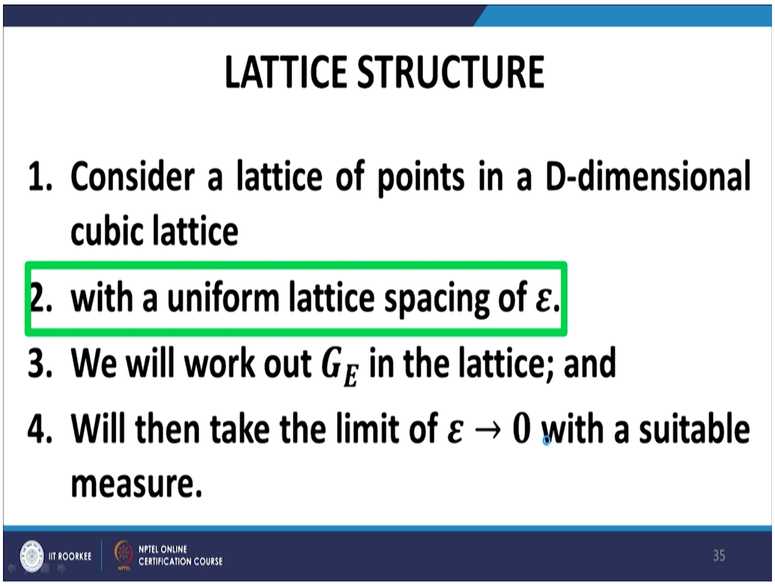

Please note this, X E are now Euclidean space coordinates. What we are do now is, as we were done, in fact, earlier also. As we did in the case, when I introduced the concept of path integrals.

We introduce a lattice of points in a D-dimensional Euclidean space. D-dimensional cubic lattice in Euclidean space. And the uniform lattice space, we assume to be a small, represented by epsilon. Of course, in the continuum case, as we move from this discrete version to the continuous version, we will gradually or we will take the limit of epsilon tending to 0.

So, we will work out the expression for the path integral on the lattice and then we will take the continuum case, we will translate to the continuum case with epsilon tending to 0. And then of course, we will analytically continue to the Lorentzian or the Minkowski space.

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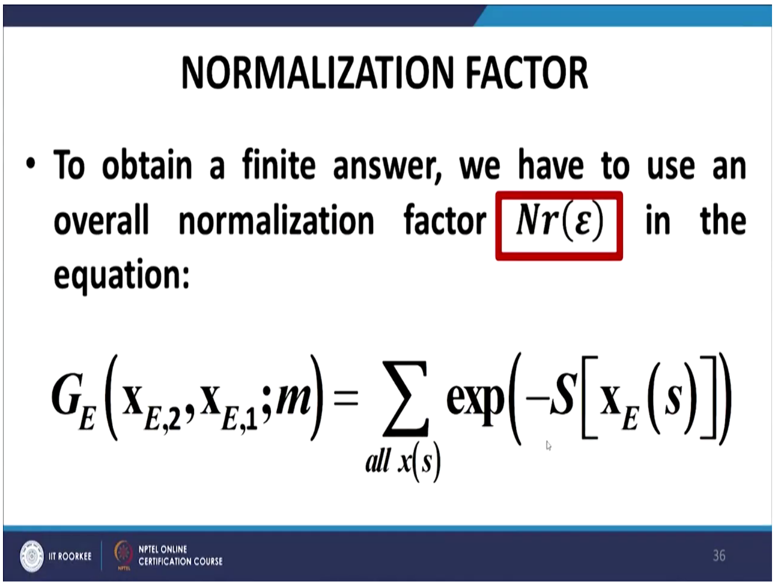

We in order to ensure, that we get a finite expression, we need to introduce a normalization factor. This normalization factor will be a function of the lattice spacings or a lattice constants, and it can be and can we will call it Nr epsilon and the relation of the Nr epsilon would be in the next equation, in this expression.

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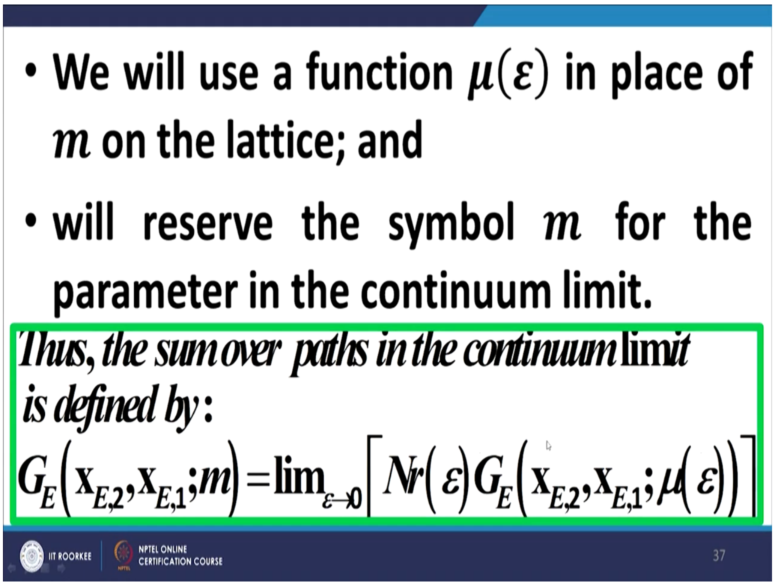

The continuum version of the path integral will be given by limit of epsilon tending to 0, this normalization factor into the discrete version.

And there is another important point, instead of using m for the mass or the function mass of the particle, we shall be using mu epsilon as the parameter as the parameter in the system space.

And, we shall resolve the, corresponding to this mu epsilon in the continuum case. In other words, m will correspond to mu epsilon, mu epsilon is a parameter that depends on or that relates to the Lorentzian space, that is essentially relating to the measure of integration.

In the discrete version that relates to the lattice constant and it would be replaced by m when we moved from the discrete version to the continuum. And that and this is what the expression would be.

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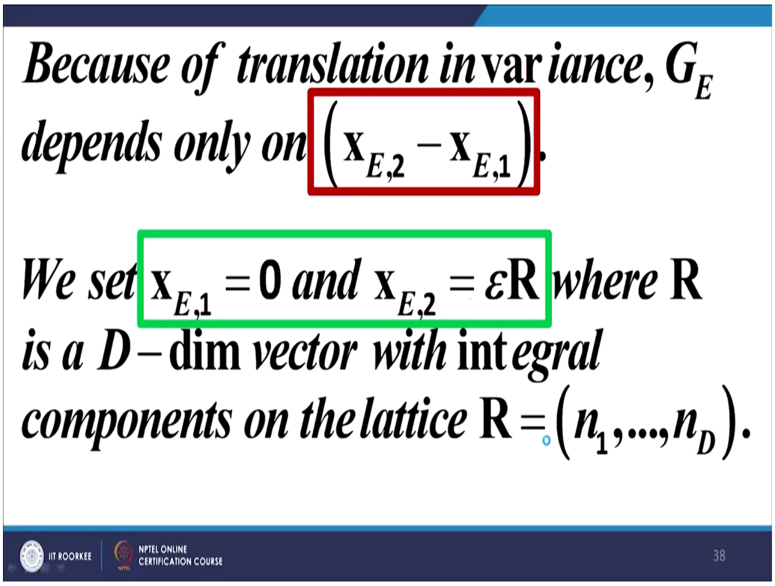

Now, because of translation invariance, because of translation invariance that discrete version can only depend on the difference of coordinates, it is natural. So, we set one of the coordinates X E 1 equal to 0 as the origin and we work. And, let us identify the other coordinate as epsilon R, where R is a particular point as a particular node on the lattice. It is a D-dimensional vector..

It is a D-dimensional vector connecting the origin to a particular node on the lattice, whose coordinates in terms of the identification by the lattice points is given by n 1. into all this will be integers naturally, because they represent the components on the lattice; on the discrete lattice.

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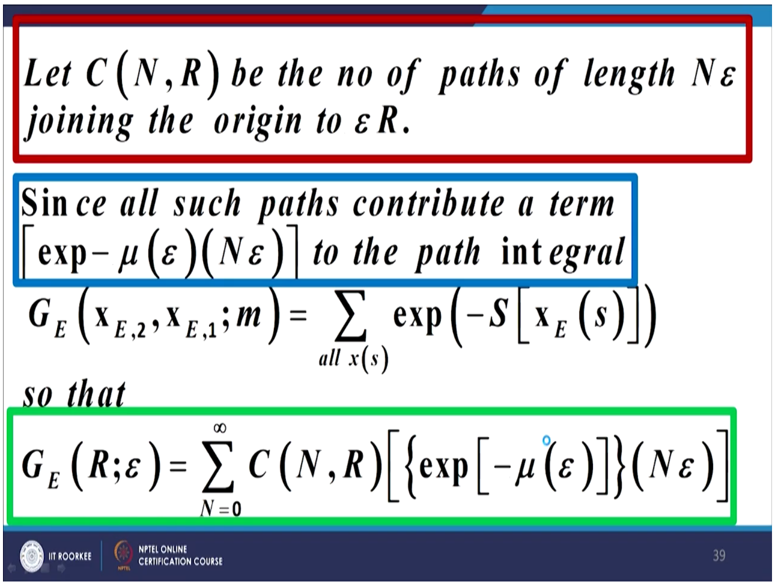

Now, C N, R are the number of paths of length N epsilon N epsilon. Now, N is variable here, epsilon is the lattice constant. So, N is variable here. So, we are not only accounting for the direct path, we are accounting for variable, all the possible paths from the origin to the point R are which are joining the origin to epsilon R epsilon R is the point which corresponds to the coordinates R which we have worked out in the last slide.

This is, epsilon R is this point and this X 2 is epsilon R and the coordinates of R at this point R is a vector, right. And each of these paths will contribute a term to the path integral, and what will be the weight of that term? The weight of the term will be exponential minus mu epsilon into the length of the path. N epsilon minus mu epsilon into the length of the path, that is what will be the contribution to the path integral of the each such path. We will continue from here in the next class.

Thank you.