Path Integral Methods in Physics & Finance
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Lecture - 25
Vacuum Persistence Amplitude

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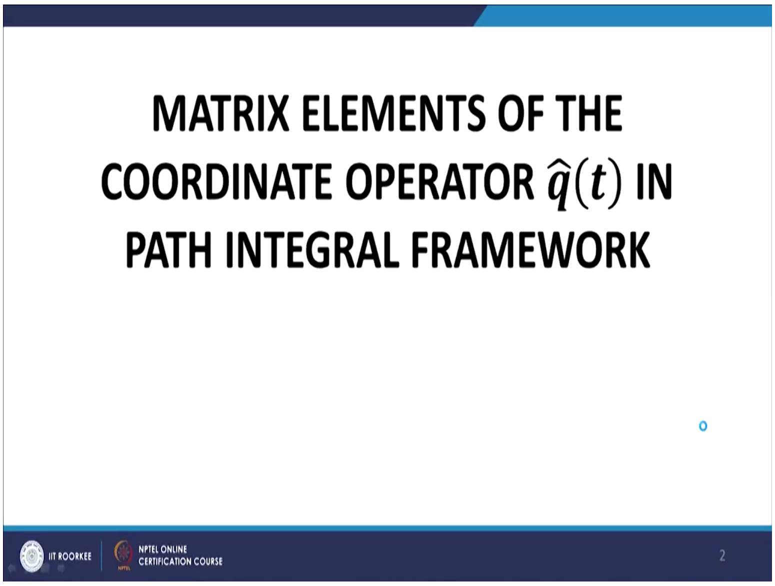

Welcome back. Let us recap what we did last time. Last time we worked out the matrix elements of the coordinate operator q t q hat t in the path integral framework within the sandwich within the initial and the final state. And we also worked out the time ordered product of operators in the two states.

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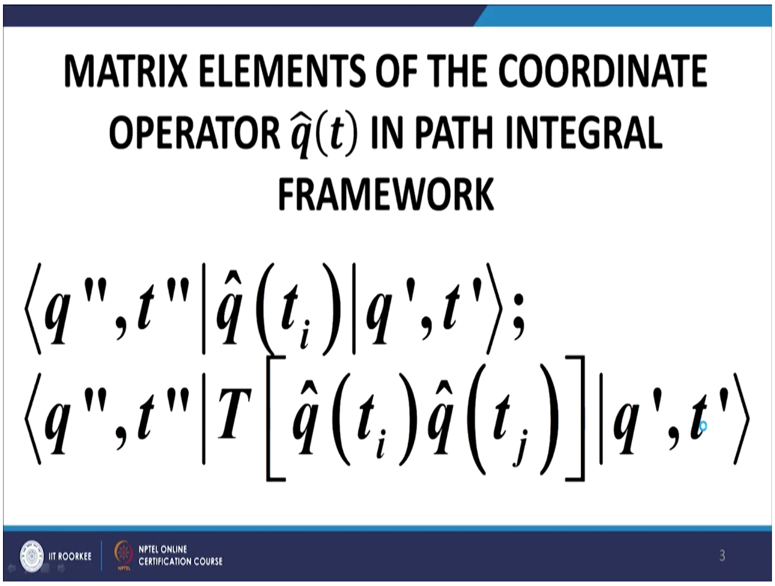

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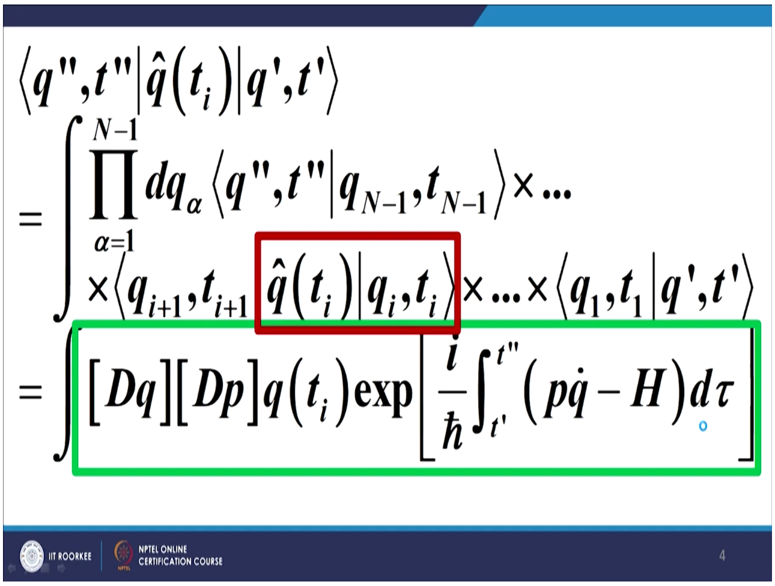

Then this is the expression that we arrived at the expression given in the green box. And the important thing I emphasized here was that, when we work out this expressions; when we work out the time order product the path integral automatically assumes in terms of indexing in terms of time. The right hand side gives you a time ordered product.

So, whether if you have q i q t i and q t j, with t i greater than t j and q t j being placed to the right then it is fine. But, if q t j is placed to the left of t i q t i and q t j occurs at a earlier point in time, then the integral would not be defined. Because it assumes that increasing grid indexing in terms of time right.

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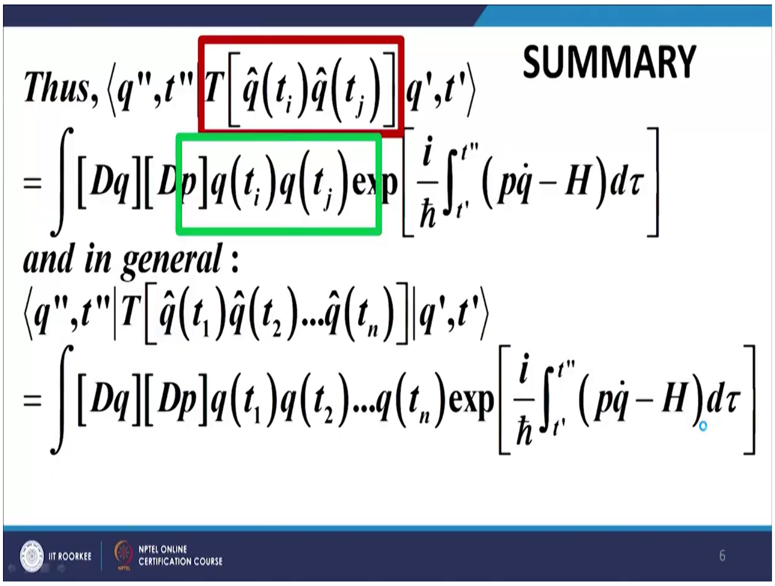

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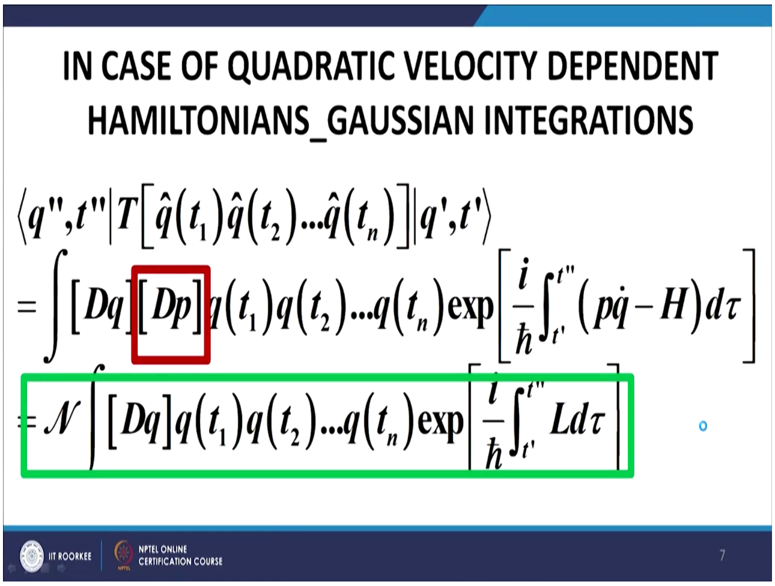

Then we worked out the operator product that is what I mentioned just now. We also worked out that in the case of a quadratic velocity dependent Hamiltonian. In the Gaussian integrals can be carried out explicitly and we arrive at the expression. And those Gaussian integrals can be incorporated in a normalization and therefore we arrive at the expression which is given in the green box in the configuration space.

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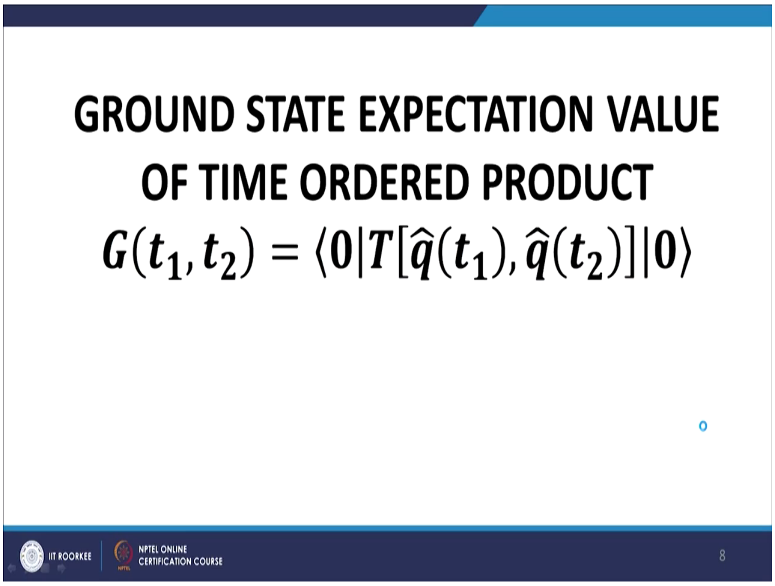

Then we worked out also worked out the ground state expectation value of the time ordered product. The expression which is given in your slide we worked out and expression for this and what we found was the expression given in the green box here.

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$$\langle 0|T\left[\hat{q}(t_1)\ldots\hat{q}(t_n)\right]|0\rangle$$

$$\boxed{= \lim_{\substack{t''\to\infty \\ t'\to-\infty}} \frac{\int [Dq][Dp]\, q(t_1)\, q(t_2)\ldots q(t_n) \exp\left[\frac{i}{\hbar}\int_{-\infty}^{\infty} d\tau\,(p\dot{q}-H)\right]}{\int [Dq][Dp] \exp\left[\frac{i}{\hbar}\int_{-\infty}^{\infty} d\tau\,(p\dot{q}-H)\right]}}$$

In the special case of quadratic velocity dependence of the Lagrangian, we have

$$= \lim_{\substack{t''\to\infty \\ t'\to-\infty}} \frac{\int [Dq]\, q(t_1)\, q(t_2)\ldots q(t_n) \exp\left[\frac{i}{\hbar}\int_{-\infty}^{\infty} d\tau\, L\right]}{\int [Dq] \exp\left[\frac{i}{\hbar}\int_{-\infty}^{\infty} d\tau\, L\right]}$$

So, this is what we did last time, this was what we had covered in ground that we covered in the last lecture.

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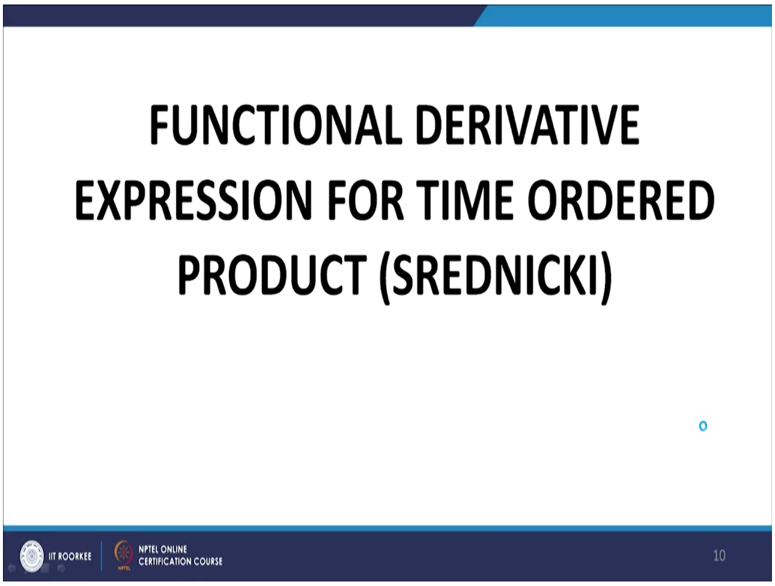

Let us proceed from here. From here on a concept called functional derivatives is going to make, repeated presence; and as a result of which because of many of the viewers may not be familiar with the concept of functional derivative. What I have done is, I have incorporated an appendix in this PPT covering functional derivatives, and the viewers can go through with the PPT together with the lecture substance.

So, let us start we work out the expression for the time ordered product using the concept of functional derivative. For, this purpose what we do is we introduced two inhomogeneous sources by adding the terms time dependence sources by adding the terms, f t and h t to our Hamiltonian, adjusting our Hamiltonian, transforming our Hamiltonian in the form given in the green box here.

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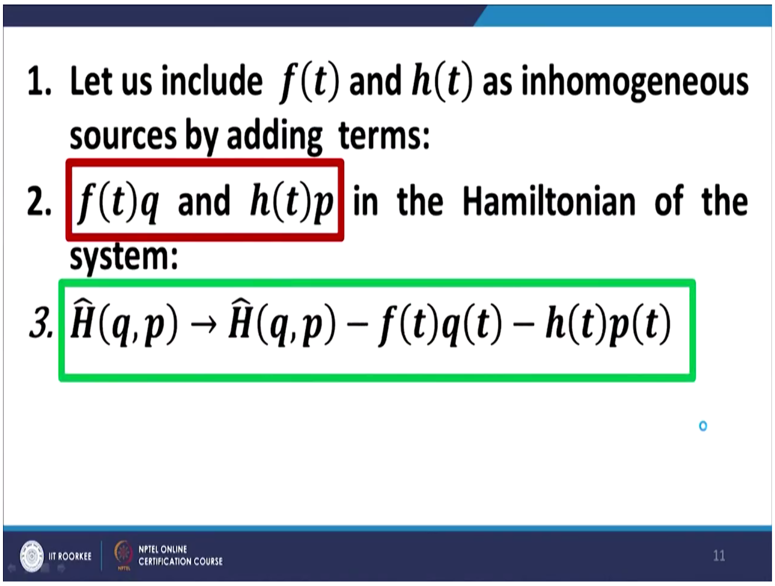

This two, in this two expressions f t and h t represent inhomogeneous sources; classical sources they are. And to incorporate the impact of this sources, we correct the or we adjust the Hamiltonian to the form given in the green box.

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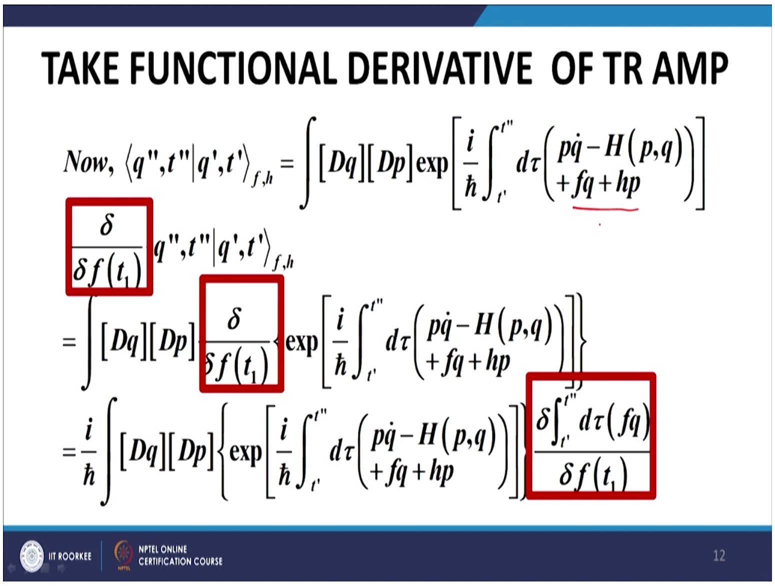

Now, we take the functional derivative of the transition amplitude. The transition amplitude is defined as, in the earlier case as in dotting of the final state with the initial state, with in this case in the presence of the sources the source terms are retained in the Hamiltonian.

As you can see here, as you can see in this expression this source terms are retained, and we take the functional derivative of this expression with respect to f, at time point t i. Now, just trace carefully or follow carefully; the moment of this functional derivative across this path integral or the functional integral.

First of all this functional integral attaches itself to the exponential term it goes inside the functional integral, and attaches itself to the exponential term. When it operates on the exponential term within the functional integral, the functional derivative takes out the exponential as it is in this form. And then it attaches to the exponent of the exponential term, which is given in the green box at the bottom equation of your slide red box at the bottom equation of your slide.

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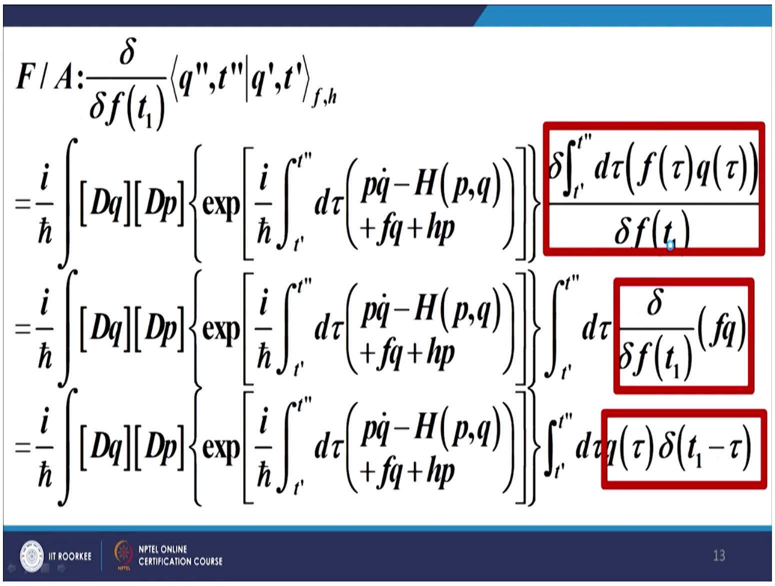

Yes. So, now what happens is that functional derivative again goes through the integral here because, the integral is with respect to tau the derivative is with respect to f and therefore, it can be commuted with the integral and it goes inside the integral and attaches itself to fq.

Now, when this functional integral operates on a f t q t, the fq abbreviated form of f t q t. So, the q goes out of the functional integral and, the functional derivative and the functional derivative operative on f t with respect to f t 1, it gives the direct delta function at the point t 1.

That is given in the red box at the bottom of your slide. So, what we have here in this integral now, the last integral is d tau as it is. The functional derivative with respect to f operating at point t 1, operating on f captures the returns the direct delta function at t 1. And the q passes through the derivative and written as q of tau, tau is the integration variable here.

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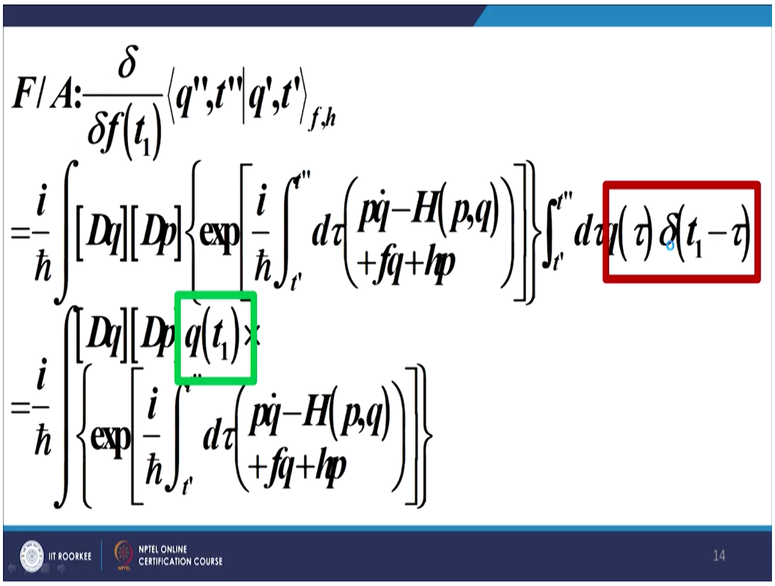

So, when we do this integral; when we do in this integral we end up with getting q of t 1, and due to the property of the direct delta function when the integral is done over the direct delta function we get q of t 1. So, the net result of all this movement of the functional derivative across the entire integral is to bring forth or capture factor of q t 1 into the functional or the path integral. We bring a factor of q t 1 into the path integral.

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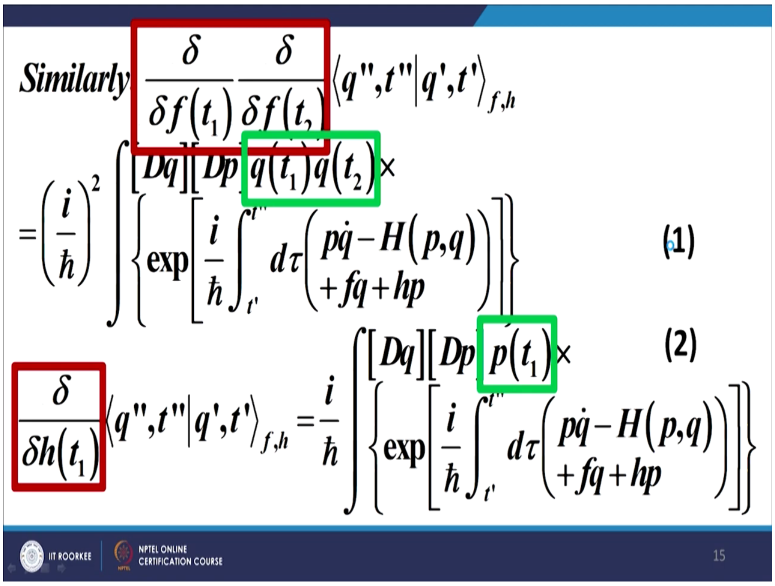

And please note, this is nothing these expression here is, nothing but the, expression for the expected value or the expression for the matrix element of q t 1 in the initial and final states.

We have been able to extract the matrix element of the of q t 1 with respect to the initial and final states. The process can be repeated; the process can be repeated again and again to pull out as many factors of q as we want in the as many point functions as you want for example, if you operate by q t 2 the functional derivative with respect to f t 2, and then you operate again with respect to f t 1 as shown in the red box here.

You end up with q t 1 and q t 2 in the path integral which is nothing but the time ordered product in the initial and final states. Similarly, by taking functional derivative with respect to h, you can put it down terms of p into the path integral.

So, the net result is to reiterate the net result is that by taking functional derivatives with respect to f and with respect to h we can pull in terms of q and p into the path integral; that is the net fall out of. And then finally, after we have pulled out as many factors of q and p as we desire we can set f equal to h equal to 0, and recover the original Hamiltonian.

So, the net scheme of things operates as follows: introduce f and h as classical sources into the Hamiltonian by writing f t q t and h t p t. Then take functional derivatives of the path integral with respect to f t 1, you will get a factor of q t 1 into the path integral; when the functional derivative operates on the exponential term and similarly, if you take a functional derivative with respect to h t 1 you get a factor of p t 1 into the path integral and that is how it operates.

And then finally, what you do is put f equal to h equal to 0. And what we end up with the original Hamiltonian; and therefore the net result of all this processes given in this important slide.

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$$\left(\frac{\hbar}{i}\right)^2 \boxed{\frac{\delta}{\delta f(t_1)}\frac{\delta}{\delta f(t_2)}} \langle q'',t''|q',t'\rangle \Big|_{\boxed{f=h=0}}$$

$$= \int [Dq][Dp]\, q(t_1)\, q(t_2) \times$$

$$\left\{\exp\left[\frac{i}{\hbar}\int_{t'}^{t''} d\tau \begin{pmatrix} p\dot{q} - H(p,q) \\ + fq + hp \end{pmatrix}\right]\right\}\Bigg|_{f=h=0}$$

$$= \langle q'',t''| \boxed{T\left[\hat{q}(t_1)\hat{q}(t_2)\right]} |q',t'\rangle$$

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This is a very important slide. The left hand side as we as I showed is the functional derivative of f t 1, functional derivative of f t 2; functional derivative with respect to f t 1, functional derivative with respect to f t 2 of the transition amplitude.

Incorporating there in f and h, and they when we do this we get the expression which is in the 2nd equation, if i put h equal to 0, and f equal to 0 in this expression I recover the original Hamiltonian. And, what I get here is simply the time ordered product of the relevant quantities.

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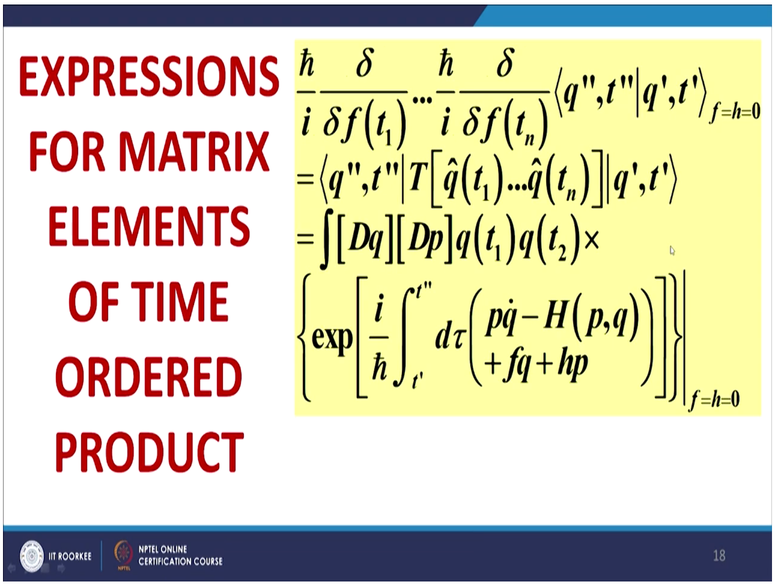

So, this is another approach another way in which one can recover the matrix elements of the of the time ordered product; through application repeated application of the functional derivatives on the transition amplitude after incorporating there in inhomogeneous source terms in the Hamiltonian and finally, putting the inhomogeneous source terms equal to 0.

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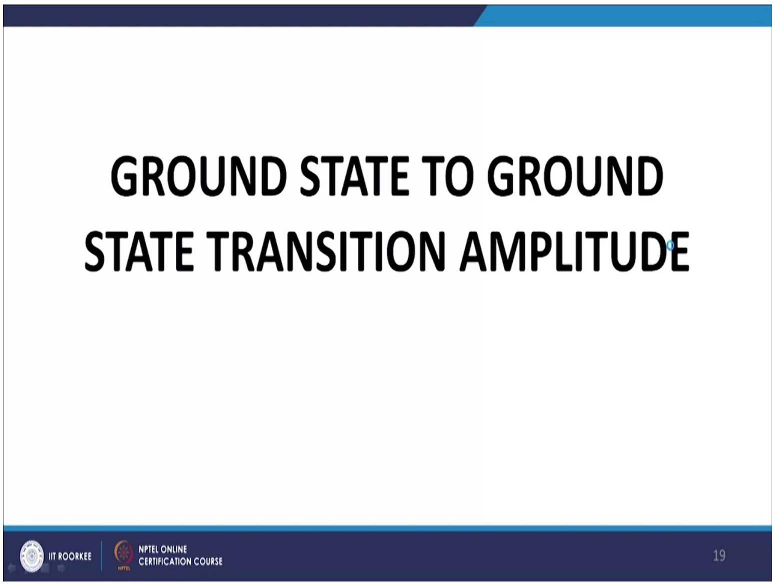

Now, we look at the ground state to ground state transition amplitude. That is very important that is you see what happens is in certain systems, when you give it give the system a kick and impulse, in some form it is very interesting and to know the intrinsic properties of the system by how the system reacts to the kick.

It in particular it is very interesting whether the impulse has generated any impact or has put it to the case system out of the ground state, or whether the system continues to remain in the ground state even after the impulses acted on it.

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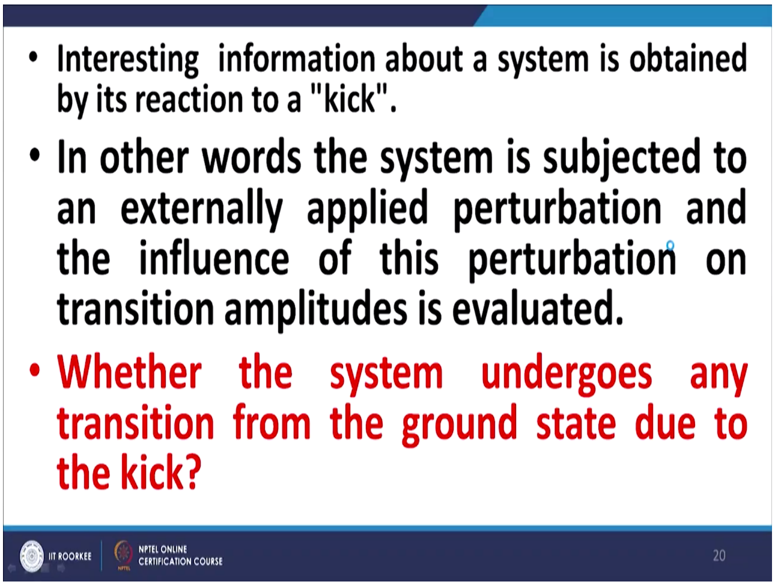

So, that is the very important property and attribute of the system which is as usually investigated in microscopic experiments scattering experiments and so on. And, which gives us a lot of information about what is the internal constitution of such microscopic systems.

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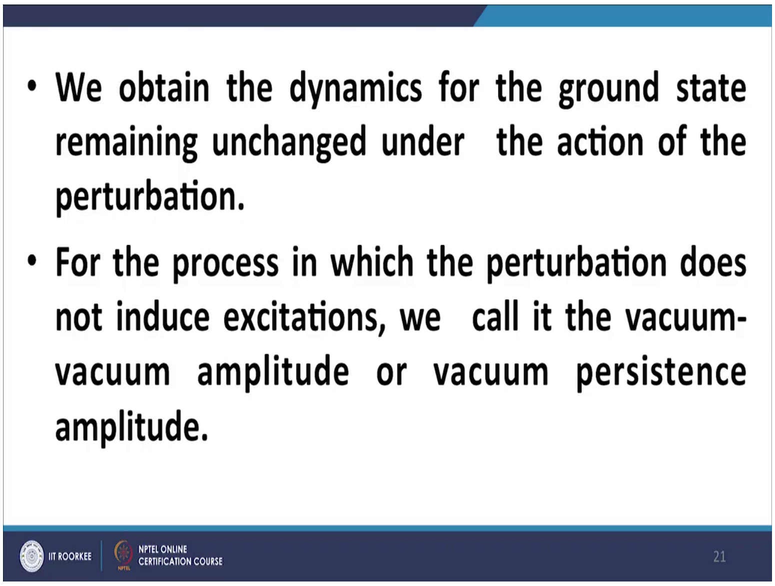

So, here what we are going to try to do is to work out the probabilities, or work out the amplitudes of the system remaining in the ground state; after being initially in the ground state that is, yet despite their being a kick, despite being exposed to external source classical source the system remains in the ground state, that is what we are going to investigate here.

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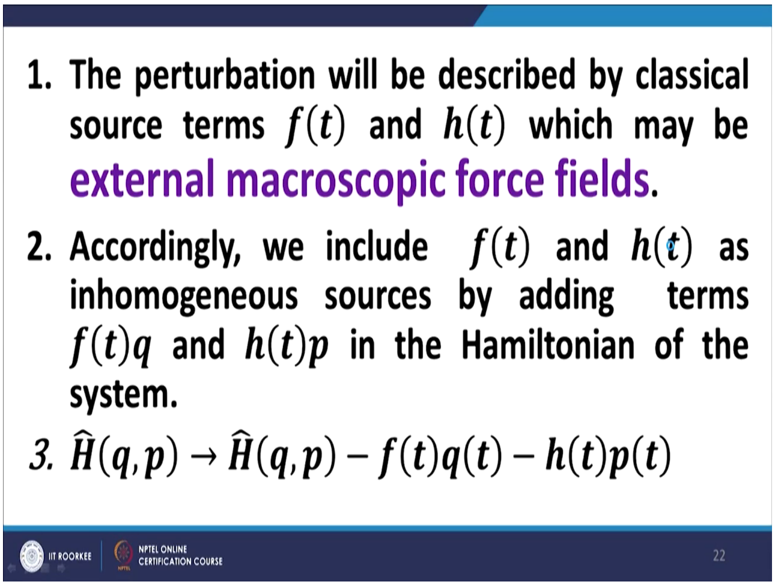

So, we describe the perturbation as we did in the earlier case by exogenous term inhomogeneous terms f t and h t which we incorporate in the Hamiltonian, and we write the Hamiltonian in the form that we are written earlier f t is transformed h q p is transformed as h q p minus f t q t minus h t p t. So, the this is incorporating as the impact of the inhomogeneous sources f t and h t.

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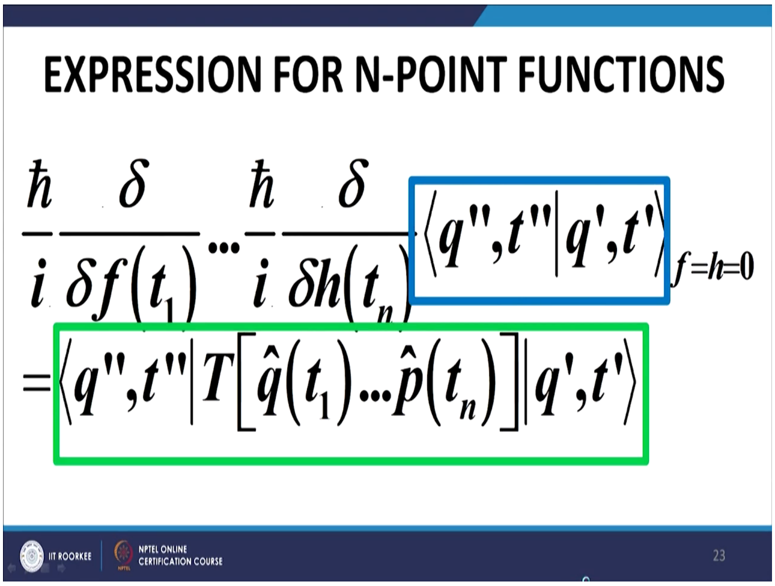

Then, when and this is again what we would done just now by taking functional derivatives with respect to f t 1, f t 2 and so on and with respect to h t 1, h t 2 and so on. Acting on the transition amplitude with f and h being therein and subsequently, f and h put equal to 0 we are able to recover the time ordered product of this quantities.

Pull down the various factors that are shown in the square bracket representing the time ordered product into the path integral and therefore, this represents the time ordered product right.

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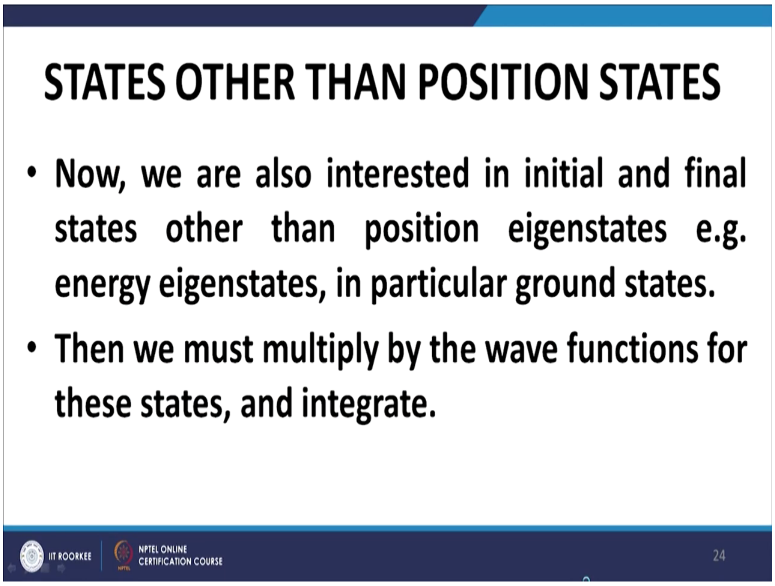

Now, suppose here you see so for what we have been focusing on, is the case of position states. If you recall this these this is the initial state and this is the final state in the position space in the coordinate space. It is interesting because, we are talking about ground states now; it is pertinent to investigate the or to examine, the various quantities in the energy space for that we have to change the bases and that is the next item that we are going to take up.

So, we now look at transforming the problem to an energy Eigen state bases and then, from there we try to recover or we try to isolate the ground state amplitudes corresponding to various quantum operations.

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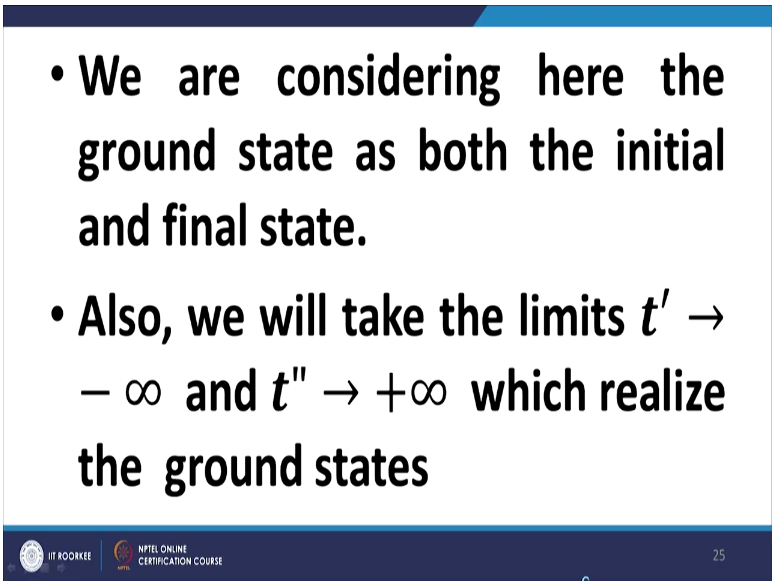

So, as I mentioned earlier we are we are considering here the ground state as the initial state and the final state; we are examining the situation where, the impulse does not impact the or does not result in the system moving from the ground state to a higher state. The system remains in the ground state and we also examine the impact as the time t dash tends to minus infinity; that is the initial time t dash tends to minus infinity and the final time t dash tends t double dash tends to plus infinity.

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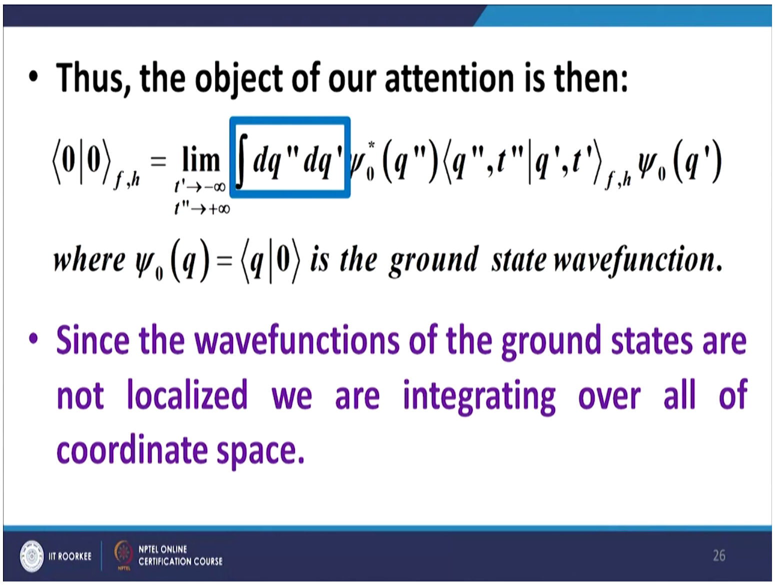

So, now the object of for an attention therefore, is the quantity that is given here is in the equation that is given here. If you look carefully of this equation the position state transition amplitude; the position state transition amplitude is multiplied by the respective Eigen functions in the energy space, and integrated and the integration is done over all position space.

Please note the integration is over all position space. See the point here is that the ground state may not be localized the ground state in energy may not be localized and therefore, we are integrating over the entire space in order to recover the complete ground state; we are integrating over the entire positions as the initial state and the as well as the final state.

Now, see the wave functions of the ground state they would not be localized in space, in coordinate space and therefore, we need to integrate over all the coordinate space in order to recover the complete wave functions.

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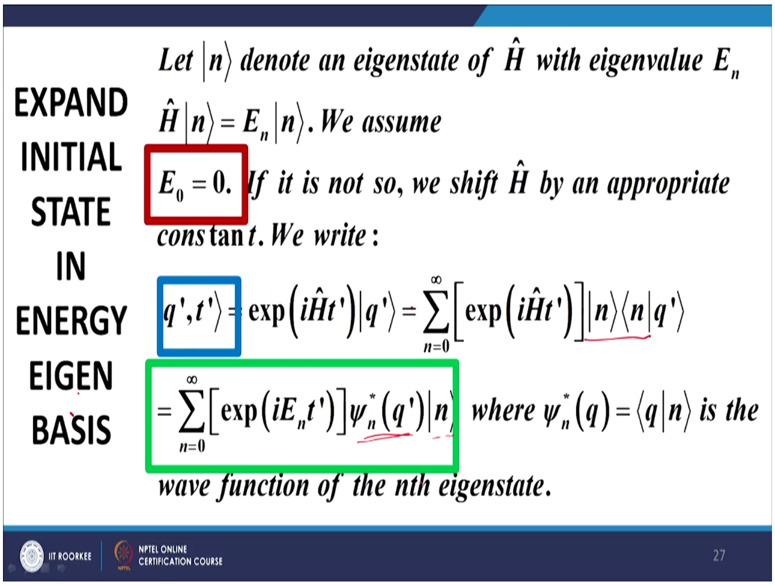

So, what we do is. First step is that we assume that the ground state energy is equal to 0, if the ground state energy is not equal to 0 it can be made equal to 0 by shifting of the Hamiltonian operator or introducing a constant into the Hamiltonian and that enable us to work on the premise or on the assumption that E 0 is equal to 0. The energy is the ground state energy is equal to 0.

And of course, the Hamiltonian the this n index the energy states now please note this and this number n indexes the energy states rather than the position states. So, the Hamiltonian operating on the i on its own eigenstate generates the corresponding energy eigenvalue. So, this equation is as far of the for the action of the Hamiltonian on its own eigenstates.

Now, we can write the initial position space state or the initial coordinate space state through, manipulation by introducing a complete set of energy states. By introducing a complete set of energy states which is in this step, and this summation together with this step and we can write it as this expression this is the wave function corresponding to the eigenvalue energy n at the point q dash and this n is the energy eigenstate and this is the effect of the Hamiltonian acting on evolution operator acting on the energy eigenstate.

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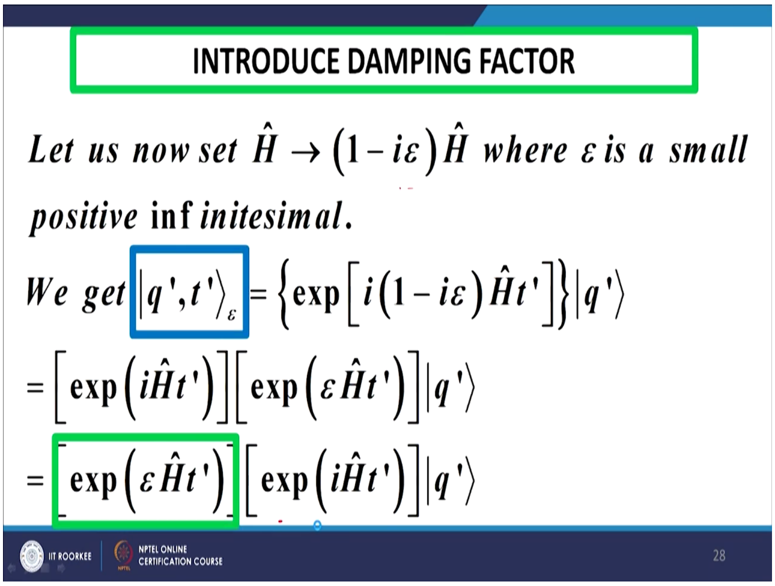

Now, what happens? Let us see what happens our problem is to recover somehow the ground state in this framework. The energy ground state in this framework that is our problem that is our objective. So, let us see how we go about it we instead of using the Hamiltonian in H, we start use we introduce the Hamiltonian, 1 minus i epsilon into H where, epsilon is some small number some small infinitesimal quantity. But it is positive; it is positive it is small.

Now, what happens? Suppose, I use this transformed Hamiltonian and try to explore the evolution of the initial state initial state what happens what do I get? I can represent it. I can represent it in the energy bases in the following form, which is given in the last equation on this is slide, but please note this extra factor which originates due to this quantity epsilon in the Hamiltonian.

This quantity epsilon coming into play into in the Hamiltonian; the importance of this extra factor will soon be apparent, but please keep that at the back of your mind and please note that this factor is the real.

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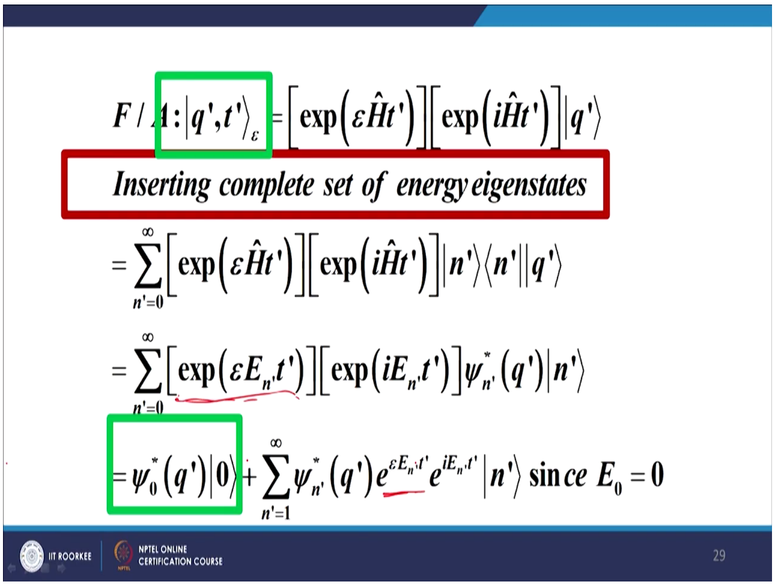

Now, what happens? When this extra factor operates on a on a basis are complete basis of energy eigenstates that is the important part. When this whole thing operates on a complete basis of energy eigenstates; what we recover is this expression and in this expression, this factor corresponds to the factor that I had mentioned earlier.

And this particular factor as I mentioned is a real. And in the this whole thing in fact, can be expanded, can be written as a term which involves the n equal to 0, n dash equal to 0 term, and the rest of the term.

In other words this summation here is from n dash equal to 0 to infinity. I am writing this summation as the term n dash equal to 0. I am segregating out this in this term for a purpose which will be known soon. I am out this n dash equal to 0 term, and I am writing the rest of the sum as n dash equal to 1 to infinity.

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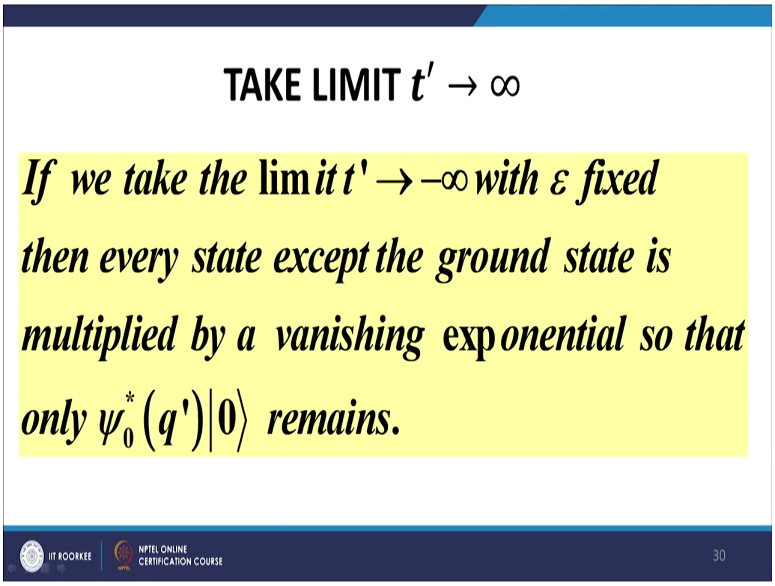

Now, what happens is, if you substitute if you take the limit t dash tending to minus infinity; if you introduce the term t dash equal to minus infinity. Then what you find is, that because of this factor this is the real factor and because of this factor, all the terms that are here; all the terms that are in this summation n dash equal to 1 to infinity they will tend to 0.

Each and every term will tend to 0 because of the existence of this factor, and this factor which does not contain this quantity this first factor which does not contain this expression this exponential damping quantity that will survive.

So, let me reiterate what is happened is we have replaced the Hamiltonian, we have replaced the Hamiltonian by the Hamiltonian 1 minus i epsilon; epsilon is positive real quantity. And the impact of this results in this expression coming into play, and this expression when it operates on a complete set of energy states; when the whole thing operates on a complete set of energy state this gives me this particular expression with the summation.

And this is again a real expression please note this and then, this expression can be split up in this whole summation can be split up into 2 parts. Because remember E 0 is equal to 0. Recall that E 0 is equal to 0. So, when we put n dash equal to 0, this expression gives me 0, this expression gives me 0, and I am left with this and this. These 2 expressions I am left with which is psi 0 star q dash into 0 this corresponds to n dash equal to 0, these 2 terms gives me 1 a piece.

Now, the rest of the terms are write as it is. Now, you take the limit t tending t dash tending to minus infinity. When you take the limit t dash tending to minus infinity, this expression the real expression e to the power exponential e n t dash with t dash tending to minus infinity, this will tend to 0. And as a result of with each of these terms; each of the terms in this summation will tend to 0, and what we will recover is this expression.

So, the net result is that every state except the ground state is multiplied by a vanishing exponential and we get only this expression this is what remain.

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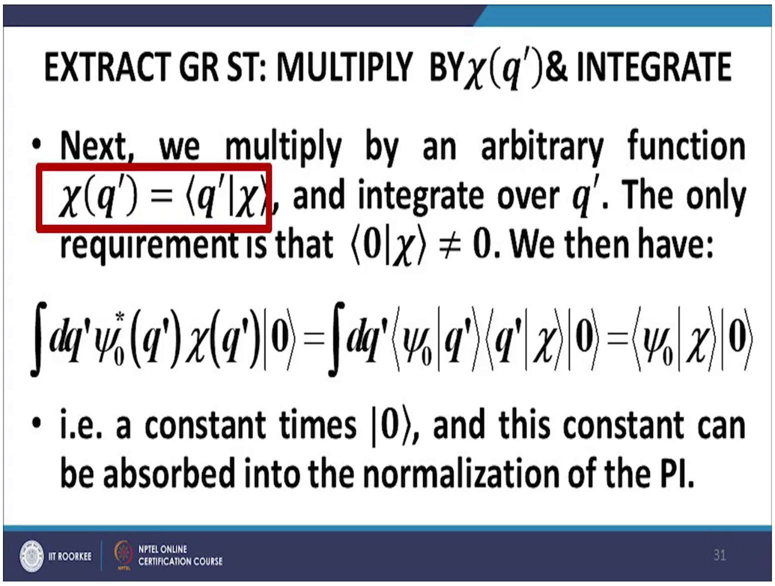

So, let us quick. Now ,what is the next step? How to now see where do where are we now? We are having this expression with us; we are having this expression with us psi 0 star at q dash with operating or with the ground state 0.

We want to recover the ground state. What we do is we multiply it by a function chi q dash and we integrate over q dash. We multiply it by a function chi q dash and we integrate over q dash. When you do this as you can see here in this in equation, what you have left is integration over q dash; with this expression now chi q dash can be written as if you note this q dash into dotted with chi, and similarly this expression can be written as psi as psi 0 dotted with q dash.

So, this whole expression if you look at it carefully, what are we left with? We are left with psi 0 and dotted with chi and the ground state. Now this expression is a constant; this expression is a constant and therefore, we get constant times the ground state and the constant can be absorbed in the normalization factor.

So, that is how; that is how by this process; by this process first manipulating the Hamiltonian. Secondly, taking the limit t tending to infinity or t dash tending to infinity and then multiplying by an arbitrary function chi q dash and then integrating over all of q q dash, we are able to recover the ground state of our problem.

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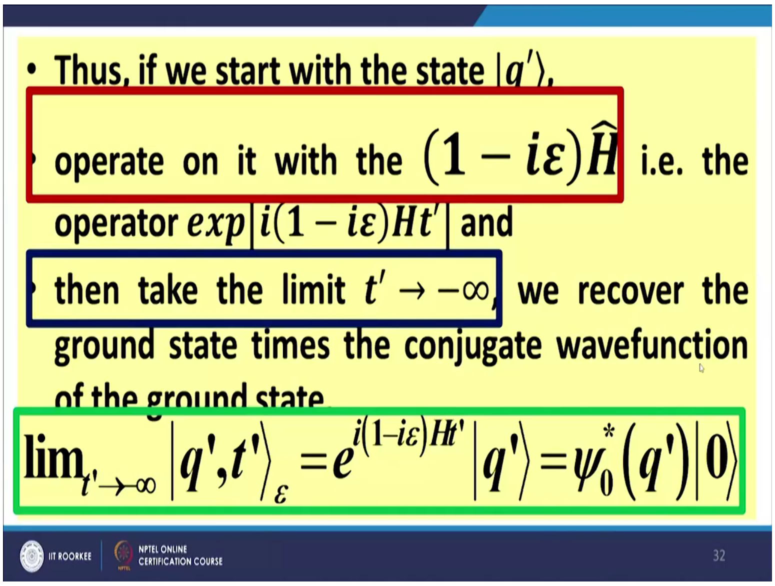

So, just to recap you start with this state q and q dash in the coordinates representation, coordinate space operate on it with a Hamiltonian instead of H, we operate with a Hamiltonian 1 minus i epsilon into H epsilon is real positive or equivalently operate by the exponential of this the evolution operator.

And then take the limit t dash tending to minus infinity, and you get the ground state; you get the ground state and the conjugate wave function of the ground state. Ground state times; ground state times the conjugate wave function of the ground state.

This is the conjugate wave function of the ground state, this is the ground state. So, after doing these 2 operations, operating on the state q dash with this Hamiltonian operator corresponding to this Hamiltonian, you get the ground state multiplied by the wave function conjugate wave function corresponding to the ground state at q dash. You multiply it with an arbitrary function as chi q dash integrate over q dash and you are able to recover the ground state right.

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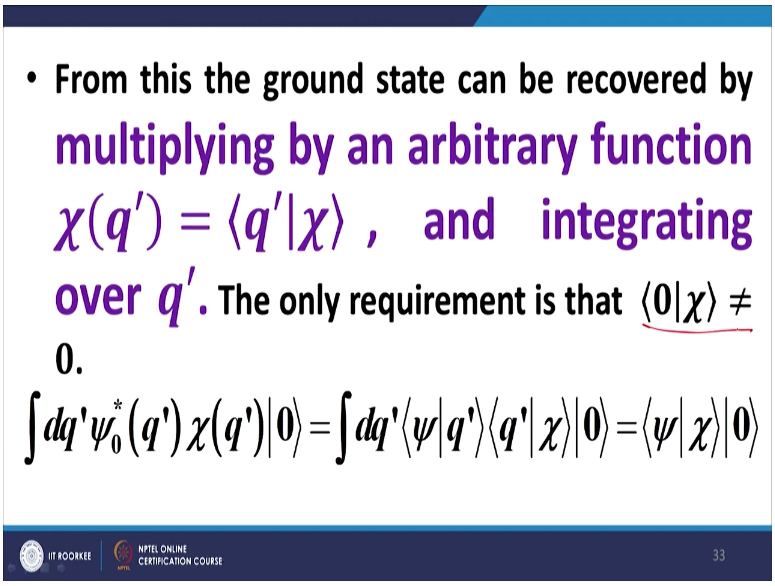

Of course, the condition the arbitrary function that you are going to use must have this particular condition that; it must not vanish when dotted with the ground state.

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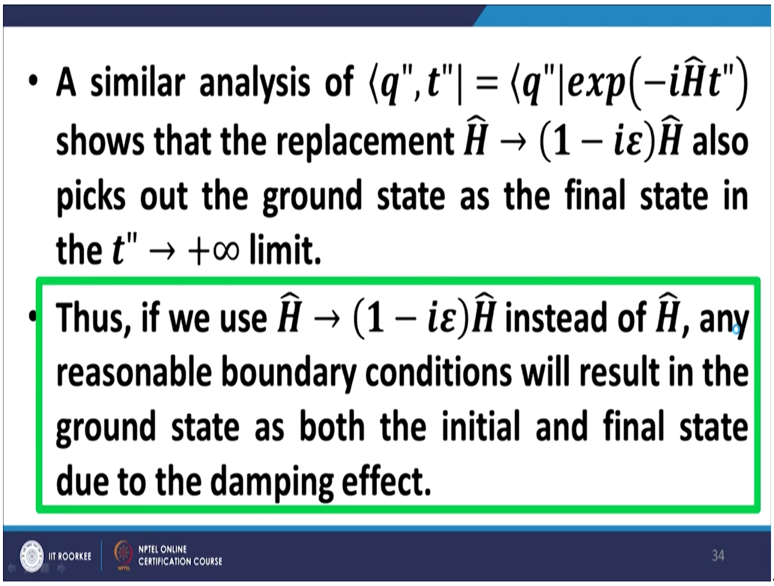

Similarly, we can and this is this was for the initial state. We similarly, we can manipulate the final state - q double dash, t double dash, absolutely on parallel lines, we can manipulate the final state q double dash, t double dash. And therefore, the net result is that if we use instead of H if we use the Hamiltonian given by this expression 1 minus i epsilon into H we recover the ground state with any reasonable boundary conditions in the limit, but the initial and final time extend significant tend to infinity.

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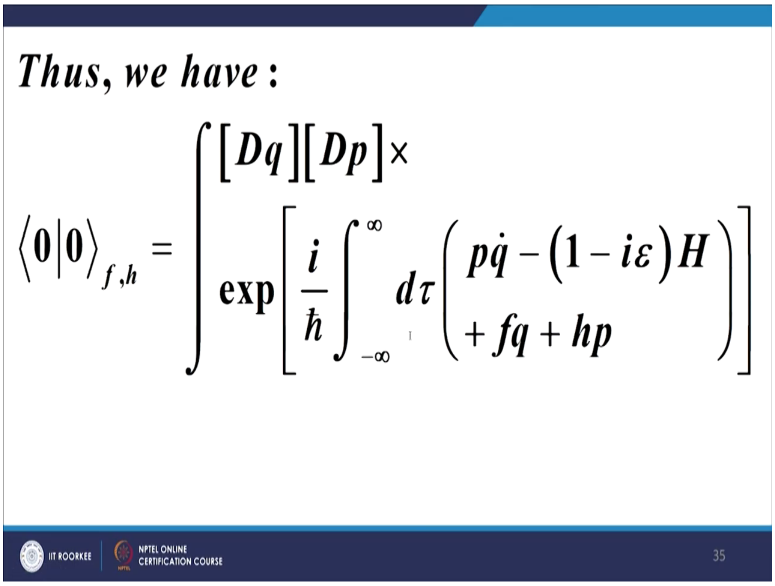

Therefore, putting all these pieces together I can write the transition amplitude of the vacuum to vacuums transition amplitude in the presence of a sources f and h in the form which is given in your slide as the path integral in the form given in your slide.

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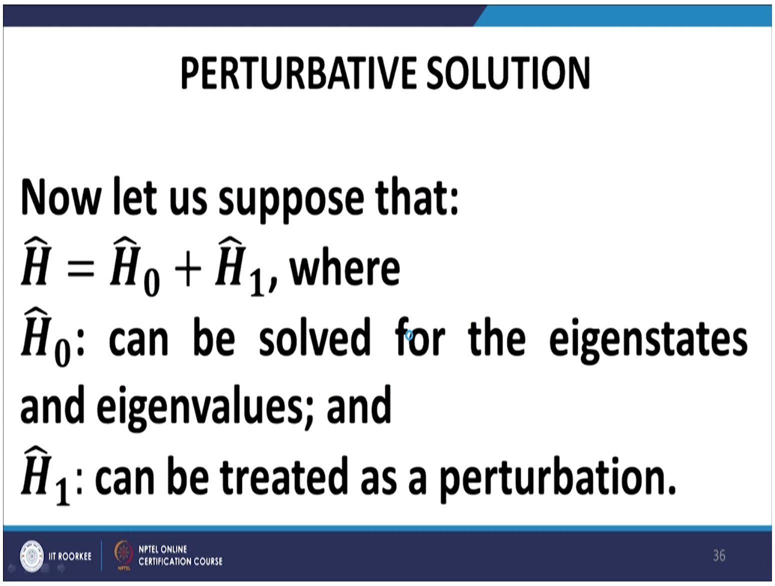

Now, we can do up further analysis on this, we can do a perturbative solution, we can attempt a perturbative solution of this expression. Now, for this what we do is we segregate the Hamiltonian, we split up the Hamiltonian into H 0 and H 1 where H 0 is in a sense and exact Hamiltonian which can be solved explicitly for the eigenvalues and the eigenstates and H 1 is to be treated as a perturbation of H 0.

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$$\langle 0|0\rangle_{f,h} = \int [Dq][Dp]\exp\left[\frac{i}{\hbar}\int_{-\infty}^{\infty} d\tau \begin{pmatrix} p\dot{q}-(1-i\varepsilon)H_0(p,q)- \\ (1-i\varepsilon)H_1(p,q)+fq+hp \end{pmatrix}\right]$$

$$= \int \begin{matrix} [Dq][Dp]\exp\left[-\frac{i}{\hbar}\int_{-\infty}^{\infty} d\tau(1-i\varepsilon)H_1\left(\frac{\hbar}{i}\frac{\delta}{\delta f(t)},\frac{\hbar}{i}\frac{\delta}{\delta h(t)}\right)\right]\times \\ \exp\left[\frac{i}{\hbar}\int_{-\infty}^{\infty} d\tau\left(p\dot{q}-(1-i\varepsilon)H_0+fq+hp\right)\right] \end{matrix}$$

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Then what we do is we start with this expression replace therein H equal to H 0 plus H 1; that is precisely what is done in the first equation nothing more absolutely nothing. And then instead of writing now H 1 is the function of p and q.

Now, instead of writing p and q as if we discussed in the first article that we discussed today that, p factors of p and q can be recovered by the action of these functional derivatives with respect to f and h. Functional derivatives with respect to f and h acting on the path integral; enable us to recover or acting on the exponential term in the path integral enable us to recover factors of q and p respectively.

That is precisely what is the underlying philosophy and therefore, in H 1 instead of representing H 1 as a function of p and q, we represent H 1 as functions of this functional derivative. Because this functional derivatives when they act on the remaining terms they put back terms factors of p and q respectively and we these two give equivalent results.

For instance, when this acts on this exponential which is given here it will pull down q. And if this acts on in the exponential which is given here it will pull down p right.

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$$F\,/\,A : \langle 0|0\rangle_{f,h} =$$

$$\int [Dq][Dp]\exp\left[-\frac{i}{h}\int_{-\infty}^{\infty} d\tau\,(1-i\varepsilon)\,H_1\left(\frac{h}{i}\frac{\delta}{\delta f(t)},\frac{h}{i}\frac{\delta}{\delta h(t)}\right)\right]\times$$

$$\exp\left[\frac{i}{h}\int_{-\infty}^{\infty} d\tau\left(p\dot{q}-(1-i\varepsilon)H_0+fq+hp\right)\right]$$

$$=\exp\left[-\frac{i}{h}\int_{-\infty}^{\infty} d\tau\,(1-i\varepsilon)\,H_1\left(\frac{\hbar}{i}\frac{\delta}{\delta f(t)},\frac{\hbar}{i}\frac{\delta}{\delta h(t)}\right)\right]\times$$

$$\int DpDq\exp\left[\frac{i}{h}\int_{-\infty}^{\infty} d\tau\left(p\dot{q}-(1-i\varepsilon)H_0+fq+hp\right)\right]$$

So, that having been done we can now you see now we can split up this and this path integral into two parts. The one part now the important part this you now the question was why did we introduce this functional derivatives? This is a very very engineers strategy that will be followed throughout when we talk about quantum field theory also.

You see what is happened is now this is this functional derivatives are with respect to f and h, the path integral that was respect to p and q; the path integration is with respect to p and q and therefore, because this functional derivatives are because H 1 is now; H 1 is now not a function of p and q, but it is a function of in this functional derivatives.

Therefore, you can take this outside the path integral that is the important part. If you look at the two equations the difference between the two equations is precisely this that the terms relating to H 1 have been pulled out of the path integral, earlier it was not possible because H 1 was a function of p and q.

Now, it is possible because H 1 has now been converted to functions of functional derivatives of f and h, and the path integrals are with respect to p and q. So, we can take this term outside the; we can take this term outside the path integral and the rest remains as it is.

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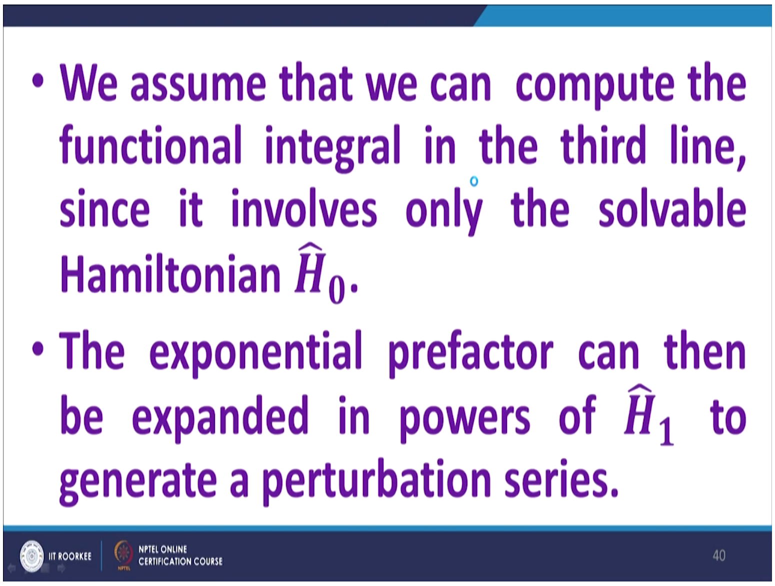

Now, things become very simple. Things become simple because, if you look at this in this expression it is containing our H 0 and by presumption we have assume that H 0 part of the Hamiltonian is exactly solvable and therefore, we should be able to solve the path integral contained in this expression the path integral expression should be solvable. And however, this exponential which is the prefactor would have to be treated as a power series or a perturbation series and then solve thereafter.

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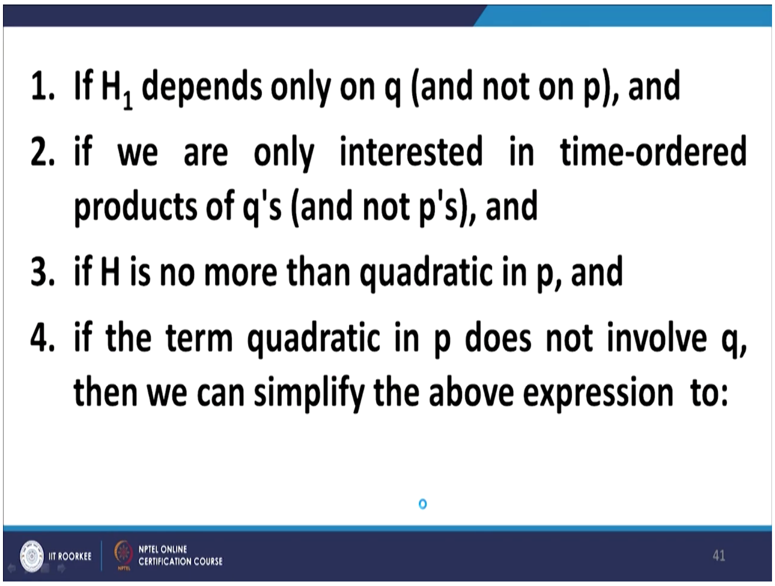

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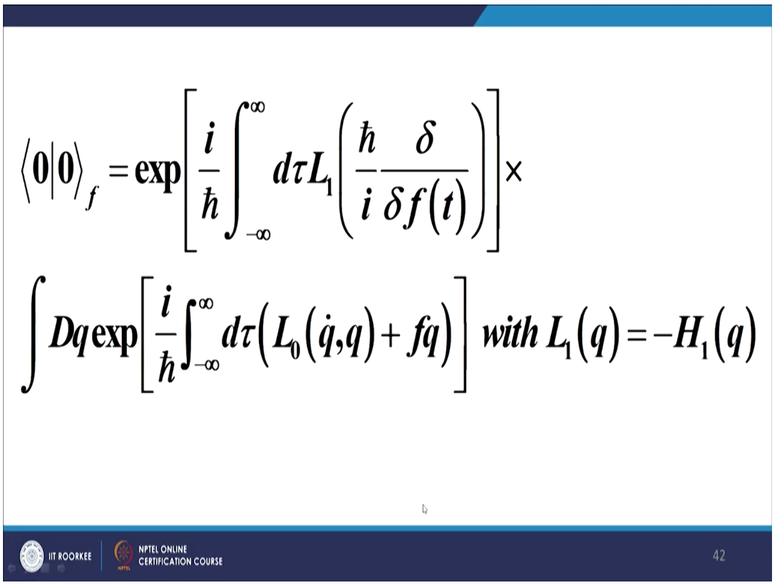

In some special cases; in some special cases we can simplify this and we can arrive at this expression for the transition amplitude between vacuum states.

Thank you, we will continue after the break.