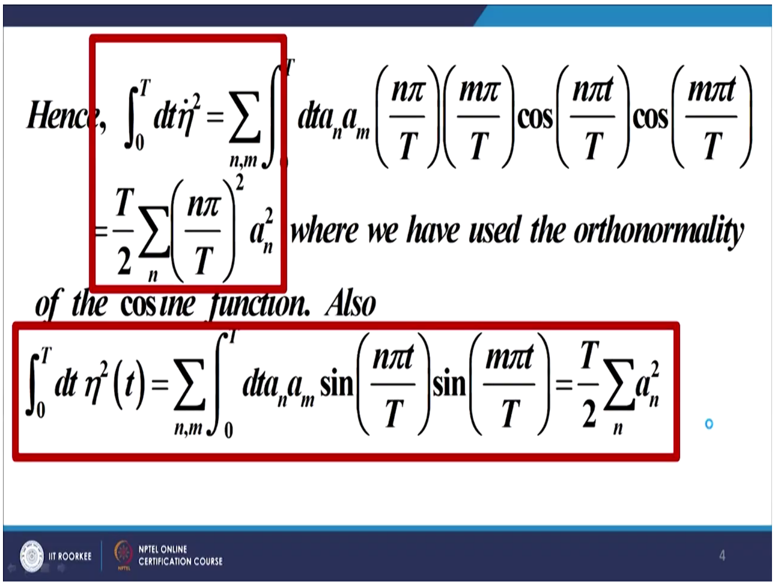
Path Integral Methods in Physics & Finance
Prof. J. P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee

Lecture – 22
Free Particle Path Integral

Right, so we were talking about calculating or computing the expression that is there in the green box, and towards that what we did was we expanded eta t as a Fourier series.

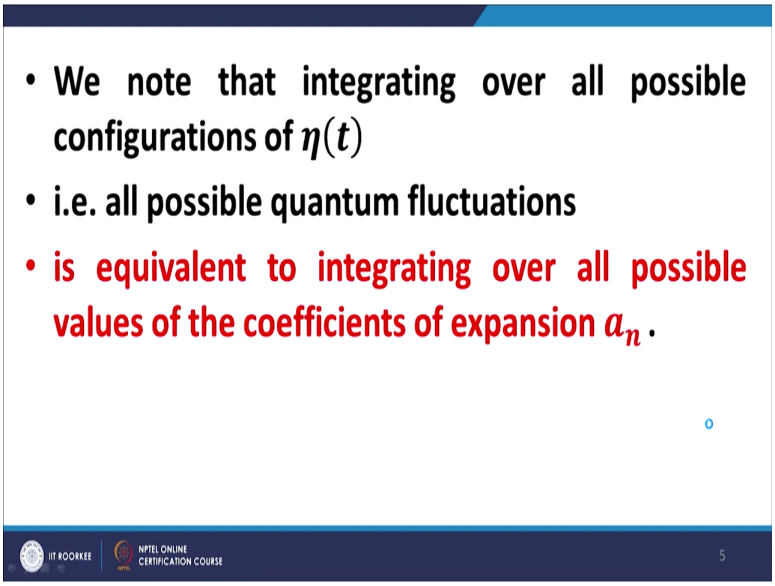
(Refer Slide Time: 00:40)



And thereafter, we worked out the two integrals; the integral of eta dot square and the integral of eta square because of the orthonormality of the cosine functions the integrals expressed the particularly simplified value, then work out to very simple expressions which are given in the two red boxes.

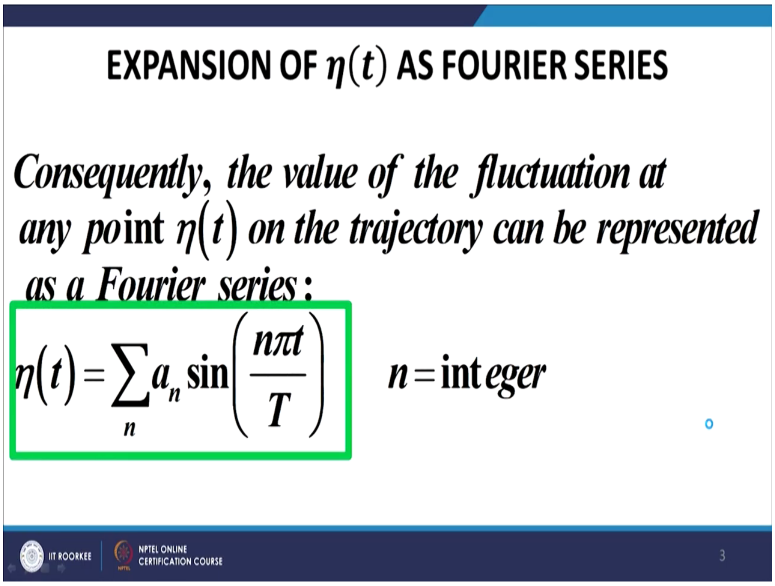
T by 2 summation n pi upon capital T whole square a n square; where a n are the Fourier coefficients and the integral of eta dot this was the integral of eta dot square. And the integral of eta square works out to even simpler; it is T by 2 summation of these squares of the Fourier coefficients. This simplification is achieved because of the orthonormality of the sine and cosine functions, right.

(Refer Slide Time: 01:34)



Now, certain important points as we recall and as has been emphasized again and again path integral involves integration of over all possible paths. In this particular situation, it will tend amount to the integration over all possible configurations of eta t that is because eta t represents the quantum for fluctuations and we therefore, need to integrate over all possible configurations of eta t. And in fact, that is the reason that we could substitute the path integral volume by the volume over eta t.

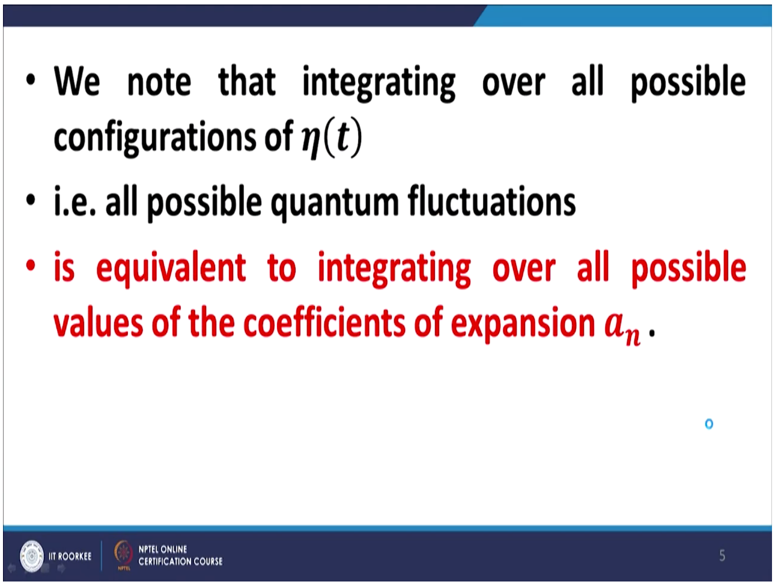
(Refer Slide Time: 02:24)



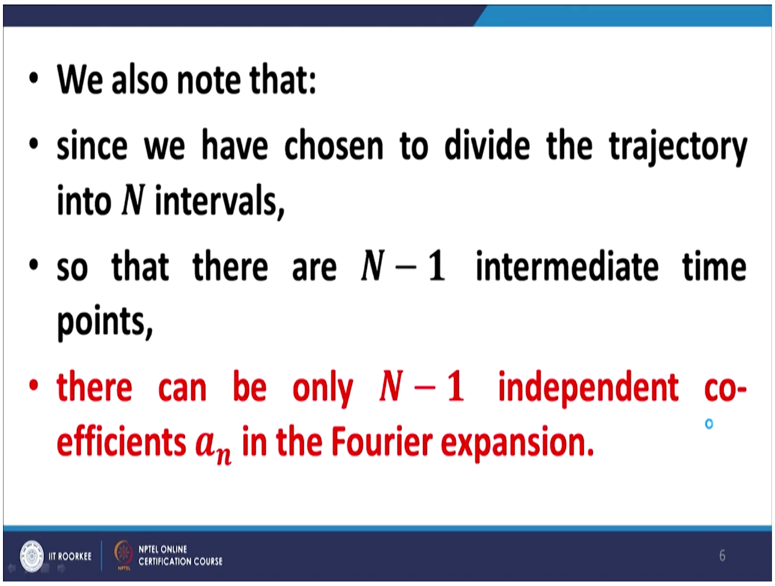
The second thing is now if you look at this, if you let us go back eta t has been expanded in terms of the Fourier coefficients. So therefore, integration over eta t would be equivalent to integration over all possible realizations of eta t would be equivalent to integration over all possible realization of these Fourier coefficients.

So, in other words what we need to do now is to the problem has been simplified from an integration over x to in path integration over x to a path integration over eta, and now to a path integration over all the Fourier coefficient; d a 1, d a 2, d a 3, and so on. So, we simplified the problem now considerably.

(Refer Slide Time: 03:10)



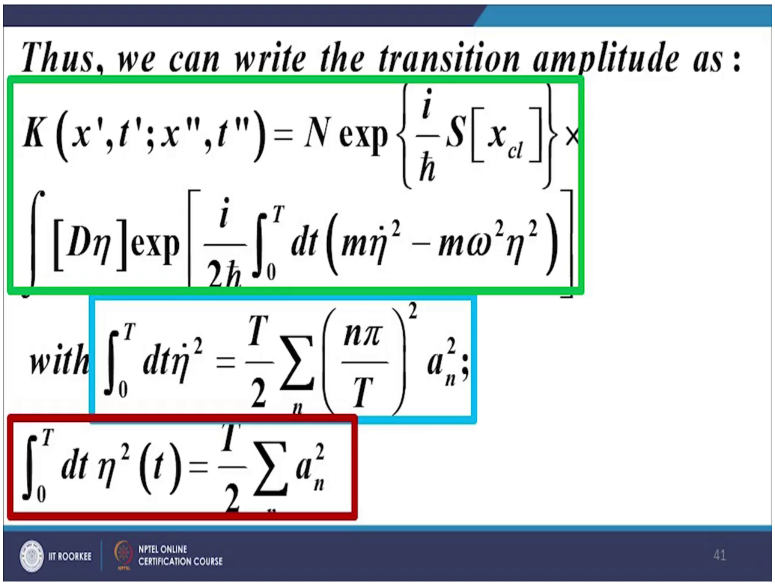
(Refer Slide Time: 03:12)



Furthermore, there is an important point; the important point is that we have divided the entire trajectory into n intervals and that implies that there are only N minus 1 time steps, intermediate time steps. You see, because the time steps at the beginning and the time final time step is are fixed.

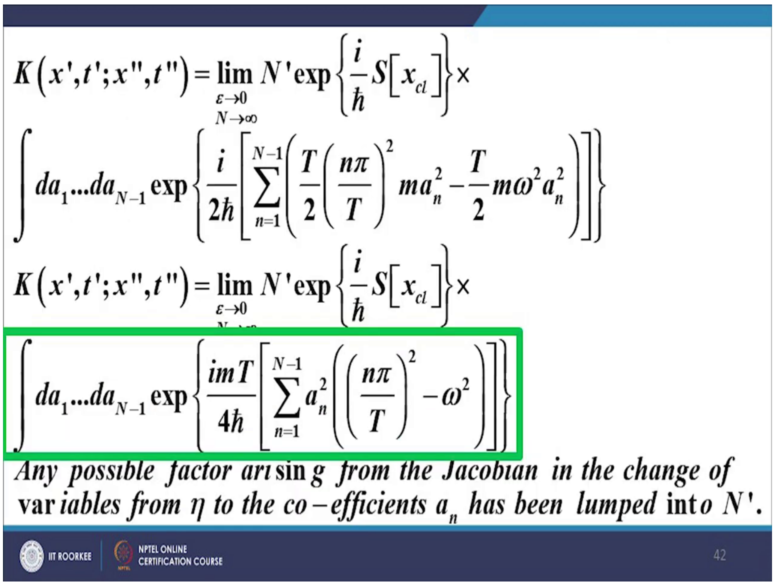
And therefore, there can only be N minus 1 intermediate time first; and that implies that there can only be N minus 1, capital N minus 1 Fourier coefficients a n that need to be integrated over. So, these are the independent Fourier coefficients that will need to be integrated over to arrive at the path integral over eta that is the summary of this argument.

(Refer Slide Time: 04:00)



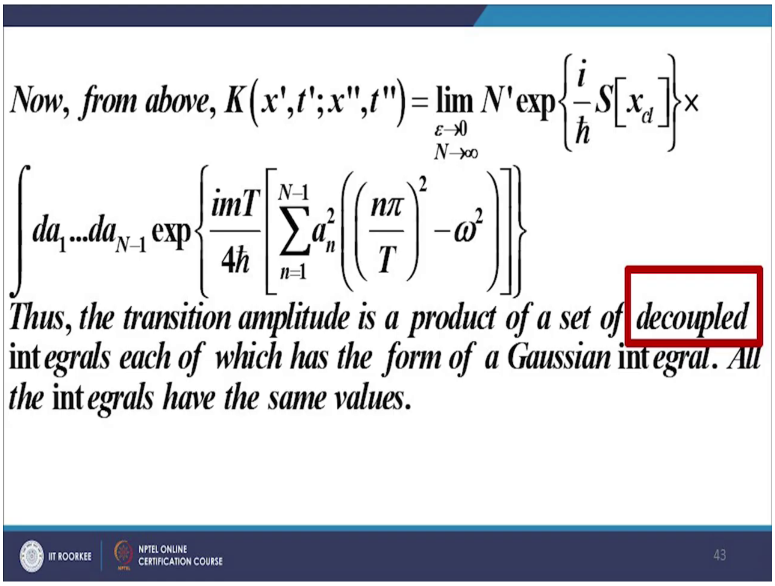
And therefore, what do we have? We can simplify the entire expression we started with the expression that is there in this green box. We worked out the values of the integral of eta dot square, we worked out the value of eta square integral, and we now substitute these values in the expression that is there in the green box.

(Refer Slide Time: 04:24)



What do we get? We get this expression with and we also replace the integration the path integration over eta by integration by multiple integration over a 1, a 2, a 3 up to a N minus 1, because there are a N minus 1 intermediate points; where the initial point and the final point are fixed. They will not form integration variables, the integration variables will be the intermediate slice points (Refer Time: 04:56) and they are N minus 1 of them, right. So, simplifying a little bit what we get is the expression in the green box.

(Refer Slide Time: 05:06)

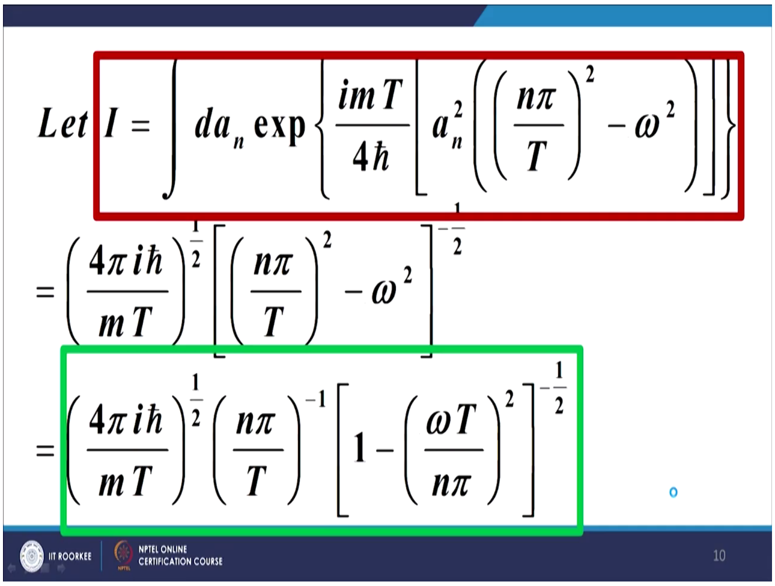


Now, if you look at it carefully, now we will just look at it carefully. There are this is a exponential, and then we have a term and then we have a summation of a and this thing, a n square and then this expression that means what? That means, it is basically exponential of a 1 this thing, a n equal to 1, then exponential n the whole expression with n equal to 2.

Then exponential the whole expression with n equal to 3, integrated with respect in the first with respect to a 1, the second term with respect to a 3, a 2; the third term with respect to a 3, and so on. In other words, what I am simply trying to say is, that this whole expression is a product of n integrals which are themselves identical.

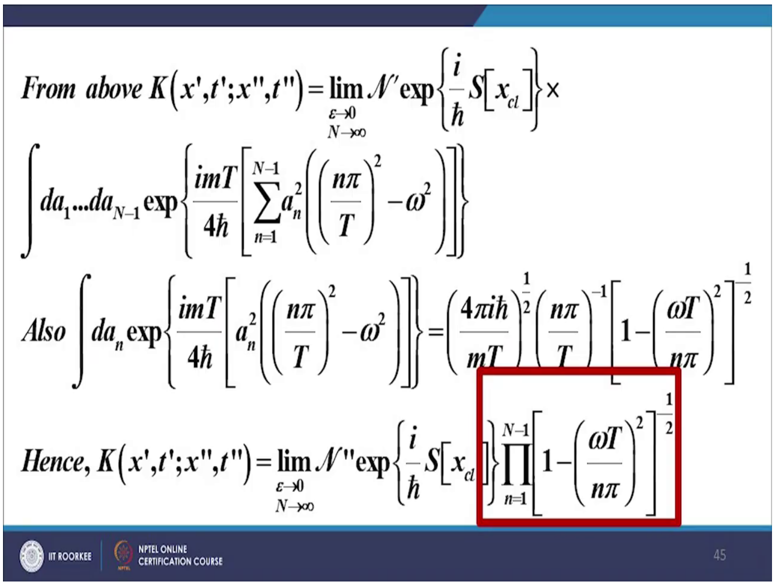
You can split this in to, because this is an exponential of a some exponential; let us say its exponential x plus y, I can write it as exponential x exponential y and in let us say, let us say it is of the form. Integral dx dy e of x plus y; obviously, I can write it as integral dx dx e x integral dy ey, and that is nothing but i square, where either one is taken as i. So, these integrals can be decoupled can be separated and each of them is the same integral, so that is the important thing there are N minus 1 such integrals.

(Refer Slide Time: 06:50)



Let us do one of them, let us take any arbitrary one of them and let us work out its value. If you look at this expression now the single integral one integral, it is clearly a Gaussian. If you look at it carefully, and the integration is with respect to a n and we have a n square in the exponential, it is clearly a Gaussian integral; it can be easily done by completing the square and what we have is the expression that is there in the green box here.

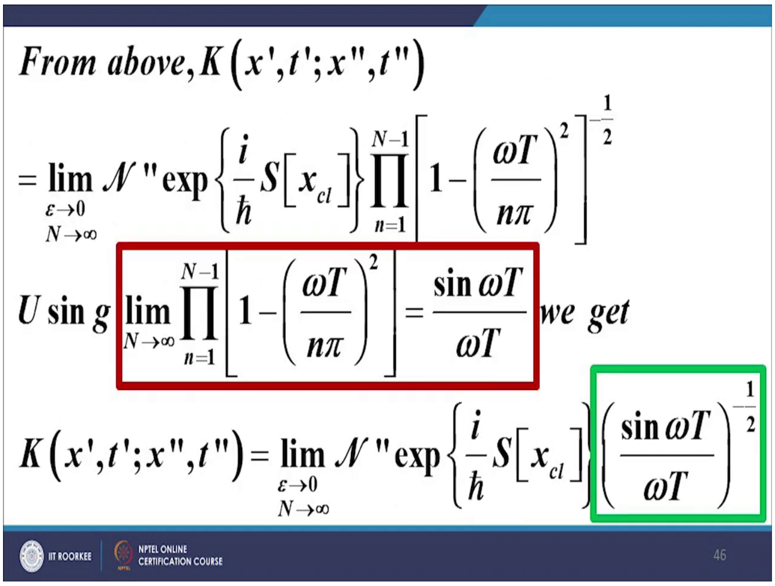
(Refer Slide Time: 07:21)



So, what do we have now? We have simplified it the whole expression and we have now got this expression, but there are there are n of and there are integrals from n equal to 1 to N minus 1 and each of this expression is equal to this expression which is here. And this coefficient 4 pi i h bar upon m T to the power 1 1 by 2, it can also be absorbed in into the normalization and we have a new normalization N double dash.

Instead of having N dash, we have the N normalization N double dash; this expression which is which does not contain any dynamics of the system really, they can be absorbed in the normalization. And we have the second expression as classical here, and the integration per integration we get a factor of this quantity.

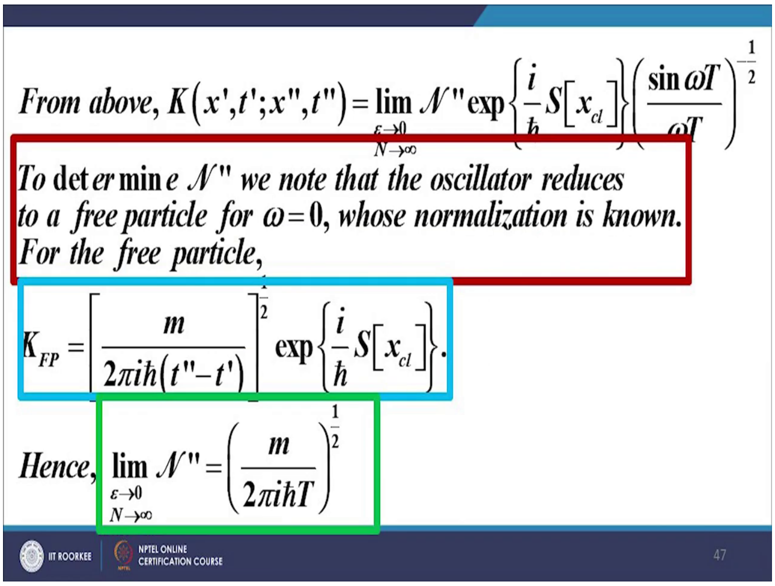
(Refer Slide Time: 08:14)



Now, these quantities in the limit that N tends to infinity, in the limit that N tends to infinity this product is nothing but sine omega T upon omega T; it can be simplified and it can be shown to be equal to sine omega T upon omega T. So, now using this the expression that is there in the red box which is a simple trigonometric relation.

Using this expression in the red box, we finally arrive at the expression for the transition amplitude, for the harmonic oscillator, quantum harmonic oscillator, using the path integral formalism, as the expression which is here. This is the quantum correction the correction that is shown in the green box is the quantum correction and this expression is of course, the classical expression, the expression of the class in terms of the classical action.

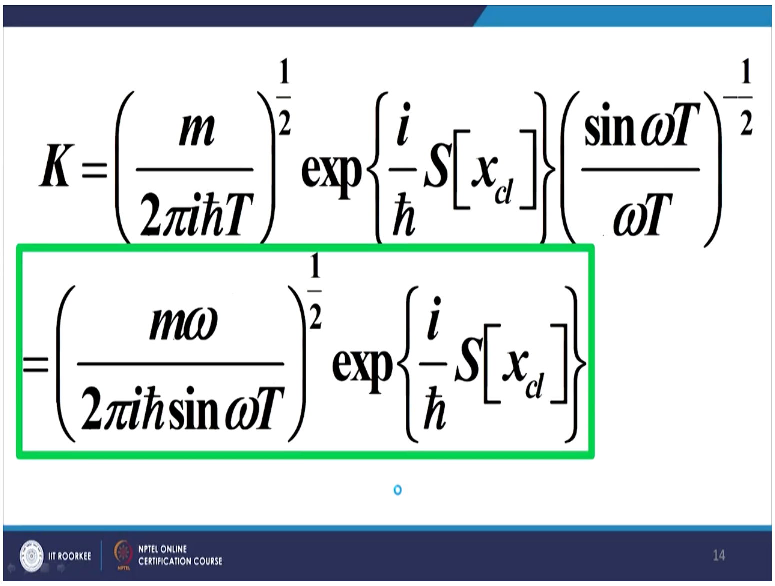
(Refer Slide Time: 09:19)



So, now our problem remains to working out the normalization constant. Now, for the normalization constant we make a note of it that the free harmonic oscillator, the free harmonic oscillator is equivalent to the free particle for omega equal to 0. The harmonic oscillator if you if I put omega equal to 0, it becomes a free particle; free harmonic oscillator for omega equal to becomes a free particle.

And the normalization of the free particle is this expression, which is in the blue box we know that the in fact will work it out also, we will work it out in today’s lecture itself. So, knowing this expression we get the value of the normalization constant as m upon 2 pi i h bar T square root.

(Refer Slide Time: 10:14)

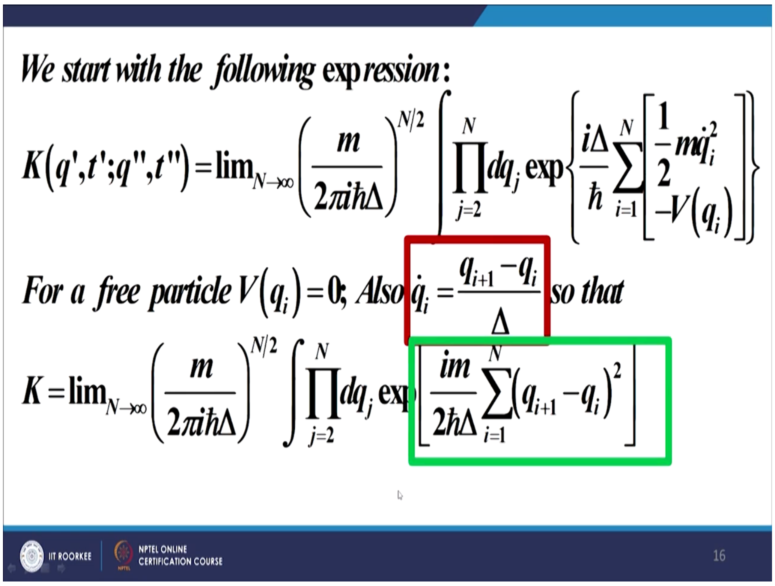


So, putting all the pieces together we have now found the expression of the transition amplitude, as the expression that is there in the green box. The final expression for the transition amplitude of the harmonic oscillator, right.

(Refer Slide Time: 10:39)

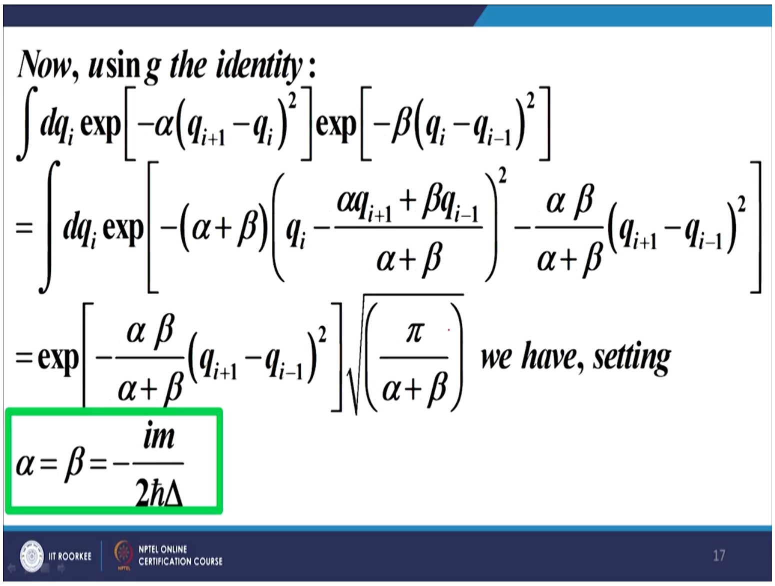
Now, we let us do one more example, another very important example. It is the free particle path integral, the free particle path integral.

(Refer Slide Time: 10:47)



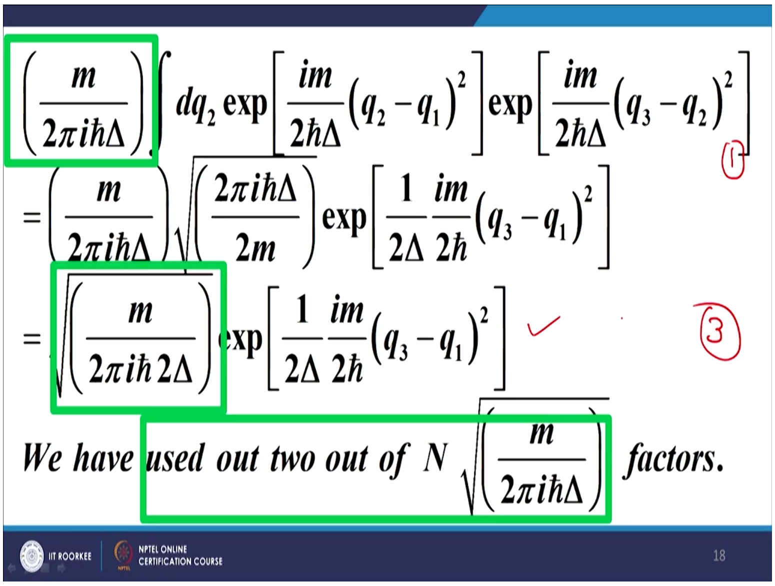
Now, clearly in this is the expression it is quite straightforward, so we will do it quickly. We start with this expression for the for the transition amplitude, which has been taken from the previous lecture. And we make the substitution that q dot is equal to q i plus 1 minus, this the divided difference we are replacing q dot by the divided difference q i plus one minus q i upon delta. So, in that part we have substituted it back, other than that we started with the expression which we had earlier and we made this particular substitution.

(Refer Slide Time: 11:32)



Now, we make use of the a very important identity, this identity is given on the slide. And it is a very important identity, we make use of this identity with the condition alpha is equal to beta is equal to minus i m upon 2 h bar delta. In others let me explain, in this particular identity I put alpha equal to minus i m upon 2 h bar delta; I put beta equal to minus i m upon 2 h bar delta; and this is my right hand side, and this is my left hand side.

(Refer Slide Time: 12:14)



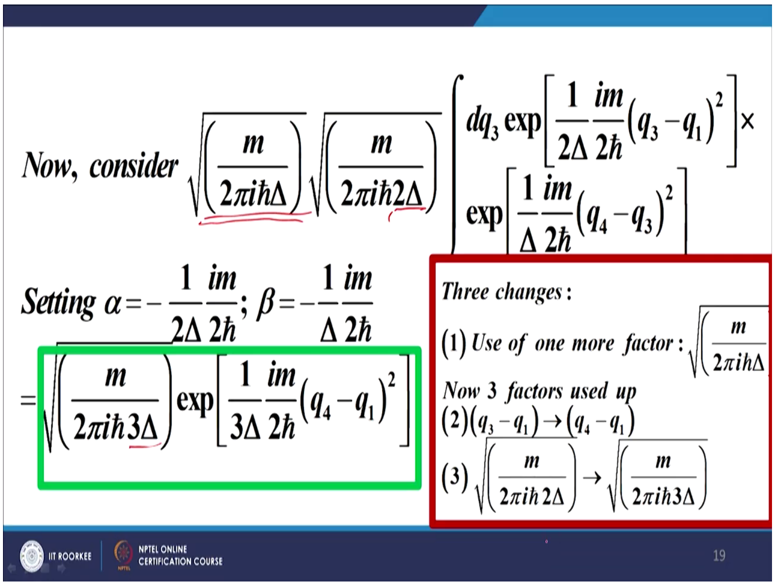
Let us see what I get after making the substitution, after making the substitution what I get is the expression that is there in the in this expression, this particular expression that is what I get. The calculations are slightly tedious, but they are straightforward there is absolutely no nothing special in them; we are simply substituting the value alpha equal to beta equal to i m upon 2 h delta and you are using q 2 minus q 1 here and q 3 minus q 2 here.

ah This is q 3 minus q 2, this is q 2 minus q 1; so these values are simply being substituted the integration is over q 2, as you can see you can see here the integration is over q 2. And this is what you get, this is what you get here this expression has been transposed to the left hand side pi upon alpha plus beta has been transferred to the left hand side, and we get this expression m upon 2 pi this expression this is on the left hand side, right.

So, other than that there is no change, absolutely parallel. Now, the important thing is so now, I can at the end of the day I can write this expression as equal to this expression, the expression number 1 is equal to expression number 3. Now, look at this; how many of these factors are here I have used m upon 2 pi h delta, this is an important consideration so let us go back a bit.

If you look at it m upon 2 pi h bar delta, there are N by 2 factors, N if you put it within the square root, there are N factors of square root m upon 2 pi h bar delta; N factors of square root m upon 2 pi i h bar delta N factors of the square root. So, here we are having two factors consumed there right, two factors we have put here right; and here of course, we have got this expression. So, this is important just keep track of this we have consumed two factors that means, out of N we have consumed two factors.

(Refer Slide Time: 14:45)



Now, let us move to the next step. Now, in the next step what we do is we write alpha equal to minus 1 upon 2 delta i m upon 2 h bar and beta equal to minus i m minus i m upon delta into 2 h bar. So, when I make these substitutions, when I make these substitutions and then do the simplification, I get the expression in the green box right, I get the expression in the green box.

Now, there are certain important observations here I have used of one more factor here now if you look at this, this factor has also been consumed. So, I had used two factors earlier; now I have consumed one factor, so in the second step I have consumed one more factor right. The second thing is q 3 minus q 1 has now been replaced by q 4 minus q 1. I started if you look at this here it was q 3 minus q 1; now I have replaced it by q 4 minus q 1, in the next iteration.

And the third is if you look at this carefully, 2 delta has been replaced by 3 delta. So, these are three changes that have taken place in the second iterative step; first one more factor has been taken up extracted, second q 3 minus q 1 has gone to q 4 minus q 1; and the 2 delta has become now 3 delta, because this iteration is the bases of working out the normalization and so this is what is given here.

(Refer Slide Time: 16:21)

Three changes:

(1) *Use of one more factor:* $\sqrt{\left(\frac{m}{2\pi i\hbar\Delta}\right)}$

Now 3 factors used up

(2) $(q_3 - q_1) \rightarrow (q_4 - q_1)$

(3) $\sqrt{\left(\frac{m}{2\pi i\hbar 2\Delta}\right)} \rightarrow \sqrt{\left(\frac{m}{2\pi i\hbar 3\Delta}\right)}$

(Refer Slide Time: 16:23)

From above, $\left(\frac{m}{2\pi i\hbar\Delta}\right)\int dq_2 \exp\left[\frac{im}{2\hbar\Delta}(q_2-q_1)^2\right]\exp\left[\frac{im}{2\hbar\Delta}(q_3-q_2)^2\right]$

$$= \sqrt{\left(\frac{m}{2\pi i\hbar 2\Delta}\right)}\exp\left[\frac{1}{2\Delta}\frac{im}{2\hbar}(q_3-q_1)^2\right] \qquad (1)$$

$$\sqrt{\left(\frac{m}{2\pi i\hbar\Delta}\right)}\sqrt{\left(\frac{m}{2\pi i\hbar 2\Delta}\right)}\int dq_3 \exp\left[\frac{1}{2\Delta}\frac{im}{2\hbar}(q_3-q_1)^2\right]\times \exp\left[\frac{1}{\Delta}\frac{im}{2\hbar}(q_4-q_3)^2\right]$$

$$= \sqrt{\left(\frac{m}{2\pi i\hbar 3\Delta}\right)}\exp\left[\frac{1}{3\Delta}\frac{im}{2\hbar}(q_4-q_1)^2\right]. \qquad (2)$$

IIT ROORKEE | NPTEL ONLINE CERTIFICATION COURSE | 21

So, now what do we have this is my equation number 1, this is my equation number 2, right.

(Refer Slide Time: 16:41)

SUBSTITUTING LHS OF EQUATION (1) IN LHS OF EQUATION (2)

$$\left[\sqrt{\left(\frac{m}{2\pi i\hbar\Delta}\right)}\right]^3 \int dq_2 dq_3 \exp\left[\frac{im}{2\hbar\Delta}(q_2-q_1)^2\right]\exp\left[\frac{im}{2\hbar\Delta}(q_3-q_2)^2\right]\times \exp\left[\frac{1}{\Delta}\frac{im}{2\hbar}(q_4-q_3)^2\right]$$

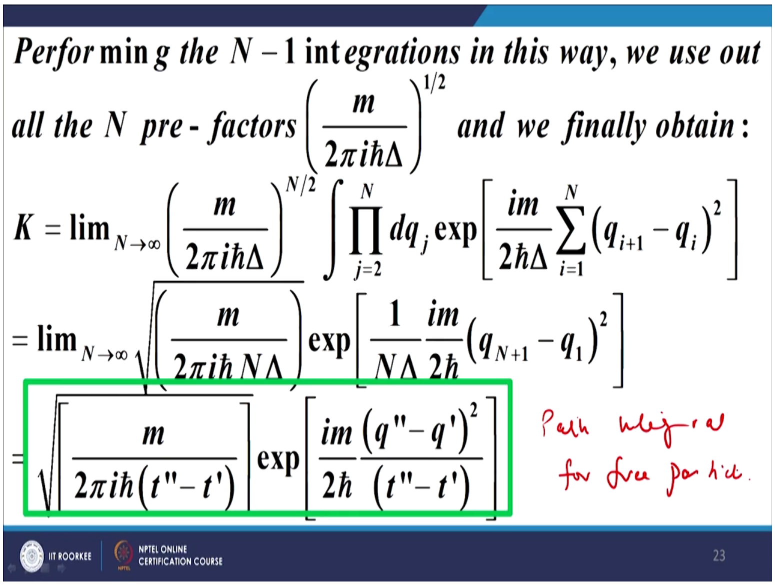
$$=\sqrt{\left(\frac{m}{2\pi i\hbar 3\Delta}\right)}\exp\left[\frac{1}{3\Delta}\frac{im}{2\hbar}(q_4-q_1)^2\right]$$

Now, if I substitute, if I substitute from equation number 1 in equation number 2, what I get is left hand side of equation; if I substitute the left hand side, left hand side of equation number 1 in the left hand side of equation number 2. If I substitute left hand side of move on in left hand side of equation number 2, see this this expression q 3 minus q 1 square; q 3 minus q 1 square is appearing here precisely this expression that means what that means, the right hand side is appearing here.

So, I replace this right hand side which is appearing here by the left hand side in equation number 2 that is precisely what I have done. Replacing the right hand side of equation number 1 by in equation number 2, by the left hand side of equation number 1 that is what I have done. And then I get the result this whole expression simplifies to this expression; q 4 minus q 1, I have mentioned you have now picked up three factors as I told you; two factors in the first iteration, one factor in the second iteration.

So, you picked up three factors and this has become 3 delta; you started from 1 delta, 1 delta here you now have 3 delta here, alright.

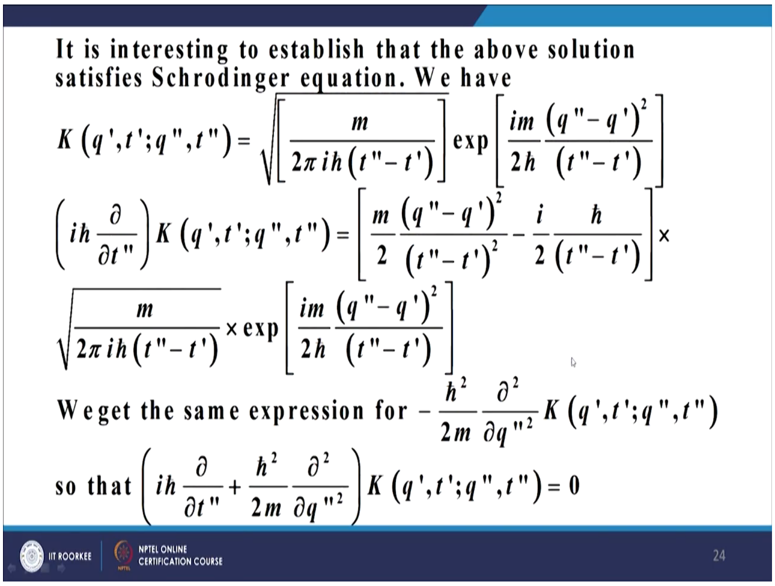
(Refer Slide Time: 17:55)



So, you continue this way and perform all the N minus 1 integration, perform all the N minus 1 integrations. Obviously, how many factors will be consumed? N minus 1 plus the extra factor that we consumed in the first iteration; so, all the N factors would be consumed. And, instead of q 4 minus q 1 we will have q N plus 1 minus q 1, and instead of 2 2 delta, 3 delta, 4 delta we will now have N delta, but N delta is nothing but t double dash minus t dash final time minus initial time, it has been split up into n partitions as you know, so it will be t double dash minus t dash.

And this so this whole thing becomes exponential, 1 upon N delta, 1 upon N delta is nothing but t double dash minus t t dash, i m is as it is, 2 h bar is as it is; and q N plus 1 is nothing but q double dash and q 1 is nothing but q dash.

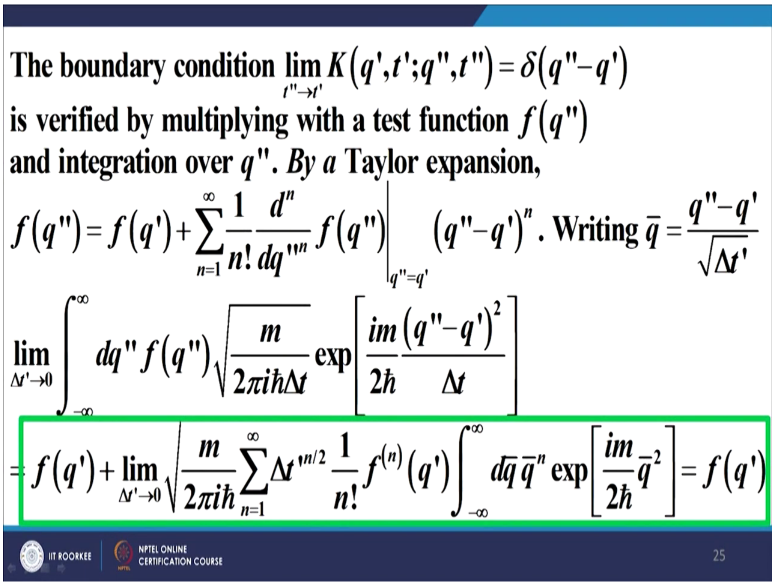
(Refer Slide Time: 19:03)



So, this is the ultimate expression for the path integral for the free particle, just recap this expression is the path integral for free particle right. So, it can be shown that this path integral expression satisfies this Schrodinger equation, it is quite straightforward you differentiate in the expressions with respect to t double dash and with respect to q double dash put their values in the Schrodinger equation.

The left hand sides and the right hand sides of the Schrodinger equation, you find that they are satisfied. Simple calculation, simple algebra and nothing work.

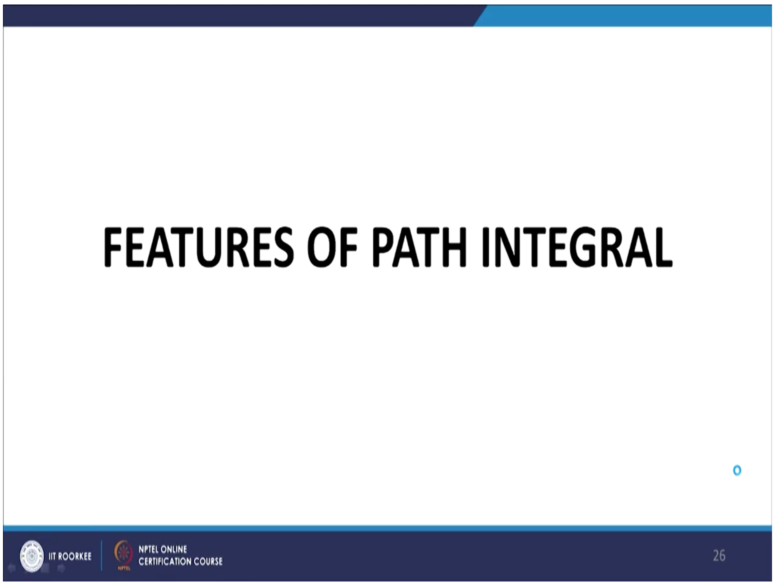
(Refer Slide Time: 19:53)



And in fact, even the delta can requirement that limit t tending to t double dash of the transition amplitude is equal to the direct delta function can also be verified. If you use a test function, let us say f q double dash multiply the expression and then integrate over d integrate with respect to q double dash, you find that you end up with the required condition. So, this is solved here slightly extended calculations, but nothing more.

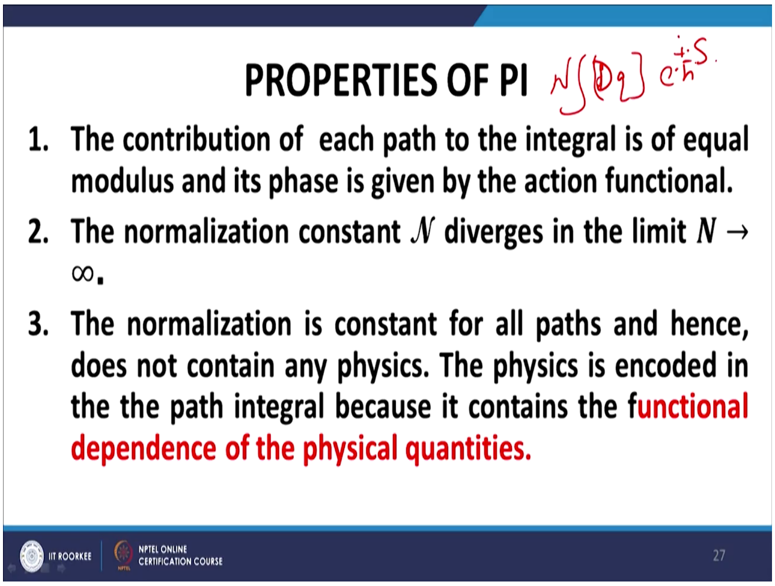
So, this is these two examples I have taken here in the context of path integral to acquaint the viewers with the concept and to familiarize them. We shall progress gradually, we now we shall what we shall go on to is to work out the expectation values the in vacuum states of various operators, we shall also work out the expressions for the time order products of operators using the path integral framework.

(Refer Slide Time: 21:07)



So, that is the agenda for the next few lectures. In the moment, let us explore few features of the path integral important properties of the path integral.

(Refer Slide Time: 21:24)



Now, the first property fundamental property is that the contribution, because you see how do we work out the path integral. We work out the path integral by integral D of q e to the power i upon h bar into S, of course we have a normalization here. So, this is the path integral that we talk about, clearly the modulus this modulus e to the power i upon h bar S is unity and therefore the weightage given to each path is the same, weightage given to each path is the same, but the phase is determined by the action S and the phase may be different.

So, this is one important property the weightage given to each path is the same, nevertheless also extravagant the path may be even then the weightage is given to the path is the same, but the phase may be different of course, phase is determined by the action of each path. Now, here we have a very important issue see as I mentioned just now the phase of the path of a particular path, it depends on the action.

Now, you would recall that how do we determine the classical path; we determine the classical path by extremizing the action. In other words, in some sense the action is stationary along the classical path. A putting it another way, if we deviate slightly or if you perturb the system slightly from the classical path, the action is not going to be disturbed significantly; because it is at an extremum along that particular path.

What does it mean? It means that even if there is an infinitesimal a small paths that are slightly away from the classical path, paths that are slightly away from the classical path would not differ greatly in terms of the impact on the action. And as a result of which as a result of which they would act coherently with each other. And whereas, what happens in the case of paths which are far away from the classical action path or the classical path.

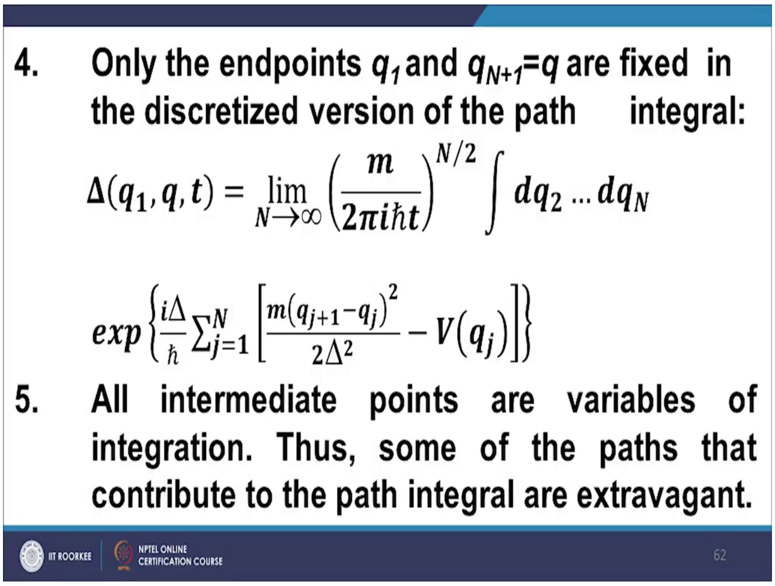
In that case what happens is even a slight perturbation would cause the action to change significantly, and because the action would change significantly what would happen is that the paths would act destructively or interfere destructively and as a result of it their impacts would be neutralized.

So, in a sense although the weightage of all the paths is the same, the impact or the why the classical path becomes significant, why the classical path shows up is because the action is extremal along that path, so that is important. And so the first thing is that the contribution of each part to the integrals of equal modulus, equal magnitude, equal modulus, equal weight and the phase is given by the phase is given by the action, so that is important.

The normalization, now another important feature is that the normalization constant diverges as N tends to infinity, nevertheless the normalization constant does not actually contain the physics of the system. The physics of the system work is contained in the path integral itself, it contains the functional dependencies and their evolution behavior that is contained in the path integral not in the normalization.

So, notwithstanding the fact that the normalization constant may diverge we still get the desired physics or the desired physical information about the evolution of the system from the path integral.

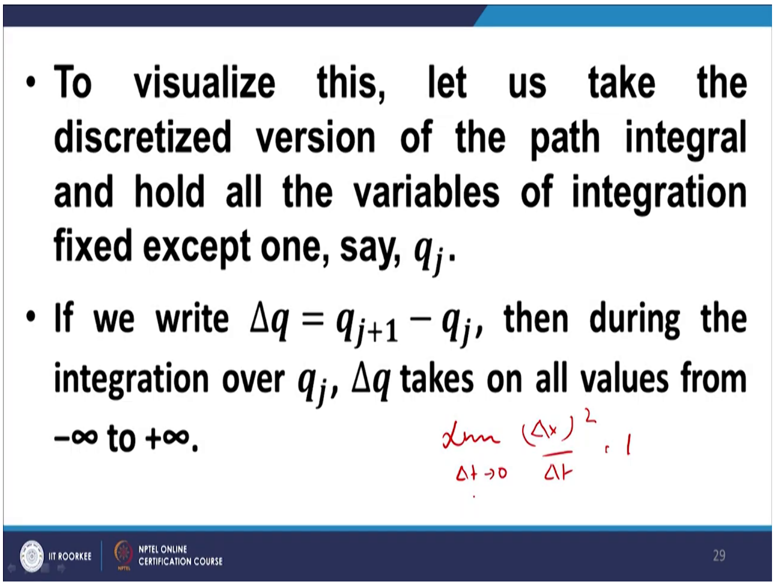
(Refer Slide Time: 25:40)



Now, there is a very interesting issue here; if we have this let us look at the discretized version of the path integral. Now, as I mentioned it all intermediate paths in the discretized version; all in only the endpoints are fixed and the endpoints are fixed, the other points are not fixed.

Then the integration is carried out with respect to all the other paths, at q 2, q 3, q up to q N or q N minus 1 as N plus 1 as the numbering may be, but the basic thing is that it is all the intermediate path or intermediate points other than the initial point and the final point.

(Refer Slide Time: 26:26)



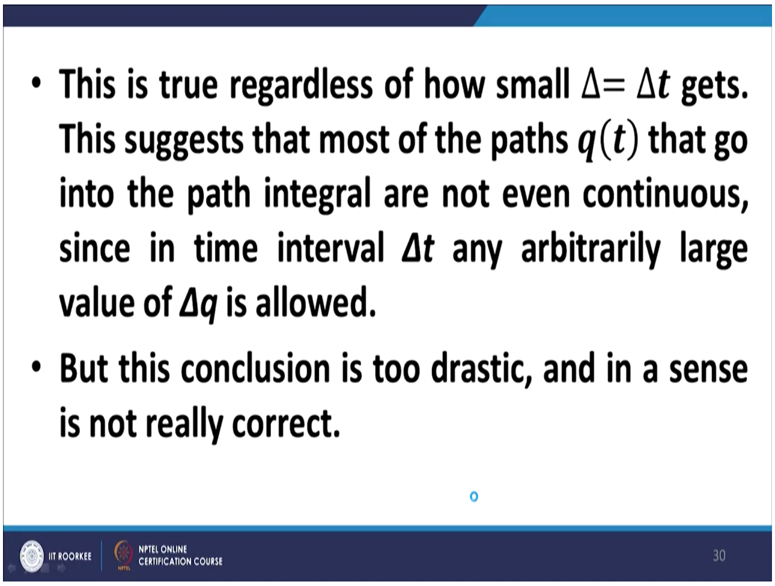
So, let us try to understand what they what is the implication for this. For this purpose, suppose we assume that all the points are fixed let us say let there be only one point which is not fixed. And therefore, integration about that from minus infinity to plus infinity needs to be done. Let us say q 2 is not fixed and therefore, integration i along q 2 will be done from minus infinity to plus infinity.

Now, what happens what does it mean? It means that all paths even paths that are that are in some sense in some rough sense discontinuous; they have some kind of a jump, there they would also need to be integrated over. But then this creates distortion as I explained in the context of Brownian motion.

We do require, you would recall that in the context of Brownian motion what did we say; we said that limit in a in a limiting scenario we have delta x square upon delta t is equal to 1; delta t tending to 0. If you recall, I discussed this in the context of Brownian motion; this was in a sense to ensure that some kind of a continuous behavior is sustained in the path that we are talking about that we are integrating over.

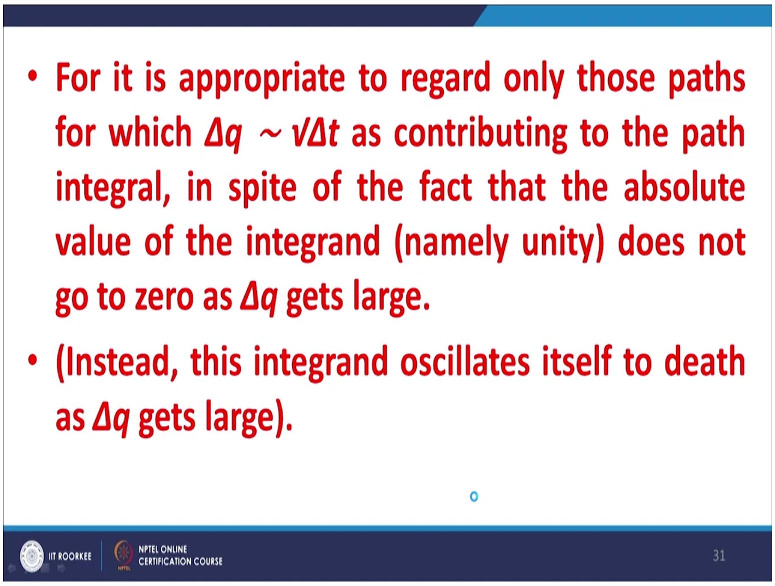
So, the important thing is that despite theoretical or a hypothetical possibility of there being discontinuous paths existing over which the integration may have to be done. And therefore, the very process of integration may be questioned, we assume that only those paths for which this kind of a condition hold which are of this which satisfy this kind of a scaling, this which satisfy this kind of a scaling which is given here delta x square upon delta t tends is of equal order, they are adopted. And this would imply that the not withstanding the fact that the paths would not be differentiable, nonetheless the paths would still be paths would still be continuous.

(Refer Slide Time: 28:52)



You see that is that is what I said you see, howsoever small delta t gets the jump could still be there that the path needs to be queued de queued, still have to be or delta q would still have to be q i plus 1 minus q i. So, even if delta t becomes very very small and this this is finite, then you are you are simply tending toward discontinuity, so that is the important point.

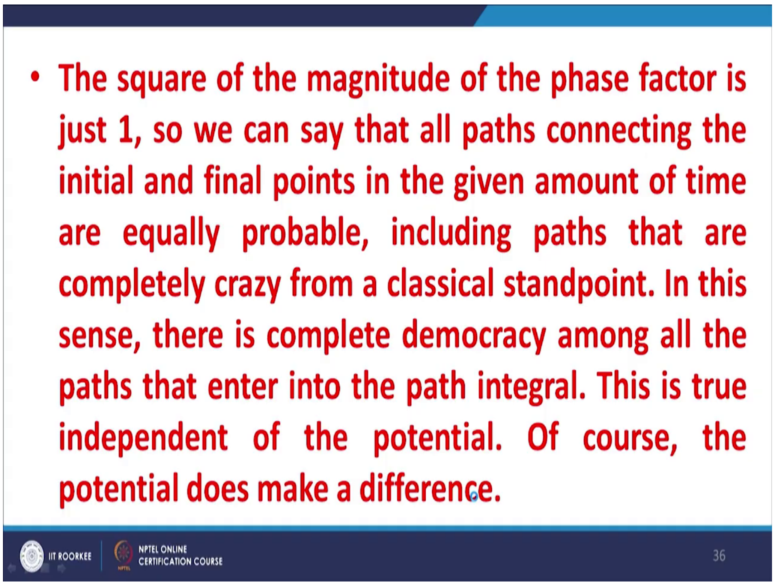
(Refer Slide Time: 29:26)



So, and this is the condition that I just explained that we had in the case of Brownian motion. In fact, as I explained in fact in the context of the diffusion equation, when I worked out the solution of the diffusion equation; we found that it turns out to be a Brownian motion.

Therefore, there is nexus there is a underlying structure behind common structure behind the concept of Brownian motion and the concept of path integral, and this forms one of the constituents of that that in the limit that delta t tends to 0 delta x square upon delta t are of equal of the order of unity.

(Refer Slide Time: 30:10)



So, this this is so this is just to reiterate the square of the magnitude of the phase factor e to the power i is just 1 right. So that means that all paths between the initial and final points are given equal amount of weightage, equal amount of importance, equal amount of probability.

However, the interaction that that is created, for example if you have any potential involved in the Hamiltonian of the system is due to the distortion of phase. Phases that thus creates phases that creates the interference constructive at some points, destructive at some points; constructive around the classical domain and destructive otherwise and that is there and that is how the path integral reproduces the quantum mechanical behavior, right.

Thank you. We will continue from here.