Path Integral Methods in Physics & Finance
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Lecture – 20
Quantum Mechanical Path Integral

Well, so let us start let us continue; from where we left off in fact. Now, one of the most important properties of propagators is the composition property.

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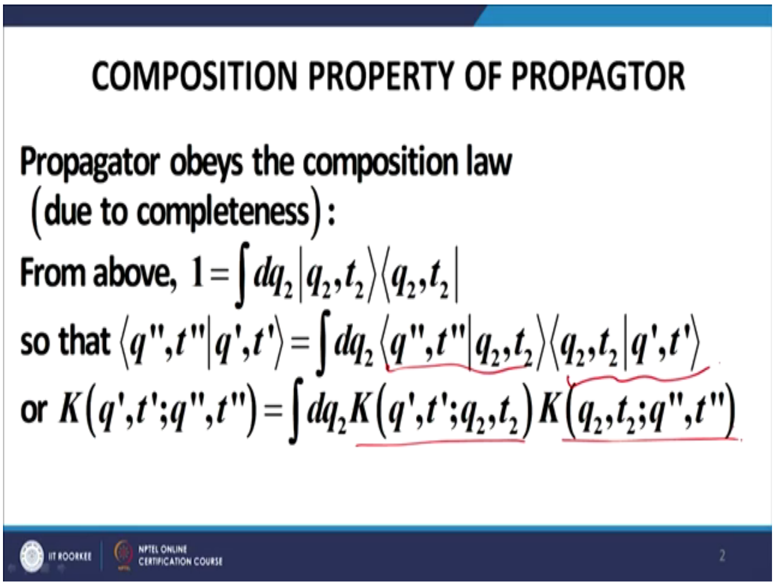

Is something related to what we had in the Chapman Kolmogorov equation let us look at it in the context of quantum mechanics. Now, we know that the basis that moving basis that we have constituted a forms a complete orthonormal basis. So, this first equation represents the resolution of identity in this basis.

Now, if we write the transition amplitude in this form as the scalar product. And we introduce incorporate the here in the resolution of unity on the right hand side of the second equation. And then we find that if we recombine terms; if we recombine terms we have simplifying that this term q gives us this and this term gives us this term.

So, what have we done? We have been able to form to derive a composition law or a composition property of the propagator. The propagator from this state q dash at time t dash to the state q double dash at time t double dash can be represented as the product of the propagators q dash at time t dash to q 2 at time t 2 and then from q 2 to time at from q 2 at time t 2 to q double dash at time t double dash integrated over all possible values of q 2.

So, we are basically we what we have done is we picked up a time slice between q dash as between t dash I am sorry between t dash and t double dash we picked up a particular arbitrary time point let us call it; t equal to t 2. Then at that point we worked out all the two legged paths starting from q dash t dash up to this slice at t equal to t 2. And they are after from the slice at t equal to t 2 to the point q dash q double dash at time t double dash.

Now, it is very similar to what we have in the Chapman Kolmogorov equation. So, this is for two legged paths please note this.

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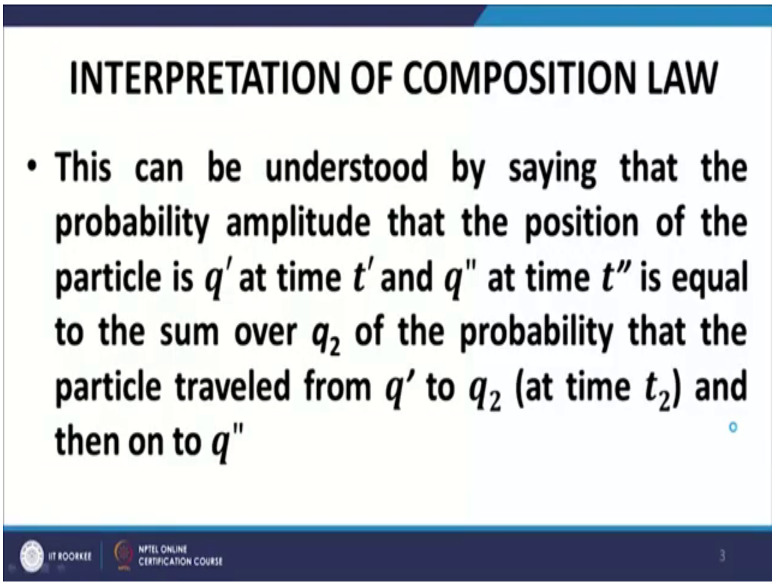

But the question is why do we need to restrict ourselves to only two legged paths? We do not need to restrict ourselves to two legged path.

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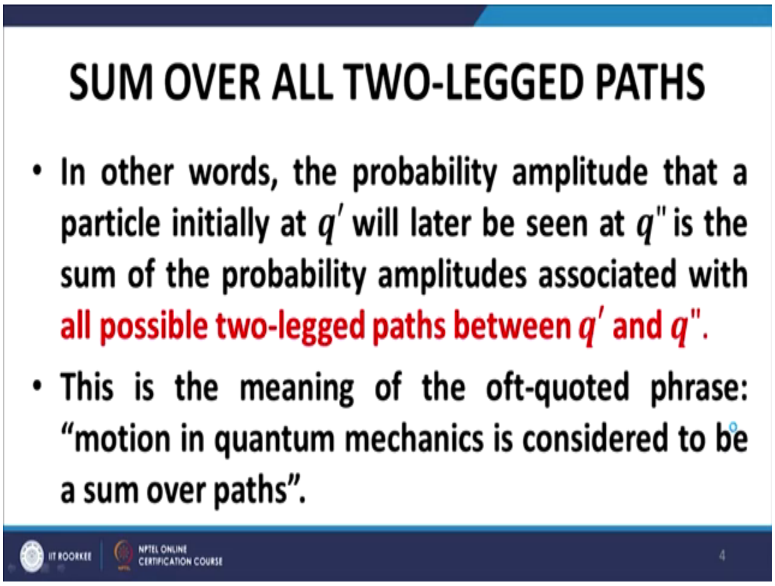

We can have; three legged paths, four legged paths, five legged paths, by picking up arbitrary time points at various points along the time interval t dash to t double dash.

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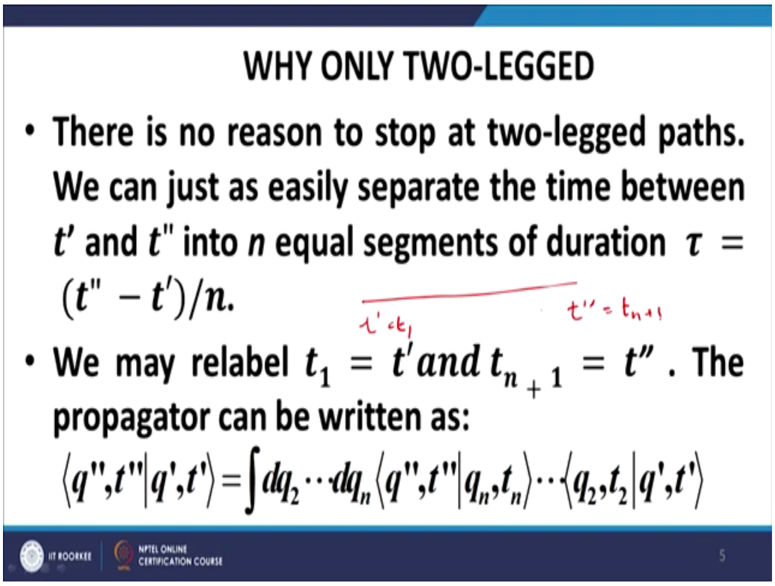

Indeed we can time slice the entire time interval between t dash and t double dash into infinitesimally small time intervals of length tau and we can call it as t double dash minus t dash upon n. In other words let me explain what I am trying to say. What I am trying to say is simply this is my point t dash, this is my point t double dash.

What I do is I slice this interval of time I partition this interval of time into extremely small time intervals non overlapping intervals each of length tau and given by each of length tau where tau is; obviously, given by t double dash minus t double and t dash divided by the number of such intervals.

And; obviously, we can relabel if we want we can relabel this t dash equal to t 1 and t double dash equal to t n plus 1 for convenience of notation nothing more and we can write the propagator. Now, instead of considering only two legged paths we can consider n legged paths all possible n legged paths; a path starting from t 1 going to t 2 from t 2 to t 3 and then to and then at a q value at t 3 q value at t 4 and so on.

It all possible paths of will be having n legs with each leg terminating at one of the time slices given in this particular fragmentation or partition. And therefore, we can write this transition amplitude in this form. And now please note why this integration? Because we are considering all possible paths on the time slice; from t let us say from t equal to t 1 which is the initial point t dash t dash is equal to t 1 which is the initial point.

So, from initial point to the first slice which is at tau which will be what? Which will be t 1 plus tau; so, that slice at t 1 plus tau all possible path from t 1 to t 1 plus tau that is t 2 and that will be represented by the first integral d q 1 and so on.

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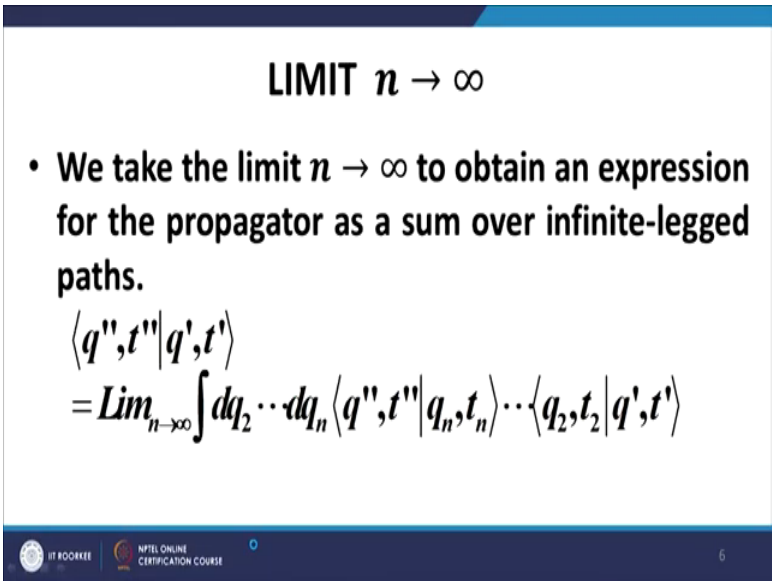

And then we can proceed towards the infinitesimal case or move towards continuum. And how can we enforce continuum? It is quite simple put take n tending to infinity take the number of partitions larger and larger and larger and as you approach n tending to infinity.

Obviously, the partitioning of the timeline becomes a continuous line or the time line becomes continuous. And we can write we can therefore, write the transition amplitude in this form and with the limit n tending to infinity the number of partitions tending to infinity ok.

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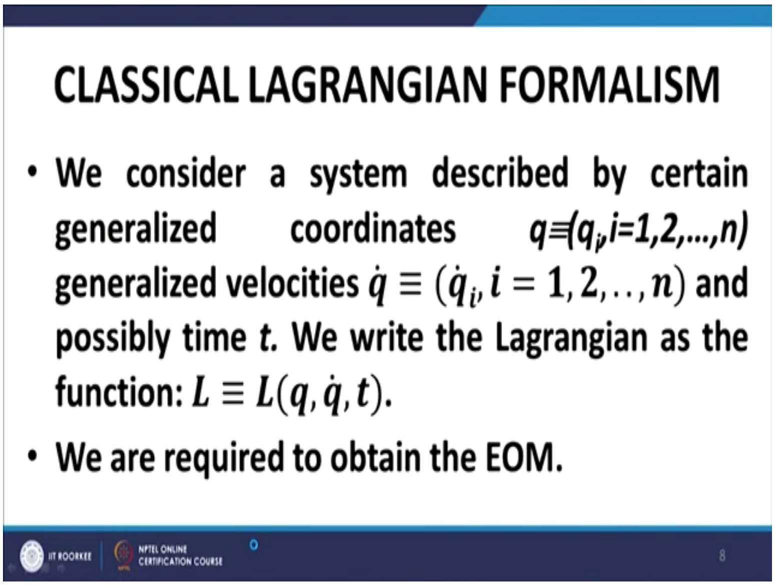

So, a quick recap on the classical Lagrangian formalism, what we have here is; we have a system in certain generalized coordinates we have called them ah we study the system in velocity phase space for the purpose of which we use generalized coordinates and generalized velocities q and q dot respectively.

We write the Lagrangian as the function of q q dot and t if there is explicit dependence on time. For the moment we will ignore the explicit dependence on time we will consider systems having no explicit dependence on time for simplicity. That means the a Lagrangian depends on q and q dot.

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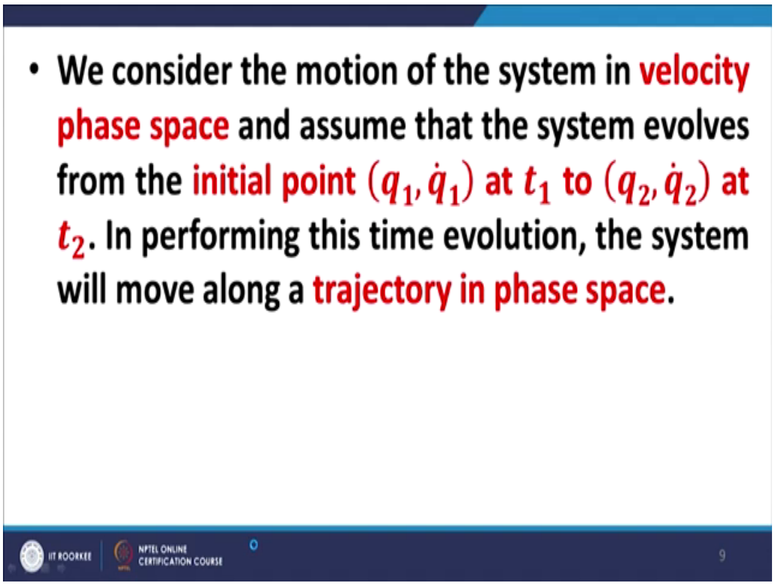

So, we consider the motion of the particle in velocity phase space. It moves from an initial point q 1 q dot 1 that is q 1 is the position q dot 1 is the velocity to another point q 2 q dot 2 in the phase space at time t 2. And this is how it is evolving in time and the this evolution is captured by a trajectory in phase space. Our issue is to determine the equation of the trajectory.

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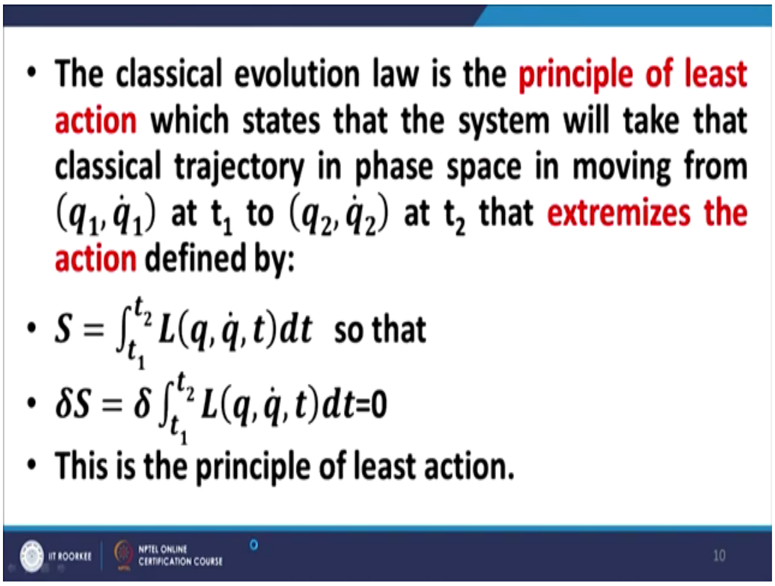

That is followed by the particle is the law of least action for the purpose. How we define the action? We define the action as the integral of the Lagrangian d t and so in order to work out the principle in order to work out the least action we take the variation in the action and we equate it to 0.

So, that is our defining law the particle or the object will follow that path in classical phase space in which the variation in the action is 0 the action is an extremal.

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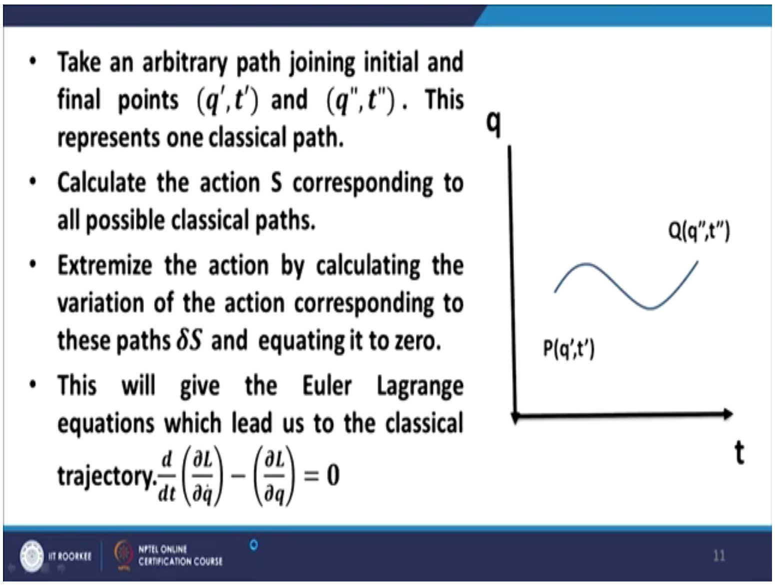

And ah when we follow this process of optimizing the action or extremizing the action we end up the you end up with the equation which is given and right in the bottom of the slide which is called the Euler Lagrangian equation. This forms a fundamental building block of lot of quantum mechanics as well. And we need to keep this at the back of our mind throughout.

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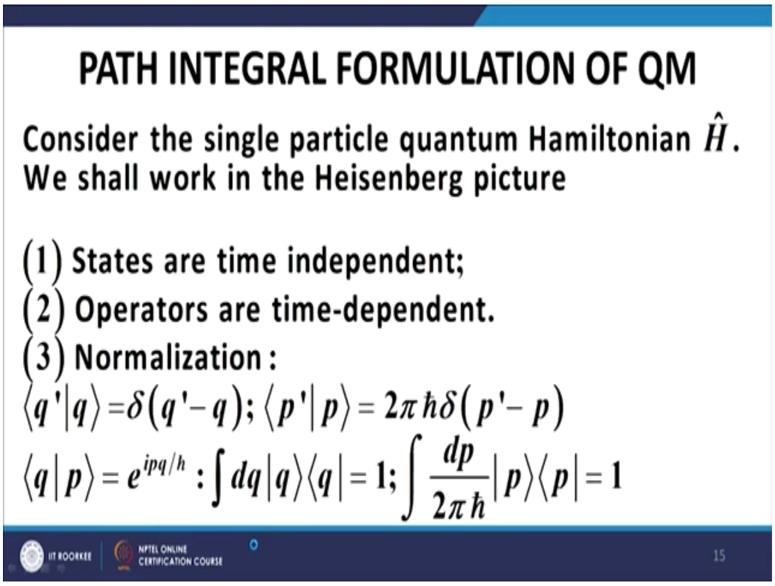

And then we move to the construction of the constitution of the path integral of a single particle physical system. We assume we work in the Heisenberg ah picture. That means, states will be time independent, operators will carry the time dependents, operators will carry the time dependents, operators will evolve will time, the states will remain time independent, the normalization that we follow is given on the slide.

This is this in case there is any deviation from this I will mention it explicitly for the moment we shall be following in this normalization.

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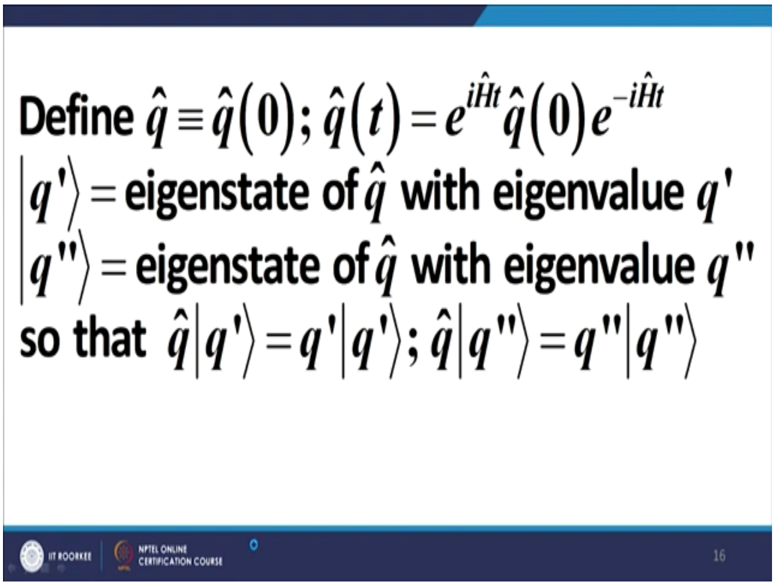

So, these we define q as the 0 time operator q 0; for abbreviation q t is the evolved operator which evolves according to this equation the Heisenberg equation. q dash is an eigenstate of q with eigenvalue q dash and q double dash the eigenstate of q with eigenvalue q double dash.

And of course, in the moving basis we can also use the moving basis. These will be the eigenvalues that the eigenvectors will change accordingly, the basis will change accordingly. The eigenvalues will remain the same when operated upon the transformed or the time evolved operators.

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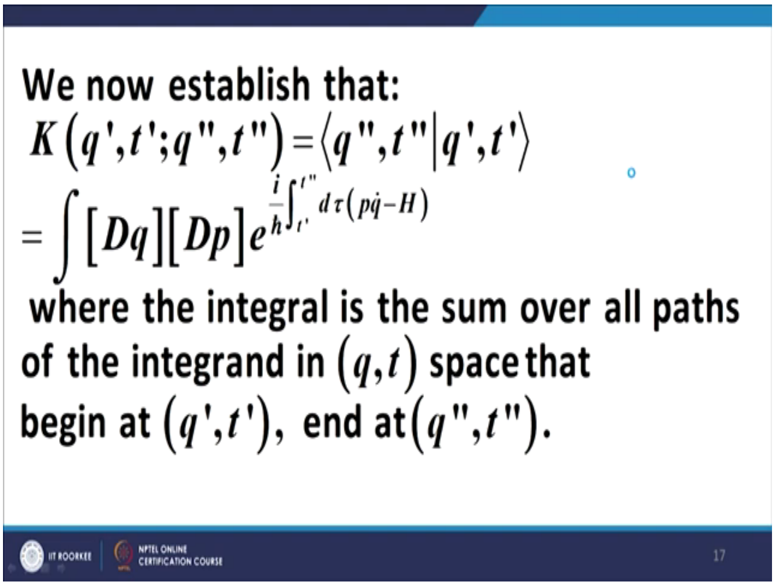

We need to establish this expression the transition amplitude which is equal to the propagator is given by the path integral over q, path integral over p, of the expression e to the power this expression integral d tau p q dot minus the Hamiltonian.

Recall these are path integral this D q and D p with within square brackets with the capital D represent path integral. That is the represent summation over all possible paths in the integrand in q t space; qt is the space.

And, we assume that the paths begin at the point in q t space at point q dash t dash and the terminated at the point q double dash t double dash.

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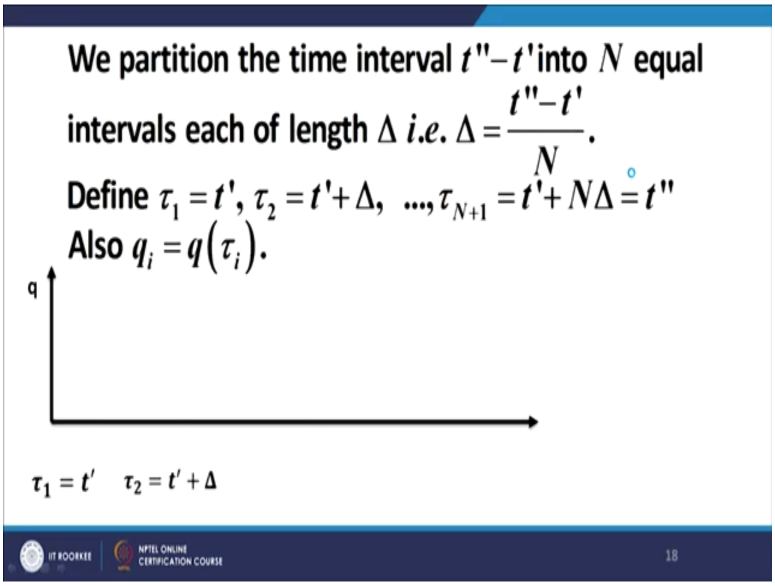

So, as and discussed in a slightly earlier slide the partition the time interval t double dash minus t dash that is the time of evolution of the system into N equal intervals each of length delta. So, we write delta equal to t double dash minus t dash.

And we identify the points of partition the coordinates of the point of partition as is given in this figure tau 1 is equal to t dash the starting point. Then we have tau 2 is equal to t dash plus delta, tau 3 is equal to t dash plus 2 delta and so on.

And tau n plus 1 is equal to t dash plus N delta that is equal to t double dash that will be the point at which the system is being observed or the probabilities is to be determined.

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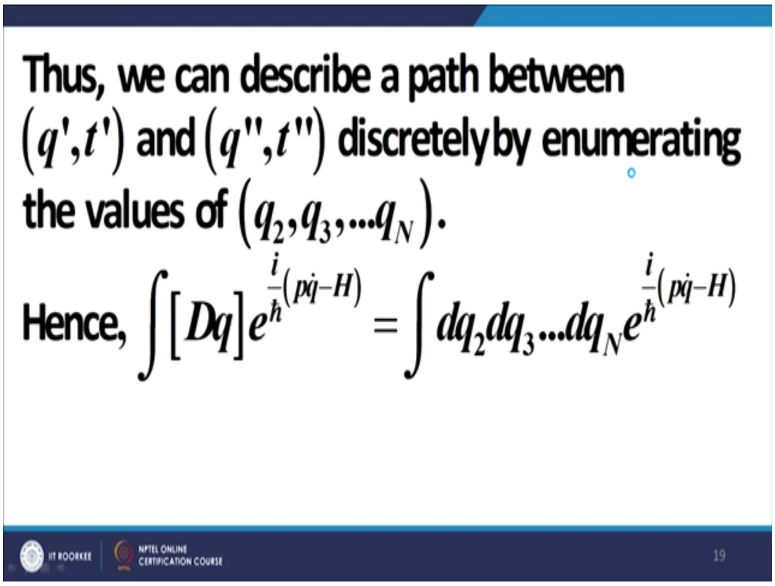

Now, we consider a path starting from q dash t dash and culminating at t q double dash t double dash. Now, as you can see from this diagram the path can be represented by a line or a curve in the space. Now, if we take if this the delta values is small enough infinitesimally small then the paths the discretization of the paths can be assumed to be an accurate representation of the continuous path.

And as a result of which we can represent the paths or we can describe the path in terms of the values of q that the q; variable takes at various coordinates of tau of the value of q at tau 1,the value of q at tau 2 the value of q at tau 3 and so on.

That is reasonable or a good approximation of the complete description of the path if the discretization is small enough at the infinitesimal level. And in that situation this path integral translates to integration over the discrete variables q 2, q 3 or this path integral can now be segregated into a product of variables q 2, q 3, q and q n.

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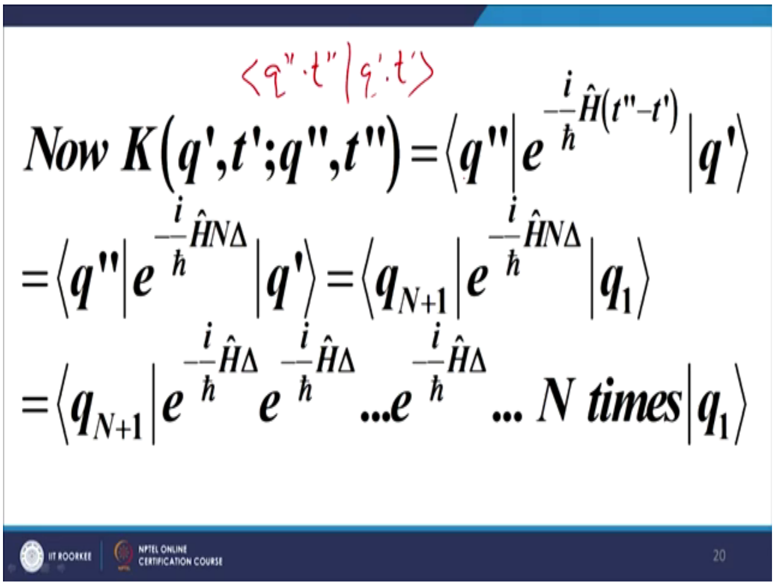

Now, the propagator or the transition amplitude is what? It is; let us write it here it is q double dash t double dash q dash t dash. And this expression can in can be written in this form is quite straightforward because q dash t dash is nothing, but the evolved version of the state vector q dash which is the state vector at t equal to 0.

And similarly q double dash state vector is nothing, but the evolved version of q dash evolved up to time t and double dash. And the Hamiltonian or the exponential of the Hamiltonian in this form minus i upon h represents the time evolution of the quantum state. Now, t double dash minus t dash; this quantity this factor recall that this upon N is equal to delta.

So, t double dash minus t dash is nothing but N delta and that is precisely what we have written here. Instead of t double dash minus t dash we have written at N delta. And now, we have simply substituted the values of q double dash as q N plus 1 q dash as q 1 just to simplify notation nothing else.

And now we write this N delta as products as n products of delta. In other words minus i h i h bar i upon h bar i am sorry minus i upon h bar H N delta has been written as n products of minus i upon h H delta; that is the rest is as earlier.

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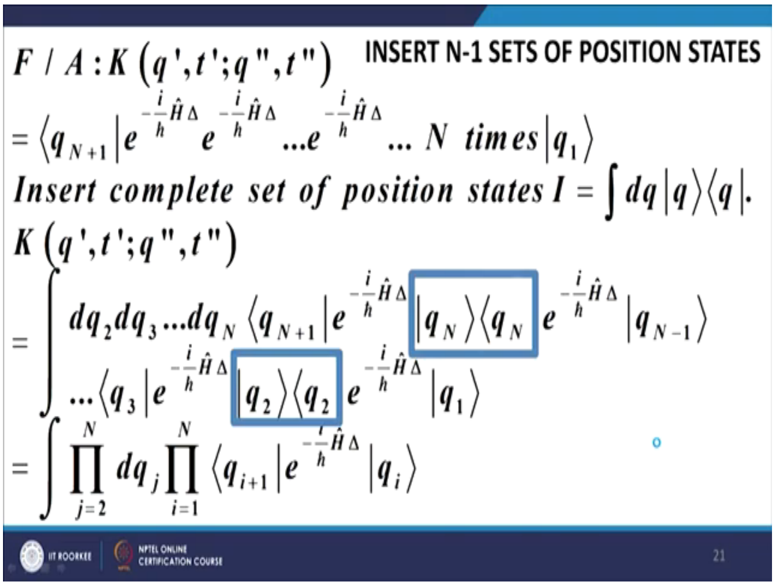

Now, we come to the very interesting part. We introduce complete set of states in the position representation. We introduce N minus 1 complete set of states and we introduce each of them after each factor of e. We let us say we start with the with the right hand side and the right hand side is q 1; q 1 and then we have a factor of q 1 and then we have as you can look here.

We can q 1 and then I have a factor of H delta. And then I introduce as complete set of states q 2 q with q 2 and I have integration q 2 so, to ensure completeness that the set is complete then and this way we proceed further.

And for the last q N plus 1; what do I have? I introduce a state q N q N and I integrate over q N. So, all this N minus ones introductions at complete set of states and these are represented by these integrations over q 2 q 3 q and q n and with this is set of states here. Now, and this is an average notation of what we have in the above expression right.

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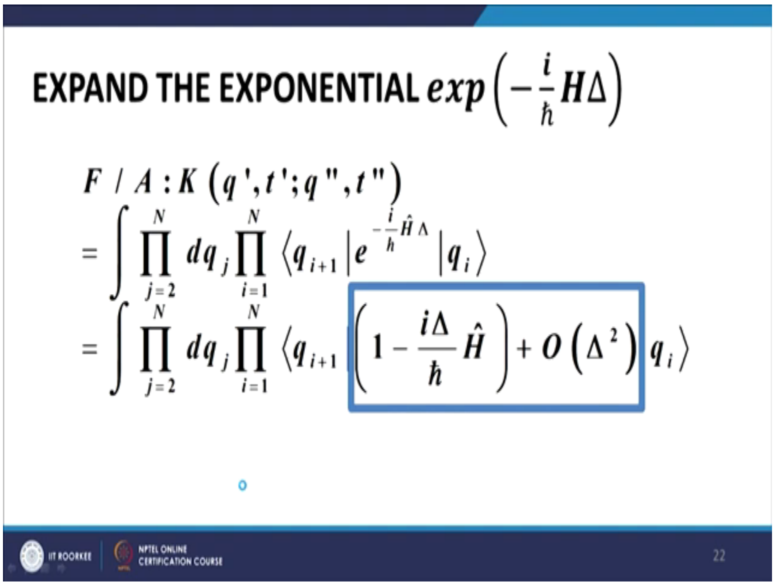

Now, we expand the exponential minus i upon h H delta; what do I get? When I expand this expression, I get 1 minus i delta upon h bar into H plus terms of the order of delta square and higher orders. This 1 has no problem the problem is with this Hamiltonian operator here it is sandwiched between the states q i plus 1 and q i. So, we need to study this particular part this is the path that we carrying forward from here; q i plus 1 H q i; q i plus 1 H q i.

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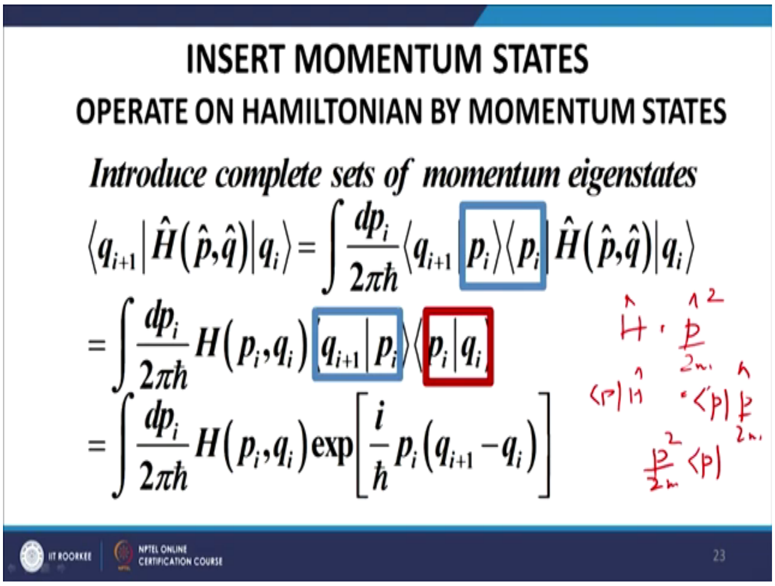

What we do here is; we introduce a complete set of momentum states and to simplify this expression we introduce a complete set of momentum states in the manner as you can see ;as you can see in this blue box.

Now, when this momentum state momentum bra acts on the on the Hamiltonian h it gives us the eigenvalues of the momentum operators contained in the Hamiltonian. For example, if the Hamiltonian consists of p square up let us say Hamiltonian consists of p square upon 2 n. Then the momentum state acting on this will simply pull out this moment this the momentum state where the momentum k bra vector when it is act on this.

This way what do this do? This will simply pull out the eigenvalue p square upon 2 m and we will have this expression. But the important differences here we have p in the form of an operator the p occurring in the Hamiltonian is in the form of an operator; however, the p that is occurring after the action of the momentum state on the Hamiltonian is the number is the c number.

Therefore, now this Hamiltonian entire Hamiltonian has been converted to a function of c numbers. It is not a function of operators anymore by the action of the momentum states p. So, that is an important difference and therefore, it this enables us because it is no longer an operator it is a number it is a function of number. Therefore, it can be shifted away from this now it can come here as a pre factor and what we are left with is q i plus 1 p i and p i q i.

Now, these are in scalar products which are well known and which we have seen earlier in the slide also and this is given by this expression. So, this particular quantity by the introduction of this complete set of states gets simplified to this expression together with of course, this integration which is required to in because the states need to be a complete set of states resolution of unity.

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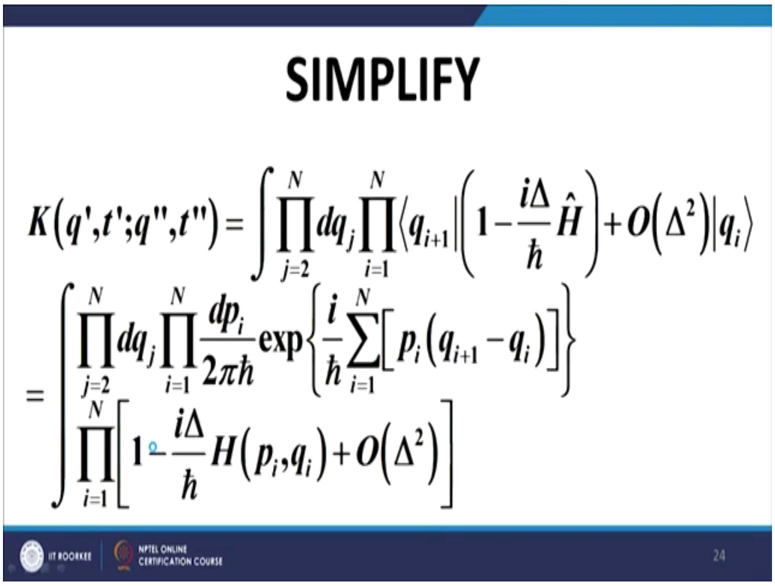

Now, we simplify this making use of what we have done earlier. Now, this h poses no problem because it is no longer an operator it is a function simple function it can commute with anything. And the fact that it can be it can commute with anything enables us to write the expression in this form; now, another important observation.

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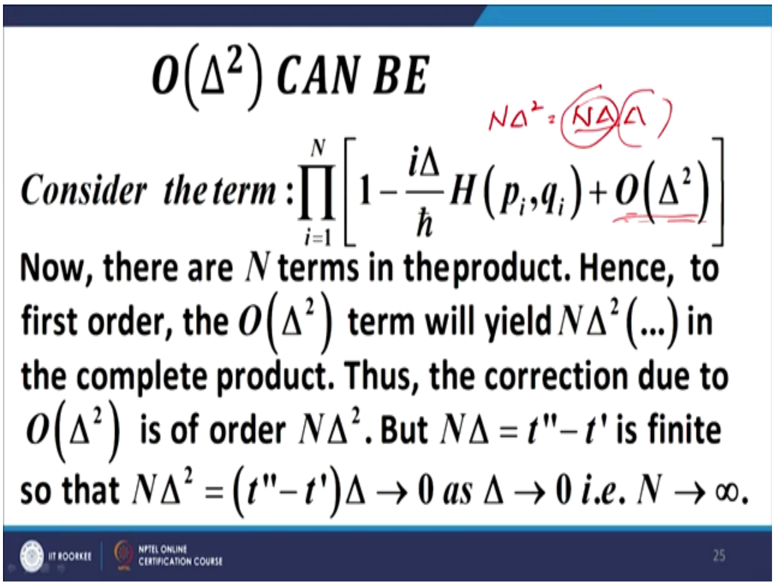

If you look at this expression and let us go back. Now, you look at this expression how many products are there product ranges over N factors. There are N factors of this quantity there are N factors of this quantity. So, because there are N factors of this quantity to first order this expression will be of the order of N delta square to first order other terms will be of higher orders.

The lowest order that we can have in delta square or when this term becomes relevant is N delta square; let us look at this. N delta square can be written as N delta into delta N delta square is equal to N delta into delta. N delta is nothing, but t double dash minus t dash recall that we partition the time slice t double dash minus t dash upon N is equal to delta. So, t double dash minus t dash is equal to N delta.

In other words this part is finite and this part is tending to 0 that is our requirement as N tending to infinity. So, what happens to this expression? As N tends to infinity this expression approaches 0 and so this expression we need not to worry about. What we are concerned with is up to this expression.

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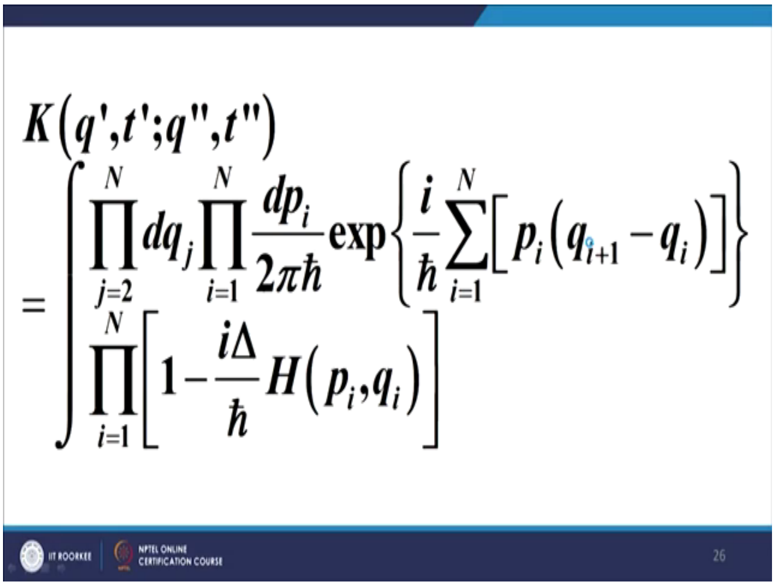

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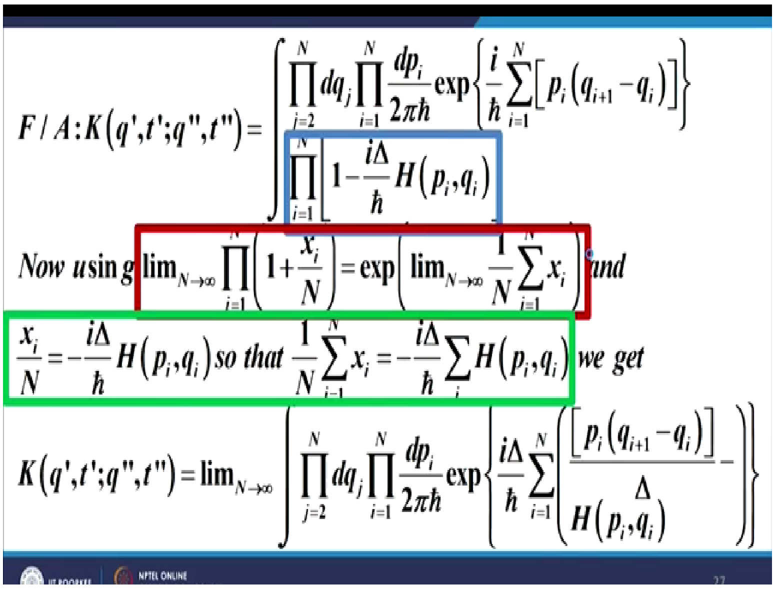

So, now we write it in this form right. So, the order delta squared term has gone and we are left with this term and we need to simplify this. The next step in the simplification comes from the use of this formula which is there in the red box. This is a modification of a very standard formula that we have in mathematics or finance as well.

In fact, it can be easily proved by binomial ah expansion, but the basic thing is that this is an expansion and this is an extension of that. So, by this formula if we apply this formula and we make use of x i upon N is equal to minus i delta upon h bar into H. Remember H is no longer an operator H is no longer an operator.

So, minus i h i delta upon h bar into H this is my x i upon N. If I use this on the left hand side I get this expression this expression this is my left hand side sorry and this is my left hand side here, this is my left hand side on making this substitution. And on the right hand side I get limit

N tending to infinity 1 upon N summation x i. Summation x i in my case will become summation minus i upon H delta all these are common and what I get is summation H.

So, when I make the substitution here, when I make the substitution with this exponential so this expression now gets added to the exponential we already have these terms in the exponential. So, I have this in the exponential already. This whole term gives me another exponential term which is this expression in the exponential which is here after taking i delta upon h bar common I get this expression in the exponential.

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$$K(q',t';q'',t'') = \lim_{N\to\infty} \int \prod_{j=2}^{N} dq_j \prod_{i=1}^{N} \frac{dp_i}{2\pi\hbar} \exp\left\{ \frac{i\Delta}{\hbar} \sum_{i=1}^{N} \left(\frac{\left[p_i\left(q_{i+1}-q_i\right)\right]}{\Delta} - H\left(p_i, q_i\right) \right) \right\}$$

Setting as $N \to \infty,\ q \equiv q(t);\ p \equiv p(t);\ q_i = q(t_i);\ p_i = p(t_i);$

$$\int \prod_{j=2}^{N} dq_j = \int [Dq];\ \int \prod_{i=1}^{N} \frac{dp_i}{2\pi\hbar} = \int [Dp];\ \frac{\left(q_{i+1}-q_i\right)}{\Delta} = \dot{q}(t_i);$$

$\Delta \sum_{i=1}^{N} f(t_i) = \int_{t'}^{t''} d\tau f(\tau)$, we get $K(q',t';q'',t'')$

$$K(q',t';q'',t'') = \int [Dq][Dp] \exp\left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[p\dot{q} - H(p,q)\right] d\tau \right\}$$

Now, we take the limits. We take a huge simplification of notation subsequent to taking the limits N tending to infinity. We as N tending to infinity this expression goes to q dot. This expression becomes it becomes a function of t, p becomes a function of t, q dot becomes a function of q i becomes a function of q t i, p i becomes a function of p t i, p p p dot pi becomes a function of t i, q i dot becomes a function of t i.

And these simplifications the integration volume and D 1, D 2, D 3, D 4 has been represented by this path integral in the continuous case. Similarly, the path integral over a momentum and this expression becomes q dot i and q dot i is nothing, but q at q dot at t i. So, after making all these substitutions replacing the summation by the integral here what we end up with is; this expression.

This is our transition amplitude; transition amplitude or the propagator is given by the path integral over q, path integral over p exponential i upon h integral p q dot minus H and so integrated with respect to tau or tau as a proxy for time right.

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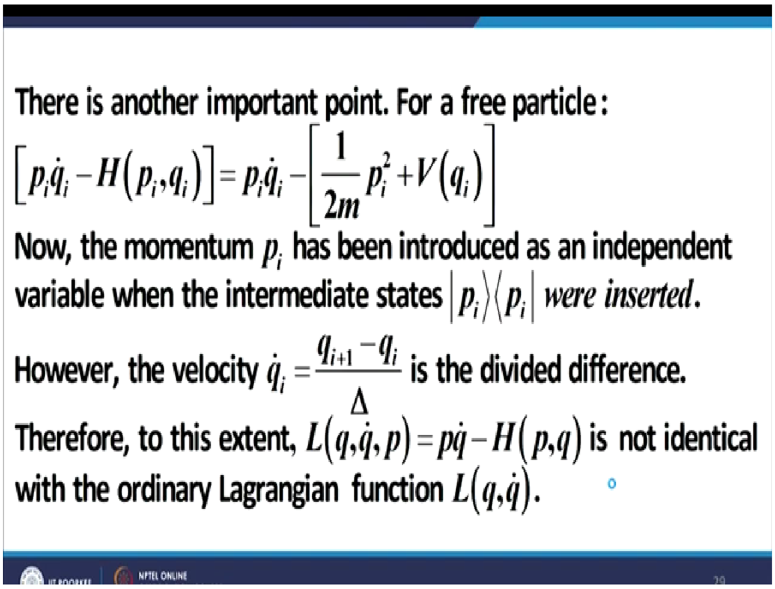

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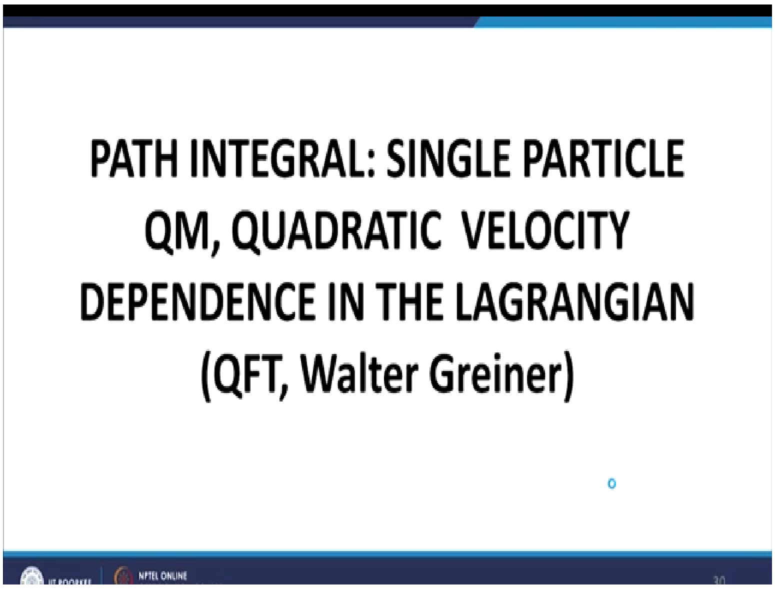

Now, what happens? There is a simplification here. The simplification can be done in special cases where the dependence in the Lagrangian is quadratic in the velocity.

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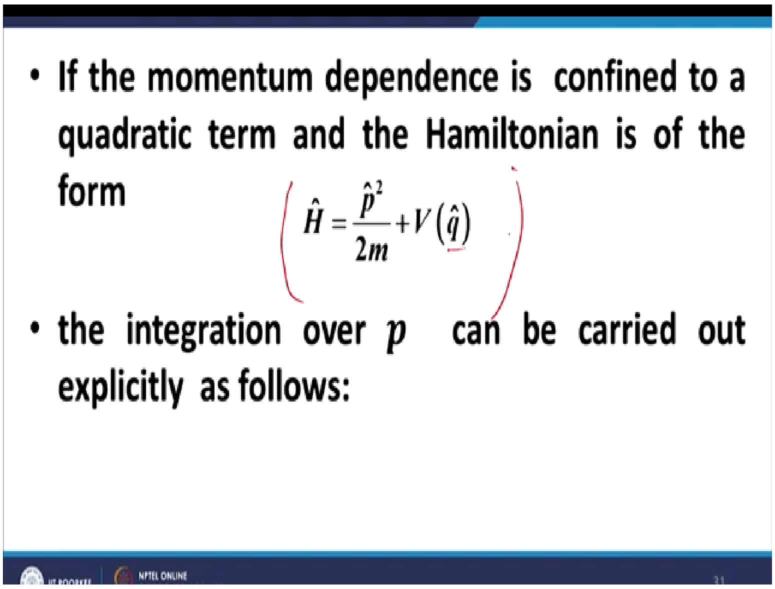

Dependence in the Lagrangian is quadratic in the velocity. In other words we have a have Hamiltonian which is of this form H is equal to p square upon 2 m is the momentum operator. So, H is equal to p square upon 2 m plus V q; V is the function of q the generalized the position of operators.

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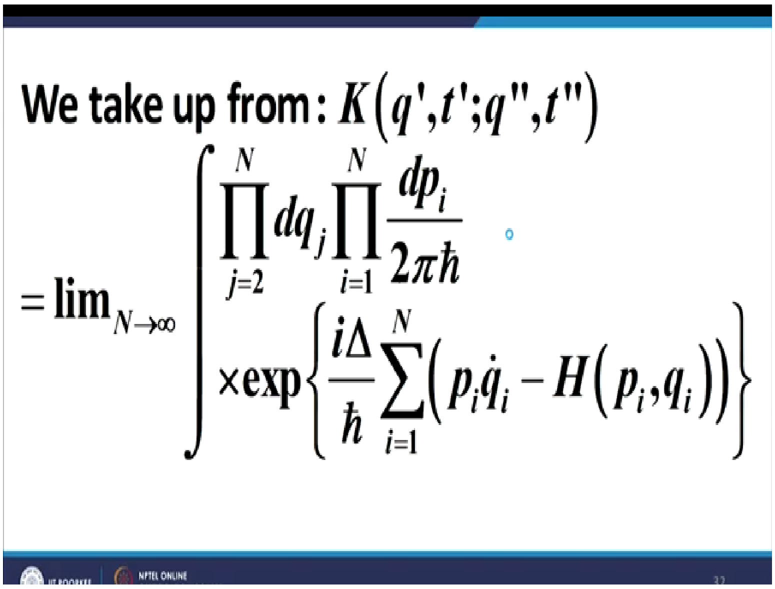

So, this is where we were last time when we concluded the last section. We revised it a bit we go back a bit and we write this in a discrete form.

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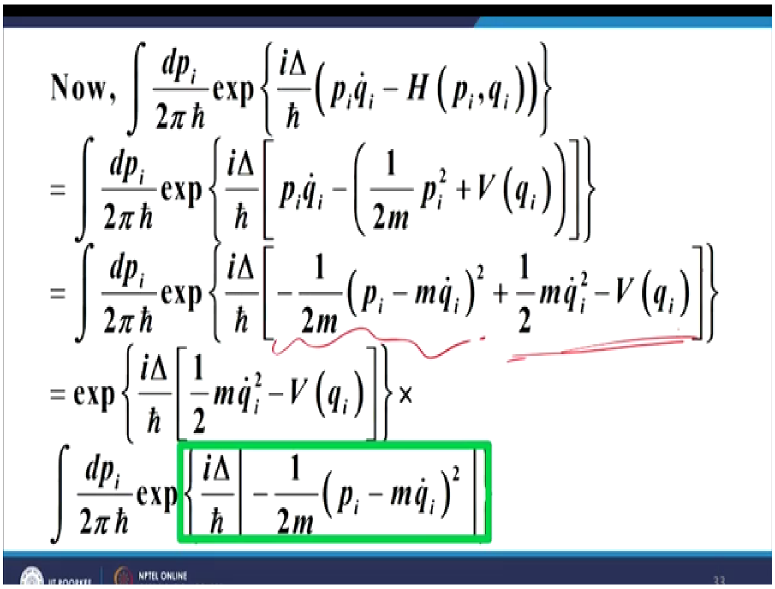

When we write this in a discrete form; what we what we get is p x naught into this is as it is. We substitute the value of the Hamiltonian we write it as 1 upon 2 m into p i square plus V q i simply substitute the value of the Hamiltonian.

And now what we do is; we do this expression in fact, represents a Gaussian integral. We can do a Gaussian integral here and by computing the square as it is the standard practice. And we can write this whole expression in the square bracket whole expression in the square bracket in this form.

And this expression in this form when a when a Gaussian integral is done. And now the important thing before we do the Gaussian integral if you look at this part of the expression.

This part is independent of our integration recall; that integration is with respect to p i and this expression there is no p i here.

So, we can take this outside the integration and within the integration we are left with this portion the first part; minus 1 upon 2 m p i minus m q i dot whole square this is what we have with inside the integration. And because of the p i presence here and the other part 1 upon 2 m q i dot square plus V q i this expression we take it outside the integration.

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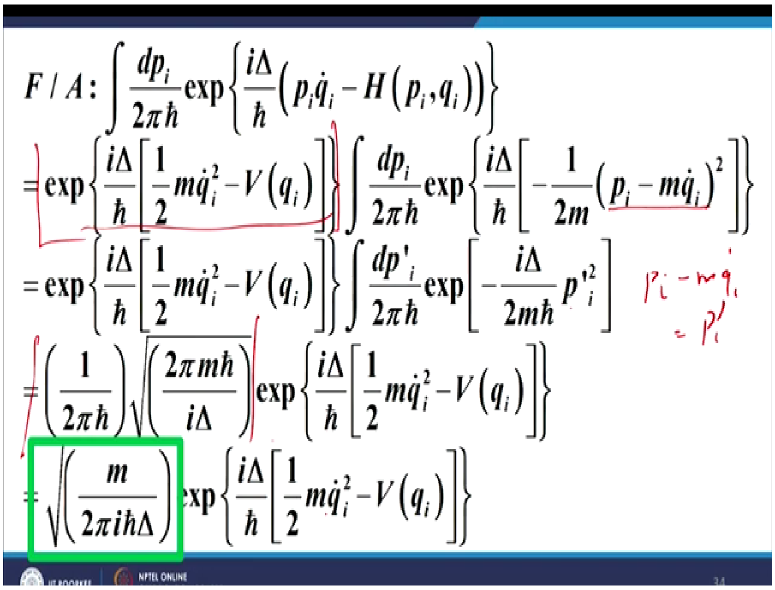

Now, let us look at this expression. Now, this is the part that is yet to be integrated. The this is; this factor this is this factor the factor in the green box, the factor in the other factor for the moment we are keeping it separate the factor in the green box we are carrying forward.

If you look at this factor in the green box now sorry this is the factor; this is the factor that is outside the integration which we brought forward from the earlier slide. And this is the factor that is yet to be integrated.

Now, by writing or by substituting p i dash as p i minus m q dot i we simplify this integral notation rather or substitution. And we now have an integral over d over p dash and this expression is after the substitution it becomes this one. We are simply substituted p i dash p i sorry p i minus m q i dot is equal to is equal to p i dash this is the substitution that we have done.

When we do this substitution we get this integral; obviously, d p i is equal to d p i dash. So, the integration volume does not change the Jacobean is one and therefore, we can write it in this form. Now, this is clearly a simple straightforward Gaussian integral.

And when we do the Gaussian integral what we end up this in this expression here. And we are left with this as it is integral outside the integral which was already there always there. And the simplifying of this factor gives us the factor in the green box at the other factor is carried forward from where we left off.

And this gives us the; so this gives us what? This gives us the value of the propagator in the case in the case when the Lagrangian has a quadratic velocity dependence. Or the Hamiltonian can be written as p p i square upon 2 m plus q i.

(Refer Slide Time: 31:45)

$$F/A: K(q',t';q'',t'') = \lim_{N\to\infty} \int \prod_{j=2}^{N} dq_j \prod_{i=1}^{N} \frac{dp_i}{2\pi\hbar} \times \exp\left\{\frac{i\Delta}{\hbar}\sum_{i=1}^{N}\left(p_i\dot{q}_i - H(p_i,q_i)\right)\right\}$$

$$= \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\Delta}\right)^{N/2}\int\prod_{j=2}^{N} dq_j \exp\left\{\frac{i\Delta}{\hbar}\sum_{i=1}^{N}\left[\frac{1}{2}m\dot{q}_i^2 - V(q_i)\right]\right\}$$

$$= \mathcal{N}\int[Dq]\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''} d\tau\left[\frac{1}{2}m\dot{q}_i^2 - V(q_i)\right]\right\}$$

$$= \mathcal{N}\int[Dq]\exp\left(\frac{i}{\hbar}\int_{t'}^{t''} L\,d\tau\right) = \mathcal{N}\int[Dq]\exp\left\{\frac{i}{\hbar}S(q,\dot{q})\right\}.$$

And we can easily identify this expression that we have a this expression that we have here with the action. We see it here; we see it here, when we substitute. When we substitute this part; this integral when we substitute this integral into the original expression for the kernel.

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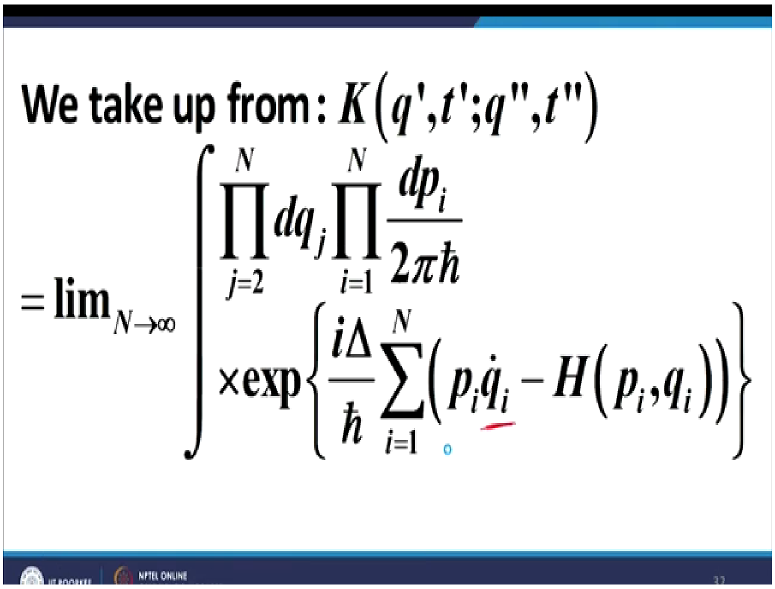

See what we have done now what we have done until now is the integration over the p i's. We have done the Gaussian integral over the p i's right, but the q i still remains q i part we have not done yet right. So, yes so the q i part we have not done yet, this part we have done we substitute this part the Gaussian integral after completing the Gaussian integral.

When we make the substitution when we make the substitution the q i part remains and this is what remains from the pi integral. This part comes from the previous slide, this slide, this slide this part comes from here and this is here we have limit n tending to infinity so this is what we get.

Now, on taking the control continuum limit I can write this integral over D2, D3 D3 as the integral D q over all paths. I replace this expression the pre factor by a normalization number or a normalization term which you write as N script.

And what we have here is exponential in and the summation is replaced by the integral; the summation is replaced by the integral. And what do I get here? I get here i upon h integral d tau; d tau comes with their due to the presence of this delta with the summation 1 upon 2 m q dot square minus V q this. And this expression which is within the integral is clearly the Lagrangian.

And therefore, we get this expression as N integral D q path exponential i upon h integral L d tau. Now, integral L d tau is nothing, but the action. So, we write it in the form of the action which is the equation in the green box.

Thank you.