Path Integral Methods in Physics & Finance
Course No 110107146

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Lecture 02: Introduction to the Path Integral

ORDINARY INTEGRATION

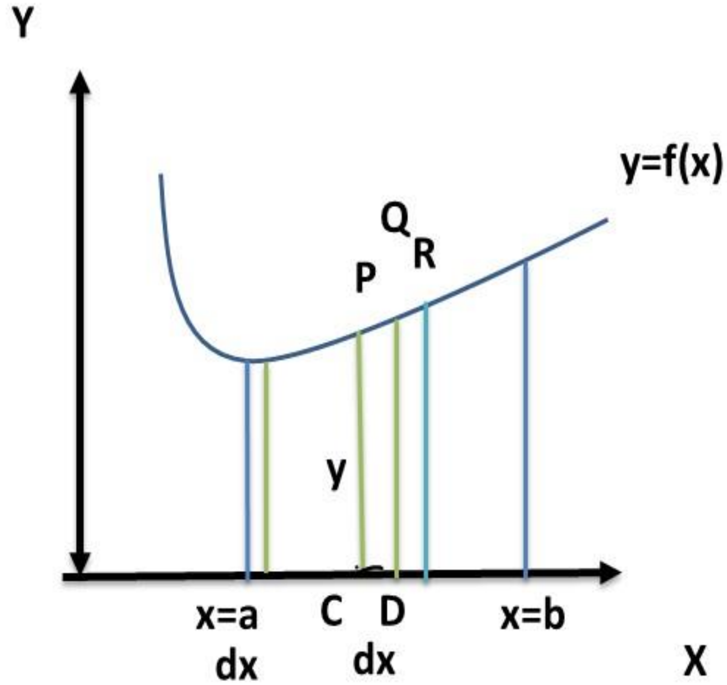
Let us start by considering the case of ordinary integration in $2-D$, $\mathbb{R}\times\mathbb{R}$ space.

Consider the function $f:\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ defined by $x\mapsto(x,y=f(x))$ or simply $y=f(x)$.

We define the integral: $A=\int_{x=a}^{x=b} f(x)dx$ as:

$$A=\int_{x=a}^{x=b} f(x)dx=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)_i\,dx_i$$

Thus, we partition the interval (a,b) into n disjoint intervals each of length dx. In the limit $n\to\infty$, these partitions become infinitesimally small and we make the assumption that at this infinitesimal level, the curve $y=f(x)$ is constructed by an assortment of infinitesimal straight lines. In other words, we assume that the variation in the value of $y=f(x)$ over each of these infinitesimal partitions is sufficiently small to be ignored. Equivalently, we assume that the value of $y=f(x)$ is constant over each of these infinitesimal partitions.



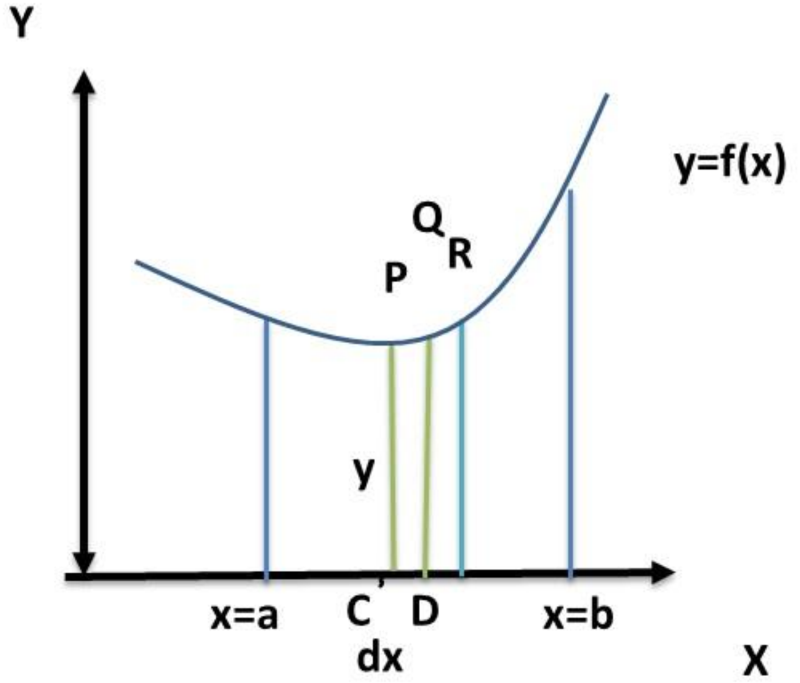
$$f(x_a)dx$$

$$f(x_a + dx)dx$$

In the figure, the points P, Q on the curve $y = f(x)$ are assumed sufficiently close to each other i.e. the thickness of the strip CD (dx) is assumed sufficiently small, that PQ approx. a horizontal straight line.

The value of y is assumed approx. constant along this region. Further, this constant value is assumed equal to the value at the initial point of the partition i.e. point C. Thus, $y(P)dx = f(x_C)dx$ is the area of the strip CPQD. This is integrated (summed over) the given range of $(x = a, x = b)$ to obtain the entire area.

Similarly, for obtaining arc length, we have:



$$L=\int_{x=a}^{x=b} ds=\int_{x=a}^{x=b}\sqrt{\left(dx^2+dy^2\right)}=\int_{x=a}^{x=b} dx\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\left[1+\left(\frac{dy_i}{dx_i}\right)^2\right]}dx_i$$

FUNCTIONAL (PATH) INTEGRATION

Function of function:

The Lagrangian $L(q(t),\dot{q}(t))$ of a particle system is typically a function of the particle's position q and its derivative $\dot{q}$ (velocity).

Both the position $q(t)$ and derivative $\dot{q}(t)$ are themselves functions of time of evolution.

Thus, $L(q(t),\dot{q}(t))$ is, in essence, a function of a function.

Functional:

Mathematical Definition:

Mathematically, a functional is a function of a vector space to a scalar field i.e. a functional maps a vector to a scalar.

Physical Interpretation & Illustration:

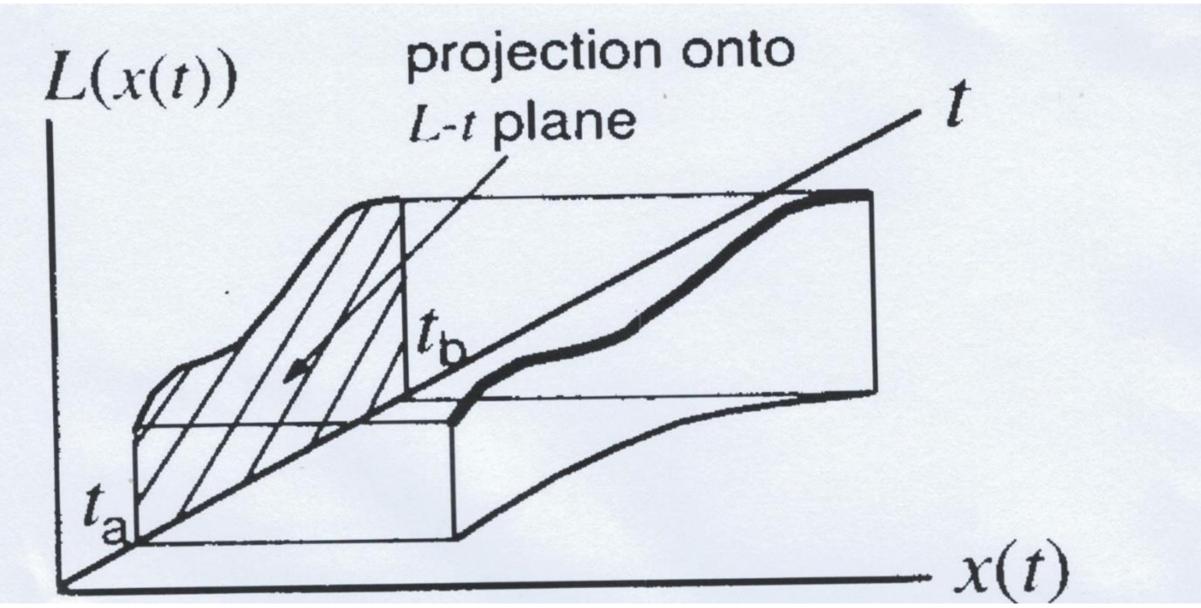
The integral of $L(q(t), \dot{q}(t))$ (function of a function) with respect to the independent variable t between fixed limits t_i, t_f is a functional F.

Thus, $F = \int_{t_i}^{t_f} dt L(q(t), \dot{q(t)})$.

The functional is a number that depends on:

(i) the function $L(q, \dot{q})$
(ii) the form of function $q(t)$
(iii) the integration limits t_i, t_f

It is different for different paths.



Since the spatial functions of time i.e. $q(t)$ are paths that form a vector space by themselves and the numbers obtained on integration $F = \int_{t_i}^{t_f} dt L(q(t), \dot{q(t)})$ form a scalar field, the physics oriented definition is a restricted case of the broader mathematical definition.

PATH INTEGRALS: GEOMETRY

Path integrals are integrals representing sum over all paths satisfying some boundary conditions.

They can be understood as extensions to an infinite number of integration variables of usual multi-dimensional integrals.

As stated above, a functional is a definite integral i.e. a number obtained by integrating between the end points of a certain path.

Yet, because we get a different such number for each different path in $q - t$ space, we can integrate those numbers over all possible paths.

They are called functional (path) integrals.

ILLUSTRATIONS OF PATH INTEGRALS

Thus, the functional which itself is an integral can be further integrated e.g. (2) & (5) below:

$$(1)\quad \sum_{n=1}^{N} F_n = \sum_{n=1}^{N}\left[\int_{t_i}^{t_f} L_n dt\right];\ (2)\quad \int_{q_i}^{q_f}\left[Dq(t)\right]F;\ \left[Dq(t)\right]\to all\ \ paths$$

$$(3)\quad \exp\left\{iF\left[q(t)\right]\right\};\ (4)\quad \sum_{n=1}^{N}\exp\left[iF_n\right]=\sum_{n=1}^{N}\exp\left[i\left[\int_{t_i}^{t_f} L_n dt\right]\right]$$

$$(5)\quad \int_{q_i}^{q_f}\exp\left[iF\right]\left[Dq(t)\right]=\int_{q_i}^{q_f}\left[Dq(t)\right]\exp\left[i\left[\int_{t_i}^{t_f} Ldt\right]\right]$$

APPLICATIONS: QM

The cardinal application of PI occurs in QM. In QM physical outputs of interest are averages of physical quantities over all possible paths weighted by the exponential of a term proportional to the ratio of the classical action S associated to each path, divided by the Planck's constant $\hbar$.

$$\langle q'',t''|q',t'\rangle = \int [Dq][Dp]\exp\left[\frac{i}{\hbar}S(q)\right]$$

$$\langle q'',t''|\hat{q}(t_1)|q',t'\rangle = \int [Dq][Dp]q(t_1)\exp\left[\frac{i}{\hbar}S(q)\right]$$

$$\langle 0|\varphi(z_1)...\varphi(z_N)|0\rangle = \frac{\int [D\varphi]\varphi(z_1)...\varphi(z_N)\exp\left[\frac{i}{\hbar}S[\varphi]\right]}{\int [D\varphi]\exp\left[\frac{i}{\hbar}S[\varphi]\right]}$$

In the semi-classical limit $S/\hbar \rightarrow \infty$, the leading contributions in the average come from paths close to classical paths, which are stationary points of the action.

The paths far away from the classical ones fluctuate widely and thereby the corresponding amplitudes tend to cancel out amongst themselves.

Appendix

FUNCTIONAL DERIVATIVES

Functional differentiation refers to the differentiation of a functional with respect to its argument.

Let $F[\phi]$ be a functional, i.e., a mapping from a normed linear space of functions (a Banach space) $M = \{\phi(x): x \in \mathbb{R}\}$ to the field of real or complex numbers, i.e. $F: M \to \mathbb{R}\ or\ C$.

The object $\frac{\delta F[\phi]}{\delta \phi(x)}$ tells us how the value of the functional $F[\phi]$ changes if the function $\phi(x)$ is changed at the point x.

Thus, the functional derivative (also known as the Frechet derivative) itself is an ordinary function depending on x.

We define: $\delta F[\phi] = \int dx \frac{\delta F[\phi]}{\delta\ \phi(x)} \delta\ \phi(x)$

which implies that the total change in $F[\phi]$ upon variation of the function $\phi(x)$ is a linear superposition of the local changes summed over the whole range of x values.
As in ordinary differentiation, the functional derivative can also be represented as the limit of divided differences.

To see this we construct a variation of the "independent variable", i.e., the function $\phi(x)$ which is localized at the point y having strength ε:

$$\delta\ \phi(x) = \varepsilon\ \delta(x-y)$$

We have:

$$\delta F[\phi] = F[\phi + \delta\ \phi(x)] - F[\phi] = F[\phi + \varepsilon\ \delta(x-y)] - F[\phi] = \int dx \frac{\delta F[\phi]}{\delta\ \phi(x)} \delta\ \phi(x)$$
$$= \int dx \frac{\delta F[\phi]}{\delta\ \phi(x)} [\varepsilon\ \delta(x-y)] = \varepsilon \frac{\delta F}{\delta\ \phi(y)}$$

For vanishing ε, we have $$\frac{\delta F[\phi]}{\delta\ \phi(y)} = \lim_{\varepsilon \to 0} \frac{F\left[\phi + \varepsilon\ \delta(x-y)\right] - F[\phi]}{\varepsilon}$$

To arrive at unique results we must specify the order of the mathematical operations.

We introduce the rule that the limit $\varepsilon \to 0$ has to be taken first, before other possible limiting operations.

Note that the x dependence on the right-hand side of:

$$\frac{\delta F[\phi]}{\delta\ \phi(y)} = \lim_{\varepsilon \to 0} \frac{F\left[\phi + \varepsilon\ \delta(x-y)\right] - F[\phi]}{\varepsilon}$$

is only a formal one; we could as well have used any arbitrary argument $(.)$ and written $\delta(.-y)$ for the variational function, in keeping with the notation $\phi(.)$ instead of $\phi(x)$, which is used sometimes in the mathematical literature if x is a "silent" argument.

Most of the rules of ordinary differential calculus also apply to functional derivatives.

(i) **The operation is linear.**

(ii) **For the product of two functionals $G[\phi]$ and $H[\phi]$ the product rule applies i.e.**

$$\frac{\delta F[\phi]}{\delta\ \phi(x)} = \frac{\delta G[\phi]}{\delta\ \phi(x)} H[\phi] + G[\phi] \frac{\delta H[\phi]}{\delta\ \phi(x)}$$

(iii) **For the chain rule:** $$\frac{\delta}{\delta\ \phi(y)} F\left[G[\phi]\right] = \int dx \frac{\delta F[G]}{\delta G(x)} \frac{\delta G[\phi]}{\delta\ \phi(y)}$$

(iv) **If $G[\phi]$is replaced by an ordinary function $g(\phi)$ that is localized at the point x, then the integration disappears and we have:**

$$\frac{\delta}{\delta\ \phi(y)}F\left[g(\phi)\right]=\frac{\delta F}{\delta g\left(\phi(y)\right)}\frac{dg(\phi)}{d\phi(y)}$$

EXAMPLES OF FUNCTIONAL DERIVATIVES

Example 1

Consider $F[\phi]=\int dx\left(\phi(x)\right)^{n}$ **. We have:**

$$\frac{\delta F[\phi]}{\delta\ \phi(y)}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int dx\left(\phi(x)+\varepsilon\ \ \delta(x-y)\right)^{n}-\int dx\left(\phi(x)\right)^{n}\right)$$
$$=\int dxn\left(\phi(x)\right)^{n-1}\delta(x-y)=n\left(\phi(y)\right)^{n-1}\textbf{since}$$
$$\left(\phi(x)+\varepsilon\ \ \delta(x-y)\right)^{n}=\left(\phi(x)\right)^{n}+n\left(\phi(x)\right)^{n-1}\varepsilon\ \ \delta(x-y)+\textbf{higher powers of }\varepsilon$$

Example 2

Consider $F[\phi]=\int dxg\left(\phi(x)\right)$**. We have:**

$$\frac{\delta}{\delta\ \phi(y)}\int dxg\left(\phi(x)\right)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int dxg\left(\phi(x)+\varepsilon\ \ \delta(x-y)\right)-\int dxg\left(\phi(x)\right)\right)$$
$$=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int dx\left[g\left(\phi(x)\right)+g'\left(\phi(x)\right)\varepsilon\ \ \delta(x-y)+\ldots\right]-\int dxg\left(\phi(x)\right)\right)$$
$$=\int dx\left[g'\left(\phi(x)\right)\delta(x-y)\right]=g'\left(\phi(y)\right)=\frac{dg\left(\phi(y)\right)}{d\phi(y)}$$

Example 3

Consider $F[\phi]=\int dx\left(\frac{d\phi(x)}{dx}\right)^{n}$ **. We have:**

$$\frac{\delta F[\phi]}{\delta\ \phi(y)}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int dx\left(\frac{d}{dx}\left[\phi(x)+\varepsilon\ \ \delta(x-y)\right]\right)^{n}-\int dx\left(\frac{d\phi(x)}{dx}\right)^{n}\right)$$
$$=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int dx\left(\left(\frac{d\phi}{dx}\right)^{n}+n\varepsilon\left(\frac{d\phi}{dx}\right)^{n-1}\frac{d}{dx}\delta(x-y)+O\left(\varepsilon^{2}\right)-\left(\frac{d\phi}{dx}\right)^{n}\right)$$
$$=\int dx\left(n\left(\frac{d\phi}{dx}\right)^{n-1}\frac{d}{dx}\delta(x-y)\right)=-n\frac{d}{dx}\left(\frac{d\phi}{dx}\right)^{n-1}\Bigg|_{y}$$

on **int***egration by parts*.

Example 4

Consider $F[\phi]=\int dxh\left(\frac{d\phi}{dx}\right)$**. We have:**

$$\frac{\delta}{\delta\ \phi(y)}\int dxh\left(\frac{d\phi}{dx}\right)=-\frac{d}{dx}\frac{dh}{d\left(\frac{d\phi}{dx}\right)}\Bigg|_y$$

$$\frac{\delta F[\phi]}{\delta\ \phi(y)}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int dxh\left(\frac{d}{dx}\left[\phi+\varepsilon\ \delta(x-y)\right]\right)-\int dxh\left(\frac{d\phi(x)}{dx}\right)\right)$$

$$=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int dx\left(h\left(\frac{d\phi}{dx}\right)+\varepsilon h'\left(\frac{d\phi}{dx}\right)\frac{d}{dx}\delta(x-y)+O\left(\varepsilon^2\right)-h\left(\frac{d\phi}{dx}\right)\right)$$

$$=\int dx\left(h'\left(\frac{d\phi}{dx}\right)\frac{d}{dx}\delta(x-y)\right)=-\frac{d}{dx}h'\left(\frac{d\phi}{dx}\right)\Bigg|_y=-\frac{d}{dx}\frac{dh}{d\left(\frac{d\phi}{dx}\right)}\Bigg|_y$$

Example 5

Consider $F_y[\phi]=\int dx'K(y,x')\phi(x')$ **where the subscript** y **indicates additional functional dependence on** y**. Then,**

$$\frac{\delta F_y[\phi]}{\delta\ \phi(x)}=\frac{\delta}{\delta\ \phi(x)}\int dx'K(y,x')\phi(x')$$

$$=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int dx'K(y,x')\left(\phi(x')+\varepsilon\ \delta(x'-x)\right)-\int dx'K(y,x')\left(\phi(x')\right)\right)$$

$$=\int dx'K(y,x')\delta(x'-x)=K(y,x)$$

Example 6

Similarly, for $F_x[\phi]=\phi(x)$**, wehave:**

$$\frac{\delta F_x[\phi]}{\delta\ \phi(y)}=\frac{\delta\ \phi(x)}{\delta\ \phi(y)}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\phi(x)+\varepsilon\ \delta(x-y)-\phi(x)\right)=\delta(x-y)$$

Example 7

For $F_x[\phi] = \nabla\ \phi(x)$**, we have:**

$$\frac{\delta F_x[\phi]}{\delta\ \phi(y)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\nabla_x \left(\phi(x) + \varepsilon\ \ \delta(x-y) \right) - \nabla_x \phi(x) \right) = \nabla_x \delta(x-y)$$