Path Integral Methods in Physics & Finance
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Lecture - 18
Langevin & Fokker Planck Equation: CLT Example

Welcome back. So, before the break I explained how to get the solution or the path integral solution for the Langevin equation. Let us explore the Langevin equation further.

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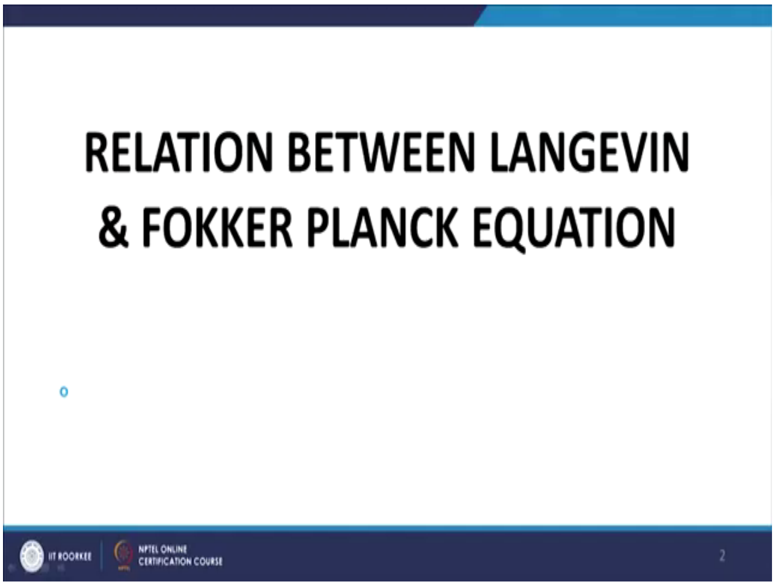

Let us try to look at the existence of a relationship between the Langevin equation and the Fokker Planck equation. Langevin equation is a dynamical equation. It relates to the explicit in a sense the Newtonian dynamics of this of the stochastic system. The Fokker Planck equation on the other hand is a probabilistic equation.

Now, let us see how we can arrive at or what if there is and if there is, then what is the relationship between the Langevin equation and the Fokker Planck equation.

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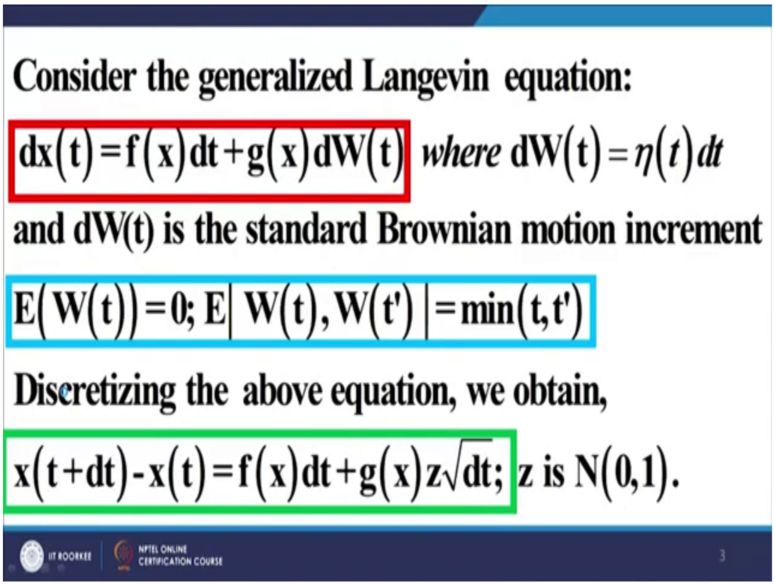

We start with the Langevin equation in the form that is given in the red box here dx t is equal to fx dt plus gx dW t.

dW t is as usual the infinitesimal Brownian motion increment which can also be written in terms of the white noise in the form given in the right hand corner. Here dW t is equal to eta t dt where eta t is white noise and Wt has the following properties fundamental properties defining properties in some sense. E W t is equal to 0 and the expected value of the expected values of the W t at different points in time is equal to minimum t, t dash.

The first step is to discretize the above equation. In fact, this is the familiar process that we follow when we start working with the Langevin equation. We discretize the equation Discretizing the equation leads us to the expression that is given in the green box right at the bottom of the slide x t plus dt minus x t that is dx t here on the left hand side and gives us fx dt plus G x and dW t can now be written as z, where z is the standard normal variate standard Gaussian variate if you like under root dt.

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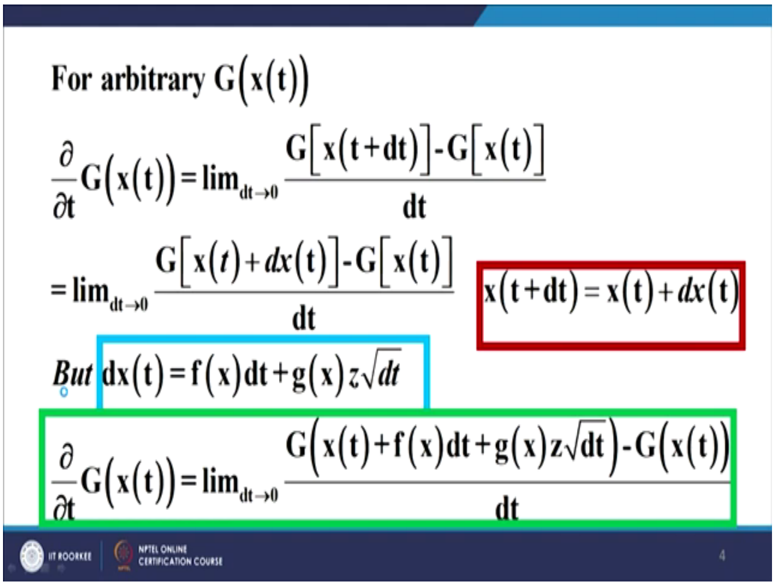

And for arbitrary for any arbitrary function G x of t we can write d of t of G x of t as limit dt tending to 0 G x t plus dt minus G x t.

This can be simplified and be can be written in the form by using the expression x t plus dt is equal to x t plus dx t expanding as t plus dt around xt. We can write to first order x t plus dt is equal to x t plus d x t that is precisely what we write here G x t plus dt minus G x t upon dt and the limit dt tending to 0.

But dx t recall dx t that appears here is nothing from the previous equation; dxt is equal to fx dt plus G x z under root dt this is from the previous equation. You can see it here. It is here in the red box here and also together with the expression in the green box here. The left hand side is nothing, but d xt, so dx t is equal to fx dt plus g x z dt under root, right.

So, we use that expression and we substitute it in our expression for d by dt G x of t and what we get is G x t plus dx t is substituted by the expression in the blue box. So, we get fx t fx dt plus G x z under root dt minus G xt upon dt and with the limit dt tending to 0.

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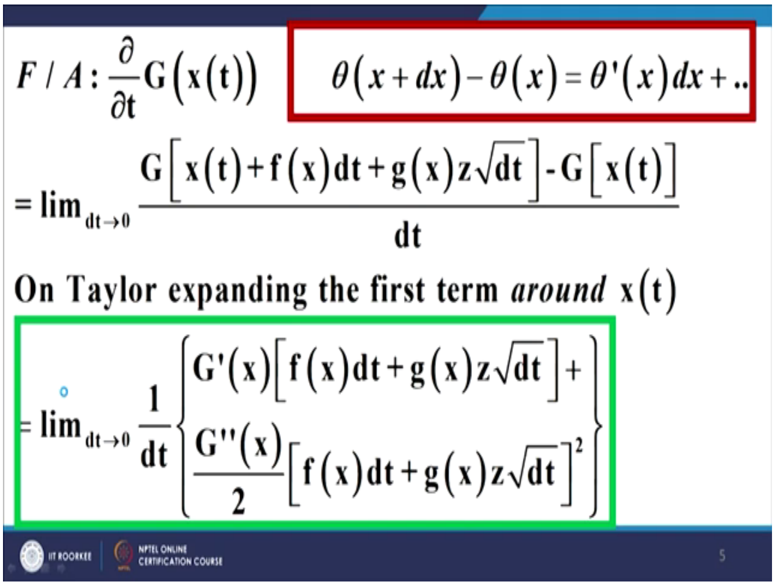

So, this is what we have from the previous equation. The first equation that we have on the top and we do the Taylor expansion, we do the Taylor expansion of G x t plus f x dt plus g x z under root dt. We do a Taylor expansion of this term around x t. So, what we and then we please note we also adjust this expression G x of t.

So, when you do the Taylor expansion of the expression in the square bracket and you deduct G x of t the what remains is g dash x of t g dash x is equal to fx dt plus gx z dt plus g double dash x upon 2 fx dt plus g x z under root dt squared of course 1 upon dt and dt tending to 0 is also brought forward.

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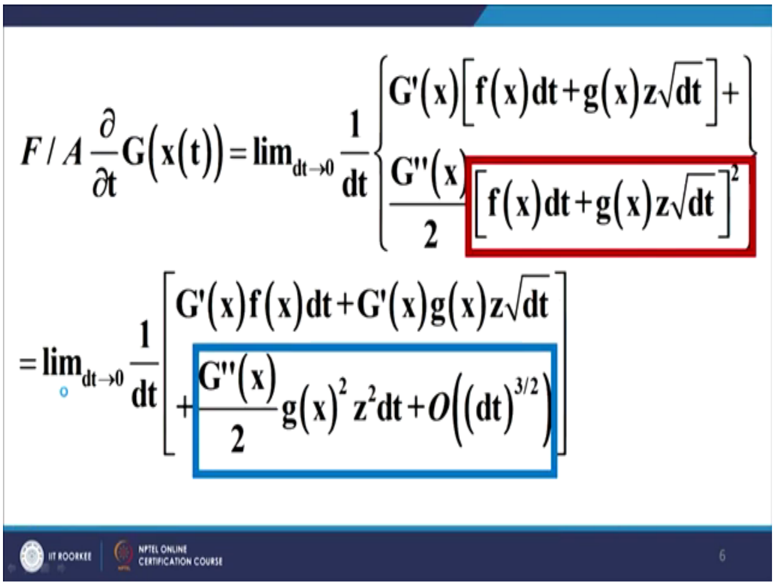

Now, let us look at the expression fx dt plus gx z under root dt. Now, we need to retain here the important thing here the manoeuvre that we are going to do here is that we need to retain only those terms up to first order in dt.

So, because we are going to retain only those terms up to first order in dt, what this expression gives us you see when you square this, the fx square term with dt square will be thrown away. The g x square z square dt will be retained because it is a first order in dt and the cross term will also go out because it is higher than first order in dt.

So, the only term that contributes to the square that we need to retain for our purpose that is linear in dt is gx square z square dt. The first term remains unchanged; we have not disturbed it.

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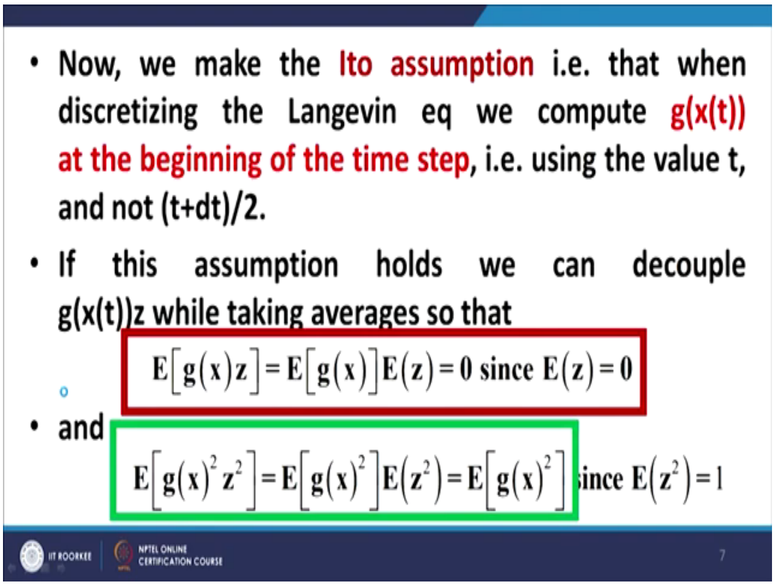

Now, we make the Ito assumption. What is the Ito assumption? You see when we do in; when we do an integral we take the value of the integral, for example y at a particular at the beginning of the interval and then we multiply it by dx or a delta x that is the slice the in the along the x axis in the and that in a sense gives us the area.

In other words, the important thing that I want to mention here is that we are taking the value of the integral at the beginning of that any interval which we first of all we partition our interval here x equal to a to x equal to b into a partition. So, let us say of n distinct distinct points intervals of each of length delta x and then for each strip that we get of delta x.

We take the value of the value of y at the beginning of that particular strip and then, multiply it by delta x and then sum over all the values of n and then limit take limit n tending to infinity that is we reduce the size of the strips and that is what gives us the integral in when we talk about integration of a or a obtaining area by the process of integration.

Now, the important thing that I want to emphasise here is that we take the value of the integrand at the beginning of each partition and then multiply it by dx. Now, there is nothing really sacrosanct about it. We could as well as taken the value at the middle of the strip and then integrated around the two points constituting the strip or multiply it by dx or we could have taken the terminal value also. It is only a convention that when we do the integration, you take the value at the beginning of each time slice or partition and then multiply it by dx. This is in a sense what we call in the case of deterministic curves. It is the impact is not significant.

However when we talk about integration or calculus of stochastic variables, this becomes a significant issue because if we use one convention the Ito convention which is precisely what we have been doing so far that is we use the use the value of the integrand at the beginning of the partition.

And we or we use the stand in which convention which uses the midpoint value, we arrive at two different results and they differ by a drift term. So, we do not get exactly the same results.

So, that is the important part here. We make the Ito assumption. In other words we make the assumption that we are taking the value of the integrand at the beginning point of the various strips that constitute the and that they constitute the area to be worked out in through integration.

So, that being the case. Now, what is the implication of that? The implication of this is that we can decouple. We can decouple for example we can de couple g and z when we take the averages we can write E g with the expression that we have in the red box will hold. If we use the E2 assumption Eg x into z can be written as E gx Ez because at that point the z and gx become independent and therefore, E x y is equal to E x E y and, but E of z is equal to 0. The expected value of z is 0 and therefore, we are the net result of this expression is 0.

But the I reiterate this holds only when the Ito assumption is used. Similarly, Egx square z square is equal to Eg x square Ez square Ez square is equal to 1 and therefore, we get here E of g x square. So, the two results that we are going to use; we are going to use the result in the red box and the result in the green box hold only under the Ito assumption.

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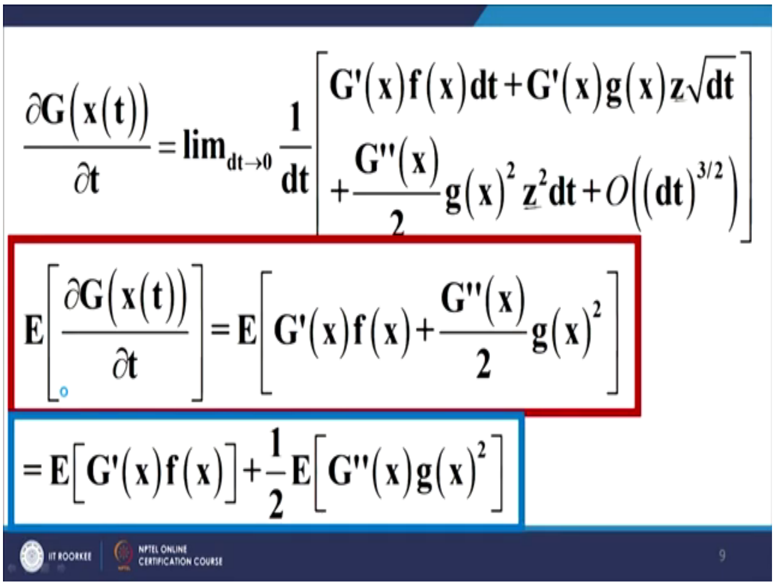

So, making use of this; making use of this what we get is if you simplify this expression, the z the second term here in the first equation the second term goes, the first term remains and the third term remains. So, the first and the third term remains and we get the expression in the blue box at the bottom of the slide.

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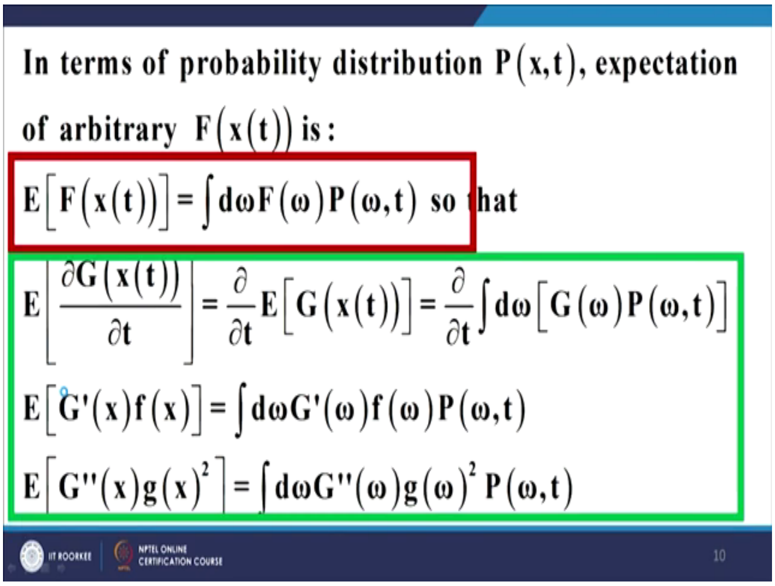

In terms of probability distributions how do we define expected value? Probability distributions expected value of any arbitrary function is defined by the expression that is given in the red box here. Expected value of F xt is equal to integral d omega F omega P omega t. This is the definition of the expected value of F x which has a probability distribution.

A continuous probability distribution P omega t and that gives us the that gives us the results that are given in the green box at the bottom of your slide. The results are quite straightforward yow get them right away E del G by del t is equal to del by del t E of G xt that is equal to this expression.

And similarly we get the expressions for the other two right hand side terms that we have in the equation that we brought forward from the earlier slide that is this expression in the blue box.

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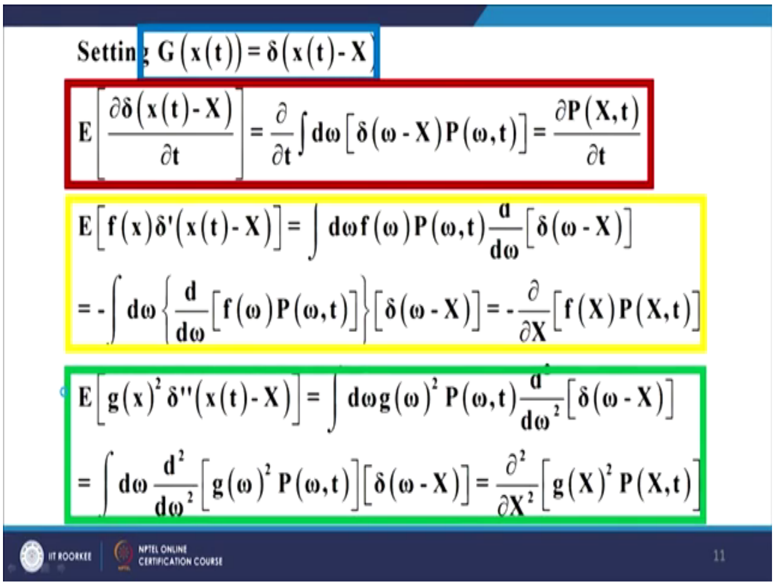

Now, we come to the most important step perhaps. Now, please note when we introduce gx t we said that it is an arbitrary function. So, we can choose that function as per our requirements as per our prescription and that is precisely what we do? We choose the function g x t as the function delta small x t minus capital X which as shown in the upper box blue box at the right at the top of your slide.

Now, if you make this choice; if this make this specific choice, let us see what results we get. The first result that we get is in the as shown in the red box here. This is quite straightforward the expected value of d by dt of delta of the gx t which is now taken as delta x minus capital

X, now it becomes when you simplify this you integrate over the delta function you get d P X t upon d t. This is simply integration over the delta function.

Now, we come to the second case. The second case is expected value of f x delta dash x t minus x writing it down in terms of the in terms introducing the terms of the expected value. We get the expression the second expression on the in the yellow box on the middle of your slide. This expression is introduced when we substitute the value of the or substitute the expression for the expected value.

Now, having done that we do an integration by parts. When we do an integration by parts, the derivative shifts itself and then we get a negative sign and the derivative shifts itself to the first term from the second term. Now, we do a delta integration integration over the delta function and what we get is the last term on the in the yellow box at the middle of your slide.

And similarly we through in through a two time path integration we get the expression for the second derivative of the delta function and we get the expression that we get is the right at the bottom of the slide in the green box.

Now, substituting all these terms in the expression in the equation that we have here in this expression in this equation, the equation in the red box and the and the blue box here what we get is the result that we are looking for.

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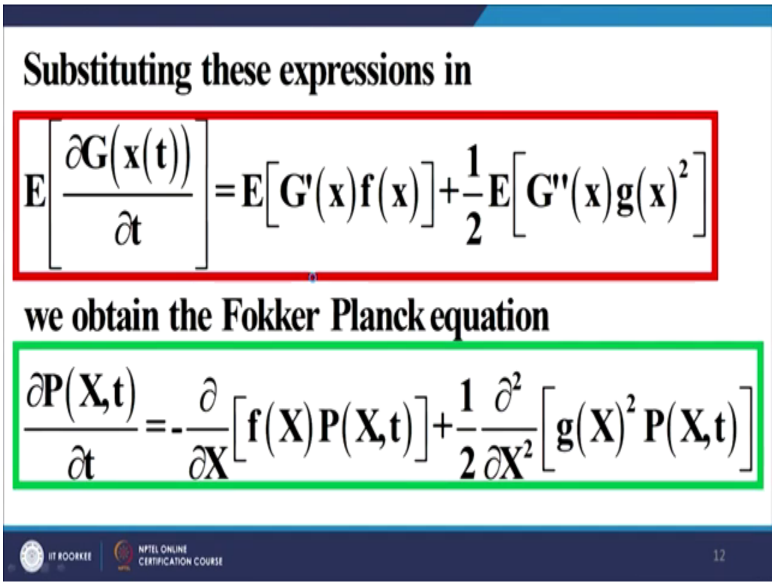

And that result is of the form in this expression which is in the red box here which can be; which can be written more explicitly in the form of the Fokker Planck equation which is given in the green box at the bottom of your slide.

We simply substitute the values of G xt. G xt was initially substituted as the delta function and then the expressions that we obtained on working out these expected values for the delta functions we make these substitutions and we get the expression for the Fokker Planck equation.

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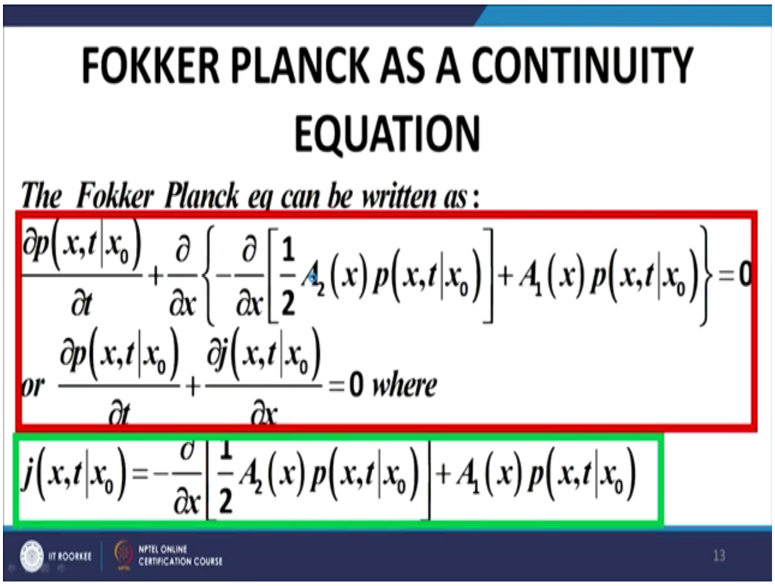

The Fokker Planck equation as a continuity equation this is a brief article very interesting. We can write the Fokker Planck equation in the form that is given in the first equation here. First equation in the red box where the expression this whole expression for the Fokker Planck equation can be summarised in the form of a two term equation.

A continuity equation in the form which is given in this second equation in the red box. This is one this has one derivative with respect to t and the other derivatives with respect to x. So, in some sense it resembles the continuity equation, right hand side is 0. So, it is a continuity equation.

So, this is where we have substituted j x t is subject to x 0 as the expression that we had within these curly brackets in the top equation on the in the red box on the in the middle of the slide.

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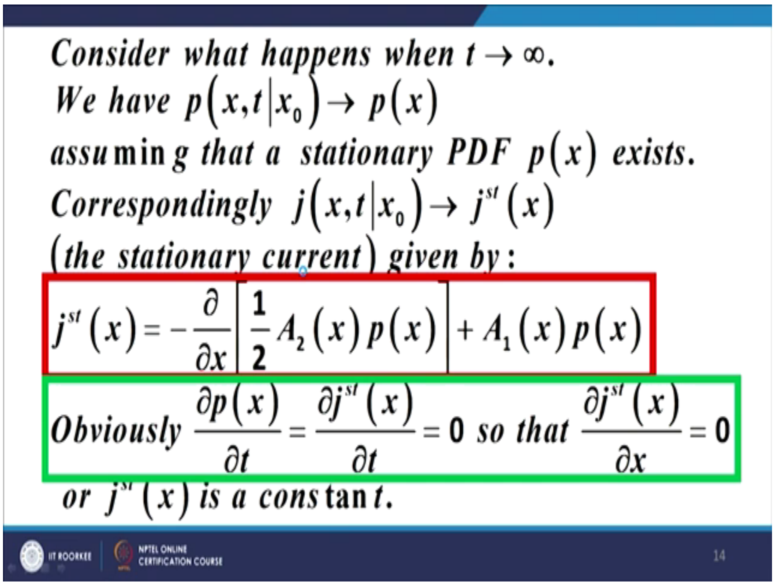

Now, when t tends to infinity if the system as t grows indefinitely approaches a steady state and it loses memory of its initial state, then we can write p x t subject to x 0 as p x and this is clearly independent of t.

So, the first thing that we get is that as t tends to infinity and p x t subject to x 0 approaches p x which is independent of t. So, dp x by dt is equal to 0 that is one conclusion that we get from here what happens to gx t subject to x 0.

Let us assume that g x t tends to or approaches j stationary x. Now, clearly j stationary x to investigate the behaviour of j stationary x we make use of the continuity equation which is here if p if dp by dt is equal to 0. That automatically implies that d j which is now j stationary dj stationary by dx is equal to 0 and obviously, j stationary is independent of t.

So, in other words what we have is j stationary is independent of x j stationary is independent to t. So, j stationary turns out to be constant provided the system reaches a steady state where it loses memory of the initial condition.

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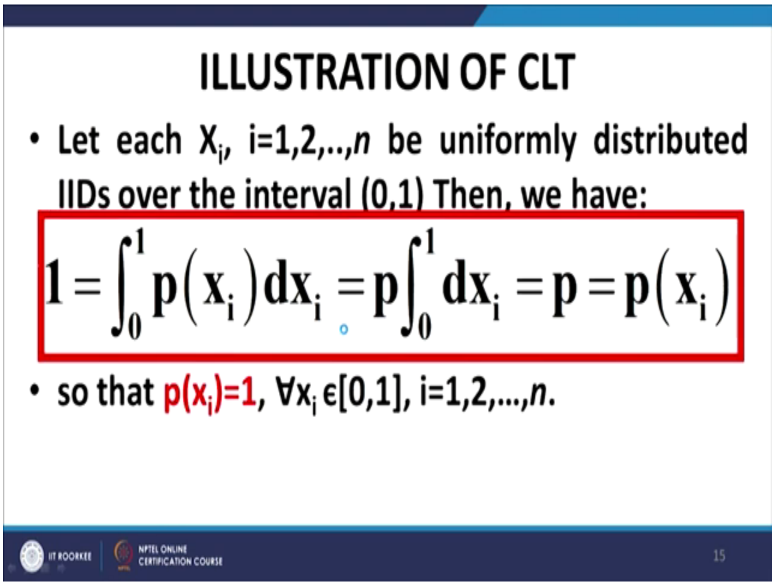

Now, I will quickly run through an example of the Central Limit Theorem. I promised at some time that we will take up the Central Limit Theorem as an example or provide an example for the illustrating the use of the Central Limit Theorem which highlights the beauty and the nuances of the theorem. So, let us just do that.

We have xi, there are n such variables and independent identically distributed variables and these variables are uniformly distributed over the interval 0 to 1 that clearly gives us the probability density over the interval as equal to 1.

And therefore, we can write this the px ip the probability density function as p of x i is equal to 1 for all i 1 2 3 4 up to n.

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$$\mu_i = E(X_i) = \int_0^1 x_i p(x_i) dx_i = \int_0^1 x_i .1.dx_i = \frac{1}{2}$$

$$E(X_i^2) = \int_0^1 x_i^2 p(x_i) dx_i = \int_0^1 x_i^2 .1.dx_i = \frac{1}{3}$$

$$\sigma_i^2 = Var(X_i) = E(X_i^2) - [E(X_i)]^2 = \frac{1}{12}$$

And that leads us to the expected value being 1 by 2. The square of the expected value of the squares being 1 by 3 and the variance equal to 1 by 12.

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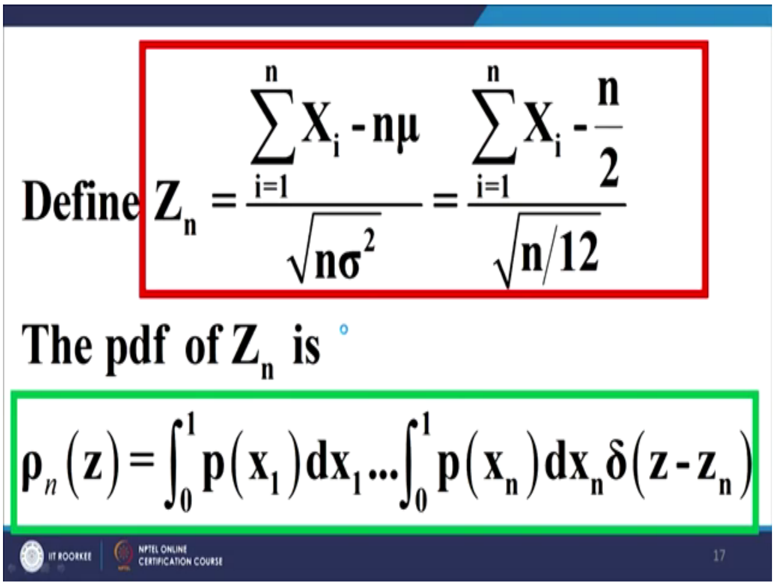

We now define the variable Z n the random variables Z n equal to summation xi minus n mu upon under root n sigma squared we write it in the form X i minus m remember mu is 1 by 2. So, we have summation X i minus n by 2 and remember sigma square is equal to 1 by 12. So, it becomes denominator becomes under root n by 12, the pdf was for Z n Zn.

Now, please note this z n is given by the expression in the green box please note the appearance of this delta function, this delta function ensures that we only include those values of xs within the integration or only those values of X is contributed to the integration which satisfy the requirements Z is equal to Z n where Z n the small zn is a realisation of capital Z n is the particular realisation of capital Z n.

So, by introducing this delta function we are ensuring that the constraint imposed by defining z n is equal to sigma x i minus n mu under root n sigma square is satisfied automatically and only those values of x 1 x 2 x 3 x n contribute to the integral where these the some of these values satisfied the requirement given in the red box, ok.

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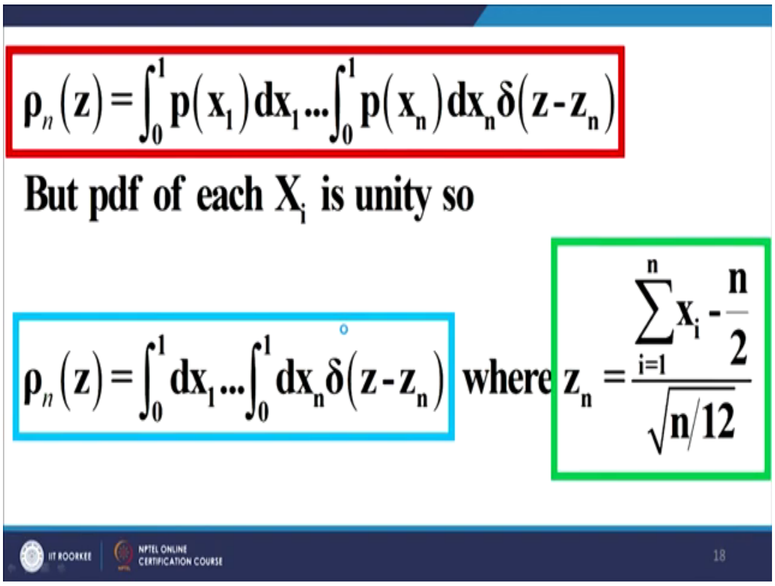

So, this is what we have from the previous slide. What we do now is because px 1 px 2 are all unity. So, we throw away the p xs and we have the expression which is given in the blue box here. We are remember z n is equal to the expression which is given in the green box right at the right hand side of the slide.

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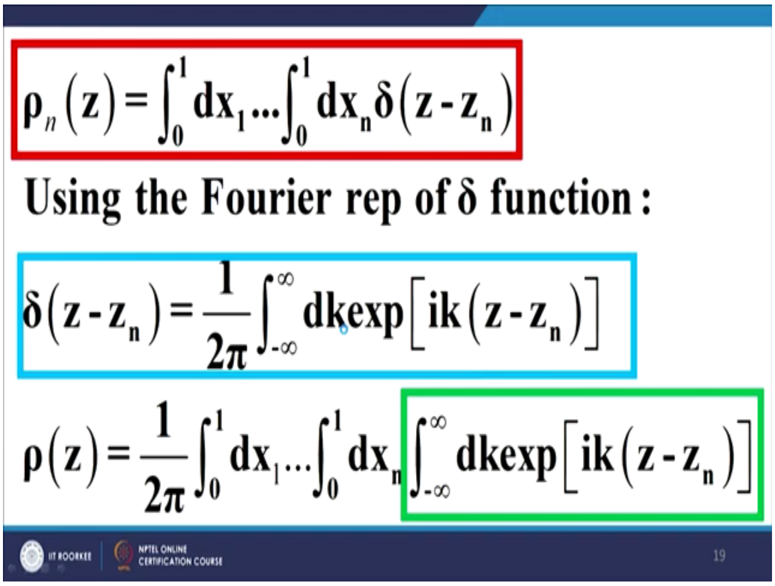

Now, let us introduce the Fourier representation for z minus zn. Introducing the Fourier representation for z minus zn in the form given in the blue box what we get is the expression at right at the bottom of the slide where this green box represents the Fourier, the Fourier transform or the Fourier representation of the delta function z minus zn.

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$$\rho(z) = \frac{1}{2\pi}\int_0^1 dx_1 \ldots \int_0^1 dx_n \int_{-\infty}^{\infty} dk \exp\left[ik\left(z - z_n\right)\right]$$

$$= \frac{1}{2\pi}\int_0^1 dx_1 \ldots \int_0^1 dx_n \int_{-\infty}^{\infty} dk \exp\left\{ik\left[z - \left(\frac{\sum_{i=1}^{n} x_i - \frac{n}{2}}{\sqrt{n/12}}\right)\right]\right\}$$

The rest is now algebra. We need to integrate this expression. The first step is to introduce z n. We write zn in its explicit form which is given here in the green box. z minus zn is written in the explicit form given in the green box here.

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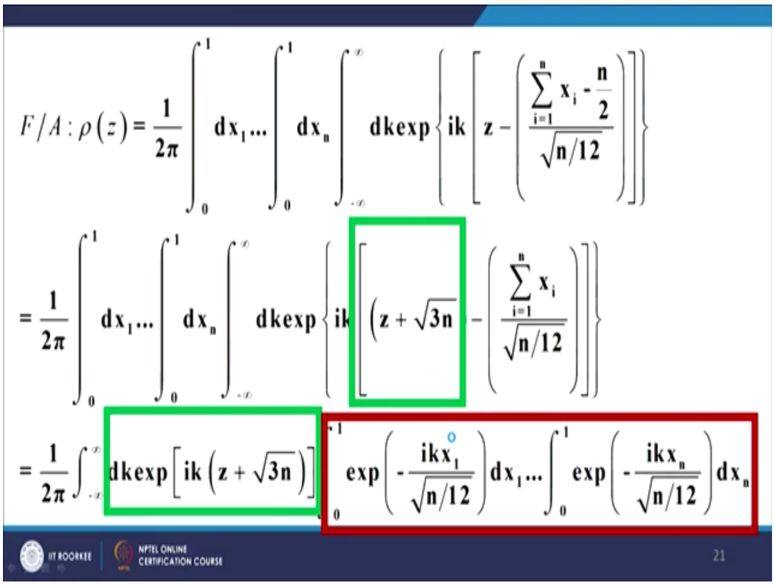

Now, our next step with this is what we obtained from the previous slide. If you look at it the second term that is minus n by 2 divided by under root n by 12 simplifies and give us under root 3 n.

Now, this expression is collected with z and written together here and the summation xis are written as a separate term. The purpose of this will be become very clear very soon. Now, that the k integral because it is independent that k integral of the terms which are independent of zs which are independent of xs. Sorry I am sorry which are independent of all the xs are taken together and are shown in the green box here and the rest of the terms which are xs integration are shown in the red box here ok.

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$$F/A:\ \rho(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\exp\left[ik\left(z+\sqrt{3n}\right)\right]\times$$

$$\int_{0}^{1}\exp\left(-\frac{ikx_1}{\sqrt{n/12}}\right)dx_1\ldots\int_{0}^{1}\exp\left(-\frac{ikx_n}{\sqrt{n/12}}\right)dx_n$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\exp\left[ik\left(z+\sqrt{3n}\right)\right]\left(\int_{0}^{1}\exp\left(-\frac{ikx}{\sqrt{n/12}}\right)dx\right)^{n}$$

Now, each of these terms if you look carefully represents an integral of exponential minus ik x upon under root n by 12 dx each of these n integral. So, in a sense they are same integrals n times over they are same integrals n time over and that is reflected in the green box at the bottom of your slide.

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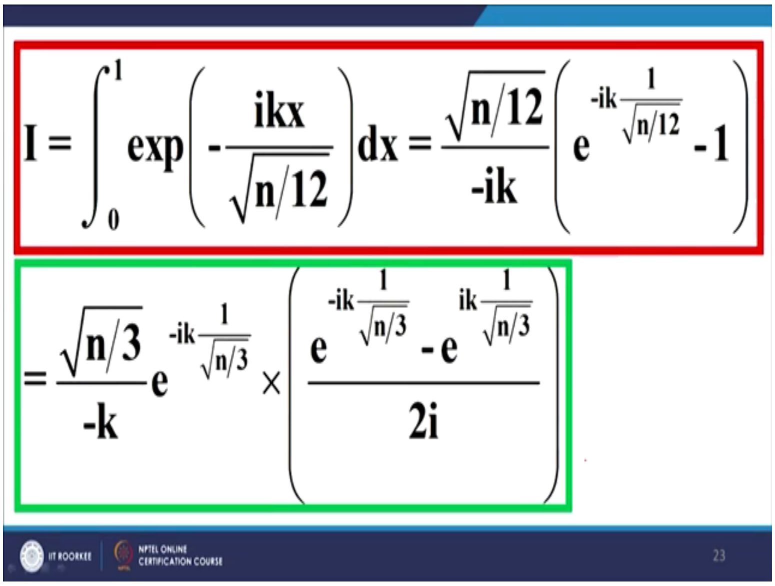

Now, we do this integration let us call that integral I. So, in a sense this expression in the green box here is i to the power n let us do the integral I. The integral I when simplified gives you the expression in the red box here which can be further simplified to the expression given in the green box here.

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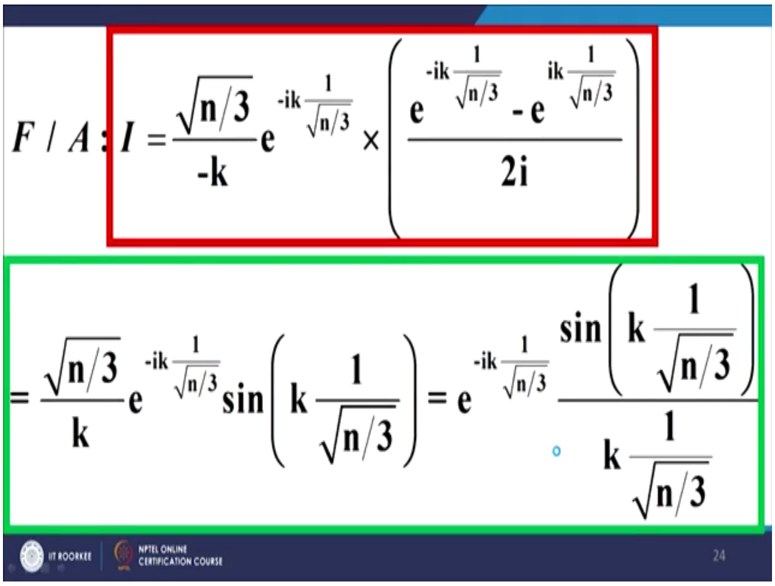

Continuing with I we continue the simplification. Further simplification enables us to write it in the form at the bottom expression on the of the slide sin exponential minus ik into 1 upon under root n by 3 sin k upon under root n by 3 divided by k upon under root n by 3 this is the expression. Please remember it is for i, but we need i to the power n.

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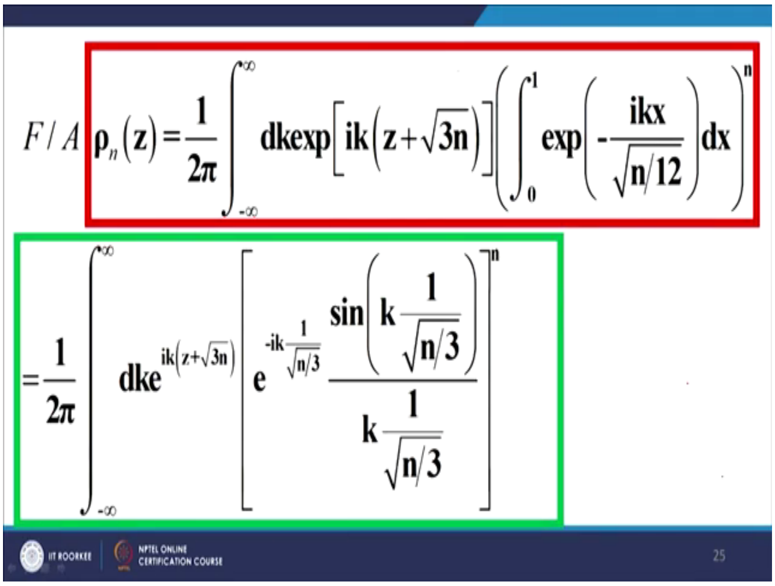

And that is precisely what we do here. We do take the nth power of this expression and when we take the nth power of this expression, we simplify this. The first thing we notice that this exponential factor within this square bracket with the power n the exponential factor here gives us precisely minus under root 3 n. So, this minus this exponential factor if you simplify this, it gives us minus ik under root 3 n.

So, this i k under root 3 n and this minus i k under root 3 n when you take this n times over, these two cancel each other and what we are left with is dk e to the power ik z and this term goes out of this nn n nth power integral. So, we have sin k upon under root n by 3 divided by k upon under root n by 3.

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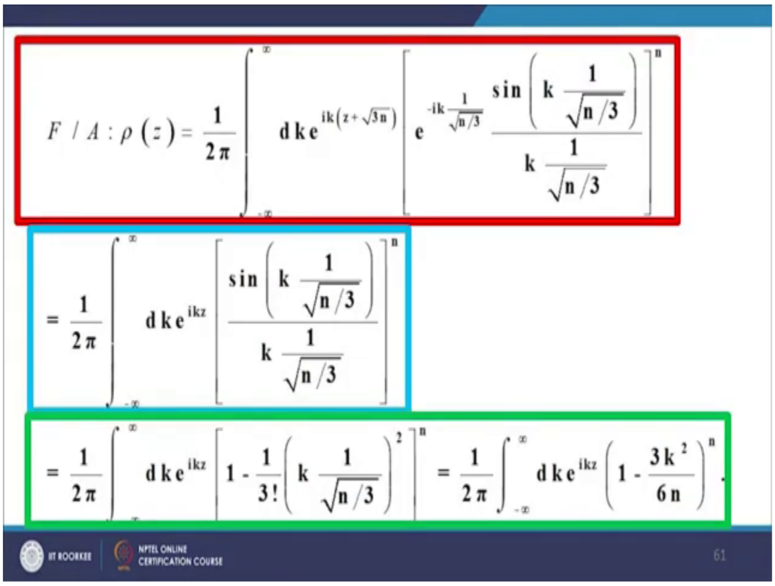

So, as I mentioned this term ik under root 3 n and this term minus ik under root 3 n with this power if you include this power n here, this nth power when it is brought with this factor it gives us minus under ik under root 3 n. So, this and this cancel out. We are left with ik z here which is retained and the rest is as it is and when we simplify it what we get is the expression that is given in the green box right at the bottom of the slide.

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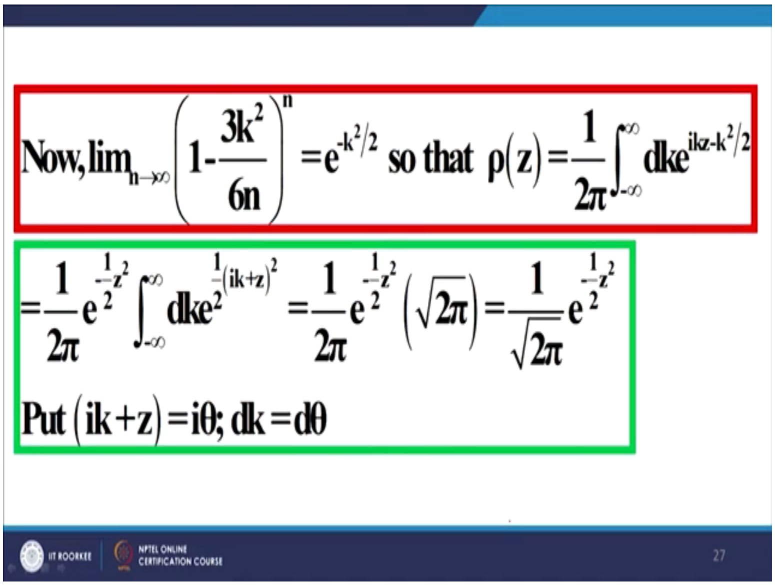

Here limit of if you look at this limit n tending to infinity of this expression, in this expression 1 minus this expression is nothing, but e to the power minus 3 k square upon 6 n minus 3 k square upon 6 n is k square upon 2. So, it is e to the power minus k square upon 2.

When this n tends to infinity 1 minus 3 k square upon 6 n to the power n when you take the limit n tending to infinity, it gives you e to the power minus k square upon 2 and therefore, we get rho z is equal to 1 upon 2 pi integral dk e to the power i k z minus k square upon 2.

Now, this is clearly a simple Gaussian integral and when we do the Gaussian integral, we end up with 1 upon root 2 pi e to the power minus 1 by 2 z square which is nothing, but a Gaussian with a mean of 0 and a variance of 1 which is what we wanted to establish.

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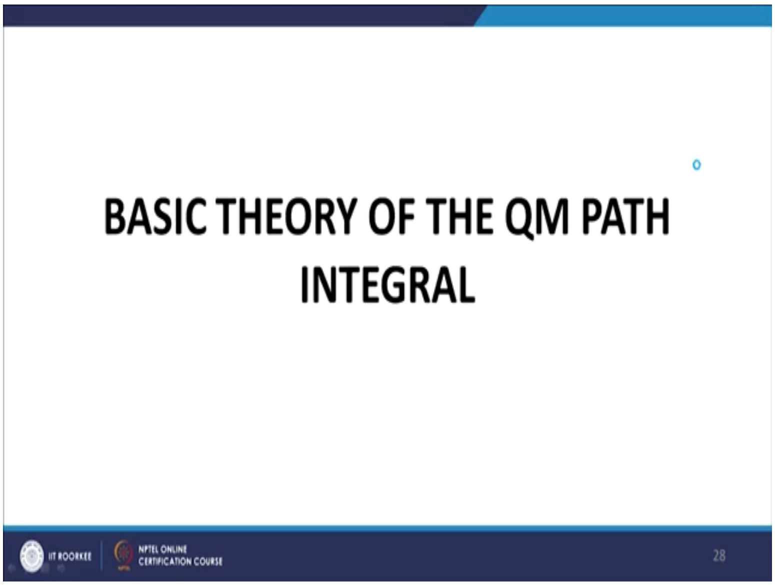

Now, we move over to the next segment of our course. We now start talking about Quantum Mechanics and before we move on to Quantum Field Theory, we shall be discussing the Path Integral approach to Quantum Mechanics and once we complete the issue of Quantum Mechanics, we shall then move over to Path Integration in the context of Quantum Field Theory.

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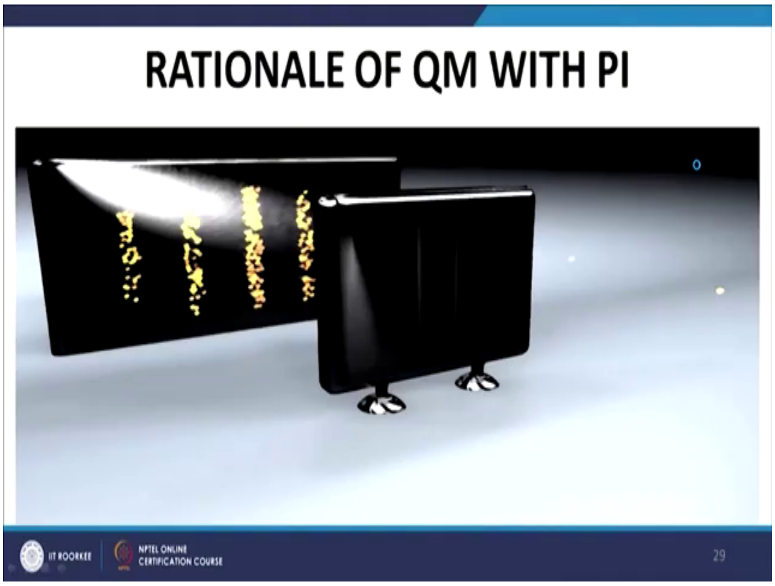

This is a recall of Davisson and Germer experiment where electrons were shot through an electron gun into screen having two slits and the interference pattern was observed on another detector screen placed behind the screen having the two slits.

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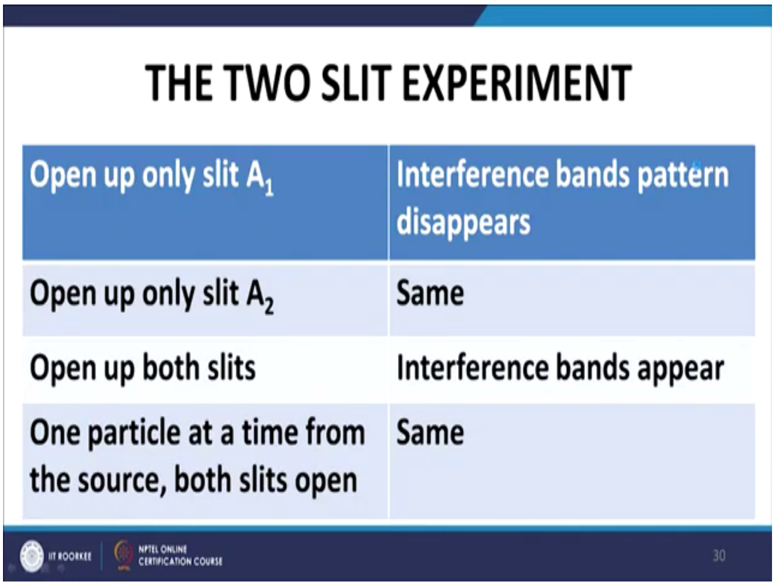

And the results that we found were just to recapitulate. When we open just the one slit, there is no interference. When we open the other slit, there is again no interference. When we open both the slits, then interference bands appears in the manner that is shown in this detector slide and when we even if we bombard one particle at a time from the electron gun, the interference pattern does appear.

So, these were certain conclusions, certain radical conclusions which completely or significantly revolutionise the theory behind Quantum Mechanics and which form the cornerstone or the background or the backdrop of the path integral mechanism.

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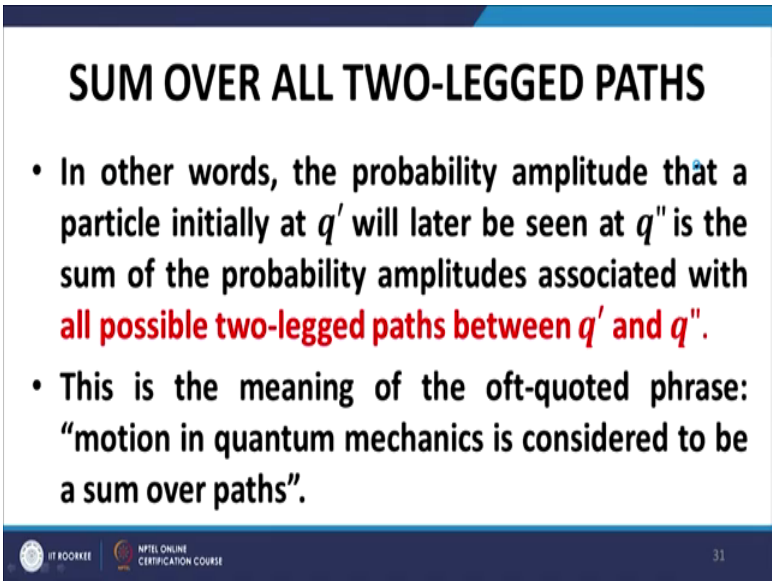

And now what was argued was you see what is clearly apparent from this is that if you look at the interface bands on this slide on this detector screen, they are the they represent interference patterns and the amplitudes are sums of the amplitudes of the waves that originate from the two slits in the middle in the screen in the screen let us call it the screen O1.

So, the in other words the interference patterns on the screen O2 that is the detector screen are due to the waves that propagate through the two different slits of the screen O1 and interfere and give rise to these waxes and wanes or the constructive and destructive interferences.

Now, what will happen if I increase the number of slits to 3 4 5 and so on? Well what will happen is the number of paths will increase firstly and secondly, the interference patterns that we will observe on the screen will be at any point at any point on the detector screen O2 will be the sum of the amplitudes of the paths emerging from each of these screens, each of these slits.

So as you increase the number of slits, the number of paths increases and the, but the net result is that as far as the intensity on any point on the detector screen is concerned, it would be determined by the sum of the amplitudes of the of all the paths that emerge from the screen O1.

Now, suppose we place another screen O3 between O1 and O2 that will again and that has a number of slits in it that will again increase the number of paths, but the principle would remain the same that the amplitude or the transition amplitude at any point on the detector screen will be the sum of all the amplitudes of all the paths that reach that particular point through various various possible various possible approach points.

So, that was the that was this is what was the rationale of the quantum mechanics with the path integral formulation. The argument was that if I place an infinite number of screens between the source and the receptor S and the receptor O2 and in each of those screens if I pierce an infinite number of slits have an infinite number of slits, then the result is.

Now, the two things happened. Number 1 the as per the proposition that is established the amplitude or the intensity at any point on the on the detector screen O2 would be the sum of all possible paths starting from the source O1 ending at that particular point on the detector screen which is being investigated and why we use the word all possible paths, now we define what we mean by all possible path.

If we have an infinite number of screens and each screen has an infinite number of holes, it randomize to the situation that we are simply having no screens at all. So, in a sense what we conclude here from this argument from this argument what we conclude here.

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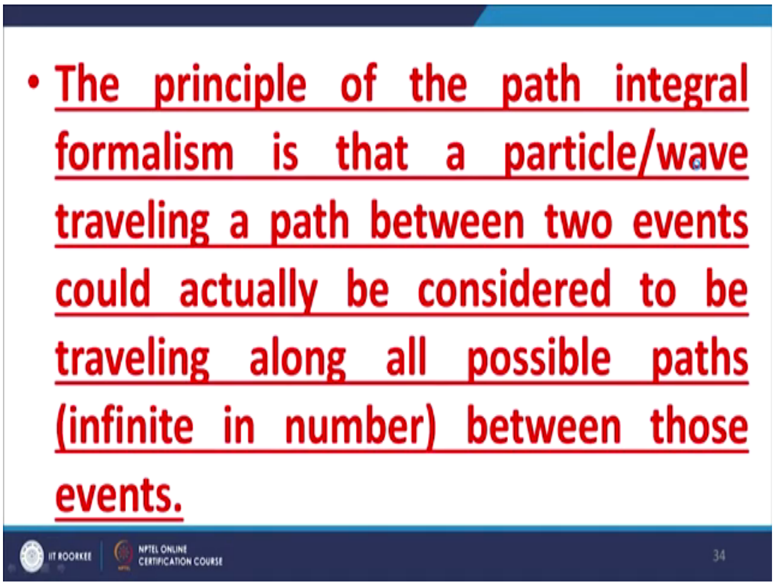

The conclude the conclusion that we draw is that the path integral formalism is that where a particle or a wave travelling a path between two events could actually be considered to be travelling along all possible paths infinite in number between those two events.

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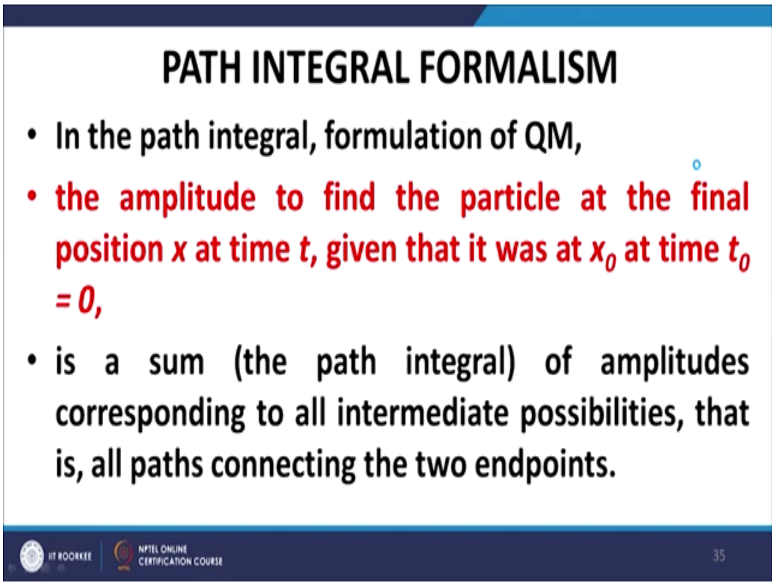

Of course the important thing is also there that the amplitude to find. In other words, in other words stating the same thing again the amplitude to find the particle at a final detector point at a time t given that it was at a well defined point x 0 at time t 0 is the sum of amplitudes corresponding to all paths is that originate from the source and reach the destination.

That is the input that is the fundamental principle of the path integral formalism. Of course, each paths has you will see later has to be weighted by a particular weight factor or a phase factor rather, but we will be talking more about it. For the moment this is the principle of the Path Integral Formalism. We shall get into the nitty gritty of it from the next lecture.

Thank you.