Path Integral Methods in Physics & Finance
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Lecture – 17
Langevin Equation Path Integral [2]

Welcome back. So, in the last lecture we were in the process of deriving the Path Integral solution of the Langevin equation.

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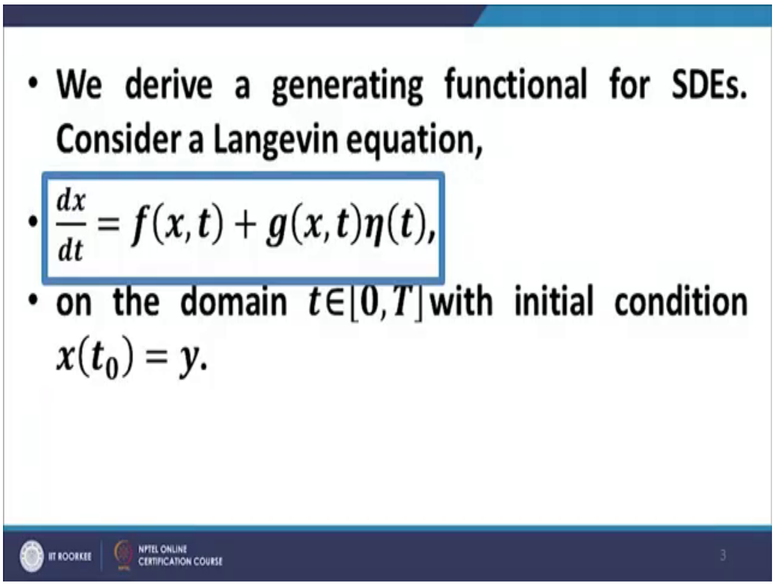

Let us recap what we have done so far and then we will continue there on. The Langevin equation can be written in the form that is given in the blue box here right in the middle of your slide, where the initial condition is written as x of t naught is equal to y this is the initial condition. And this is that first order differential equation, which represents the Langevin equation. You know right hand side the eta t term is in fact, the stochastic driving force.

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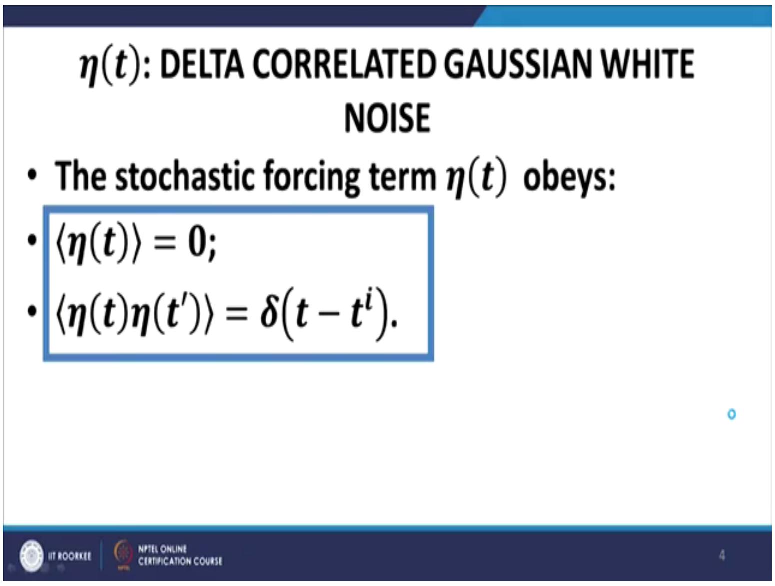

And eta t is the white noise factor and the white noise obeys the standard white noise delta correlation conditions which are given in the blue box in this slide.

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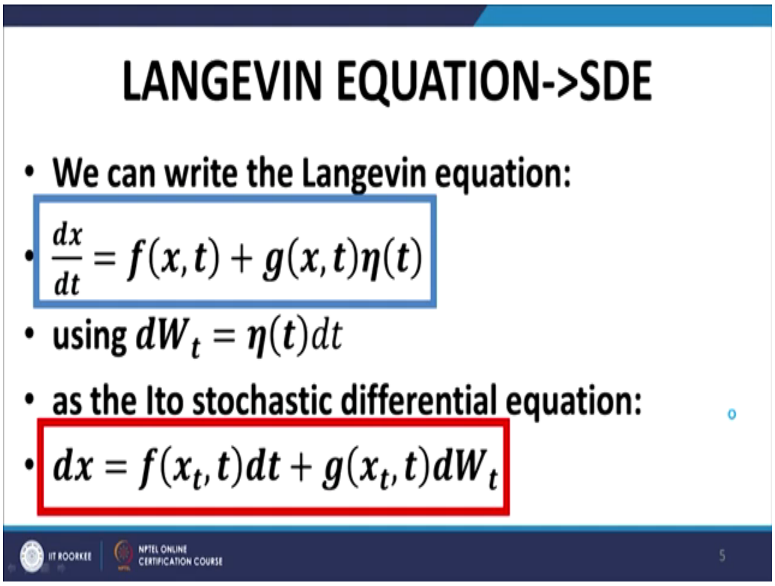

Now, this Langevin equation can also be written in the form of involving the increment of the Brownian motion. The infinitesimal increment of the Brownian motion by writing dW t is equal to eta t dt. And then it takes the form of an Ito stochastic differential equation which is written in the red box right at the bottom of your slide. This is an equivalent version of the equation that we have in the blue box at the top of this slide which has been obtained simply by writing dW t is equal to eta t dt.

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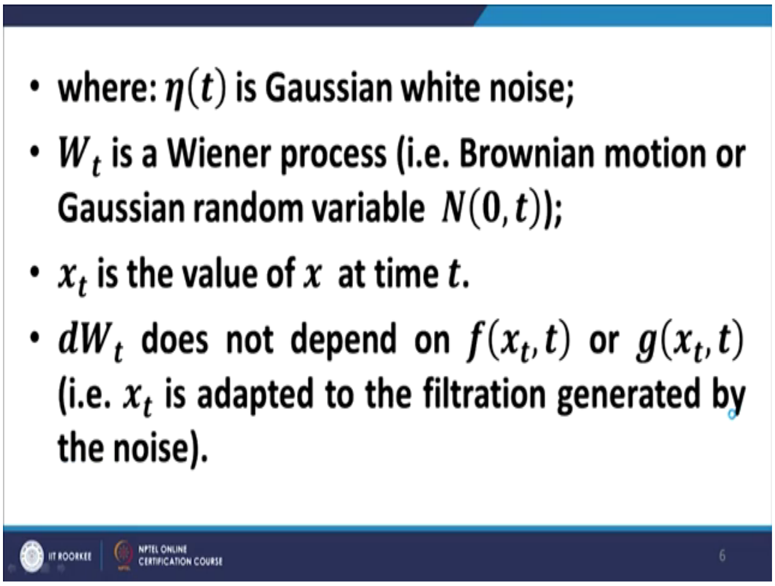

Of course, there is a question mark regarding the mathematical structure of eta t. Because strictly speaking Brownian motion is not differentiable or Brownian motion is differentiable with probability 0. And because of the zigzagging that it involves in its path and therefore, eta t is not really a very strictly well-defined mathematical object nevertheless its serves many useful purposes to view eta t as the derivative of Brownian motion.

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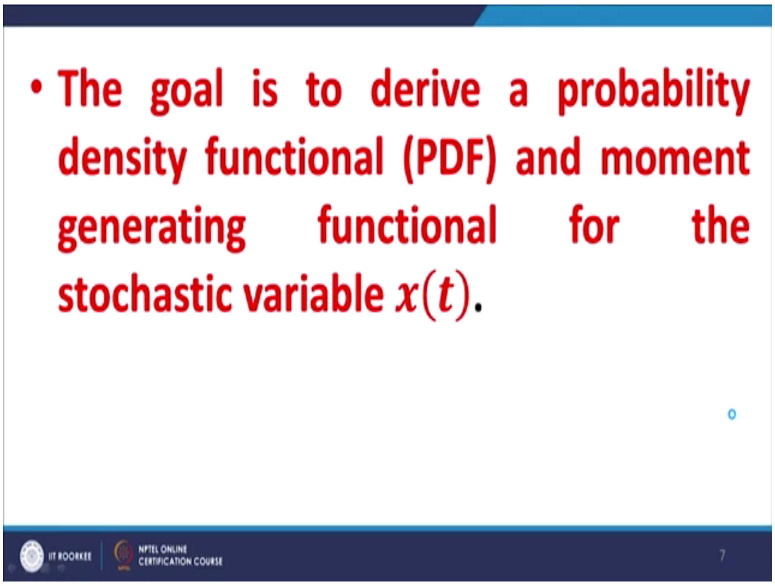

Our objective now is to obtain a probability density functional and the moment generating functional for the stochastic variable x of t.

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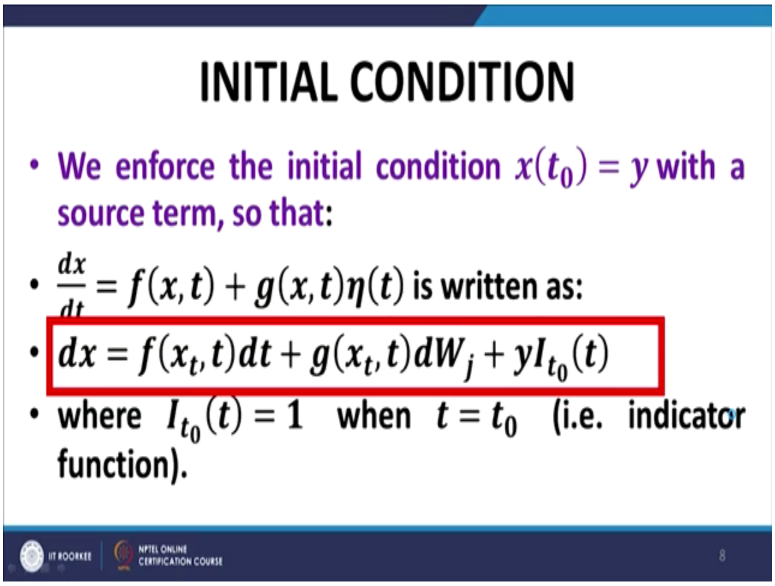

And the initial condition that, I mentioned was taken as x at t 0 is equal to y, but we shall incorporate the initial condition in our analysis in the Langevin equation by writing a source term by incorporating there in a source term which is the last term on the right hand side of the equation shown in the red box. And the 2 are equivalent because, I is the indicator function, which takes the value of 1 at t equal to t 0. And therefore, we get the initial condition equivalent to what initial condition we had originally envisaged.

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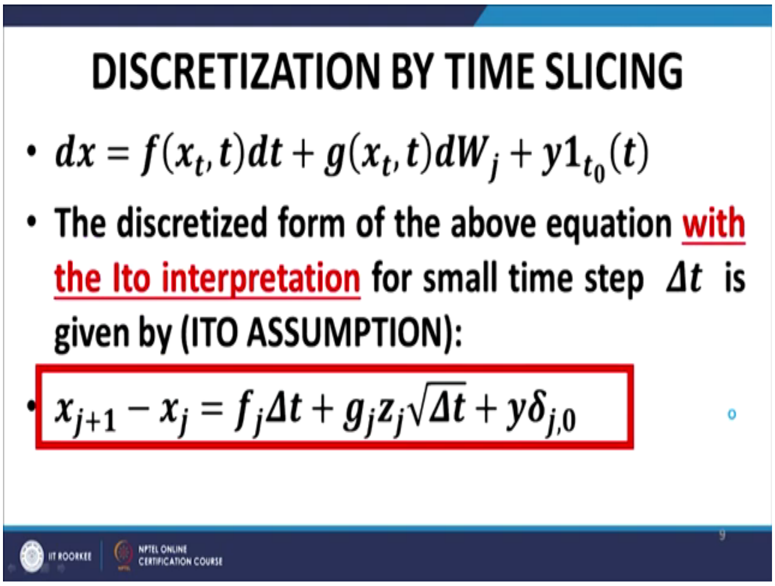

Now, we do a discretization of this Langevin equation by time slicing; which is this standard procedure which has also been elucidated earlier. We do a discretization of this by cutting the time interval t 0 equal to 0 to capital T into small time slices infinitesimal time slices that in time points. And then we shall revert back to the continuum case by reducing by increasing the number of time slices.

For the moment let us do the time slicing. We do the time slicing; in other words we convert the time axis into a lattice into a one dimensional lattice, where each point represents time point time step rather.

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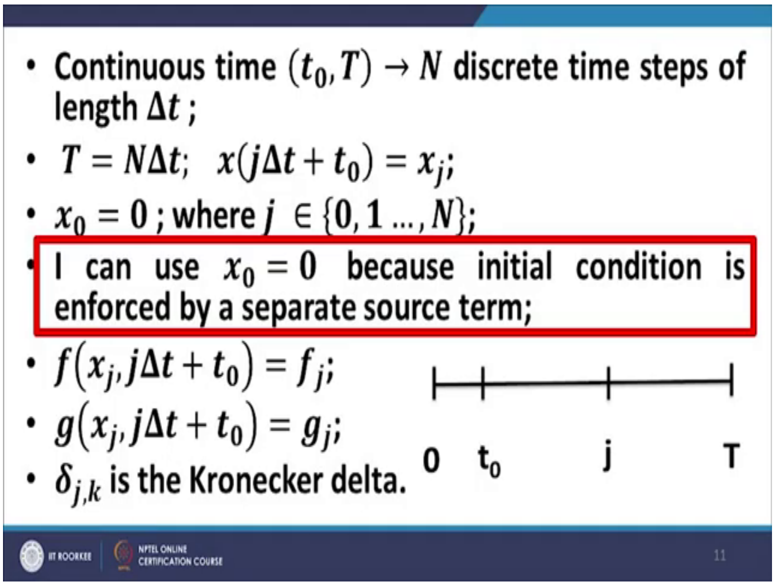

We partition this entire thing into N capital N, discrete time steps each of length say delta t, each of length delta t, say there are N delta t steps and N delta t must be equal to the total time length. And any point here for example, any point here on the lattice numerated or a indicated by j in terms of continuous time will relate to t 0 plus j into delta t, where j will be the number of time steps up to the point j.

So, in terms of continuous time it would relate 2 time j delta t and we at the initial t equal to t 0 here to the timing. And we get j in terms of continuous time is equal to j delta t plus t 0. So, the rest of the things automatically define themselves; the x at j delta t plus t 0 becomes j x j. And f at x j j delta t plus t 0 becomes f j, j at x j delta t plus t 0 becomes j.

And delta is of the of course, the Kronecker delta function. So, these things are the common abbreviations, standard abbreviations in the context of this one dimensional lattice and j being an arbitrary point on this one dimensional lattice, which corresponds to t 0 plus j delta t on the continuous time line.

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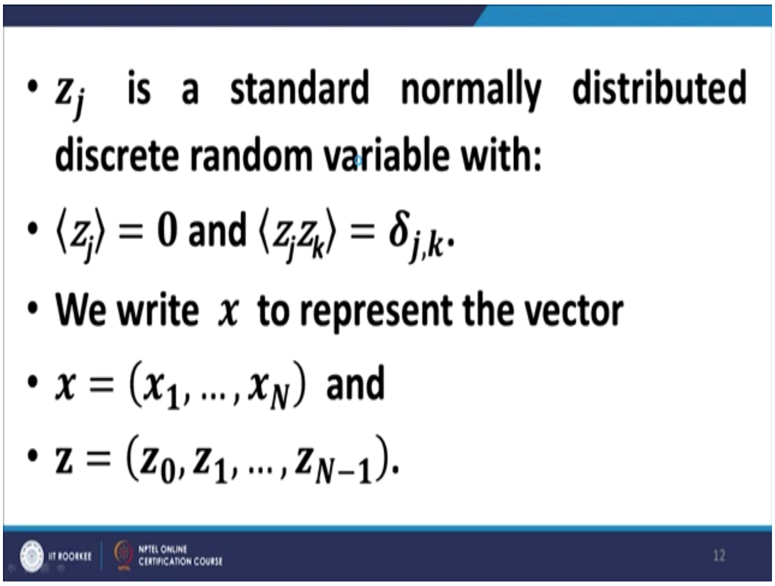

Z j as I mentioned as is abbreviated as the standard normally distributed or Gaussian discrete random variable. It has a mean of 0 and it has a correlation given by delta j k, which is also given where in the slide. And we use the let us x without suffixes to indicate the N component vector for example, the letter x represents the N component vector.

And the letter z represents the N component set of standard and Gaussian variables. Each of them being discrete and each of them having mean of 0 and a variance of 1. Both of them are random incidentally.

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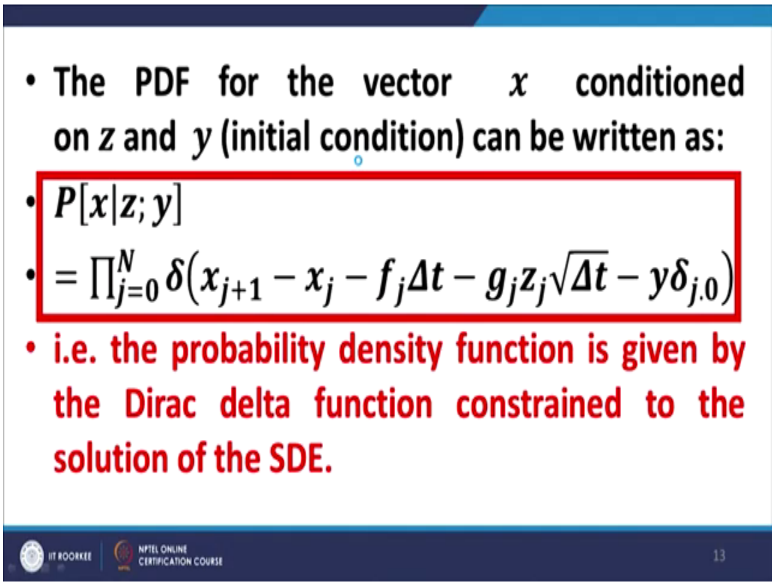

So, now we write down the PDF of the vector x. We write down the PDF of the vector x, conditioned on z and y, in other words condition on the initial condition which is y and condition on z following a certain realization or the vector z having a certain realization.

Now, clearly the P x subject to z y takes the form of a delta function or a product of delta functions where at each time step we require, that this stochastic differential equation corresponding to that time step must necessarily be obeyed by x, corresponding to the realization of z. And if it is obeyed by x at each and every time step, then we get the probability of occurrence of x and if it if at any of the time steps the condition is not obeyed, the condition imposed by the constraint imposed by the, stochastic differential equation is not obeyed, then we do not get x as the outcome.

So, it tent amounts to representing the probability of x subject to z and y; in terms of the delta function which is given here which is nothing but the, constraint imposed by the stochastic, given stochastic differential equation and discretized version of that.

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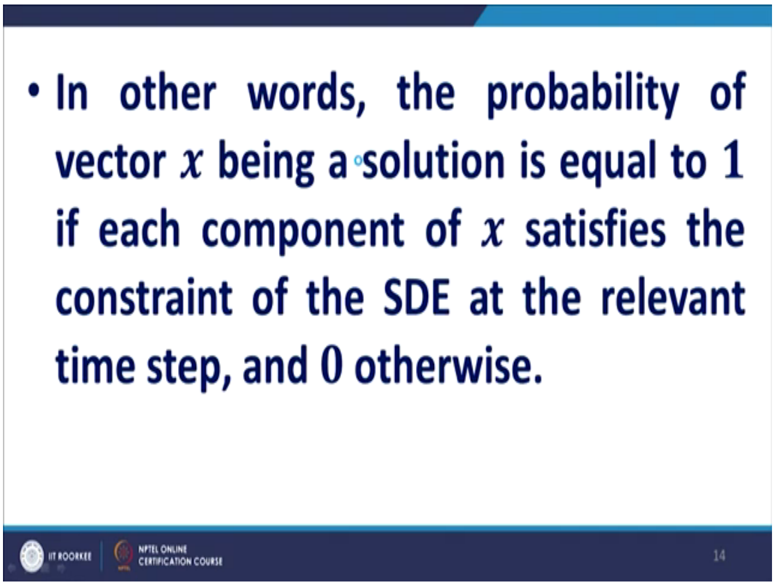

So, in other words the probability of x being a solution is equal to 1 if each component of x satisfies the constraint of this stochastic differential equation, at the relevant time step, and 0 otherwise. That is what the delta function is supposed to ensure to mandate.

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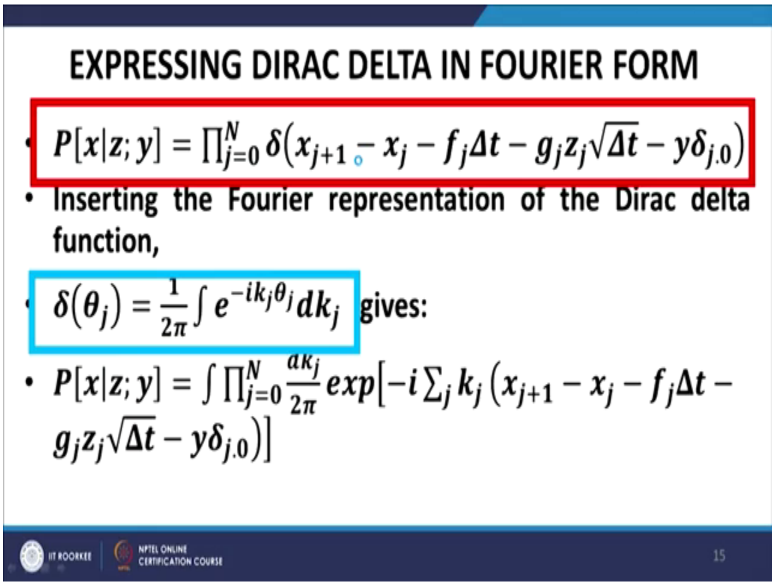

So, now we can write the delta function in Fourier form in Fourier notation, and we get the expression which is here in the blue box. So, putting using this expression in the blue box we can write P x subject to z and y, in terms of an exponential and which is shown in the last equation bottom equation of the slide.

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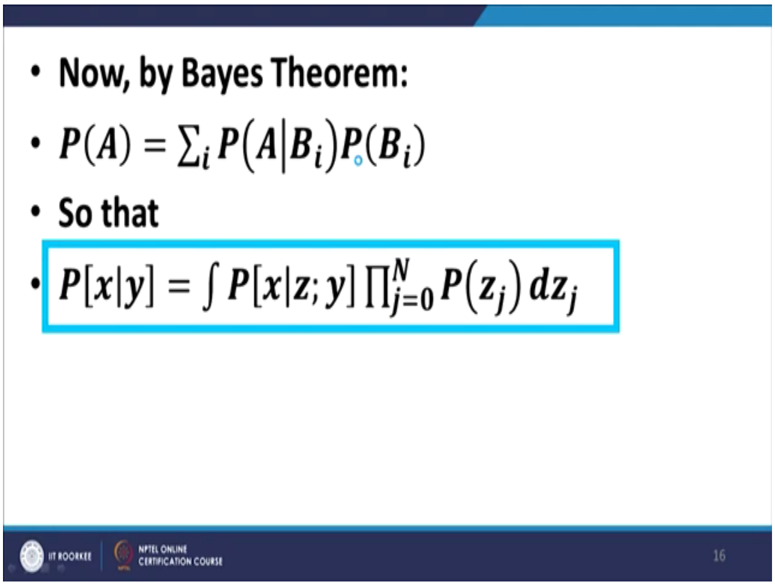

Now, we use Bayes theorem. Bayes theorem is given here in the expression for the Bayes theorem can be written in the form given in the first equation on this slide. P A is equal to summation P A conditioned on B 1 B i into P of B i this is Bayes theorem.

We use this Bayes theorem and we write P with in other words we sum over, or integrate over all possible values of z and we can then write down, P x subject to y; P x subject to y summing over all values of possible values of z or the probabilities of all possible values of z, we can eliminate z from the left hand side and we get the expression on the right hand side. This is nothing, but a representation or an expression of the Bayes theorem which is written above in the simple form.

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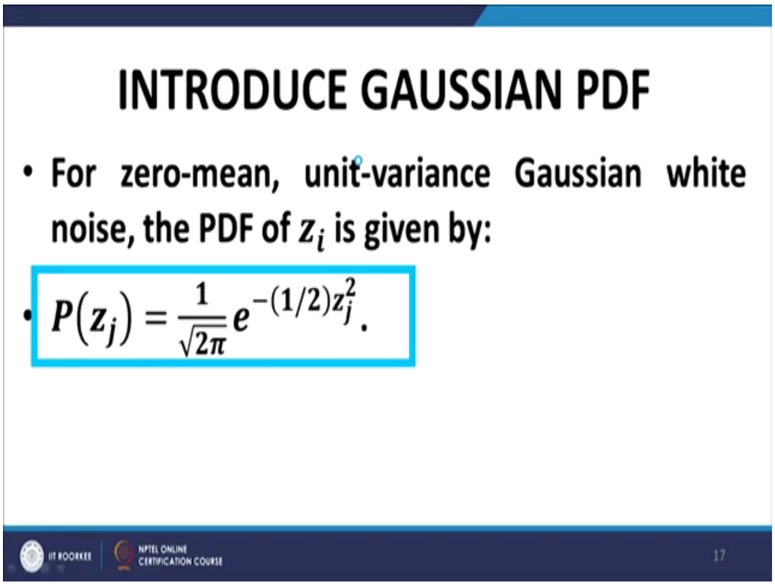

Now, please note each of these z's, as I mentioned earlier is the standard Gaussian variate with a mean of 0 and a variance of 1, and therefore, each of these has a probability distribution given by the expression in the blue box and the bottom of your slide here.

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- $$\boxed{P[x|z;y] = \int \prod_{j=0}^{N} \frac{dk_j}{2\pi} exp\left[-i \sum_j k_j \left(x_{j+1} - x_j - f_j \Delta t - g_j z_j \sqrt{\Delta t} - y\delta_{j,0}\right)\right]}$$
- $$\boxed{P(z_j) = \frac{1}{\sqrt{2\pi}} exp\left[-(1/2) z_j^2\right]}$$
- $$\boxed{P[x|y] = \int P[x|z;y] \prod_{j=0}^{N} P(z_j)\, dz_j}$$
- $$P[x|y] = \int \prod_{j=0}^{N} \frac{dk_j}{2\pi} exp\left[-i \sum_j k_j \left(x_{j+1} - x_j - f_j \Delta t - y\delta_{j,0}\right)\right] \times \int \prod_{j=0}^{N} \frac{dz_j}{\sqrt{2\pi}} exp\left(ik_j g_j z_j \sqrt{\Delta t} - \frac{1}{2} z_j^2\right)$$

We substitute this expression in the expression for P in the expression for P x subject to y; we had this expression from by using Bayes theorem the expression in the yellow box, we obtain this expression by application of Bayes theorem.

We have the expression for P x subject to z and y as a delta function got it in the earlier slides. By imposing the condition that the our variable x must satisfy, this stochastic differential equation at every discrete point at every node at every point on the lattice; corresponding to each z and each time step that was imposed by the delta condition, we converted the delta condition to the exponential form and that gave us the equation in the red box.

Then we imposed, then we choose the Gaussian PDF and we introduce that into our using the Bayes theorem, and using the Gaussian PDF we arrive at the condition that is given in the green box at the bottom of your slide. In other words the equations in the red box, the blue box and the yellow box all combined together give to give us the equation in the green box right at the bottom of your slide.

We start with equation in the red box, we use the equation in the yellow box to manipulate the equation in the red box and write it over using Bayes theorem and then, we introduce for each z j we introduce the probability distribution and that leads us to the equation that is given in the green box at the bottom of your slide.

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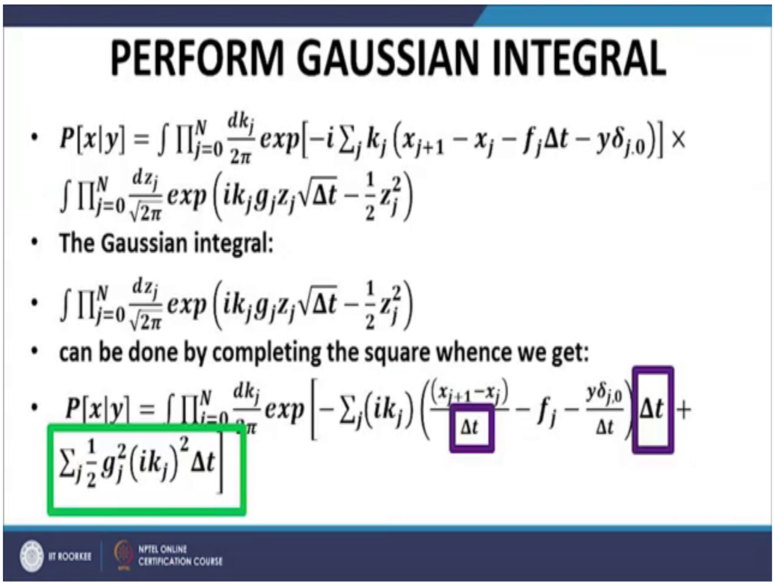

Now, if you look at this, if you look at the first equation that is the red box equation, which is brought forward from the earlier slide it can be written in this form the 2 x the z exponentials can be collected together.

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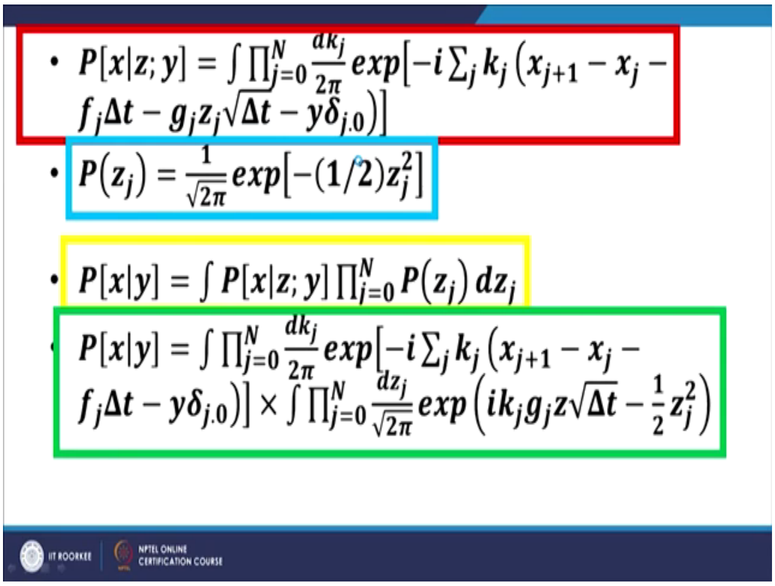

The z exponentials the last term here can be collected together and they form a Gaussian and therefore, they can be integrated over integrating over this Gaussian variables we and simplifying we get the expression which is here in the bottom equation of your slide.

Of course, the factor of delta t has been if you look at it carefully a factor of delta t has been extracted from here. For the purpose which will be apparent in the next slide. And delta t has been taken common and extracted outside the brackets to facilitate moment into the continuous continuum regime aware this delta t will help in replacing the summation by integration. So, that is the objective as you will see in the next slide.

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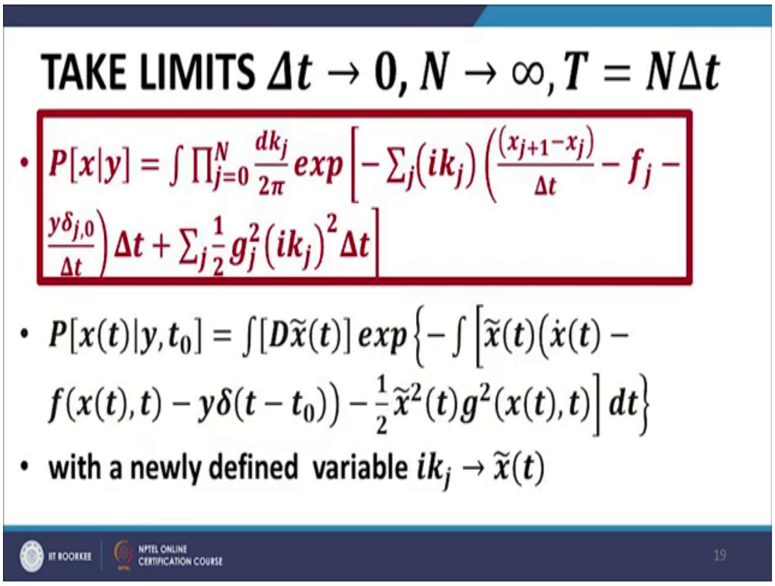

Now, we take limits delta t tending to 0, N tends to infinity, and t tends to N delta t. As you recall these were the things that we introduce when we discretize the Langevin equation.

Now, we are doing the reverse we are moving from the discretization version to the continuum version and the when we move to the continuum version the expression that we get is given in the green box right at the bottom of the slide. Where we also introduce a change in variable of ik j goes to, x tilde t; ik j is going to, x tilde t with the if the product integral dk j is now, becoming a path integral capital D x tilde t over at over all possible paths.

And the rest of the things explained themselves the x j plus 1 minus x j upon delta t represents the change in x j over with respect to unit time which is taken as x dot t f j now replace back as f of a x t. And similarly, the discrete delta function is replaced by the direct delta function, the continuum delta function. And in this summation is replaced by integration as far as the

Gaussian integral term is concerned which we added obtained on the basis of Gaussian integration.

Please note there is also a change of variable incorporated here i k j is transform to x tilde of t. The reason why an i does not appear here will become (Refer Time: 15:13) later, but for the moment just bear with this that, the change in variable is given by ik j is goes to, x of x tilde of t.

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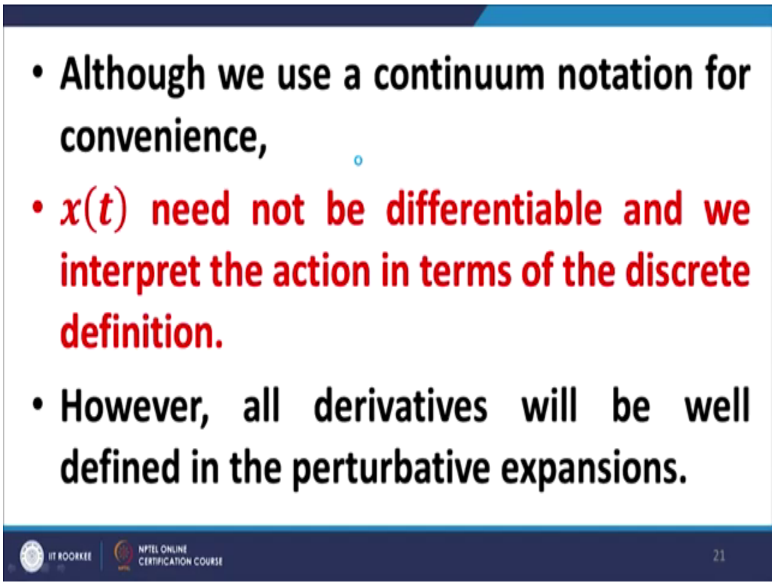

Now, x of t need not be differentiable as we have discussed earlier; x tilde t need not be differentiable and we can interpret the action in terms of the discrete definition. And all derivatives will be well defined if you do a perturbative expansion which you will found a cornerstone of solving the path integrals in the context of quantum field theory.

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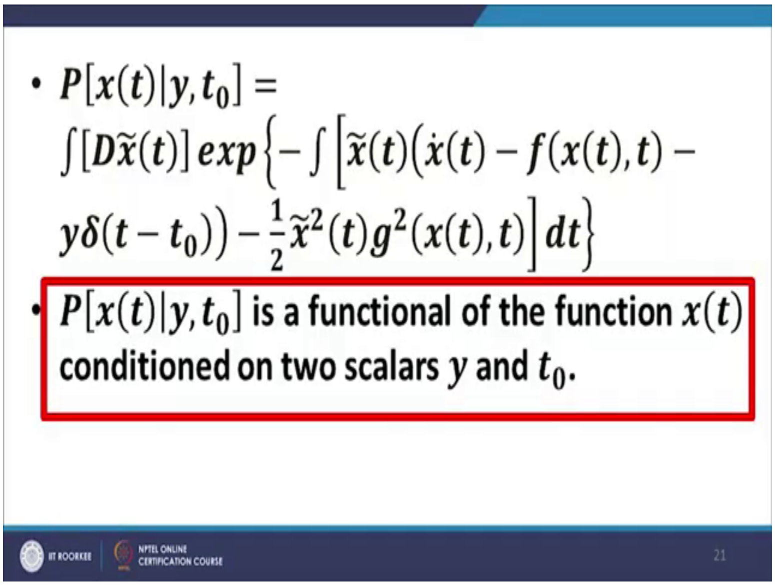

So, this is what we have from the previous slide. And from the previous slide what we find is that, P x of t y t 0, is now become a functional P of x of t is become a functional of x t; x t condition on two scalars y which is the initial condition and t 0 which is the initial time; y is the initial value of x and t 0 is the initial time.

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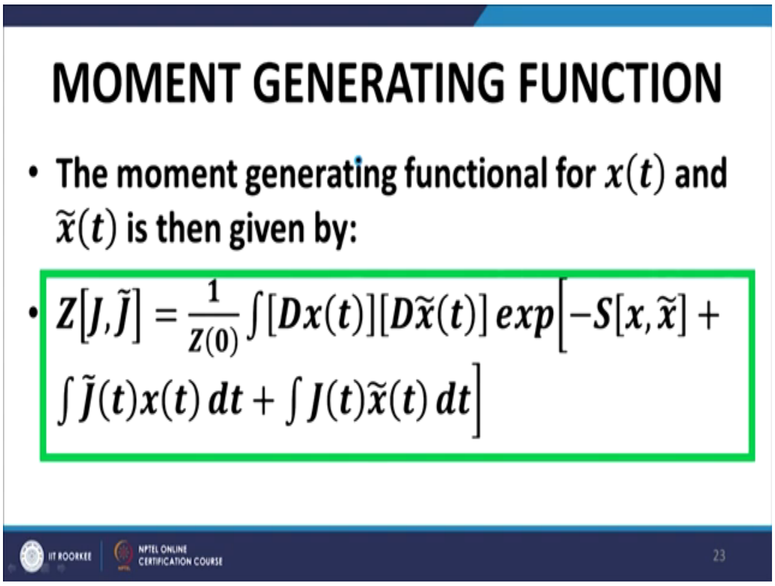

Now, we talk about the moment generating function. The moment generating function corresponding to the sum of probability or a PDF, conditional PDF is given by the expression given in the green box at the bottom of your slide.

Remember Z 0 is a normalization factor. And the moment generating function is now a function of 2 J and J tilde 2 variables J and J tilde 2 source variables J and J tilde, and it can be written ah in the form which is given in the green equation at the bottom of your slide.

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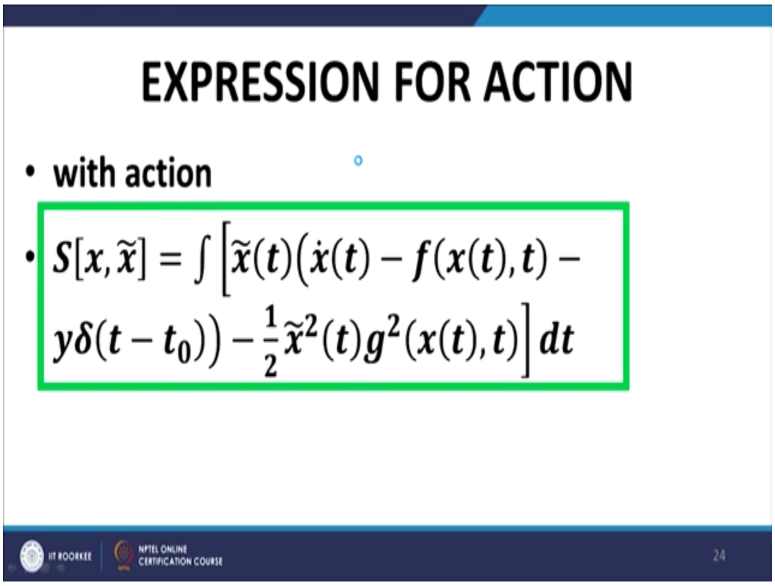

And about the expression for the action, the action indeed can be read off; the action can be read off from the expression for this moment generating function. If you the action is given here minus S x, x tilde. And if you compare that what we have earlier explicit for expression for x S x and x tilde is given by this expression which you can obtain from the earlier equation, this equation.

If you look at the term within the exponential, this is precisely nothing but the within the integral in the exponential within the square brackets in the integral in the exponential this represents the action corresponding to the moment generating function that we are talking about.

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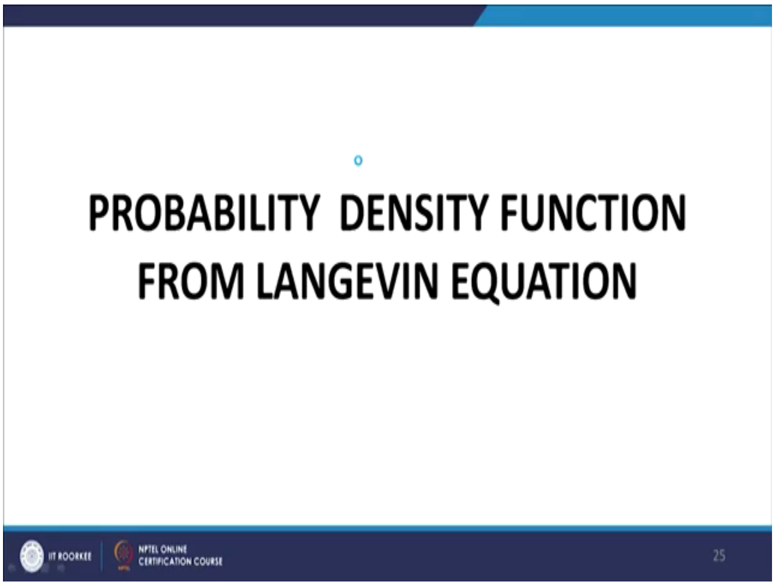

Now, we talk about the probability density function; probability density function from the Langevin equation.

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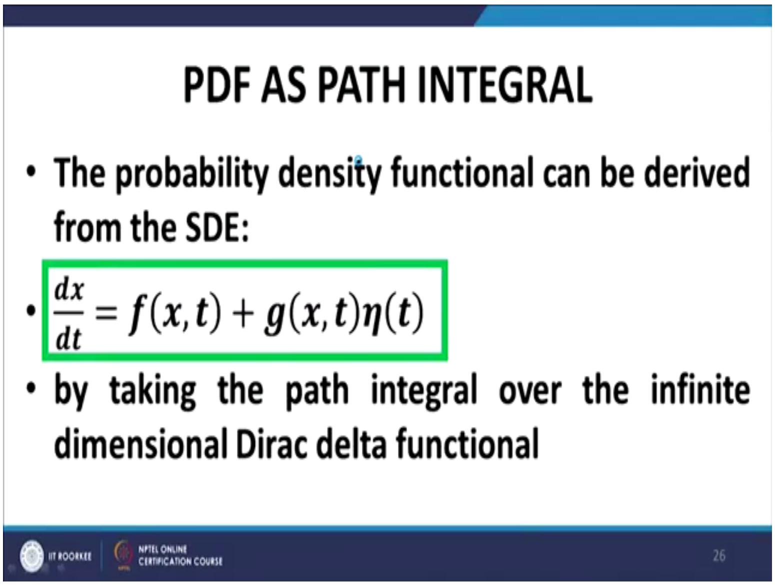

The probability density function of corresponding to this equation can be taken can be obtained directly from the direct delta function that give us the conditional probability you would recall in the earlier exposition. In the previous exposition relating to the moment generating function, we can we obtain the probability density function of x subject to, y and z in the form of a delta function we shall be using that.

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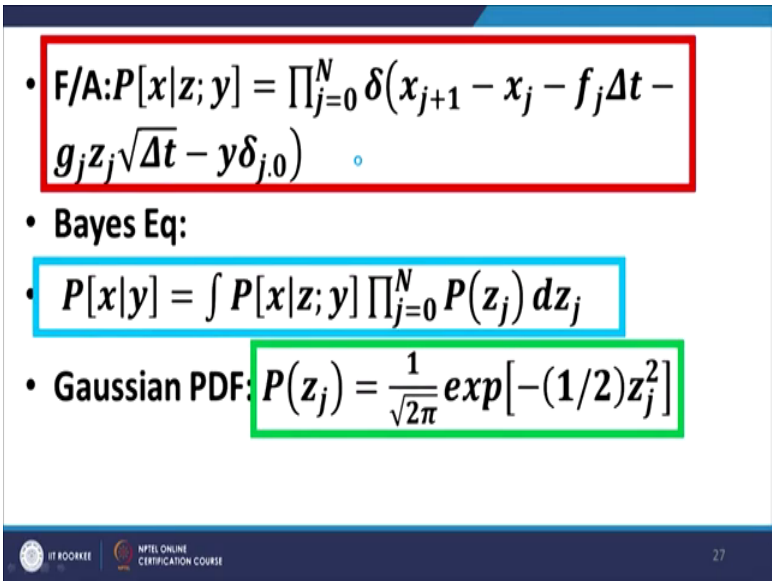

So, we start with that point, probability density function of x subject to z and y, is obtained as the delta function. As I explained earlier this represents the condition, that this stochastic differential given stochastic differential equation is satisfied by the values of x, corresponding to values of z at every node of the one dimensional lattice representing the timeline at various points and various nodes of the timeline.

Now, we use the Bayes theorem and that also we have discussed earlier. The Bayes theorem can be written in the form in the blue box here and the Gaussian PDF, because the z j is standard normal variates or Gaussian standard Gaussians, they have the PDF which is given in the green box right at the bottom of your slide.

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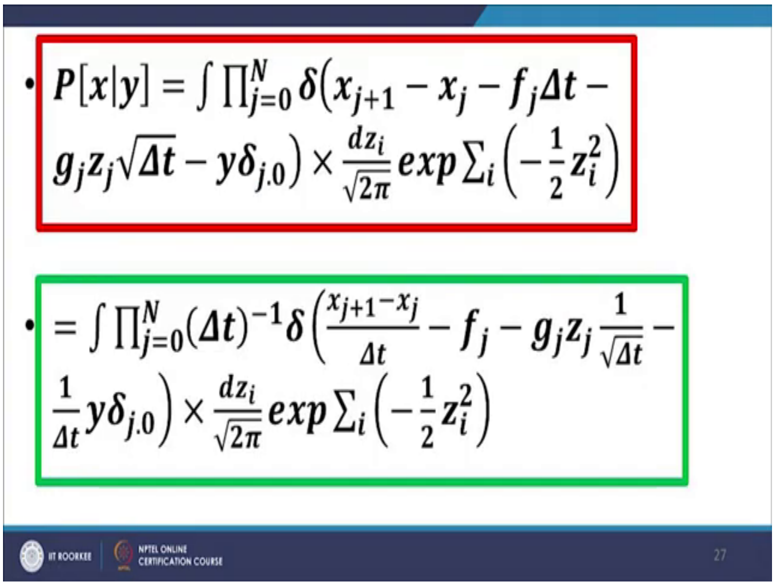

So, making use of these 3 expressions we can move further. In this case, please note in the previous case when we used or when we arrived at the moment generating function, we at this point we moved over to the exponential representation of the delta function.

Here we are continuing with the current representation of the delta function, and we are multiplying it by the probabilities of various z's z i's, which are as I mentioned standard normal variates.

Now, we take delta t in common and again it is going to serve the same purpose as I did earlier taking delta t common; enables us to write the expression within the delta, in a form which is compatible for moving into the continuum framework and that is precisely what we want to do so, we take delta t common and outside the delta term.

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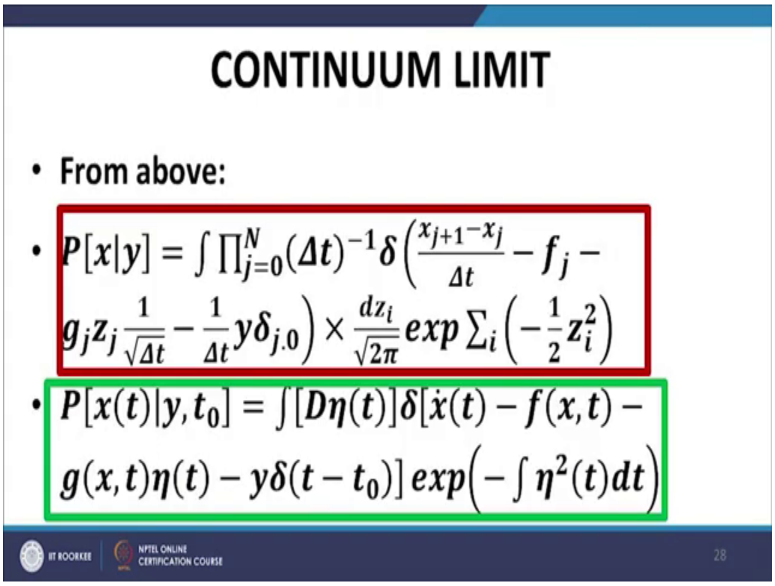

Now, we do take up the continuum limits this is the discrete expression is given in the red box on this slide and we move to the continuum limit when we move to the continuum limit we get the expression that is given in the green box at the bottom of the slide. Please note that, we the new z j or z i square has now been transformed to a continuum the white noise variable eta square t.

And the and this summation over z i square has now been shifted to a summation over eta square t dt. The rest of it the Kronecker delta has been shifted to or has been converted to the direct delta and the rate of change of x, that is x j plus 1 minus x j upon delta t has been converted to x dot j in the continuum framework. The first derivative and the rest of which it is more or less absolutely same as earlier.

And if you look at the coefficient of g, g has again now g j that is the value of g at a particular node on the lattice is now written as g x t eta t; eta t comes from the factor of z j being present here, in the red box in the in the discrete discretized version. Please note the relationship here.

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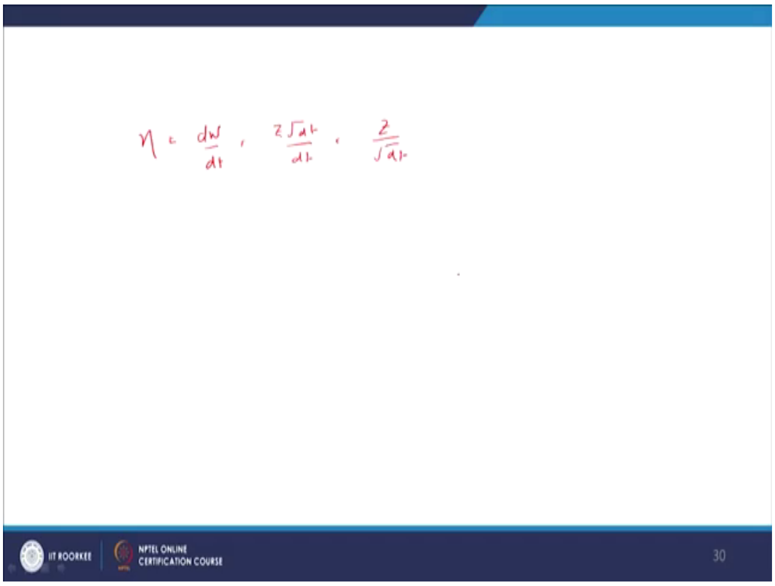

Between eta and z, eta as I mentioned; eta is in some sense can be written as dw upon dt. And this can be written as, z under root dt upon dt, and that is equal to z upon under root dt. This is the expression that we make use of in writing the expression for eta or substituting z upon under root t by eta here.

And also in the exponential summation minus 1 by 2 z i square, we do the same substitution except that we substitute the summation by the integral with respect to t.

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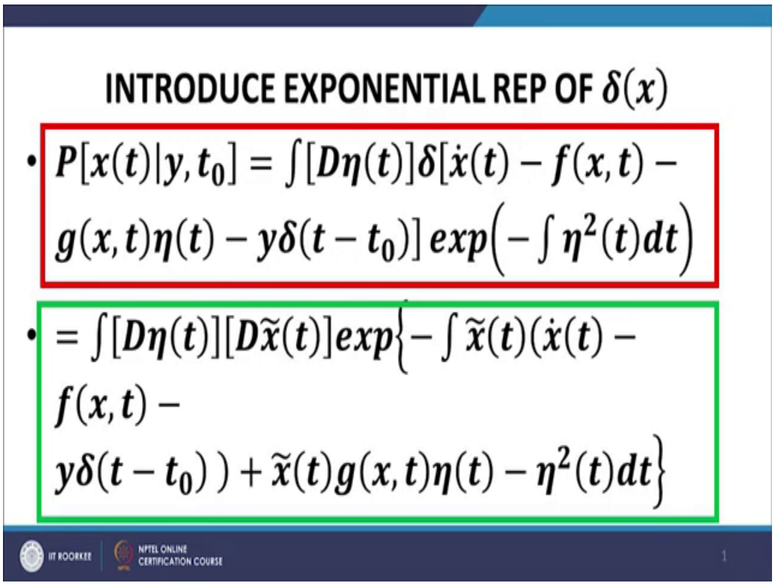

Now, we introduce the exponential representation of the delta x and we get the expression. This is the expression that we brought forward from earlier and when we use these substitutions when we introduce these substitutions we get a double path integral and a path integral over eta t and a path integral over x tilde t and we get the expression which is there in the green box at the bottom of your slide.

We have introduce this new variable x tilde t as we did in the earlier case and that enables us to write the expression for the probability density in terms of double path integral path integral over eta t, a path integral with respect to x tilde t of the expression; that is given in the green box at the bottom of your slide.

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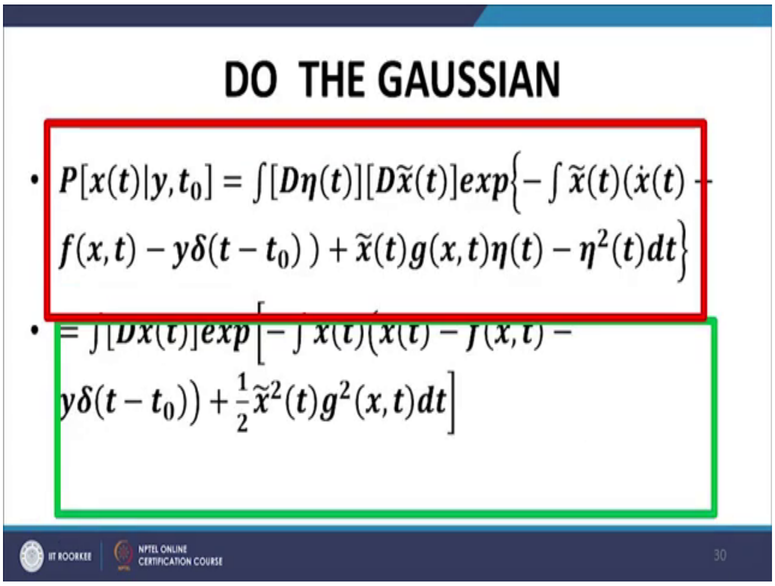

So, right. So, now we do the if you do the Gaussian integral now, please note the this expression g x t eta t minus eta square d t this expression is available to a Gaussian integration. And we can do this Gaussian integration and when we do the Gaussian integration what we arrive at is the expression that is given in the green box at the bottom of you slide.

If you do this Gaussian integral here we get the expression that is in the bottom of bottom green box of the slide. By doing this Gaussian integration exponential x tilde g x t eta t minus eta square t and doing this Gaussian integral, we get the expression that is 1 by 2 eta square t g square x t dt here right.

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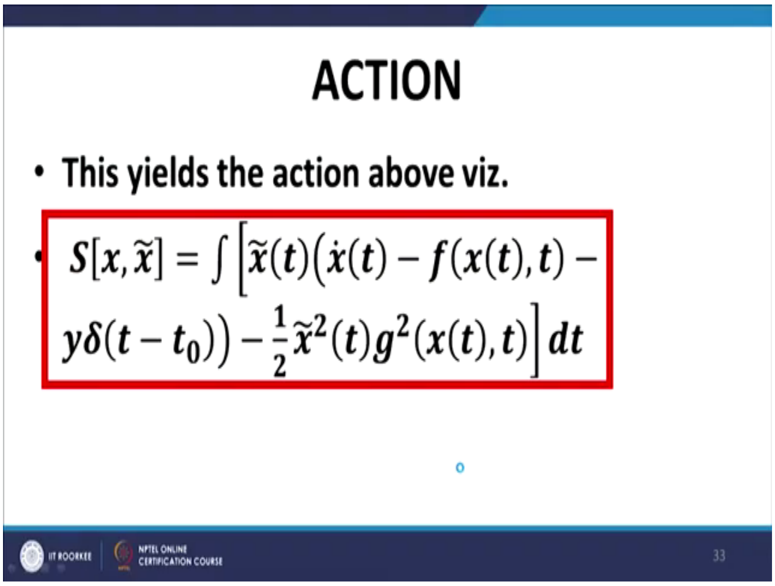

Now, this enables us to identify the action, this enables us to identify the action, as this expression which is given here in the red box on the slide. This is the action corresponding to the probability density function that we have obtained in the green box at the bottom equation of your slide.

If you look carefully the action is given by the expression in the expression that is integrated in the exponential x tilde t bracket x dot minus f and so on. This is the expression x tilde t x dot minus f and so on. And this whole term appears as the action in the context of the probability density function.

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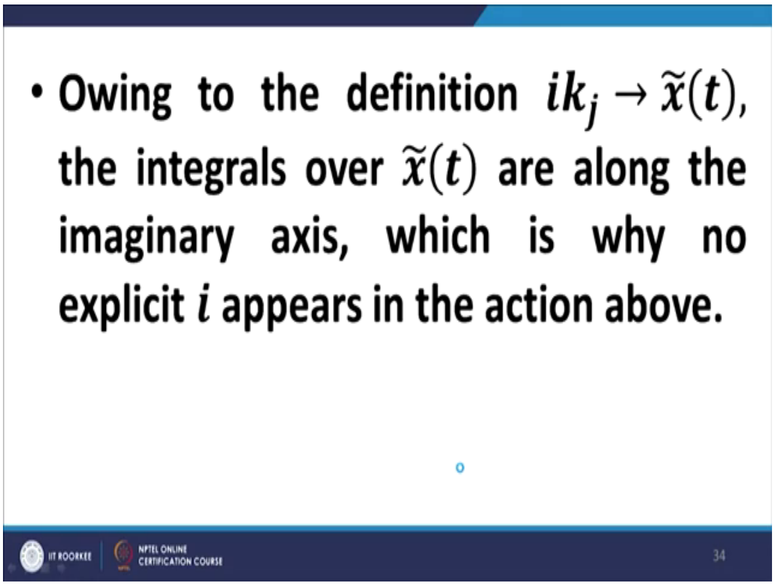

Now, you would recall that when we made this substitution ik j goes to, x tilde t, we noted that the factor i did not prefactor the integral, as it should I have this is because, the integrals over x tilde t are along the imaginary axis. And therefore, no explicit i appears in the action it is implied that x tilde t would be integrated over in the imaginary axis.

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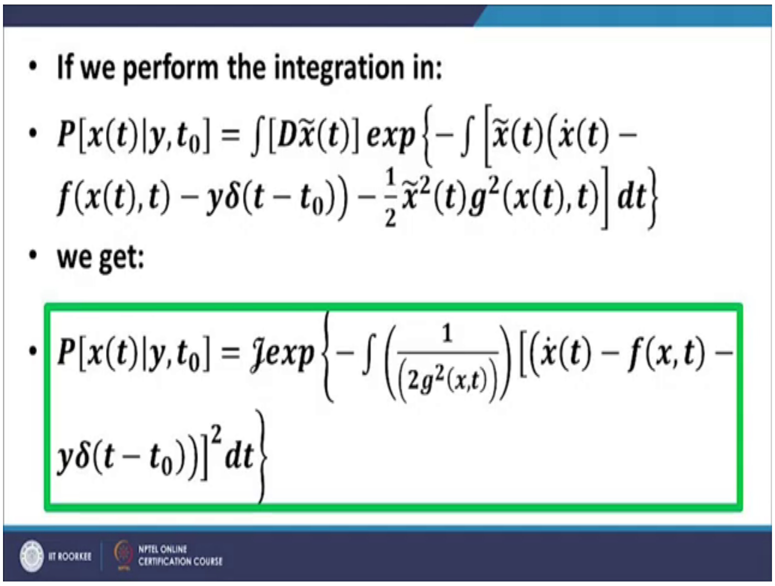

Now, if we perform the Gaussian integral here in the expression that is here in the exponentially if you perform the Gaussian integral, we get the expression that is at the bottom green box of your slide; that gives us the when if you do the Gaussian integral with respect to D x tilde t, the path integration with respect to x tilde t, if you do this integration explicitly the expression that we get is given in the blue box in the green box at the bottom of your slide.

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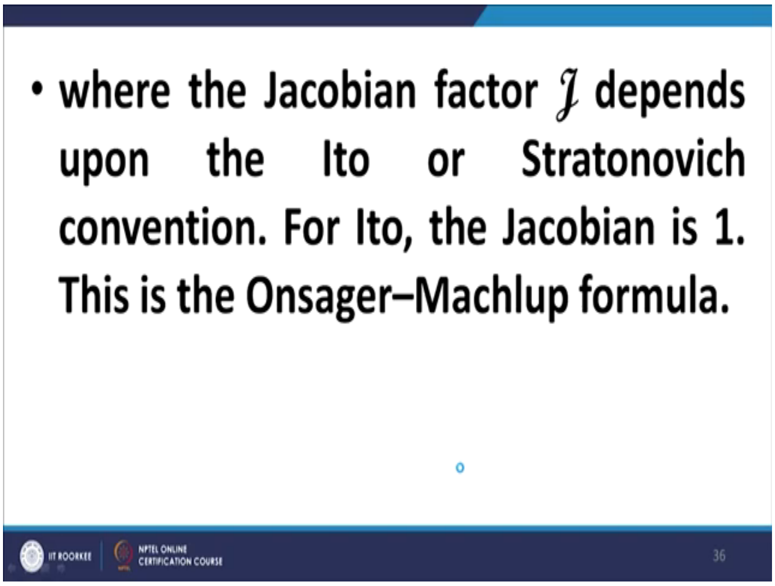

Where, J is the Jacobian and it depends on the convention that we follow whether the Ito convention or the Stratonovich convention. And the Jacobian is plus 1 for the Ito convention. And this formula that we have here is the Onsager-Machlup formula, this is called the Onsager-Machlup formula and this represents the path integral for the Langevin equation.

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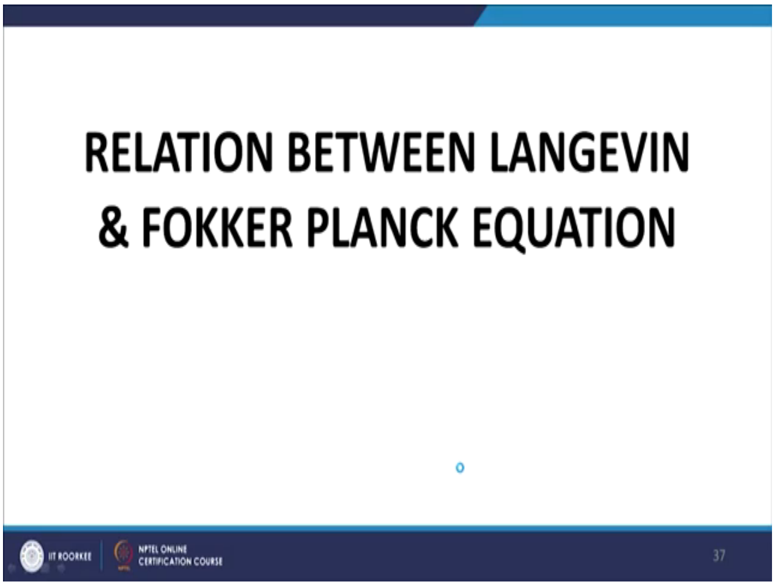

Now, we will talk about the relation between the Langevin and the Fokker Planck equation, we will take it up after the break.

Thank you.