Path Integral Methods in Physics & Finance
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Lecture – 16
Langevin Path Integral (1)

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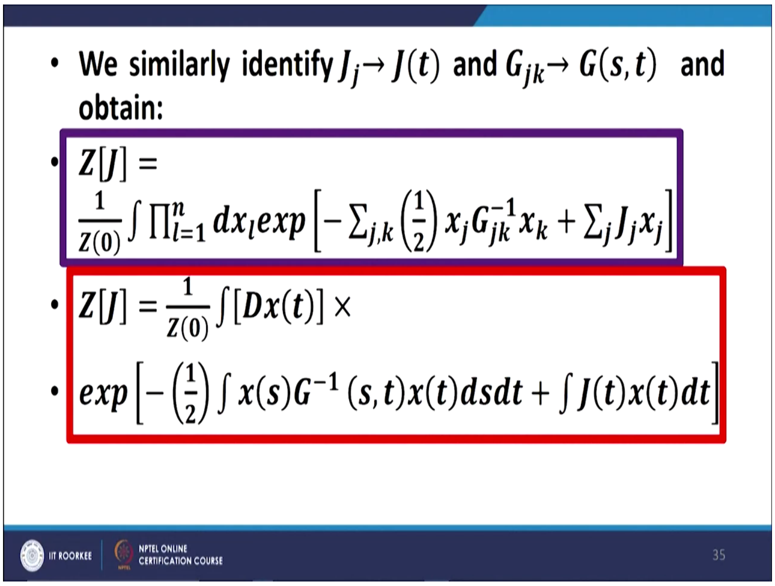

So, continuing from where we left off before the break the expression for the generating functional, now takes the form which is given in the red box, in the continuous case in the continuum; when we take the limit n tend to infinity.

The most of the part is self explanatory; the one thing that needs I mention is the integration volume; the capital D x; t within this square bracket. Now, this is the standard notation when we talk about path integrals, it represents the integration over all paths connecting any two desired points; at the initial point and the final point, all the paths that move from that move from the initial point to the final point; integration has to be carried out over each of these points, each of these paths.

And that is why in fact, the path integral has been coined for this kind of integration; it is also of course, known as functional integration.

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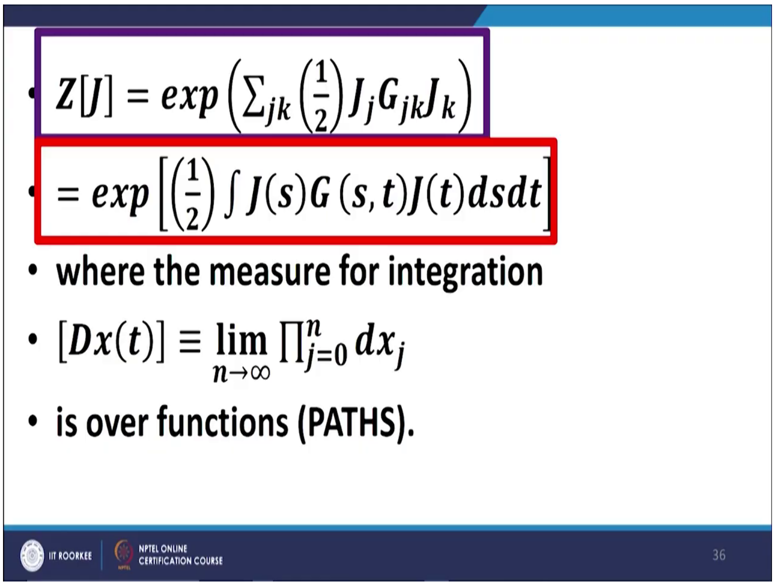

So, that you can see here in the formula D; capital D x of t is equal limit; n tending to infinity, the products of D x 1, D x 2, D 3 up to; D n with n tending to infinity. So, that simply amounts to an integration over all paths; connecting the two points under reference.

Now, as the in the earlier case as in the case of discrete variable discrete random variable when we had the generating function and we could obtain the moment simply by differentiating.

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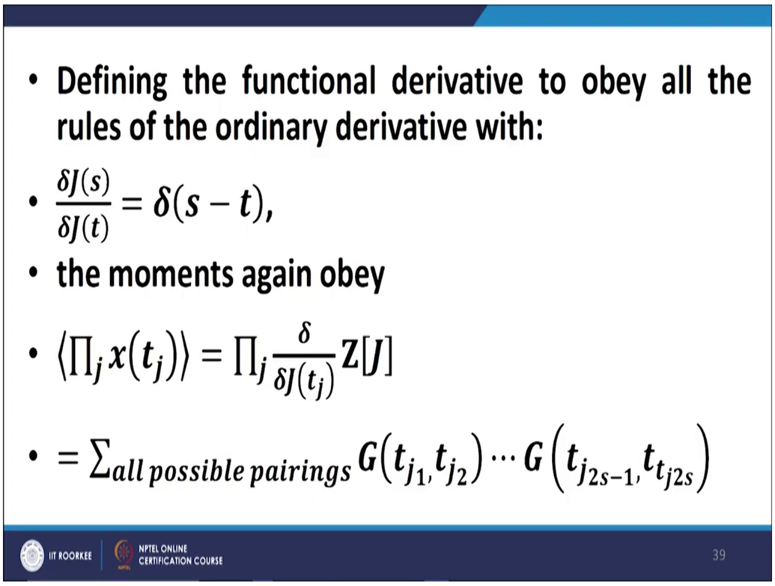

Here of course, we use the concept of functional differentiation which is the extension of the standard calculus of; standard differential calculus its slight variant of standard differential calculus.

In here in, we talk about differentiation of functionals and the primary difference between the contemporary differential calculus and the Newtonian calculus that is and the functional calculus is this first equation that you have. Delta G of s differentiated with respect to delta G of t gives you the direct delta function. You know had it been Newtonian calculus, it would have given you the Kronecker delta function; the discrete delta function.

However, we replace that with the direct delta function; most of the other rules are same. So, the moments as obtained, are obtained as earlier, except for the fact that instead of conventional differentiation; we use functional differentiation, we will use the functional derivatives.

I shall try to incorporate some examples of functional derivative because that forms the fundamental premise, when we move on to quantum field theory in one of the next few lectures; that is what is being applied in the current context.

Now, our vectors x and j were functions of an underlying variable time t. We could also have a situation where these vectors x and j; instead of being functions of time t are functions of a vector. For example, the standard situation that we encounter in field theory is when the field is defined at every point in space time.

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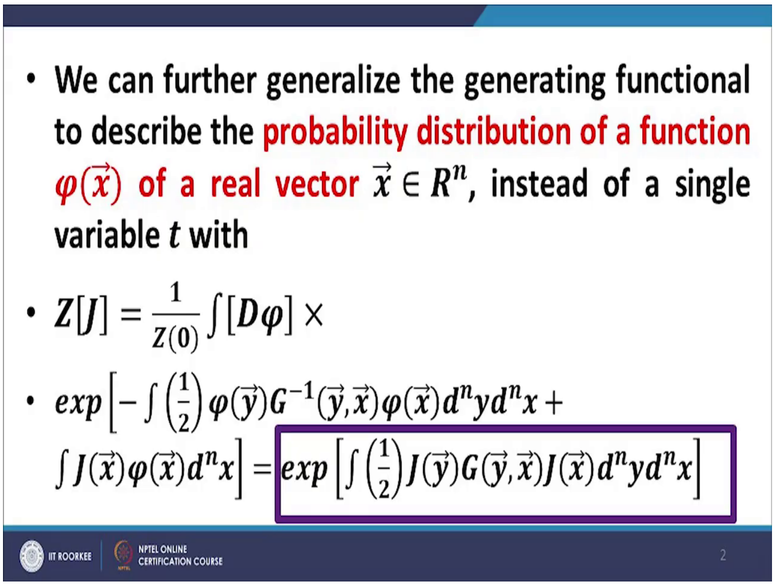

So, the field variables; then become functions of space time vectors and that is the next level of generalization that we shall be talking about. In this case, if you look at this carefully; the path integration is being; has been replaced from x; t to the field variable phi. Now, phi itself is a function of x the; and x is some space time point. So, that is the next level of extension; x is a vector in itself and phi is a function of this vector.

So, that is the next level of extension that we shall invoke, when we talk about more about this path integral framework; particularly when we work out the path integrals of various quantities in quantum field theory.

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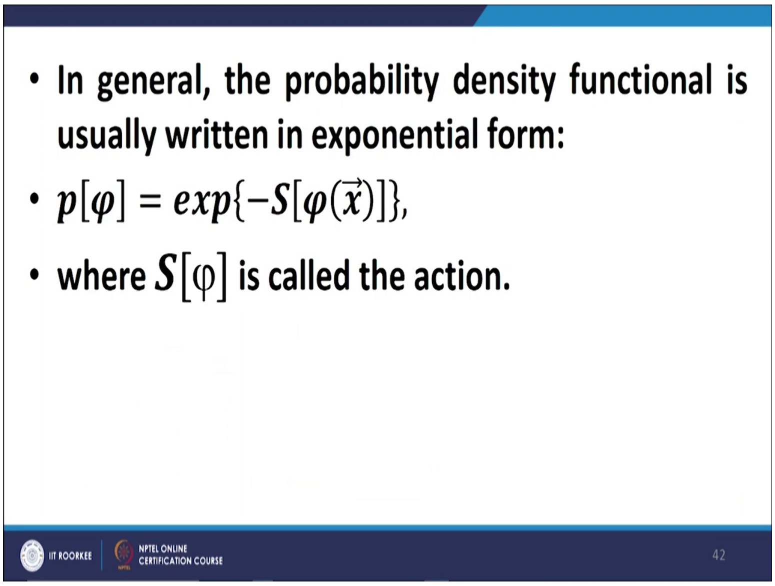

Now, again this has been borrowed from physics; in the sense that the probability density function is usually expressed as the exponential of the negative of S. What is S? S is the action functional and the action functional itself a function of phi. Phi is a field variable and x; phi is a field variable which itself is a function of a space time variable.

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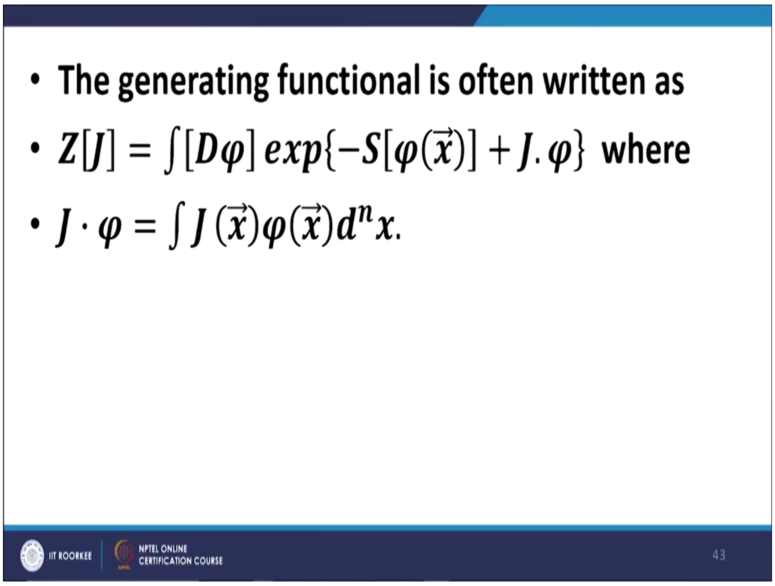

So, in that under that premise or with that assumption; the generating functional can be expressed as an integral; D phi exponential minus S; phi of x plus j phi. Now, recall instead of x of t; we now have phi of x, where x itself is a vector where there we had x of t where t is a scalar. Now, we have phi of x, where x is now taken over as a vector position and phi of course, can have a number of components as well right.

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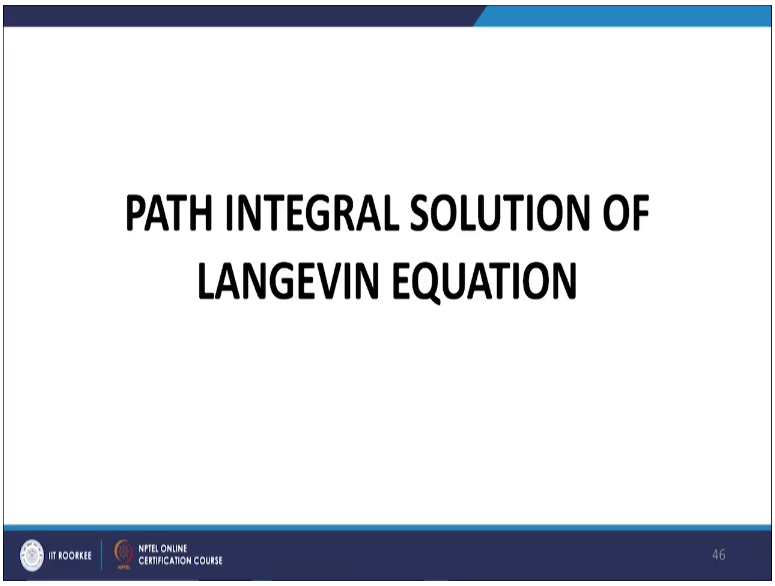

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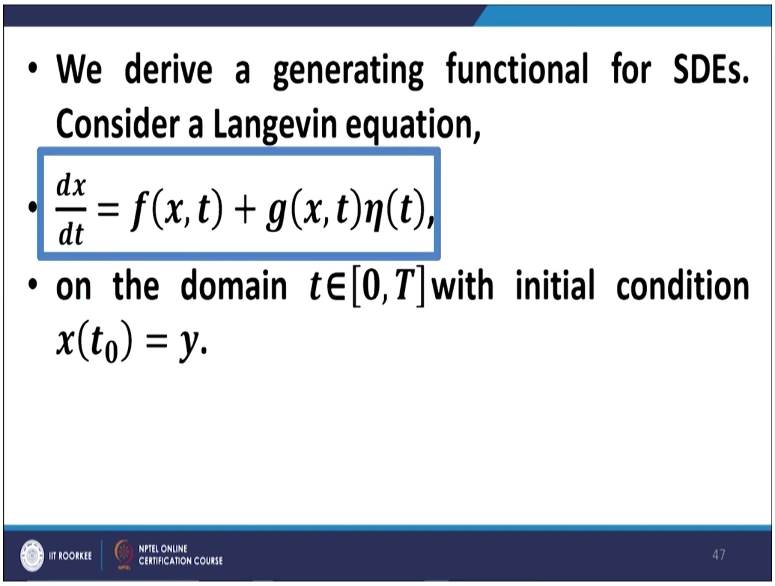

Now, we approach the path integral solution of the Langevin equation. We have got this equation; d x upon d t is equal to f; x t. This is the general form of Langevin equation; d x upon d t is equal to f; x, t plus g; x, t; eta t, eta t is Gaussian white noise and we have the initial condition x; t 0 is equal to y, we need to solve this equation.

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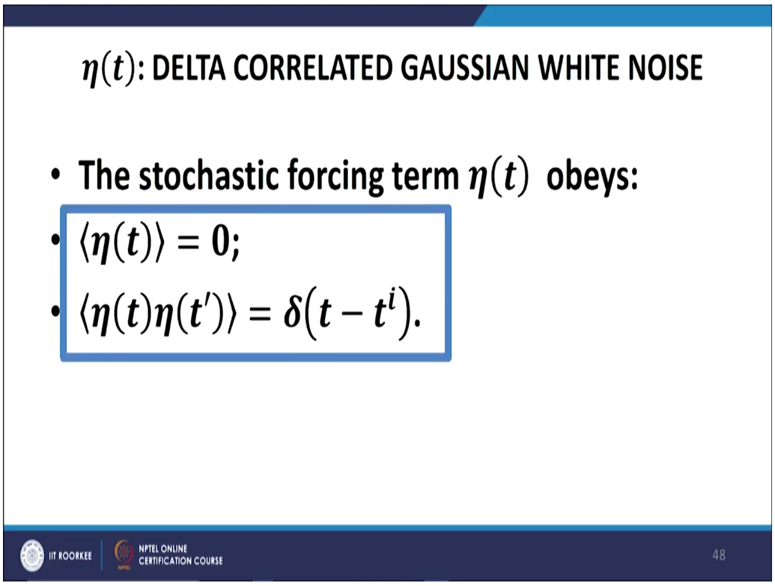

The white noise term; obviously, standard Gaussian relations the average eta t is equal to 0 over the ensemble and the auto correlation functions are delta correlations; direct delta function, they are delta correlated.

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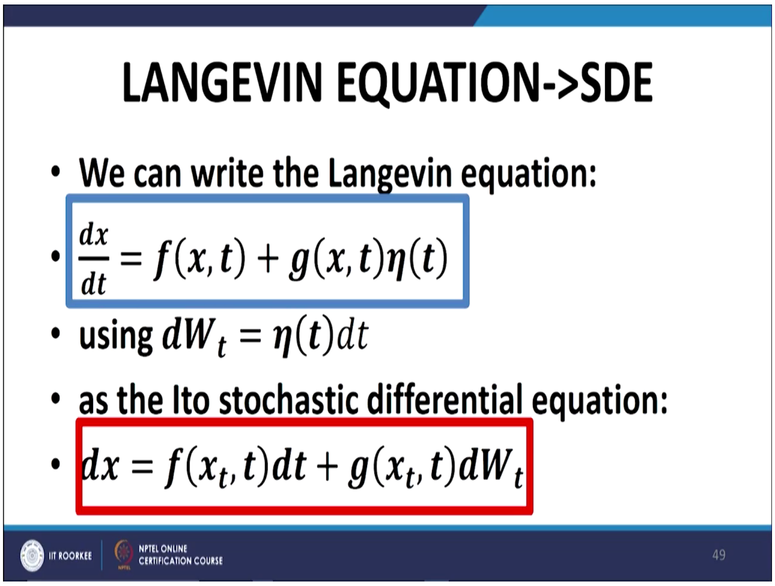

Now, how do we define eta t? Well, eta t in some sense is defined as the derivative of Brownian motion with respect to time. Now, the important thing I must emphasize here is that; as we discussed when we talked about Brownian motion, the zigzagging of a Brownian motion is so much that the Brownian motion is with probability 1, it is differential nowhere.

However, we invoke this expression which amounts to, in some sense; a non rigorous sense eta t represents the derivative of Brownian motion with respect to time; in a very non rigorous sense, in a very rough sense. And that is what is good enough for a lot of work which is done on; which is which involves the use of Brownian motion, it is called white noise; Gaussian white noise.

The Ito equation corresponding to this Langevin equation is; it can be written as d x is equal to f is simply multiplying throughout by d t and we write eta t; d t is equal to d W t, the Brownian motion; infinitesimal increment right.

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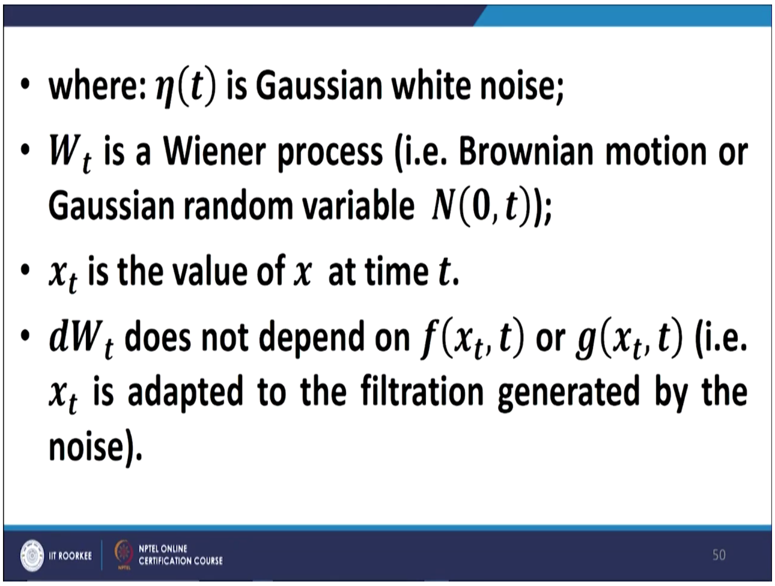

So, our objective now the important thing there is one more thing d W t does not depend on f; x t or g; x t and this is important. And that means what? That means, that x t is adapted to the filtration means in some sense in a rough sense it represents a path; x t is adapted to the filtration generated by the noise eta t, our objective is to arrive at an expression for x t right.

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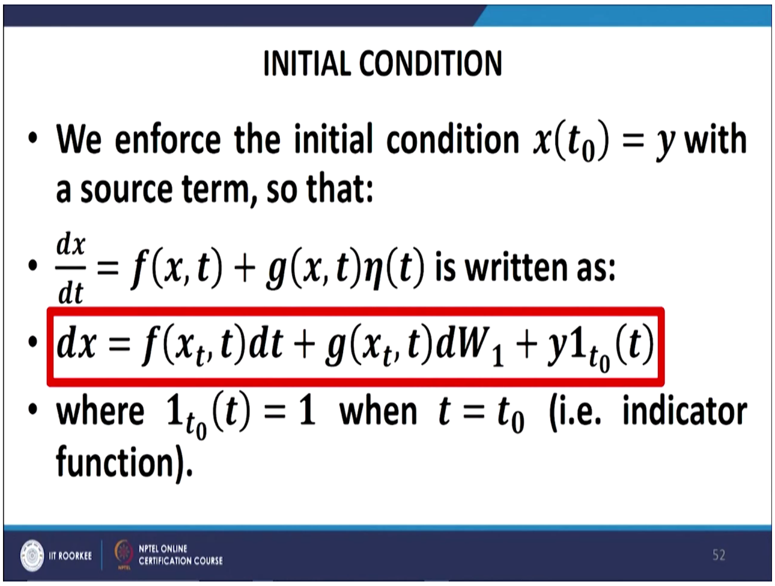

Now, we have this initial condition x; t 0 equal to y; we impose the say we assume that we use the initial condition x; t 0 is equal to 0, but to incorporate this initial condition, we introduce another term into the eta equation which tent amounts to this initial condition by and this is a source term which can we write it as y; indicator function at t 0 of t. In other words, what is this indicator function? I 1 up to t; 0 t, this takes the value 1 at t equal to t 0 and 0 elsewhere.

So, when t is equal to t 0; this factor of y will automatically become a part of this equation and when t is not equal to t 0, this will not contribute to this equation. So, this is another form of introducing this initial condition which we had; which we demanded that the system need to follow. Instead of using x; t 0 equal to y, we use it we represented in this form in the SDE.

Now, we come to discretization; we move from discretization to continuum just few minutes back.

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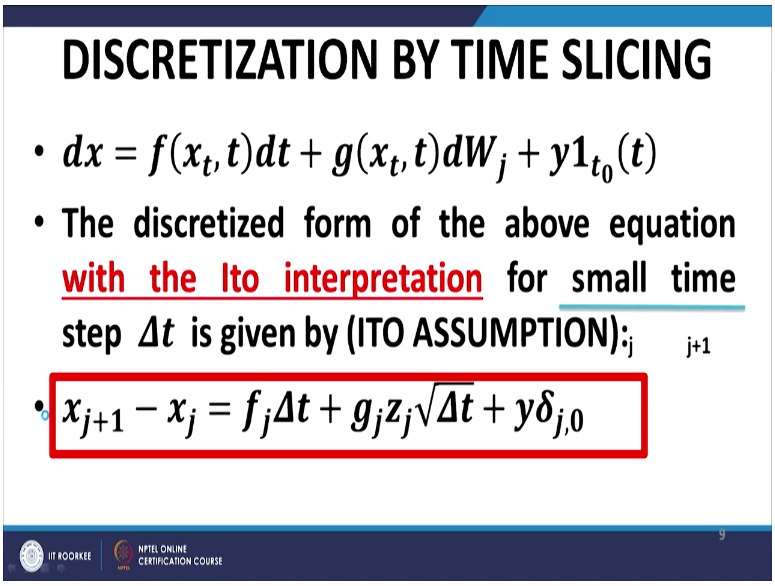

Now, we move backwards; we discretize this stochastic differential equation right. So, having impose the initial condition; we now do a discretization of the; of the problem through try time slicing. As is the usual procedure, we split up the timeline into small infinitesimal, but discrete time increments.

And we were use the; take the values at any arbitrary point in a; point on the timeline, now it is a discrete; now it comprises of discrete points. And therefore, d x will be equal to the change in value over at time slice that is equal to x j plus 1 where j is any arbitrary value between 0 and n.

So, x j plus 1 minus x j; this represents d x and similarly the other quantities are worked out; f becomes f j, the value; the initial value of the function into d t, now becomes delta t; g similarly, to similar to f; it becomes g j. Now, d W t; when we talk about d W t, d W t is the Brownian motion increment.

And Brownian motion increment as I mentioned when I talked about Brownian motion can be expressed or can be represented as z; under root delta t, where what is z? z is the standard normal variate with a mean of 0 and a variance of 1. So, that is precisely what is put here d W t is put as z j; under root delta t and the, the initial condition is written as y, delta j 0; if j is equal to 0, then we get a contribution of y to the problem to the equation and if j is unequal to 0, then this term goes out. So, this represents the initial condition; the last term in the red box, alright.

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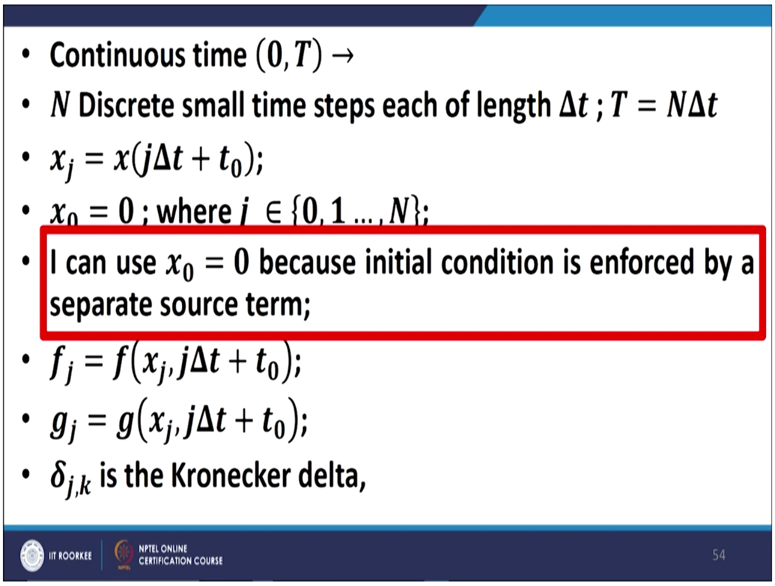

And now because each time slice is of length delta t and the total time length capital T; 0 to capital T, a total time; interval 0 to capital T consist of capital N such slices. So, we can write x j; as x, j delta t plus t 0; j delta t, t 0 is the initial value at start wherever the system starts between the interval 0 to t, wherever the system start between the interval 0 to t, that is t 0.

And j is the number of steps; since it starts that is from t 0, it is the number of steps, each step is of length delta t. So, the total distance is; total time length is j delta t plus t 0; when you look at the, when you work out the time scale or the time length from the origin from the t equal to 0 point.

Now, x 0 is equal to 0 because we have imposed the initial condition through another mechanism. We can afford to take x 0 equal to 0, we have imposed it as a separate; the initial condition has been imposed as a separate source term using an indicator function.

So, as far as x is concerned; we can take x 0 equal to 0 and f j is equal to f; x j, that is j delta t 0. And similarly, g j can be written as g; x j, j delta t comma t 0. This is you see, we are starting at the time point t 0, we are moving j steps forward in time; the time length is each step is of length delta t. So, the total length becomes j delta t and because they started at t 0, the total time elapse; since t equal to 0 is t 0 plus j delta t and that thus x value or the space lies at this point is taken as x j because we are talking about the movement from x j to x j plus 1.

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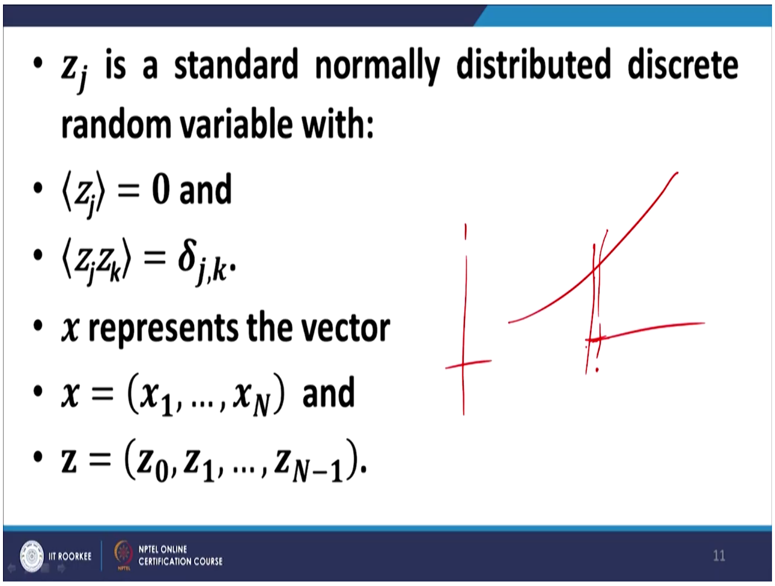

z j; as I mentioned a standard normal variate with a mean of 0 and a variance of 1 and the correlation; then all these z j's; in other words z 1, z 2, z 3, z 4 are all independent. And therefore, they have a correlation of 0, if they are unequal; of course, if z j z j has a correlation of 1 with itself.

Now, as we discussed earlier, x represents the vector x 1, x 2, x N; z represents the vector z 0; z 1 up to z N minus 1.

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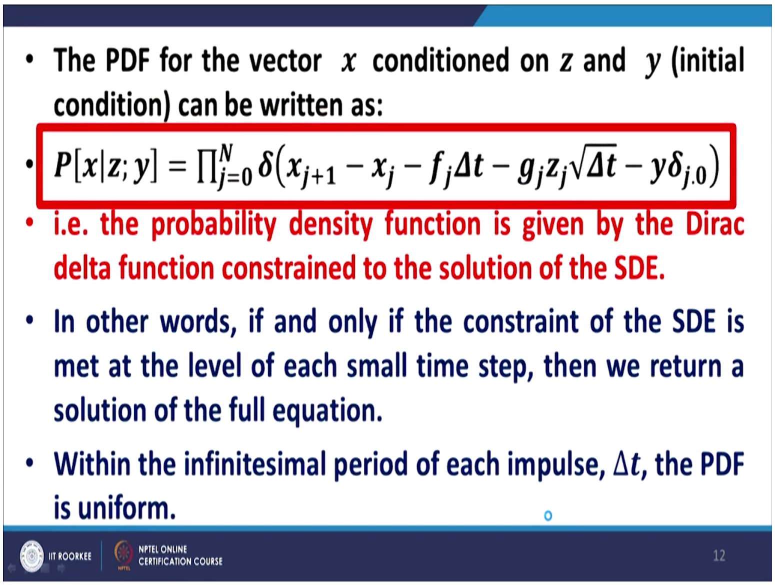

Now, the PDF for the vector x; now this is the most important slide here, the PDF for the vector x conditioned on z and y; what was y? y was the initial condition and z is the sequence, z is the vector. What does this vector consists of? This vector consists of z 1, z 2, z 3 and what are these z 1, z 2? These are the random variables corresponding to the various time slices constituting which constitute the total time line 0 to t.

Remember, we split up or partitioned the total time 0 to capital T into small infinitesimal slices of each of length delta t and we introduce this z j variables at as normal; standard normal variables, z j being the representing the random component in the process at the point j.

So, given certain respective values of z j, f and g and of course, the initial condition; the delta function will enable us to impose the constraint that the system meets that requirement, the; meets this stochastic differential equation, meets this stochastic differential equation. And therefore, we can write P; x subject to z, z taking as a values; z 0 taking z is a vector and z is component; 0; z 0, z 1, z 2, each of them may take different values, they may have different realizations.

And the probability that x takes or probability of x taking a particular value, subject to z taking a sequence of values and the initial condition y is given by the expression on the right; which is nothing, but which is nothing, but the; the requirement, the mandate that the system meets the system satisfies the prescribed stochastic differential equation.

In other words, we can say that the probability density function, the left hand side is given by the direct delta function; constrained to the solution of the SDE; correct.

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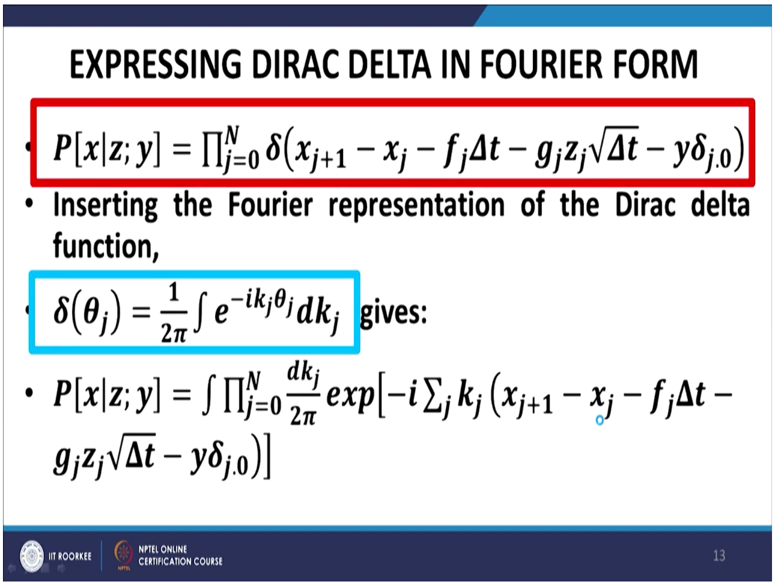

So, now our next step is that we have a direct delta function here; we have a representation of the direct delta function, as you will recall in terms of exponentials; which is given in the blue box here. Delta, theta is equal to 1 upon 2 pi integral; e to the power minus i k theta, d k. What we do is; we express this delta function in the exponential form, as given that gives us the expression in the green box.

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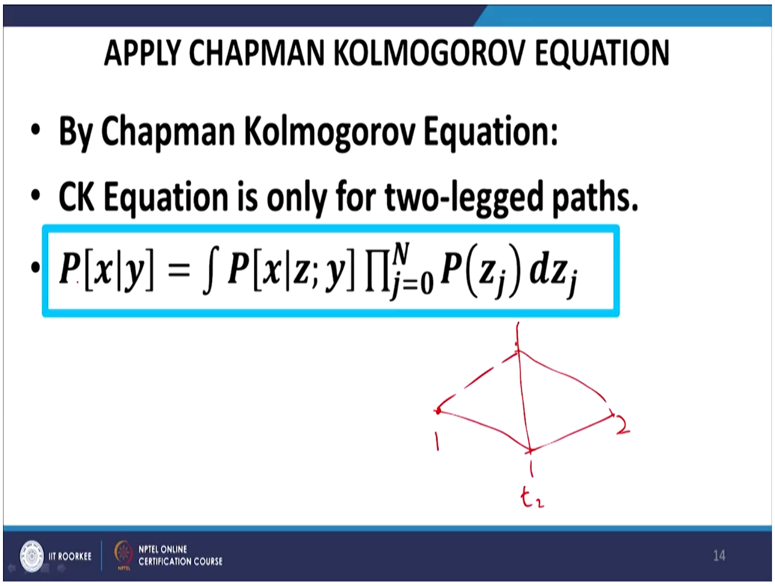

Now, we apply the Chapman Kolmogorov equation; remember, what is the Chapman Kolmogorov equation? The Chapman Kolmogorov equation is that the probability that a system moves from state 1 to a; at t equal to say 0, to a state 2, at t equal to t; say t equal to capital T is given by the sum of the aggregate of the probabilities that the system moves from the state 1 at t equal to 0 to an intermediate state.

If you take a particular time slice; let us say if you take a particular you time slice, t equal to t by 2 or t by 4, t by 6; whatever you take any arbitrary, but fixed time slice you take and you construct all the two legged paths from the origin to the desired finishing point.

And then you sum up the probabilities over all these two legged paths and that gives you the probability or the system moving from 1 to 2. So, recall when we talked about the case of the double slit experiment, we had an additional requirement that we had summation; not only over all two legged path, summation over all any legged paths.

So, summation was a two levels; number 1 2, two legged paths; that means, one screen and then summation over all; infinite number of screens and that; obviously, amounted to the system traveling through vacuum.

Now, here it is a slightly different; here it is in a sense it is a subset of that. We are talking about one screen, you are talking about one partition time slice. And you talk about all possible paths from the; as say t equal to 1, to that particular time slice and then from there to the to the desired or to the wanted outcome; let say t equal to 2. and you work out all the probabilities over all such possible paths; two legged paths this, to this alright.

So, that is what this equation represents; precisely, what this equation represents; P; x subject to y is equal to P; x subject to z into probability of z happening and integrated over; that means, all possible paths.

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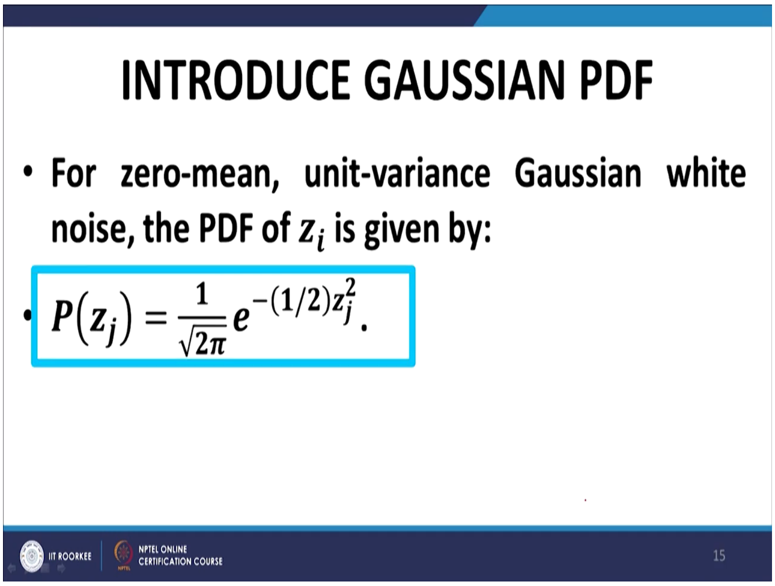

Now, this is the Gaussian PDF; so this is quite familiar issue right. So, let us before we continue a quick recap of what we have done so far.

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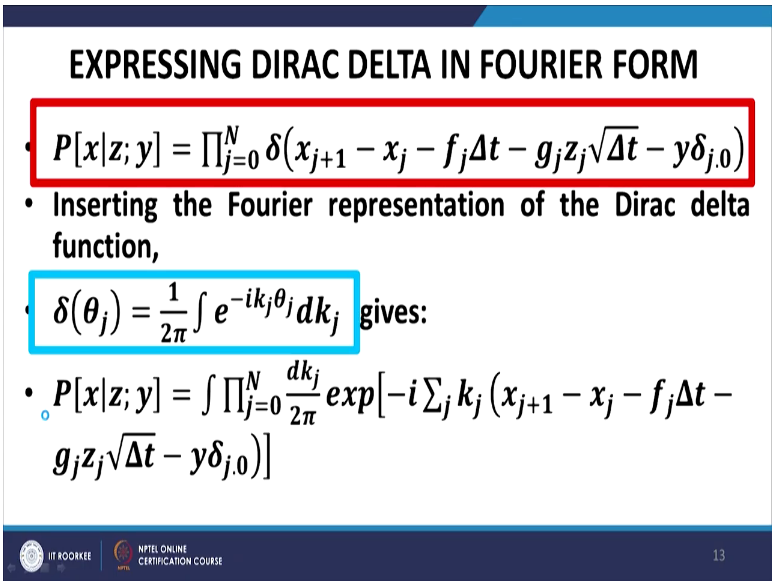

Firstly, we will introduce the initial condition in a different form by using the indicator function. And that has enabled us to rescale the in the initial condition x; t 0 equal to 0 by adding another source term to the SDE.

Then, we considered discretization of the; time line and while it is considering the discretization of the time line, just let me go back to that slide yes here it is. So, we have discretized the time line into partition; discrete partition, discrete time points; each of length delta t. And; obviously, the continuum and this discrete version of the time line correspond to each other by the limit; delta t tending to 0 or the number of partitions tending to infinity.

Now, when we do the discretization; we make certain substitutions here. For example, we may substitute x; j; delta t plus t 0 goes to x j x 0 equal to 0. I can take now because of the additional source term that I have introduced in the Langevin equation; f; x j delta t plus t 0 is equal to f t. That means, I am now expressing this in terms of the; expressing the function f, in terms of its value at a particular point on the lattice; on the discrete lattice that is, I have substituted for the continuum time line.

Similarly, for the g function, we do the substitution and we also introduce the discrete random variables, representing the randomness at the various points on the lattice. This form; they form a vector and the components of the vector represent the random variables at various time slices.

The important point that I want to make here is that; the Ito convention is followed here. See, when we talk about Newtonian differentiation; it really does not matter at what point we take the dependent variable in the infinitesimally interval.

Suppose, we are having we are integrating f x, d x; then it is simply a sum across partitions; we discretize that integral, convert into a summation and then when we take one particular layer of that summation, when we take one particular layer of that summation.

For example, if I have this random variable; these are my coordinate axis; if I take any particular layer of the summation then we assume that the variation in the function f x; across this small time slice or small slice of x is very very small it is almost horizontal line straight line. And therefore, at what point along this particular time slice of x the; we take the value of f x, makes no difference because it is assumed to be constant over this interval x and x plus d x.

Now; however, when we talk about Brownian motion or when we talk about random variables of a stochastic processes; that is not the case. As I mentioned, when I talked about Brownian motion and I had showed that a particular realization of Brownian motion. There is so much of zigzagging that howsoever small time slice you take; inspite of that the zigzagging is so much that we cannot approximate the value of the Brownian motion by taking its initial value.

We have got to follow a certain convention; either way you take its value at the initial point, the point of initiation let us say t or we take a mid point, this; these two conventions are usually followed. The first gives rise to what is call the Ito calculus, the second gives rise to what is call the Stratonovich calculus.

So, we follow the Ito assumption here; that means, we use the value of these functions, at the initial point of the time slice. Therefore, when we talk about the function f and we talk about the function g and we talk about the function g, we are taking their values at f j, not f j plus 1 or not f j plus f j plus 1 upon 2.

We are not doing that; we are taking their values at the beginning of the time slice; this is important, we need to take care of this is called the Ito convention and it shall be used again later on. But this is important point that note needs to be noted; when we talk about discretization in the context of functions that involves stochastic processes or stochastic variables right.

So, then therefore, what do we have? We have the probability distribution, the probability distribution of a vector x; subject to a certain distribution of a, of the vector z; z remember, z consist of a N components of random variables; each of which is normally distributed with a mean of 0 and a variance of 1, standard normal variates.

So, each of the z's are standard normal variates, but remember this is a vector and it has N components, just like x is a vector which has N component. And this is given by a product of delta functions which is represents the constraint condition.

So, the probability density function is the direct delta function which is constrained by the; to meet the to satisfy the stochastic differential equation. These, this is very important; in other words, across every infinitesimal time slice or time interval that stochastic differential equation needs to be met and the product of all the delta function gives us the probability density of occurrence of x; vector x subject to the vector z and the initial condition y.

Then, we express the delta; direct delta function in terms of the; its Fourier representation. The exponential form that is which is given here, in the blue box. When we substitute this, we get the expression which is given in the bottom of your slide. Nothing much, we simply substitute the Fourier representation of the delta function.

Remember, the red box shows you the expression for the probability of x subject to; conditional probability of x subject to the probability of z and the initial condition y. As a delta function, we are simply substituting this delta function by the Fourier representation of the delta function.

Now, we apply the Chapman Kolmogorov equation; recall when we talk about the Chapman Kolmogorov equation, we pick up a time slice and we sum across all the possible states that start from a certain initial point and that terminate at any point.

In other words, the probability of moving from a given initial state say x; say 1, 2 up final state 2 is equal to the probability of moving from state 1, at let us say time t 0 to a another; to another number of states say at time t 2. And then moving from this state t 2 to states at time t 2 to this state 2.

In other words, we are summing across all two legged paths; two legged paths which whose first leg terminates at a common time slice. And then from there the second leg starts and terminates at the destination that is the state 2; so, all this is my state 1, say this is my state 1, this is my state 2.

If I impose a time slice here, let us call it t 2, then the probability of moving from the state 1 to the state 2 is the sum of probabilities of all the paths; that move from state 1, to the intermediate state 2 which is represented at time t 2 and then from there; it moves to the state 2, that is precisely what the equation in the blue box represents.

Now, having when we substitute this Chapman Kolmogorov equation; we get this expression. And we also use the Gaussian distribution for the z's, just remember z's are standard normal variates with the mean of 0 and a variance of 1; so we can use this Gaussian distribution.

So, we now do two maneuvers; one maneuver is that we use this Chapman Kolmogorov equation, this particular equation. Now, please note this; now we are having the probability of occurring of x, subject to the initial condition y because we have summed over all possible z's. Therefore, now we have them on the left hand side, we have the probability of occurrence of x subject to the initial condition y and there is no mention of z because z has been summed over.

In other words, we are summing over all possible values of z. So, P; z j is the probability distribution of the standard normal variate.

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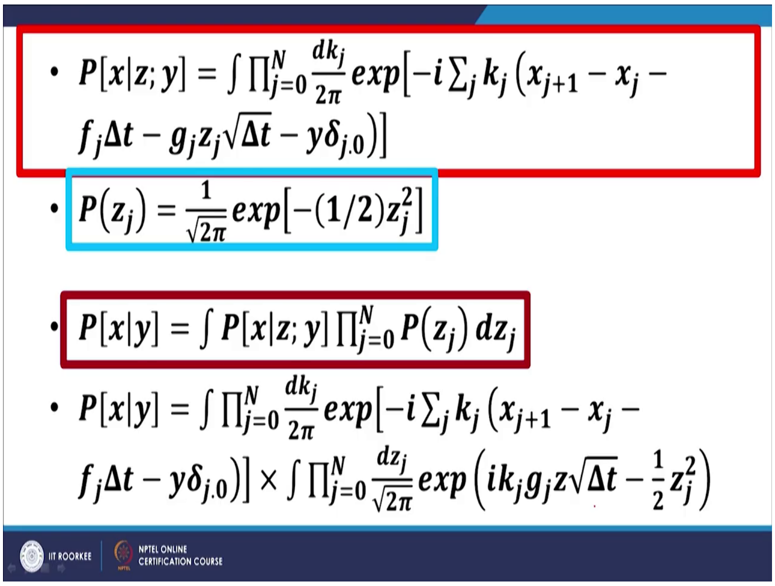

These two things, when we substitute in our first equation, in our expression for the probability P; x subject to z; y, we make use of the Chapman Kolmogorov equation. We make use of the standard probability distribution for the normal variate z's, we arrive at this expression.

Therefore, the probability of occurrence of x; subject to the initial condition y; probability of occurrence for of x subject to the initial condition y. This is what we have; the last equation on the slides gives you, this particular expression.

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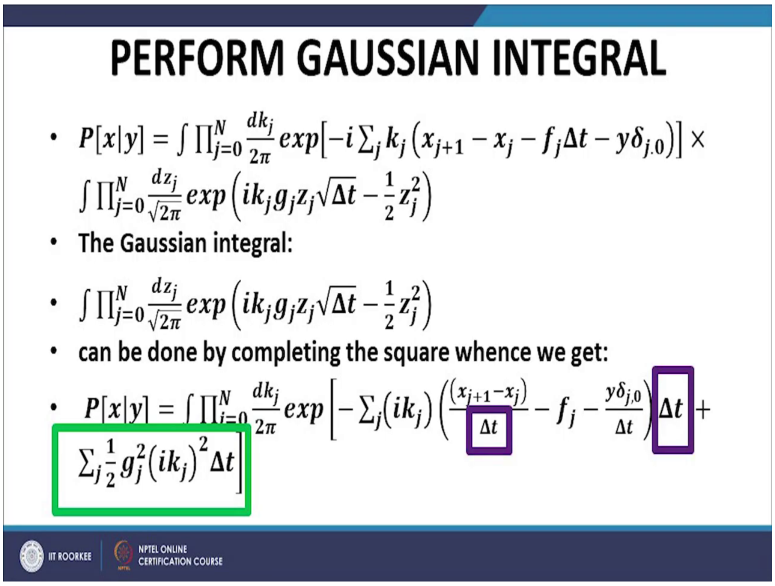

Now, now we perform the Gaussian integral; we perform the Gaussian integral. Please note, if you look at this carefully; if you look at the expression for the probability of x subject to y. We can collect the terms which contain the element z's; which contain the; remember z is the random variable, it is a Gaussian random variable; so, we can collect together the terms which contain z.

And then we can integrate over z as a Gaussian integral and what we get is this expression at the bottom of your slide. Of course, in this expression; we have multiplied and divided by delta t for a purpose which you should; which will become apparent in the next slide. But for the moment, we multiplied by delta t and we have divided by delta t.

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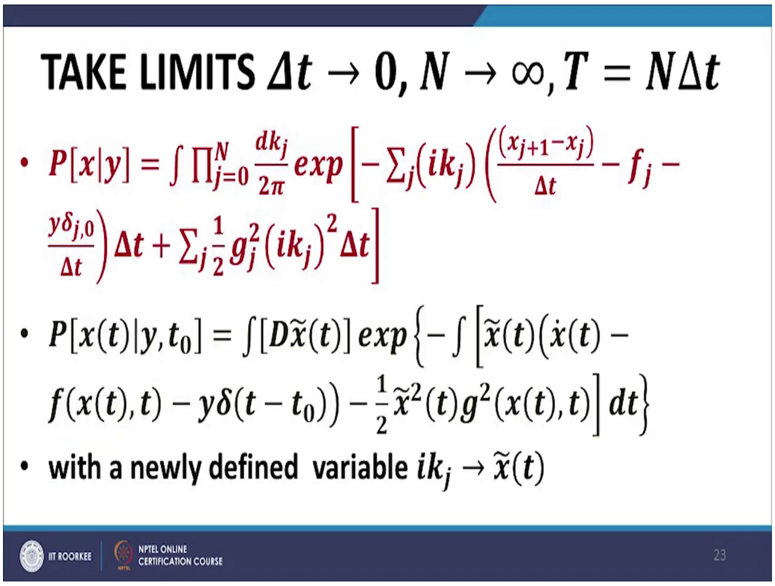

So, then when we take the limits delta t tending to 0, N tending to infinity; retaining capital T equal to N delta t, what we get is the expression on the equation on the bottom of your slide, from the expression which is at the top of the slide which we; which is carry forward from the previous slide.

You can now observe here, that x j plus 1 minus x j upon delta t is nothing, but the rate of change of x across delta t and that is therefore, being replaced by x dot. We are also making a substitution of the variable, we are introducing a new variable x tilde t to represent the; variable k; j.

And the rest of and of course, the integration volume, the integration volume is now over all paths, over all kth, as you can see here it is the product of all kth. We represent it in a different notation because we are replacing k by x tilde t and therefore, we are replacing this integration volume; product of d k j's by the path integral which is represented by this square bracket capital D; x tilde, t right. We will continue from here.