Path Integral Methods in Physics & Finance
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Lecture – 15
Statistical Formalism of Path Integral

Welcome back. In the last lecture, I introduced the Langevin equation and discussed various nuances, various issues relating to the equation, how the expected value of this square of the velocity blows up and therefore, we need to incorporate a damping term in that equation.

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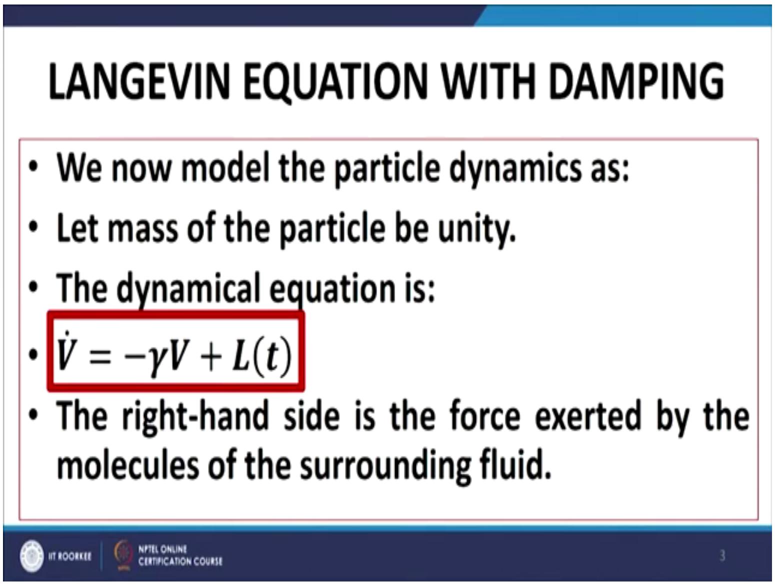

And we therefore reduce the equation to the form that is given in the red box here V dot is equal to minus gamma V plus L t, where L is the stochastic force, a force that axes random intervals on and that represents in fact the collisions between the particles in the fluid with the Brownian particle or the molecules of the fluid with the Brownian particle.

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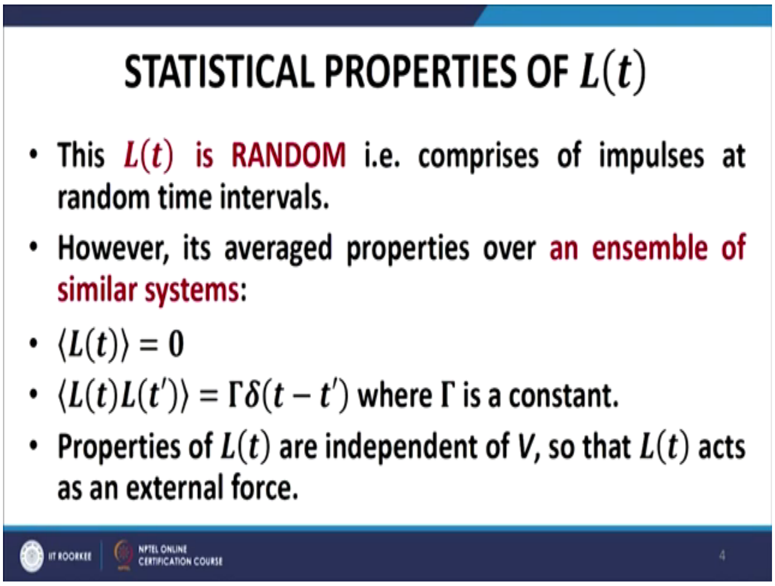

And, we express the statistical properties of L t as on the average over an ensemble. The average of L t would be 0 if it had a residual drift, the drift would be captured by the gamma term and the auto correlations at different points in time would be gamma delta correlated with a scaling that it would take care of the magnitude of the impulse of the driving force random force.

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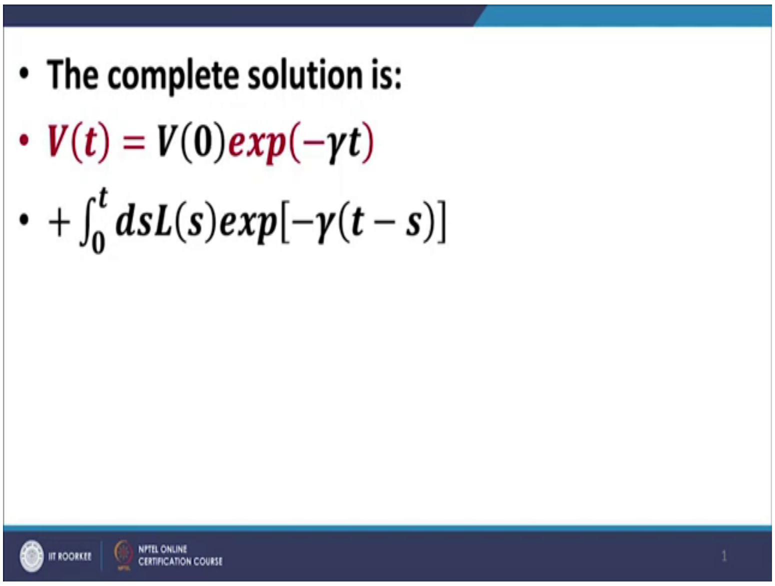

I also obtain the solution of the Langevin equation in the form that is given on the slide V t is equal to V 0 exponential minus gamma t plus integral of the noise term or the fluctuating force term with the exponential minus gamma t minus s, right.

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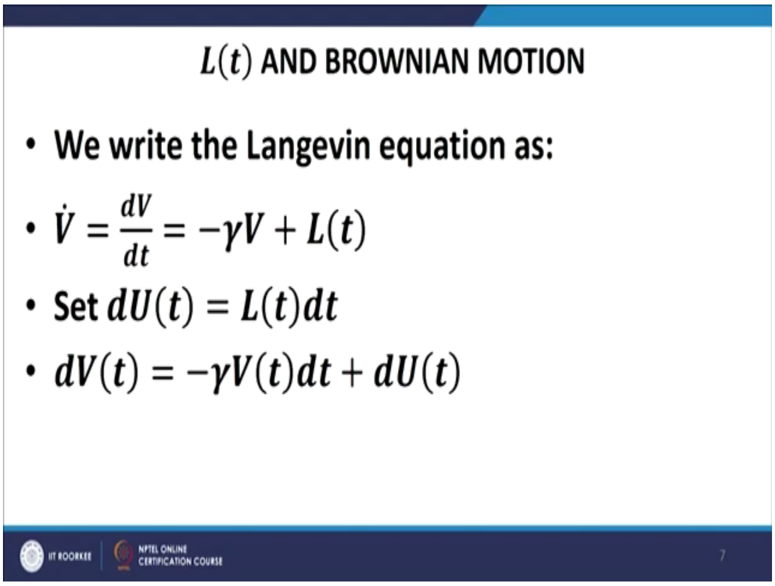

Then we proceeded also to discuss the relationship between Brownian motion and the fluctuating force L t that represented the collisions of the molecules with the Brownian particle. We expressed d U t as the L t d t and then we investigated the properties of U t and we found that U t had all the properties under the assumptions of this model, under the features of this model U t had all the properties that make it compatible with the definition of Brownian motion.

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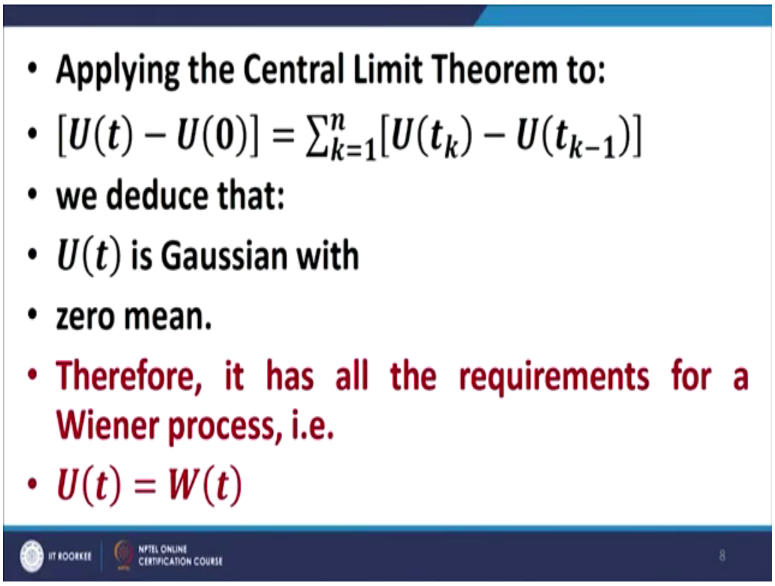

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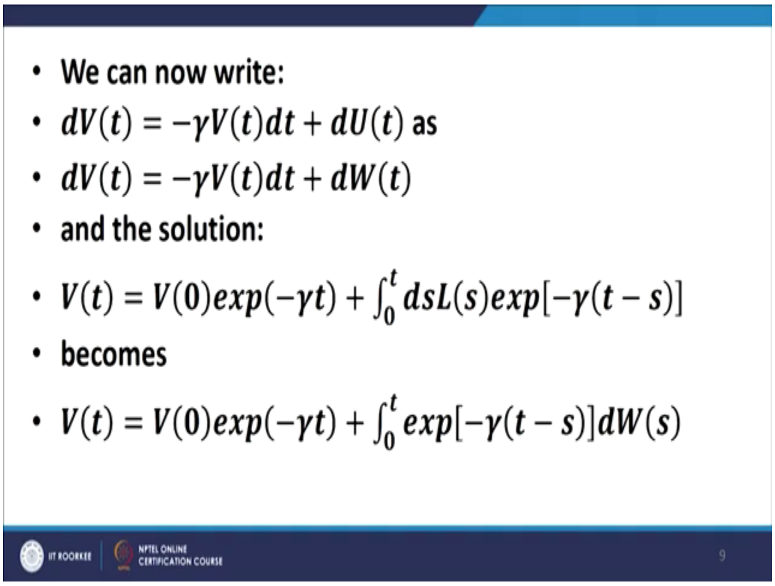

And, therefore we could it was Gaussian with zero mean and therefore, we could express the solution of the Langevin equation for the velocity with a Brownian motion integral as V t is equal to V 0 exponential minus gamma t plus integral 0 to t exponential minus gamma t minus s d W s. This d W s is nothing, but the infinitesimal Brownian motion increment.

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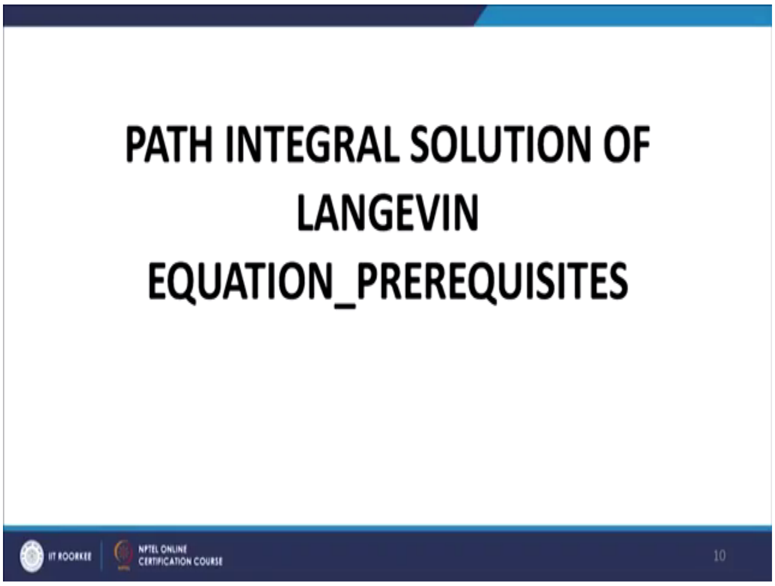

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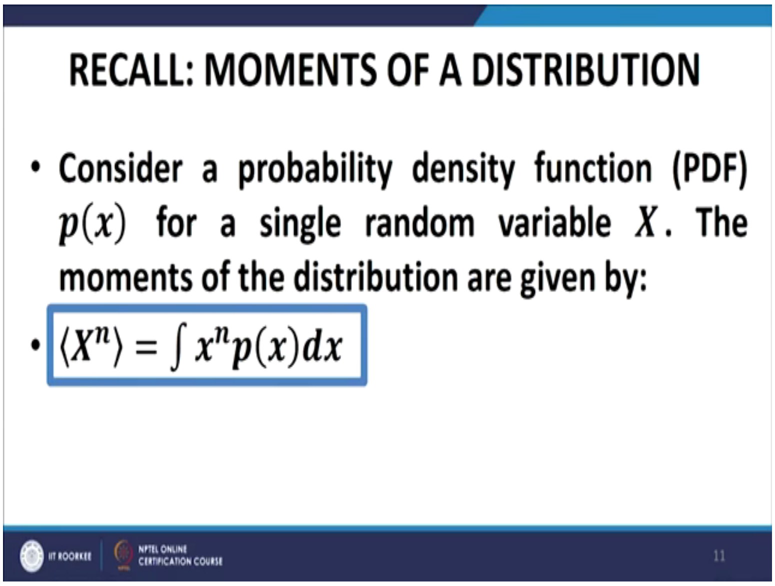

Now, we proceed further we obtain the path integral solution of the Langevin equation. We start with a few prerequisites. In fact, part of these prerequisites we have discussed earlier, but to facilitate continuity we are quickly going through or quickly recapitulating these definitions and then we will move on to the explicit derivation of the path integral for the Langevin equation.

So, as far as moments of a distribution are concerned, if the distribution is normalized, then we have it as X n expected value of x n where x is the random variable is equal to integral. Over all possible the entire spectrum of values that the random variable could take x to the power n p x, where p x is the relevant probability density function. We are talking about continuous density functions or continuous distributions which are more relevant to our context.

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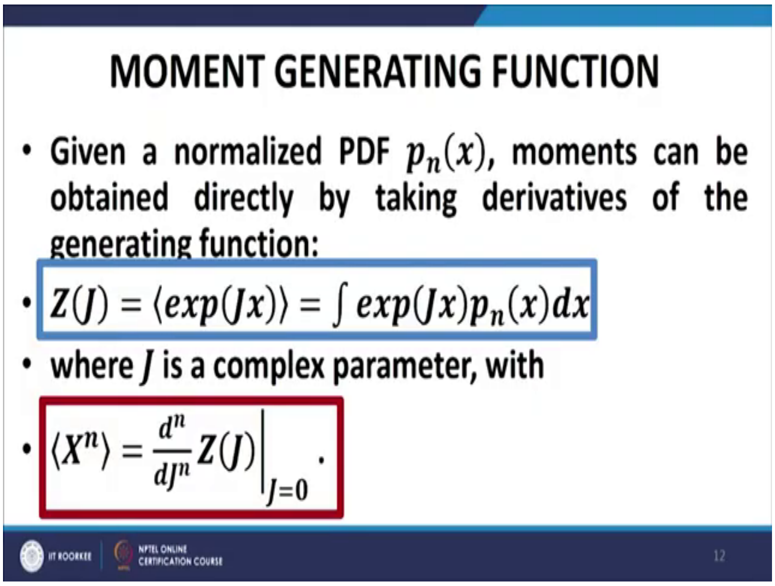

Then the moment generating function if you recall is nothing, but the expected value of J x. Earlier I used the expression t x, but now because we are having an application of we moving towards the applications and functional integrals in terms of path integrals, we as a method of as choice or as convention we use J for the source term and therefore, we write Z of J; Z of J is the moment generating function is the expected value of expectation of J x.

And that takes this form where p n x is the normalized probability distribution function. Using these two discriminate between to accommodate density functions which are not normalized, which are unnormalized density functions we encounter situations like that when we do a perturbative expansions. So, we need to be careful with that. And of course we can recover the moments by taking derivatives of Z J and then putting J equal to 0 in the resultant expression.

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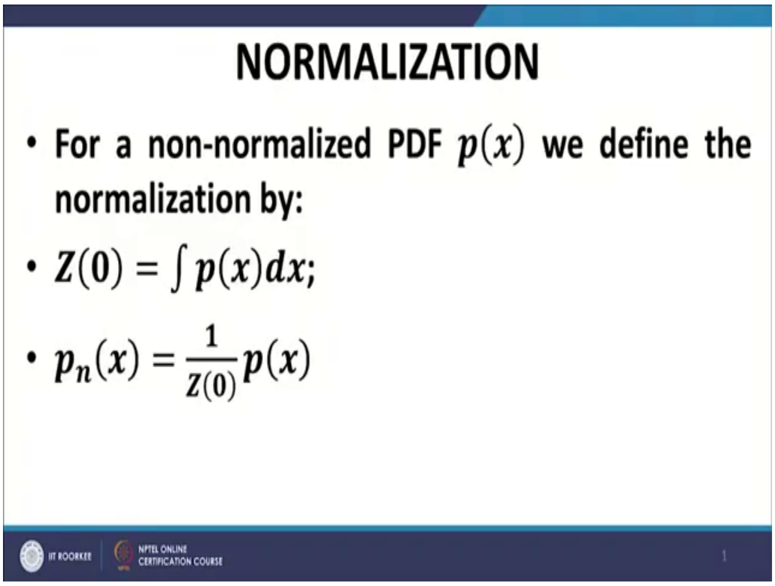

As I mentioned for a non-normalized probability density function p x, we define the normalization by Z 0 which is integral p x d x which in the normal course would be 1. If the probability density function were normalized, Z 0 would be equal to 1, but in the even that the probability density function is not normalized, we have an explicit provision for Z 0.

So, p n x the normalized density function corresponding to p x which is the unnormalized density function can be written as 1 upon Z 0 integral p x and d x.

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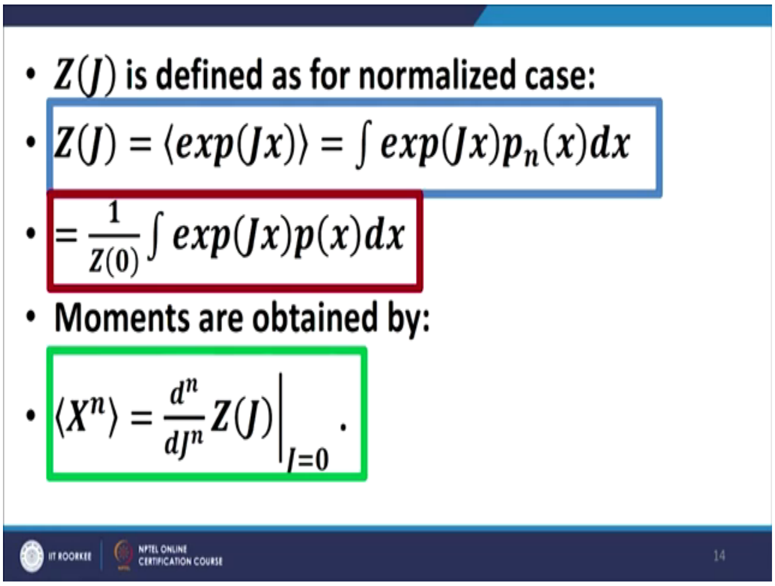

And for a non-normalized probability density function, we have another normalization factor creeping up as a pre factor when we define Z J. We define Z J as 1 upon Z 0 integral exponential J x p x and the expectation is defined with respect the normalized density function and therefore, we have a factor of 1 and 1 upon Z 0 coming up outside the integral representing normalization of the given probability distribution p x to p n x. And the moments are obtained as in the earlier case by taking the nth derivative of Z J and substituting J equal to 0.

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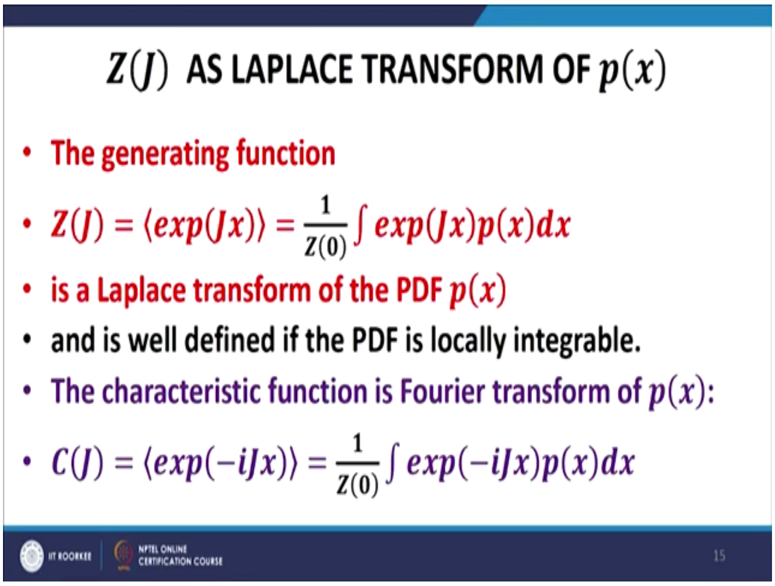

Now, as you can see from the expression for Z J, Z J is the Laplace transform of the probability density function p x, it is explicit here quite obvious in this expression for Z J that it represents the Laplace transform of p x. And if you recall we also had the characteristic function which was similar to the moment generating function which happened to be the Fourier transform of p x.

So, we have two different two variants really of the same expression Z J is the Laplace transform of p x and C J which is the characteristic function is the Fourier transform of p x.

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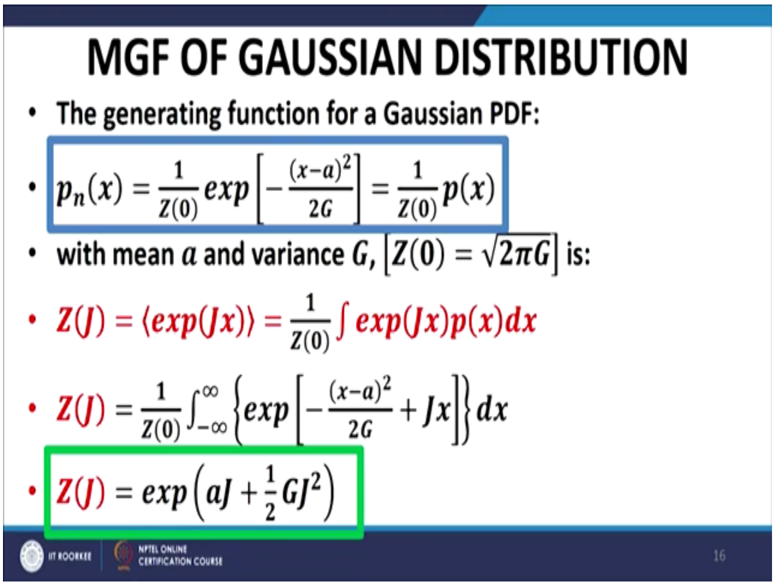

Now, for Gaussian distribution the expressions for the moment generating function and characteristic functions and take a particularly simple and a particularly valuable approach. Valuable in the sense that we phaser this in a number of applications and if we work it out by completing the square, we find that the value of Z J is found to be exponential a J plus 1 by 2 G J square, where a is the mean of the Gaussian and G is the variance of the Gaussian. So, Z J is equal to exponential a J plus 1 by 2 G J square where a is the mean of the Gaussian, G is the variance of the Gaussian.

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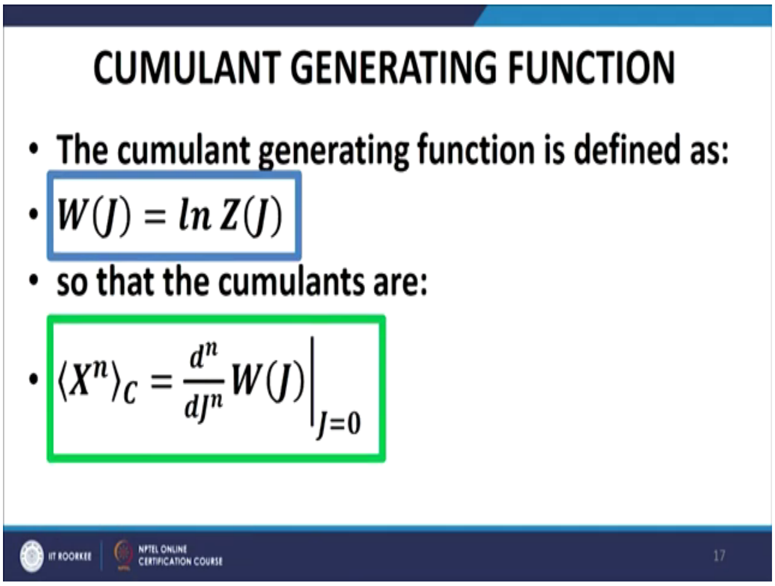

Then we have the cumulant generating function, this will also be used extensively when we study Quantum Field theory and it is nothing, but the logarithm, the normal logarithm of the moment generating function W J is nothing, but log of Z J and the cumulants are recovered in the same manner as the moments are recovered from the moment generating function by taking derivatives of the cumulant generating functions and then putting J equal to 0.

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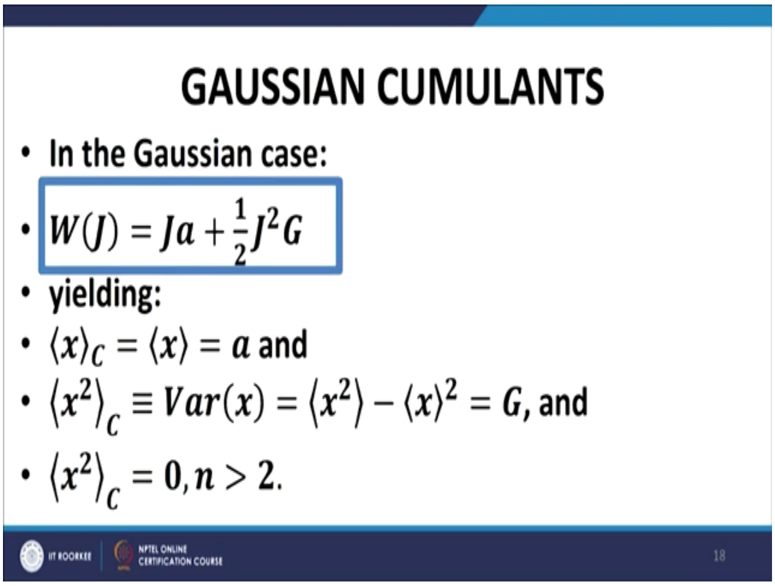

In the first few cumulants as you can see the first cumulant is the mean for the Gaussian distribution indeed the cumulant generating function for the Gaussian distribution is J a plus 1 by 2 J square G. The first cumulant as the mean, the second cumulant happens to be the variance which is G and all the other cumulants because the cumulant generating function for the Gaussian is quadratic is only quadratic in J. The remaining cumalants would vanish. We only have the first two cumulants. The first cumulant is the mean a and the second cumulant is the variance G.

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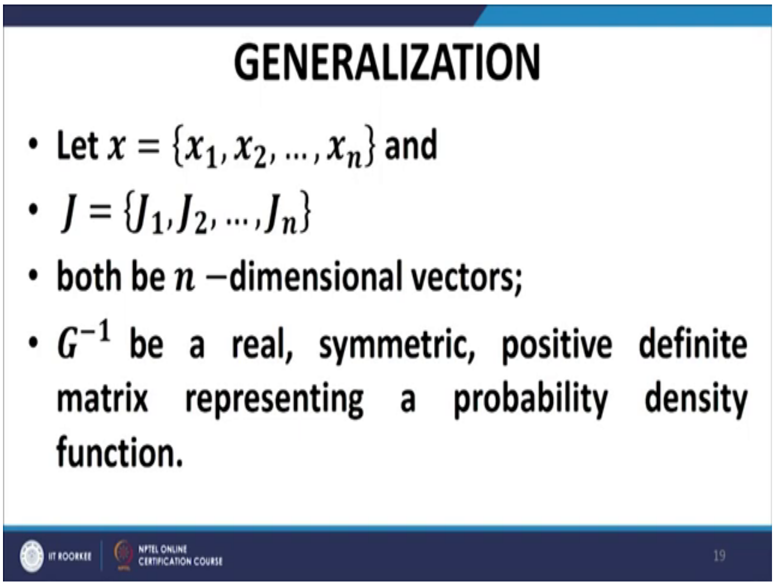

Now, we talk about generalization of these concepts that I elucidated, right. Now, instead of having a random variable x, we have a series of random variables x 1 x 2 x 3 a vector and now x is a vector and x 1 x 2 x n can be construed as the components of that vector. And similarly, J is another vector which has the components J 1 J 2 J n refer to a suitable basis. Whatever the basis may be for the moment, we shall come back to the issue of basis in a minute.

G inverse is a real symmetric positive definite matrix. Now, why real symmetric positive definite matrix? Because at the end of the day we want to establish a connection between G inverse and the probability density function. And as you know the probability density function can be represented by a matrix which has to be real, which has to be symmetric and which has to be positive definite.

So, that is an important issue and that will in fact count very significantly, that will count very heavily in the analysis what is to follow, the real symmetric and positive definite properties of G inverse would become very handy, very important in the subsequent analysis. So, to recap x is now vector with n components x 1 x 2 and up to x n. J is another vector with components J 1 J 2 J n and G inverse is an n cross n matrix which is real, which is symmetric and which is positive definite.

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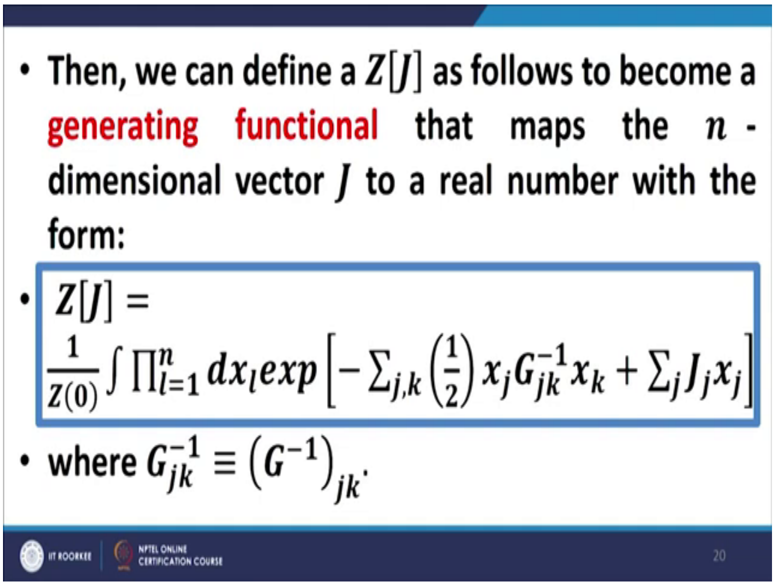

Now, we define Z J. Our objective is to define Z J. Z J is now become a functional because now it is a mapping from the space of vectors to the space of real numbers. And therefore, it now becomes a generating functional rather than a generating function it maps an n dimensional vector J to a real number and it has the form as you can see there, here it has the form which is given by this expression in the blue box.

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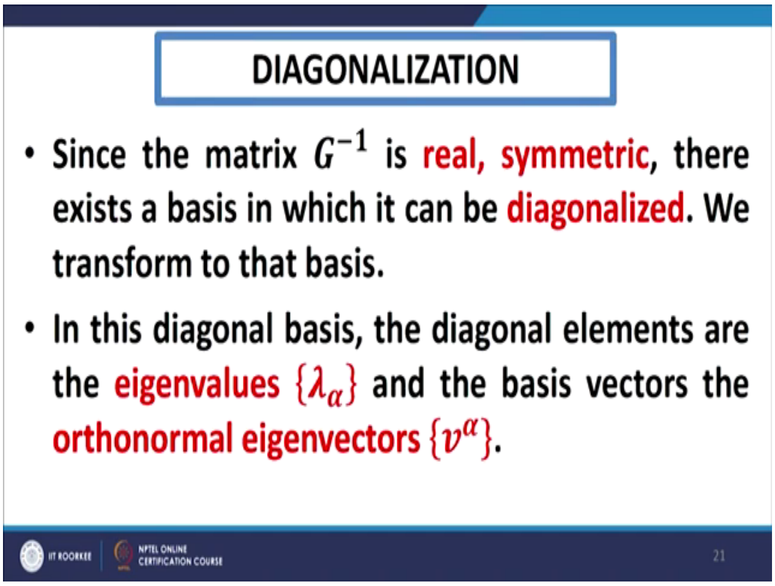

Now, because as I mentioned G inverse needs to be real and symmetric, now the very fact that it is real and symmetric should ring a bell and that various that we shall be talking about diagonalizing this particular matrix. Because, it is real and symmetric there exists a basis and we can have an orthogonal transformation from the existing basis to a new basis in which this matrix G or G inverse for that matter we will be a diagonal matrix.

And, the elements of the diagonal will be nothing else for the eigenvalues of the matrix G inverse and the corresponding vectors which will the eigenvectors corresponding to all the degenerate eigenvalues will represent the basis in which this particular matrix manifests itself as a diagonal matrix.

To repeat because G inverse is real and symmetric, it can be diagonalized, it can be diagnosed by an orthogonal transformation of basis and the basis in the diagonalized matrix corresponding to G inverse will be a matrix where the diagonal elements will be nothing, but the eigenvalues of G inverse and the corresponding basis vectors would be nothing but the eigenvectors corresponding to each of those eigenvalues.

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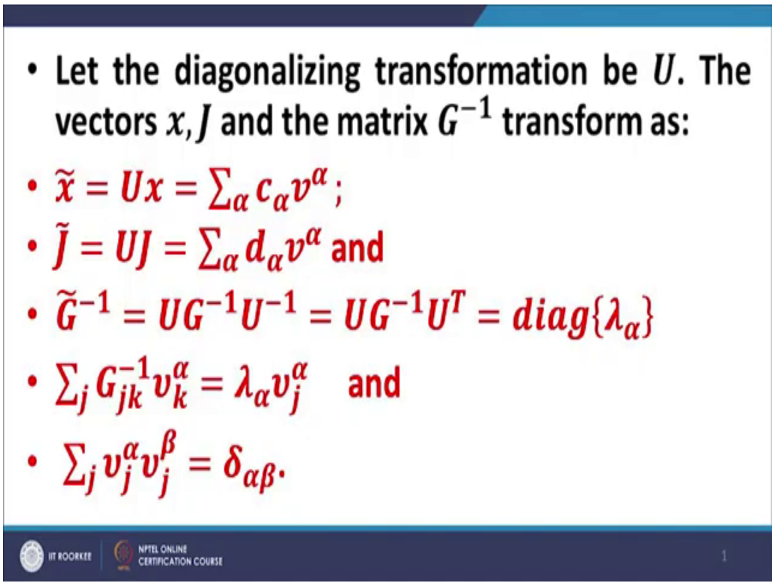

So, lambda alpha are the eigenvalues and the corresponding vectors are V alpha in the basis in which G inverse is diagonal. So, we call the diagonalizing we represent the orthogonal transformation representing the diagonalization by U and the various transformations that realize that occur as a result of the operation by the transformation matrix U will be in this form which is given on the slide, x tilde is the new vector corresponding to the original vector x when operated upon by U.

J tilde is the new vector corresponding to the original vector x in the new basis in the basis in which G inverse is diagonal, x tilde is the vector corresponding to the vector x in the basis in which G inverse is diagonal and the rest of the things also similar. So, this is the diagonal matrix G diagonal lambda, alpha which is the matrix G tilde which is the diagonal matrix which represents the diagonalize form of the matrix original matrix G inverse.

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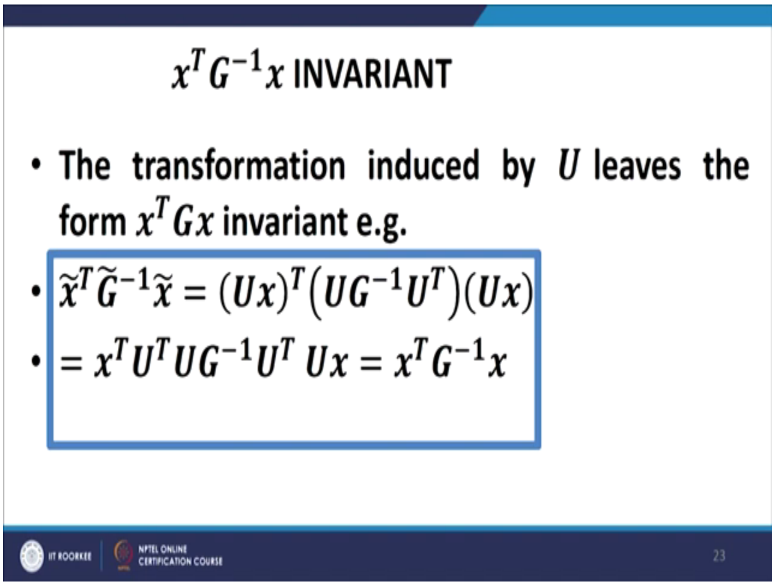

Now, the first important property is that due to this orthogonal transformation due to the diagonalization of the matrix G inverse the binary form x transpose G inverse x remains invariant. This is easy to see as you can see in the blue box here, it is quite the proof is quite straightforward. We clearly find that x tilde T G tilde inverse x tilde is equal to x transpose G inverse x. So, this transformation U leaves this binary this form x transpose G inverse x invariant. This quadratic form it leaves invariant, right.

Now, because G tilde is G tilde inverse is diagonal; because G tilde inverse is diagonal, the only elements in G tilde inverse are the diagonal elements which are nothing, but the eigenvalues of G inverse. What do we get?

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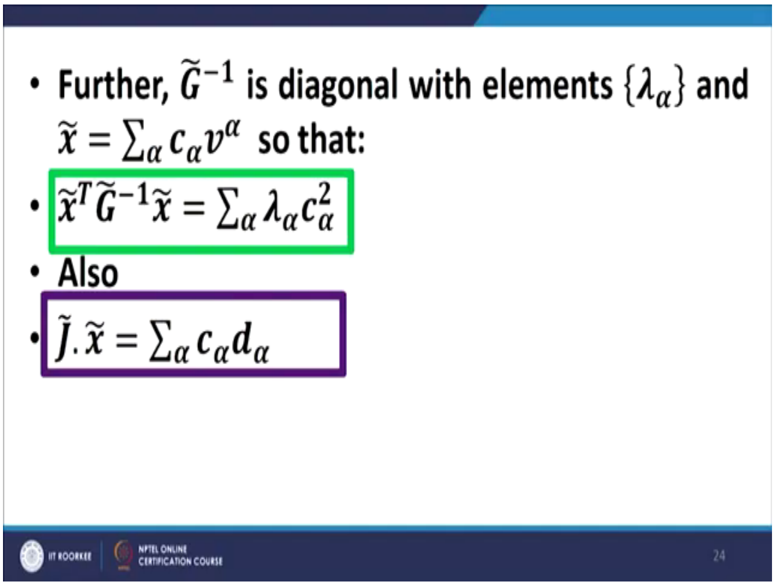

We get x transpose tilde G inverse tilde x tilde is equal to summation of lambda alpha c alpha square. This is very elementary easy to prove and this is a consequence of G inverse tilde being a diagonal matrix with the elements lambda 1 lambda 2 lambda 3 along the diagonal.

Similarly, J dot x is we represent J dot x in tilde J tilde dot x tilde in. That means, this is this inner product is also worked out in the new basis in the basis in which G inverse is diagonal and we get it as we represented as c alpha d alpha summed over alpha, right.

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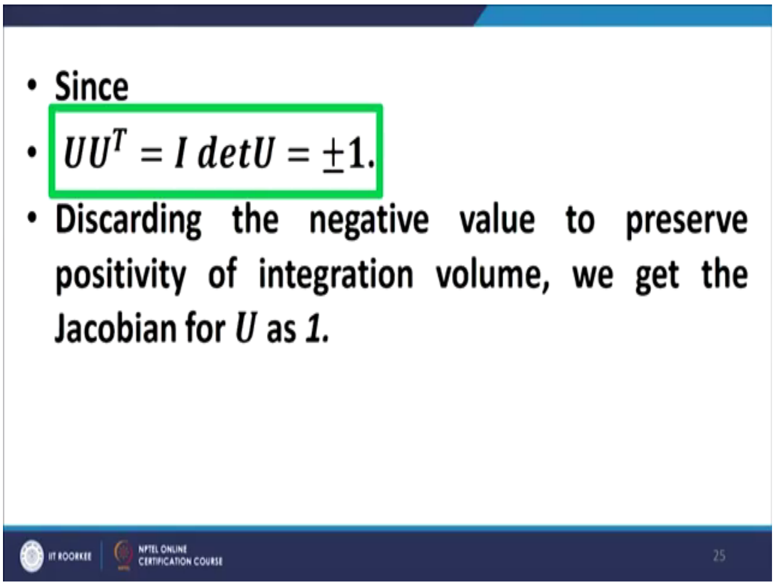

Now, because it is an orthogonal transformation, therefore U transpose is equal to identity and therefore, the terminate of U is equal to plus minus 1. We discard the value minus 1 because we want to retain the positivity of the integration volume and therefore, we restrict our choice to determinant U equal to plus 1 which implies the Jacobian is also equal to plus 1.

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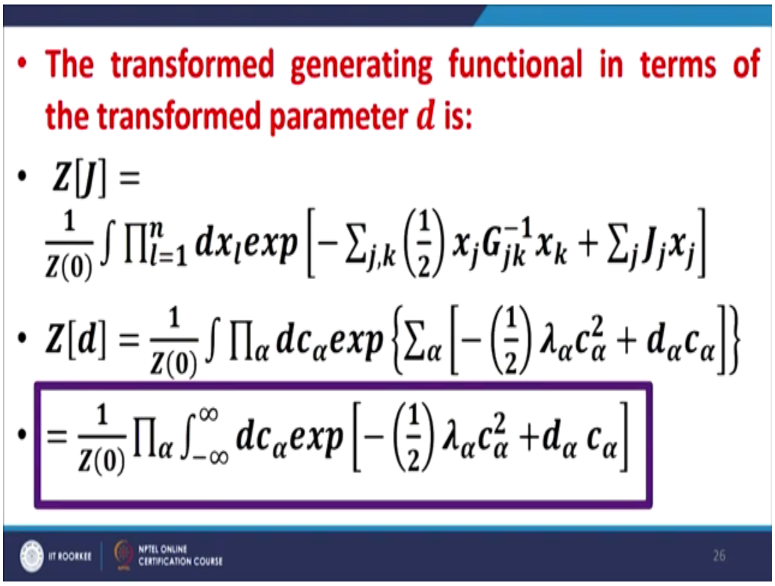

Now, what happens to the transform generating functional? And that is we have to look at it very carefully. We started with the expression for the generating functional as 1 upon z naught and in this whole expression the first equation on the slide and we when we do the transformation, when we represent Z J in the new basis in the basis in which G inverse is diagonal.

We have found that x tilde J G inverse tilde x tilde k will be equal to lambda alpha c alpha square summed over alpha and J x J that is the product of x in the scalar product of x and J would be d alpha c alpha summed over alpha. So, these two values we have already in a sense worked out earlier and these two values arise because of the transformation of the basis and we get this expression which is the second equation on the slide.

Now, we need to look at what happens. The next step is to take the product outside the integral and to leave out the summation here and that is done because we can represented the various what do you call are independent variables, random variables. Each of these exercise which constitute the random variables, the components of x are independent and therefore, we can replace this exponential of the sum of for these random variables by the product of the exponentials and that is precisely what we have done and that enables us to take the exponential outside the integration.

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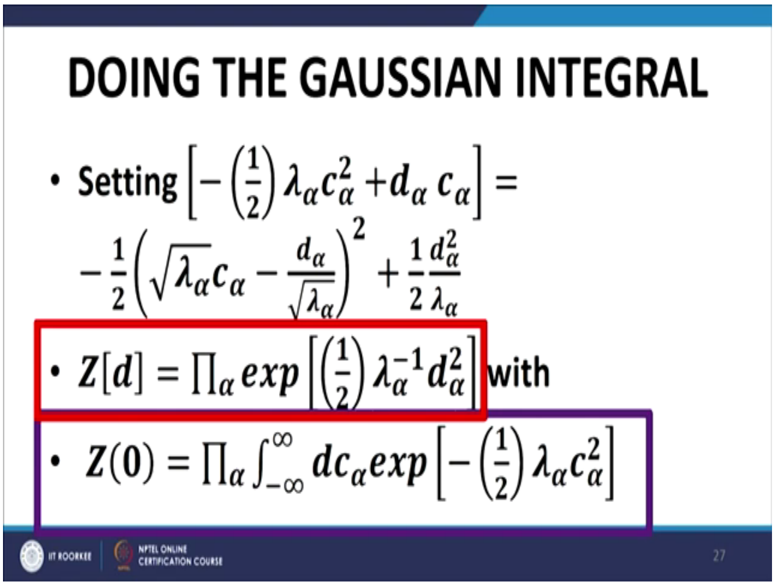

Now, we do the when we do the Gaussian integration, it is quite straightforward again we do the standard procedure of completing the square. We complete the square of this equation which is the first expression on your slide minus 1 by 2 lambda alpha c alpha square plus d alpha. We represented in the form of a perfect square plus that extra term 1 by 2 d alpha upon lambda alpha square.

And therefore, we can when we do this Gaussian integration of the expression, this expression which is there in the box in this slide. When we do the Gaussian integration of exponential of this because; this is nothing but a Gaussian integration; so, we do the Gaussian integral of this exponential and we get two terms. This particular term 1 by 2 d alpha squares upon lambda alpha will go outside the integration.

And we will get this expression in the red box as a pre factor and the Z 0 term, the normalization term will be this expression which will not contain any term containing G containing d alpha because d alpha relates to J and we have talking about Z 0 here Z 0 is nothing, but Z J which J equal to 0.

So, this integral; this integral is in a sense split up into two parts; one part the J by completing the square this integral is split up into two parts; one part that contains the J and that we can take as the pre factor and the other part that does not contain the J that represents the normalization Z 0, right.

So, at the end of the day what do we get? Z d is equal to this expression because the Z 0, Z 0 terms cancel out one Z 0 is in the definition of Z J. If you recall one Z 0 is in the definition of Z J let us go back this is this Z 0 is a there in the definition of Z J and the other Z 0 is what we have computed when we do this integration in the blue box ah. So, the two Z 0s cancel each other and we are left with this as the ultimate value of Z J.

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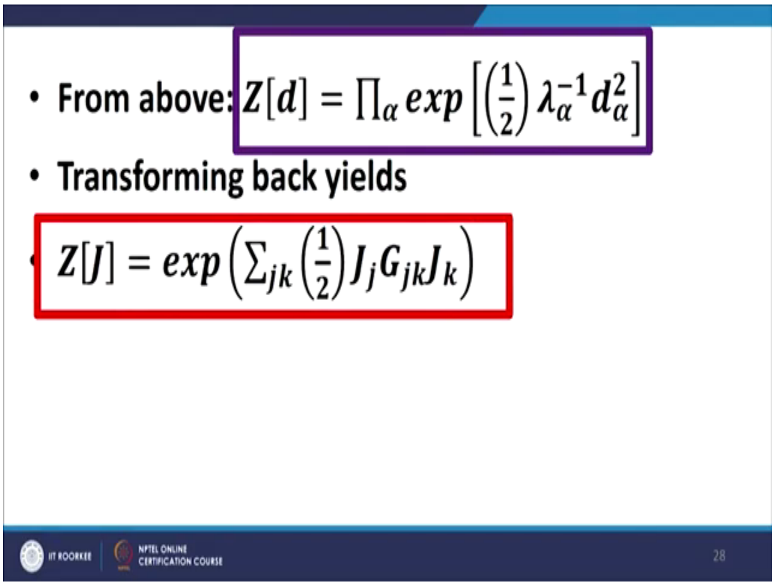

Z d; Z d as this product and when we revert back to our original coordinators through a inverse orthogonal transformation, we get Z J is equal to exponential summation over J and k 1 by 2 J G k J k. So, now this is the generating functional. Now, recall that x is now a vector J is now a vector and G J k is a matrix n into n matrix real symmetric and therefore, this expression for Z J is summed over J and k 1 by 2 J of J j. The small j is an index G j k J k.

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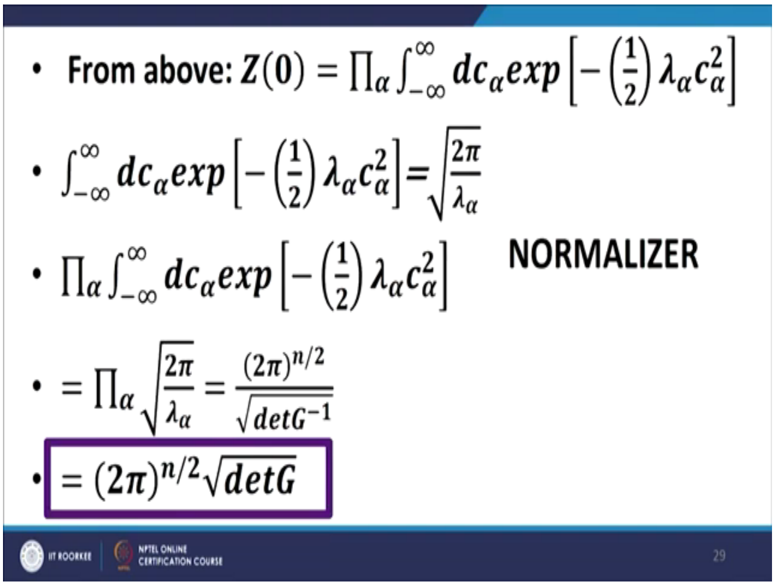

And as per as the Z 0 is concerned, Z 0 can be simplified further. Recall that this is the normalization of the Z J, Z 0 is given by the product of alpha of this expression. Now, this is this expression as this can see is a clearly a very simple Gaussian integral and it takes value under root 2 pi upon lambda alpha and therefore, when you when you have n such factors, when you take the product over alpha n such factors this under root 2 pi over lambda alpha gets multiplied by 2 pi under root 2 pi n times. So, that gives the numerator of 2 pi n by 2.

The denominator becomes under root, the product of the eigenvalues, the product of the eigenvalues is nothing, but the value of the determinant and therefore, we get this is equal to 2 pi to the power n by 2 divided by under root determinant G inverse because the product of the eigenvalues is the determinant.

And we if the determinant of the matrix of which these are the eigenvalues and our matrix is G inverse recall that and therefore, we can write it in this equation, but under root determinant G inverse is nothing, but determinant G in the numerator and therefore, we have the normalization factor Z 0 equal to 2 pi to the power n by 2 under root determinant of G.

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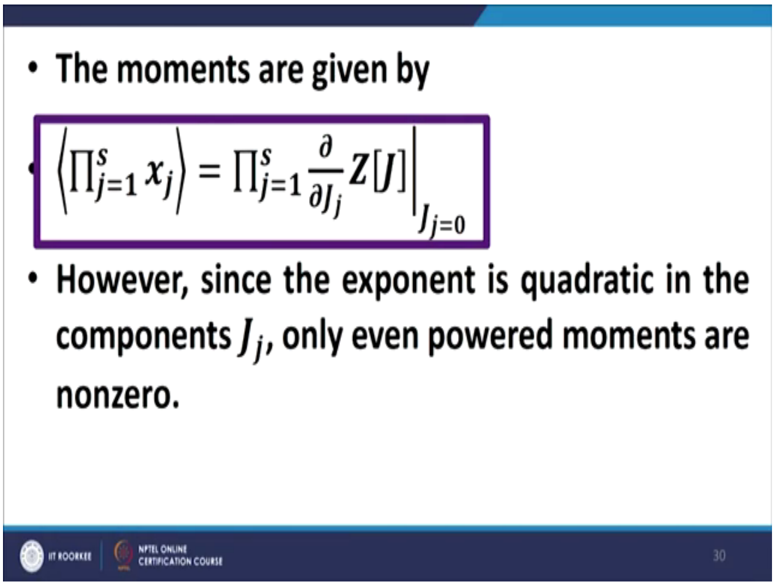

And the moments are given by this expression as usual. We take derivatives with respect to the various j s corresponding to the moments that we want whatever moments we want we take derivatives with respect to those particular j and then put j equal to 0 and the we get the expression for the moments.

The important thing is that in the case of the Gaussian only even power moments would such survive, the odd power moments would not survive because the exponent is quadratic in the components j. And therefore, the odd power odd moments do not survive, all other moments vanish in fact and only the even moments exist.

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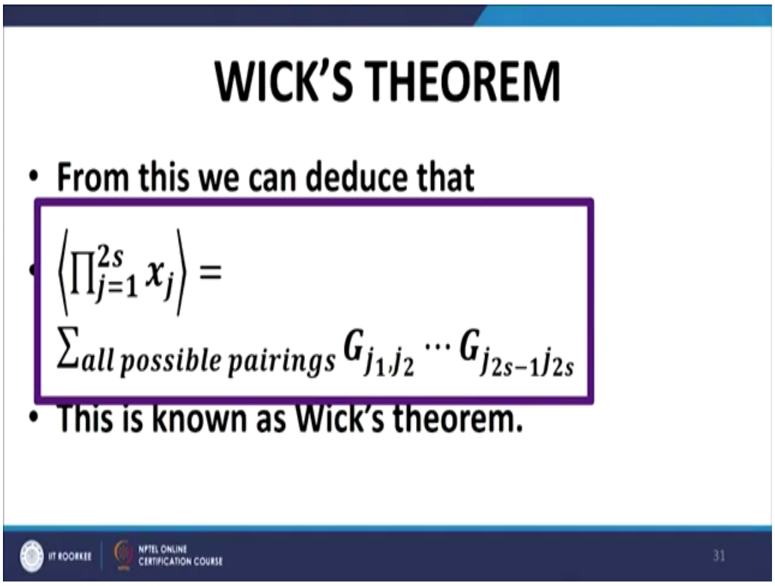

And in fact the even moments exist the expression for the even moments can be written as in the form that is given in this box and this is what is called Wick's theorem.

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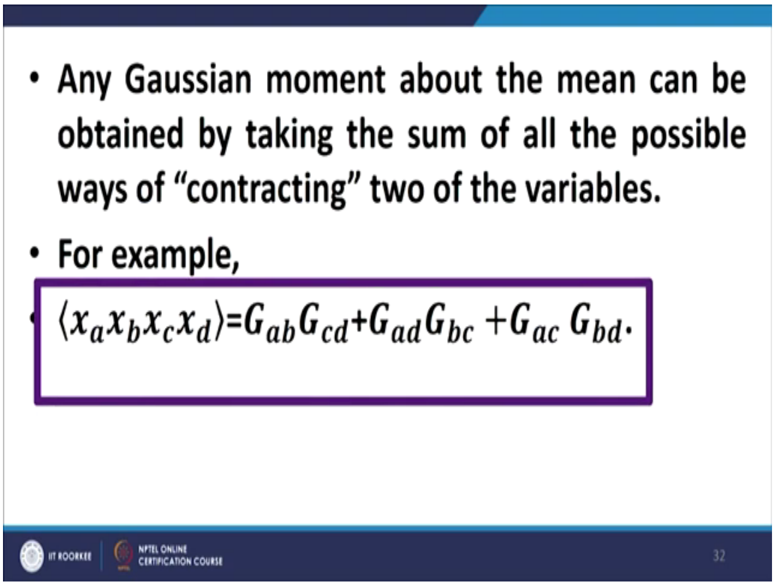

An example as an example I have this particular a moment x a x b x c x d the expectation value of this expression is given by all pairings G ab G cd G ad G bc plus G ac G bd. So, this is how we can work out the moments of complex quantities, complex statistical quantities like vectors and.

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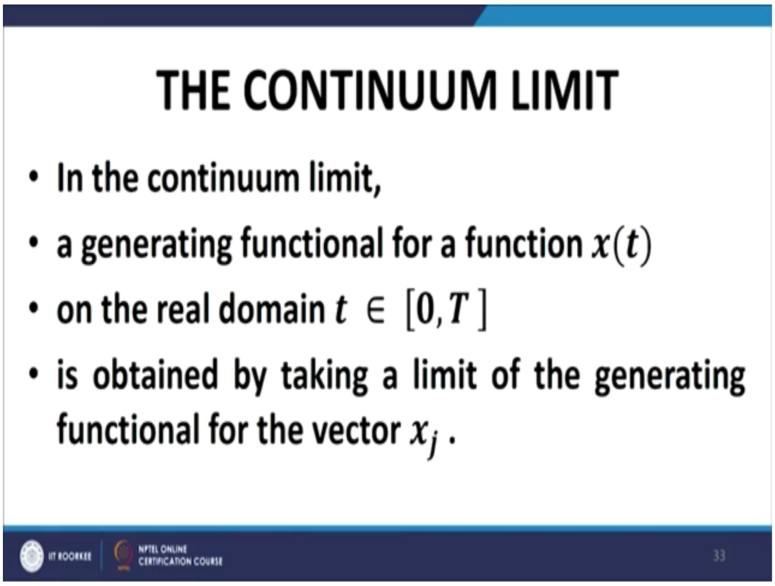

Now, we come to the continuum limit of what we have done just now. So far we have been talking simply about the discrete case where we have an x and j take discrete values, the components are discrete, we have x equal to x 1 x is the ordered triple x 1 x 2 x n and similarly j is j 1 j 2 j n. Now, we try to extend this framework to the continuum case.

Now, we are given a function x t. Let us say we are given a function x t of time in the domain t containing 0 to capital T. We need to take work out a limit obtained for the we need to work out a limit of the generating function in the for the vector x j in the continuum limit that x j approaches x t or x j takes the continuum form x t.

What are the substitutions that we have to make? Let us look at that. First of all we do a time slicing which is what we normally very common to practice. the entire time span 0 to t is split up into a very small infinitesimal time slices each of length delta t such that the total time interval t is equal to n delta t.

Now, the important thing is how will we invoke the continuity requirement or how will we make time continuous? Well we will make time continuous by taking the limit n tending to infinity because n tending to infinity or delta t tending to 0 which would mean would imply that the time slices that discreteness in time is becoming so small that s it is as far as mathematically concerned. It becomes a continuous time line.

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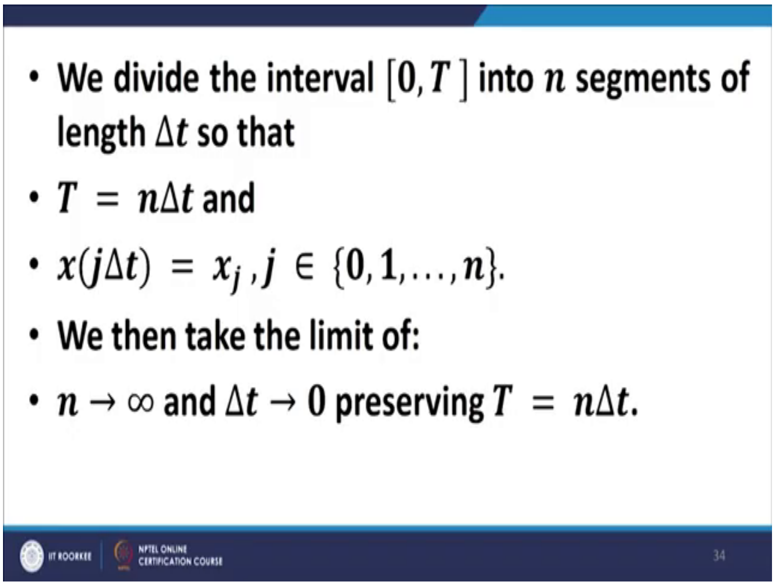

So, we make this substitutions an arbitrary x j; an arbitrary x j which was let us say it is we started with an x j or we had an x j in the discrete framework would transform as in the continuum framework as x j delta t j jth slice multiplied by the time length of each step. So, it becomes x at the at the point j delta t x at the point j delta t because we having now j. This j the index j is representing the number of time slices since the initial since t equal to 0.

So, number of time slices multiplied by delta t, the length of each time step makes it x j equal to x j delta t j is obviously can take the values because x j is arbitrary. So, j can take the value 0 to n at when x when j takes the value n, we have x n delta t which is nothing, but x t. Now, we take the limit n tending to infinity delta t tending to 0 preserving the finiteness of capital T preserving the finiteness of capital T.

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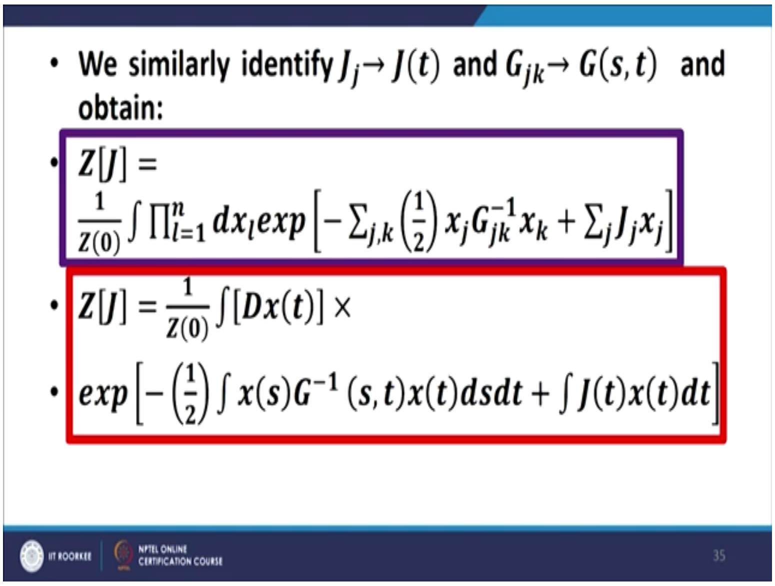

Similarly, we identify J j as J t and G jk as G s t and we obtain after making all these as substitutions, we obtain the blue box is what we had in the discrete framework. The blue box is what we had in the discrete framework and the corresponding expression in the continuous framework we get is the expression in the red box.

You can see that x j G inverse G j k x k is substituted by x s G inverse s t x t d s d t integrated, the summation is replaced by the integration and J x j j x j product is also replaced by the integral J t x t and d t, right.

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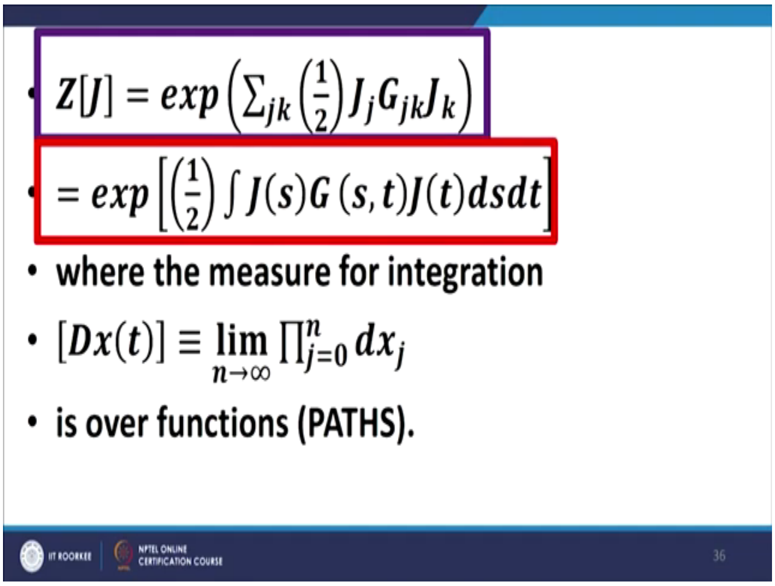

Therefore, Z J expression that we had earlier after simplification after carrying on the carrying out the Gaussian integrals, we had was the first expression the purple colored box, the first box on the top and this expression in the continuous framework it takes the expression one which is given in the red box.

And the measure of integration in this context has to be mentioned, this is a path integral measure capital D of x t in the square bracket is the path integral measure. It is given by the in the limit n tending to infinity the product of all the d x js.

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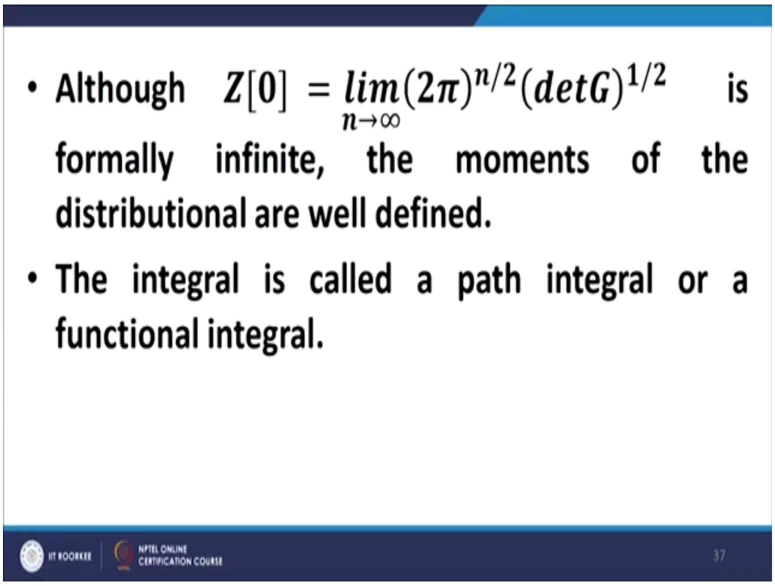

Now, and what about Z0? Z0 in our earlier case was n tending to infinity 2 pi to the power n by 2 determinant G 1 by 2. Now, this is very interesting. It becomes formerly it becomes infinite, but however the moments continue to be well defined not with standing the fact that the normalization tends to blow up or approach infinity or diverge as n tends to infinity. The integral is called the path integral or a functional integral. Now, another example we have here is of the path integral or the functional integral.

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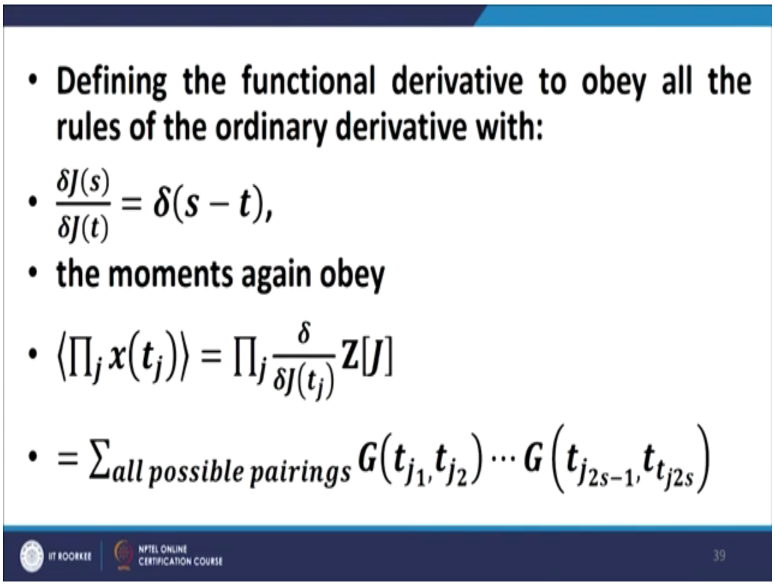

Now, let us look at something more about this. We have this the moments are obviously obtained by the same process, but now instead of taking the common derivatives, the Newtonian derivatives we now take the variational derivatives, functional derivatives whether and the functional derivatives are defined essentially by this expression.

We, the functional derivatives are defined by this expression delta J s upon delta J t is equal to delta s minus t. What we do is the Kronecker delta, this discrete delta is now replaced by the Dirac delta function that is the most important property of the these derivatives, right. We will continue after the break.