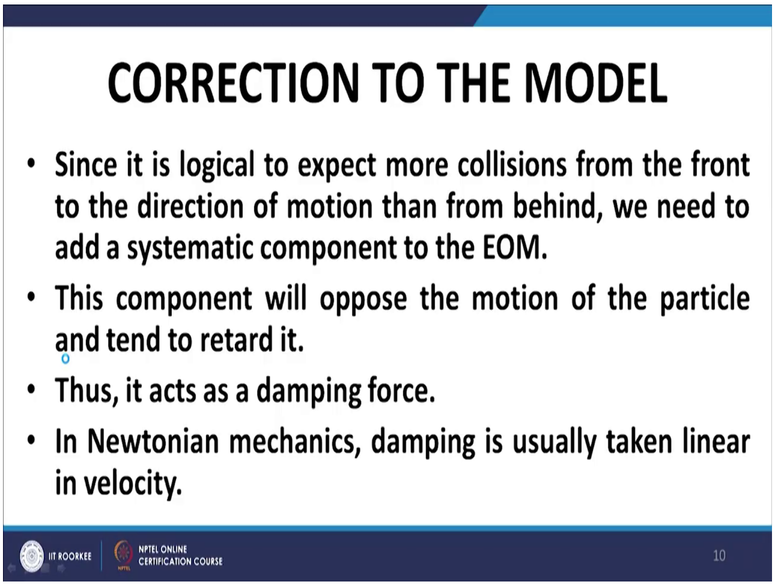
Path Integral Methods in Physics & Finance
Prof. J. P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee

Lecture - 14
Langevin Equation

So, as I mentioned before the break, our simplistic model for the dynamics of the Brownian particle has a certain limitations, it seems to be unphysical because, the squared of the velocity tends to blow up with passage of time.

(Refer Slide Time: 00:40)

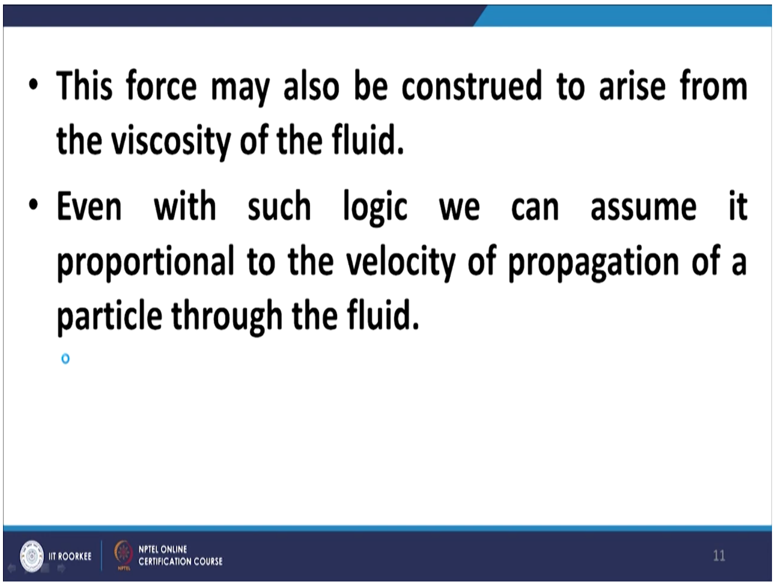


Now, the source of the flaw and we found that it one of the reasons could be, that the there is an asymmetry as far as the number of collisions are concerned; when a body is in motion in a uniform fluid. And that arises due to the fact that the number of particles hitting the Brownian;

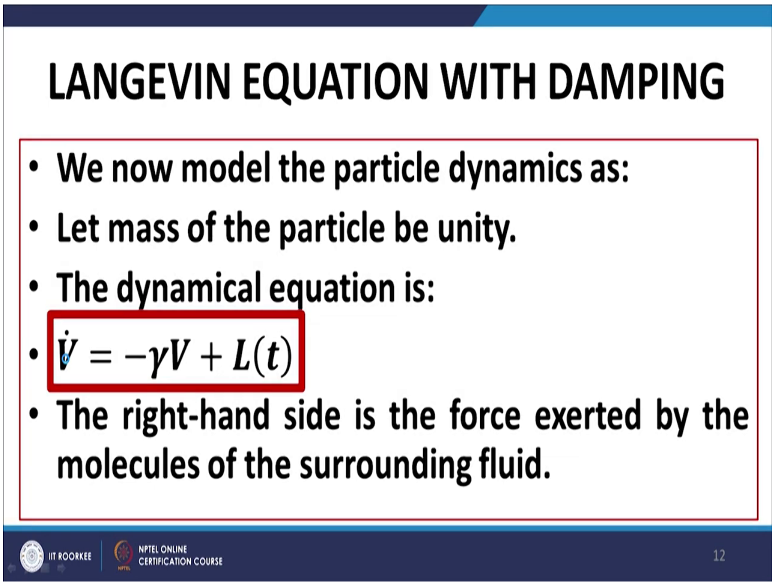
number of molecules hitting the Brownian particle at from the front as it moves forward tend to be more than the number of particles hitting the particle from behind.

And this create because of the particles hitting it from the front, I want to decelerate the motion of the part Brownian particle. And therefore, it amounts to a kind of damping and in general we model that damping as a linear terminal velocity.

(Refer Slide Time: 01:34)

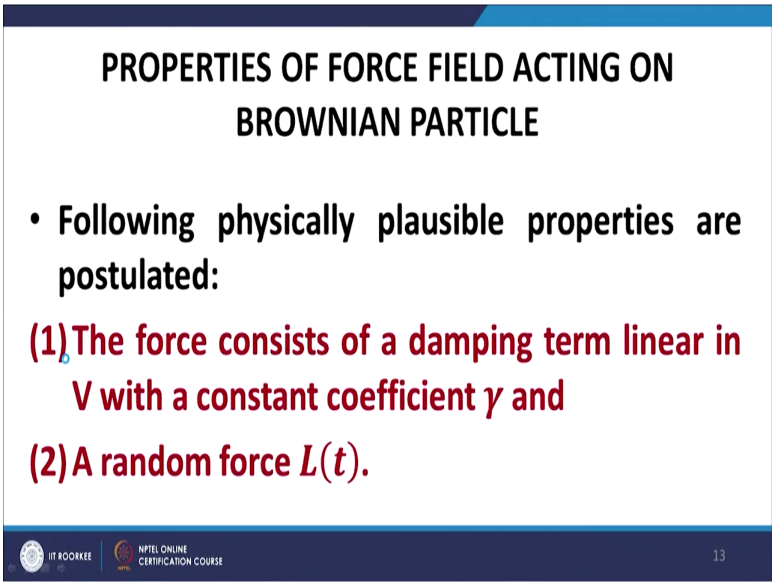


(Refer Slide Time: 01:38)

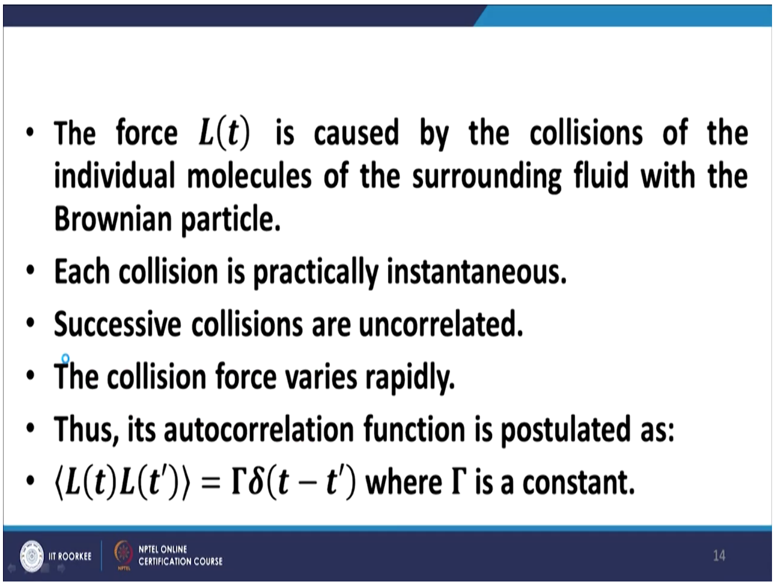


So, our Langevin equation with the stepping term gets modified. And we have the new Langevin equation that is given in the red box here, V dot is equal to minus gamma V plus L t, we shall be exploring more about L t and that will be the subject matter of most of our discussion, but for the moment gamma is the is a constant damping constant in the sense that would relate to the viscosity of the fluid in which the Brownian particle is submerged right.

(Refer Slide Time: 02:11)



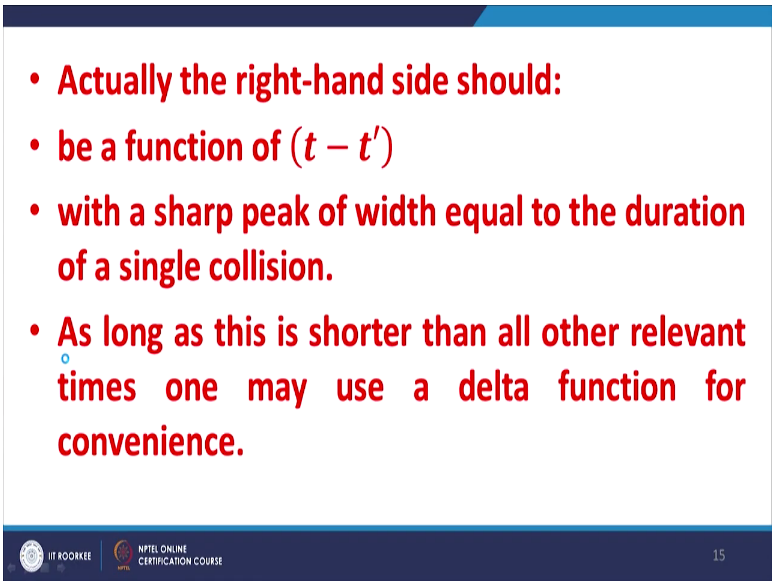
(Refer Slide Time: 02:16)



Now, the properties of the of this random force represented by L t. Now, the first thing is the axioms that we may assume, that we profess before we write the statistical properties of L t are similar to what we did earlier; the collision is all the collisions are practically instantaneous.

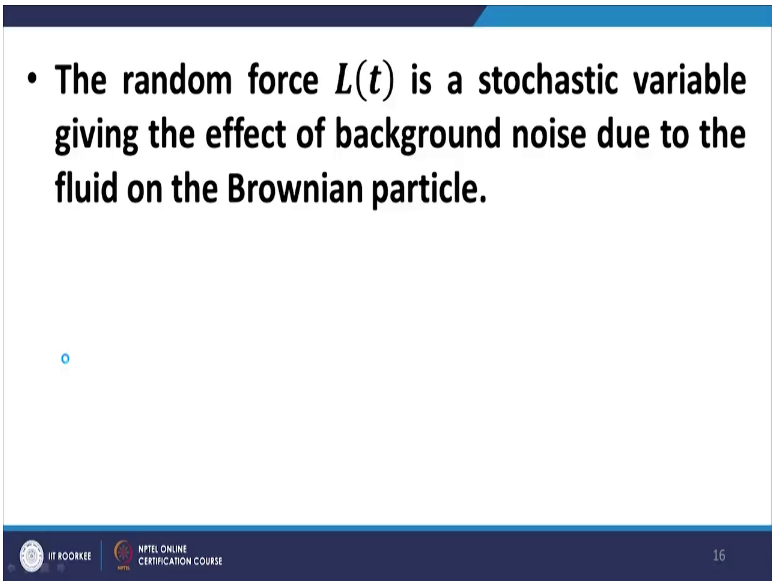
Successive collisions are unrelated uncorrelated, and the collision force varies rapidly. And therefore, keeping that in mind we can write the auto correlation function as L t; L t dash is equal to capital gamma delta t minus t dash. What capital gamma represents? Well, capital gamma represents the magnitude of the force by which the molecules collide against the Brownian particle.

(Refer Slide Time: 03:10)



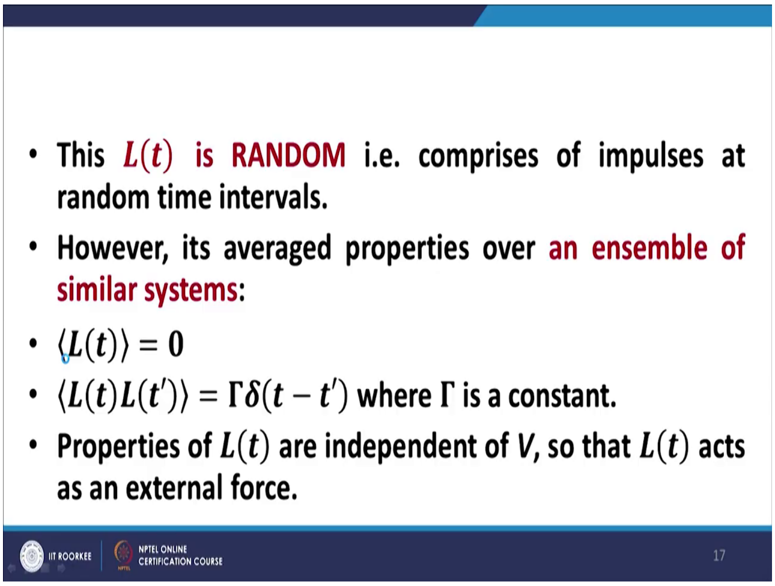
In actual fact to happen more precise the port the collision, impact of collision large for a small period of time t minus t dash. And therefore, we should have introduced a function t minus t dash because, t minus t dash is very small compared to the other scales of time that are relevant in the present problem; we assume that t minus t dash is instantaneous and we make use of the delta function in representing the auto correlation function between L t and L t dash.

(Refer Slide Time: 03:50)



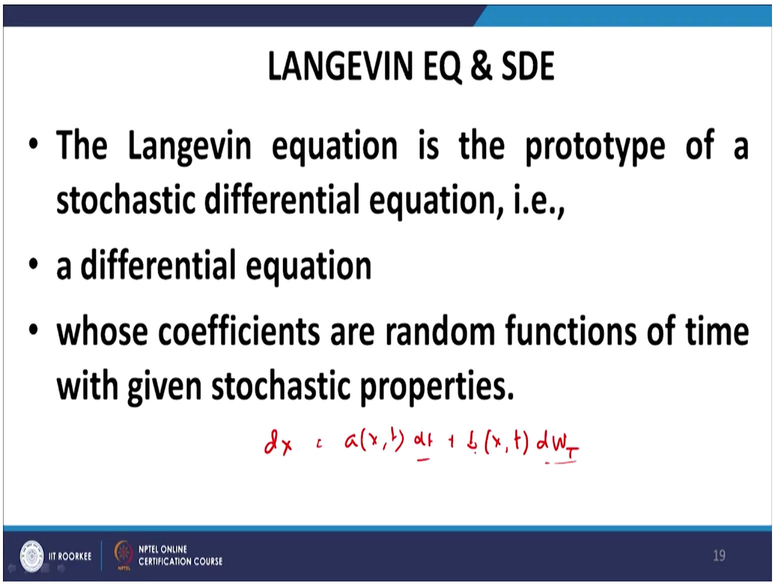
Therefore, the random force L t is a stochastic, it is a random therefore, and it is a collection of random variables, it is a sequence of random variables; therefore, it is a stochastic process. And, it incorporates the impact of the background noise of the collisions of the various ah molecules against the Brownian particle.

(Refer Slide Time: 04:11)



And these are the two fundamental properties that we come to the average of L t across all realizations will be 0 in an ensemble and the auto correlation as I mentioned is given by capital gamma delta t minus t dash.

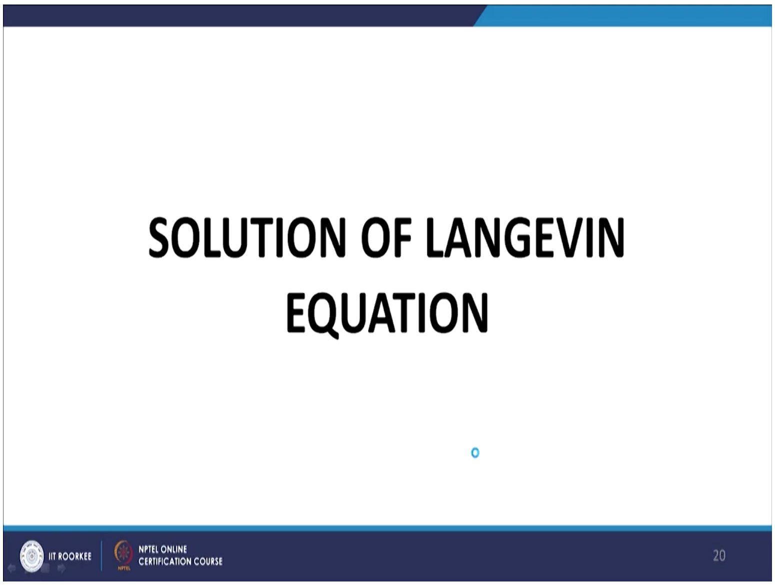
(Refer Slide Time: 04:29)



Now, the Langevin equation the which we saw just now V dot is equal to minus gamma V plus L t is a stochastic is an example of a stochastic differential equation. A stochastic differential equation is the differential equation where the coefficients of the various variables are random variables in themselves and they are also functions of time.

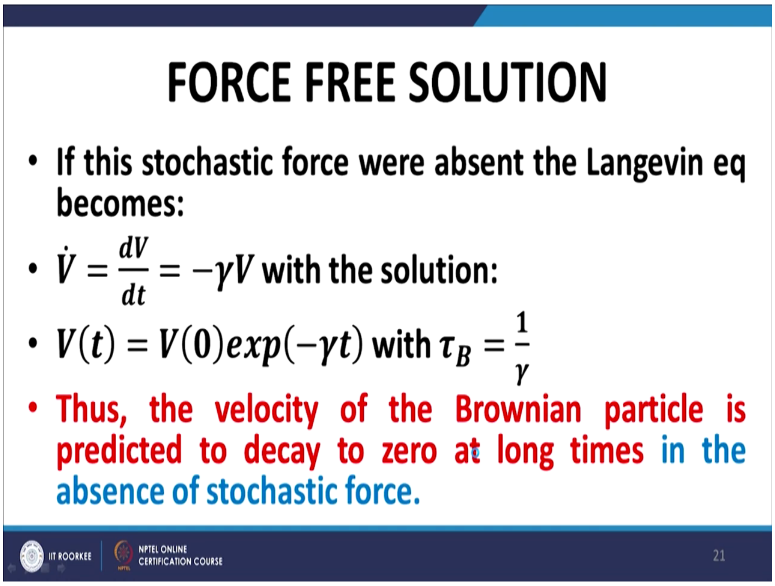
In other words, here you can write a stochastic differential equation in the form of dx is equal to a x t dt plus b x t dw T, these are the 2 variables and a and b are functions of x and t. So, this is a stochastic differential equation. The coefficients of the variables are random functions of time with given stochastic properties and the variables themselves one or more, maybe stochastic in nature, maybe Brownian motion, maybe the infinitesimal Brownian increment.

(Refer Slide Time: 05:51)



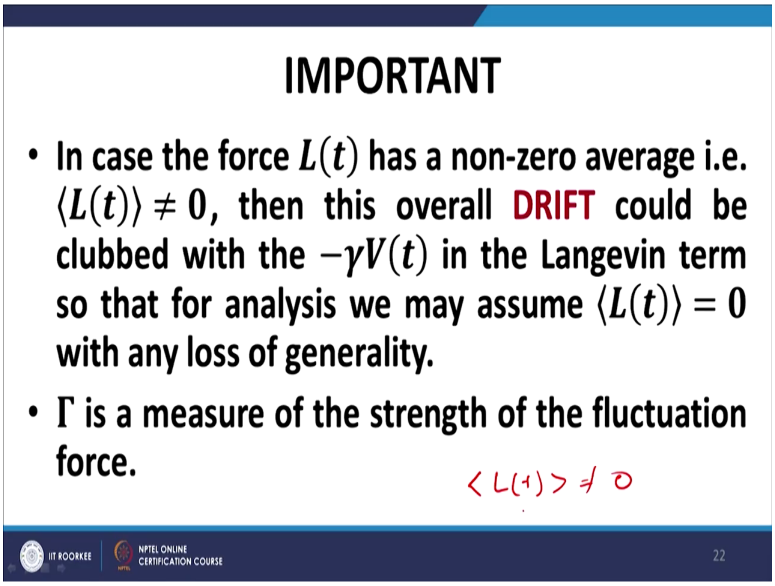
Now, we look at the solution of the Langevin equation; the generalized Langevin equation with the damping term.

(Refer Slide Time: 05:54)



Let us assume for the moment, let us start with the situation where there is no force, L t is equal to 0. The solution is easily written in that case we have V t is equal to V 0 exponential minus gamma t; this is the solution ah straightforward solution, when we have no damping no random force, no external force isolated particle subjected to only a damping force. Obviously, the damping force will tend to act on the particle in such a way that over passage of time and the particle velocity will diminish to 0.

(Refer Slide Time: 06:42)



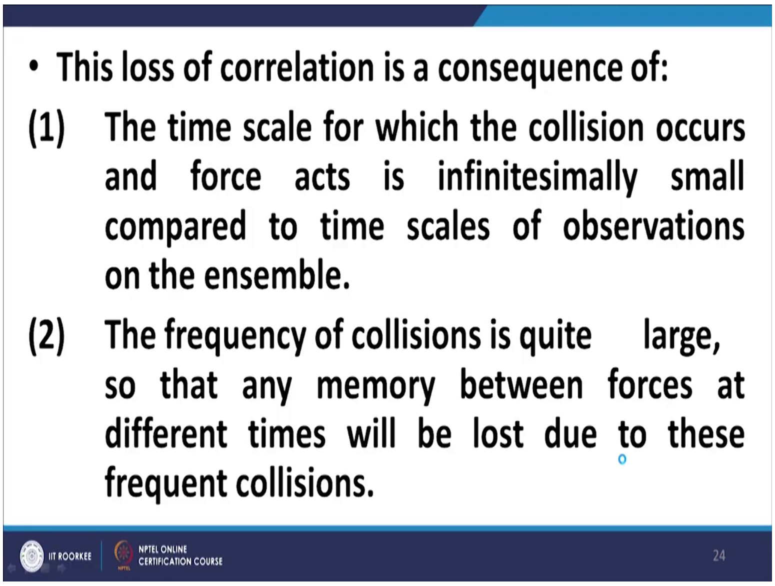
Now, before I proceed further I had assumed that the average of L t is equal to 0 and the correlations ah autocorrelation functions of L t are delta correlated in some sense although with some scaling.

Now, what happens, if the average of why is it that we can assume the average of L t has 0? The situation is that even in the case where the average of L t is not 0, the residual average can be combined with the drift term that is the gamma b term and we can thereby retain only a term on only L t which has a mean of 0.

Let me explain again suppose, we find that the average of; the average of L t is not equal to 0, then whatever that averages that that average value of L t can be added on algebraically to the gamma term and whatever remains will; obviously, have an average of 0 and that can be treated as L t.

The auto correlation functions will obviously, not change because if the even if the a auto correlation functions have a non-zero mean, we can measure the auto correlation functions in terms of deviations. So, that is not a major issue. And as I mentioned earlier gamma is a measure of the strength of the damping of the driving force of the stochastic force right.

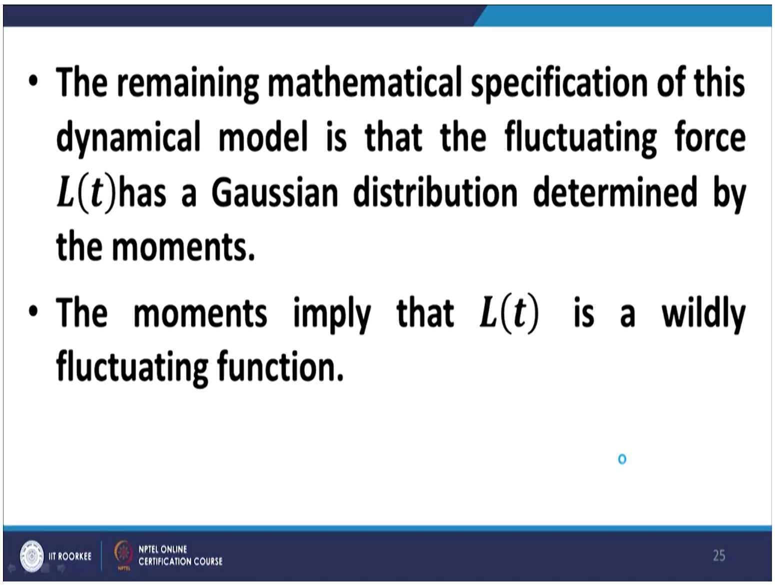
(Refer Slide Time: 08:37)



Now, why do we assume; why do we assume that there is no correlation the auto correlation is our delta correlation that is very important, there are two reasons for that. Number 1 is that the time scale over with the force that, is very very small. And because it is very very small it acts like an impulse and impulse which is limited to a particular point in time.

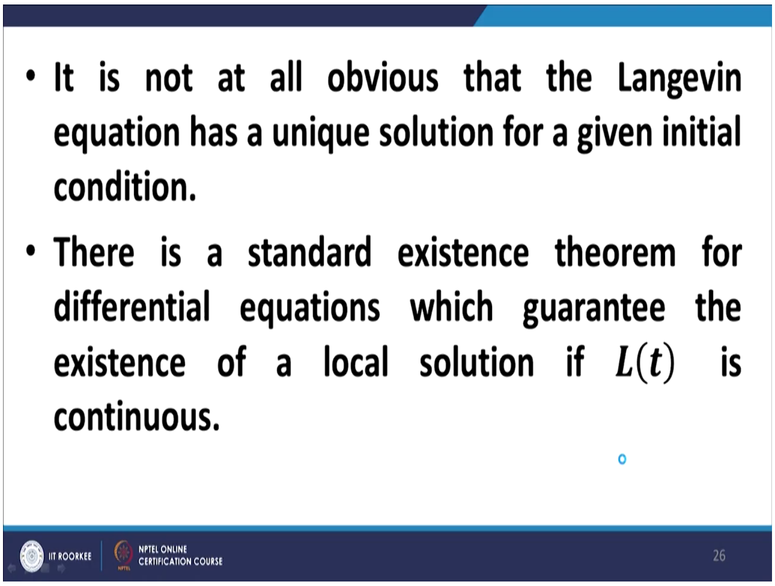
And therefore, we use the delta function in time to model this. The second thing, is that the frequency of correlations is very very large as a result of which because the Brownian particle gets hit by molecule so rapidly, that it loses memory of what had happened earlier or the impact of any earlier coalition correlation is dissipated very rapidly, because of immediately falling correlations. So, these two factors combining together result in delta function correlation being an appropriate usage for the stochastic force that forms the applied force in the Langevin model.

(Refer Slide Time: 10:01)



And of course, these are the 2 the moments the mean and the and the various we can use for describing this particular force L t and the 2 moments that are required for describing this force because we assume it to be Gaussian.

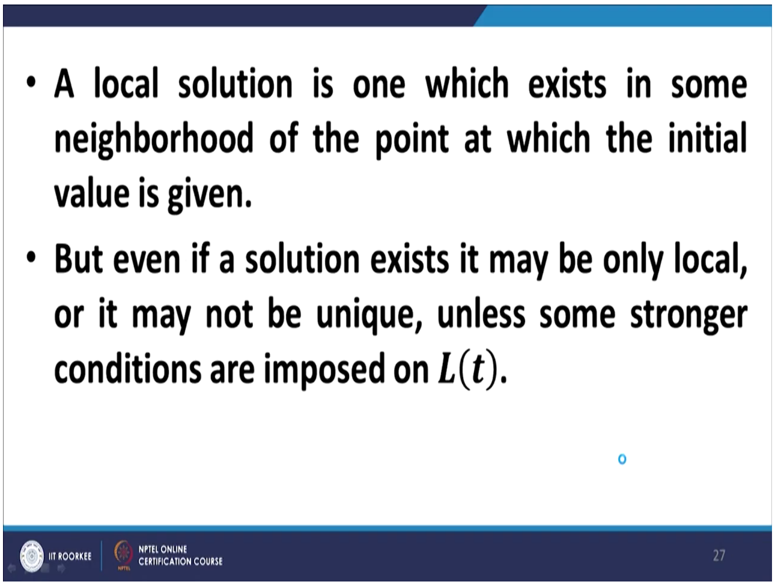
(Refer Slide Time: 10:21)



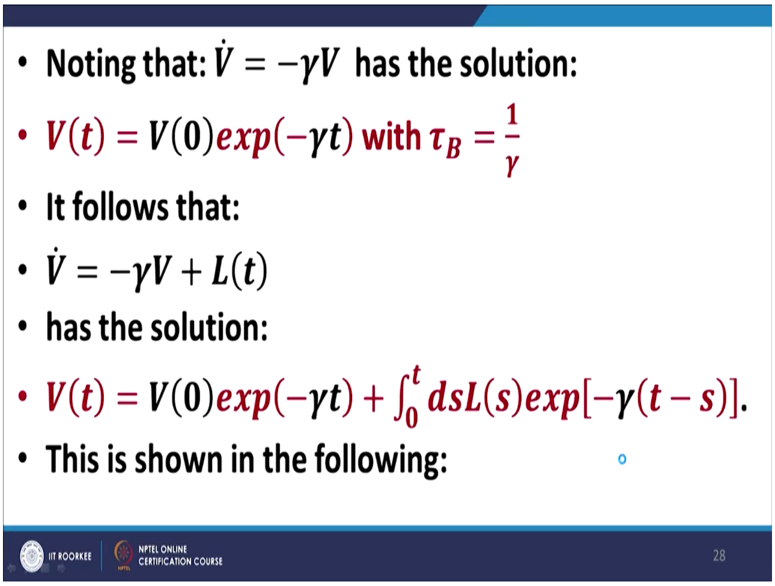
Now, there are certain technical issues which in this particular course is not really relevant, but the important thing is that the Langevin equation may not have a unique solution for a given initial condition; however, it does have a local solution. Now, the important thing is this local solution may or may not be unique.

So, what do we have? There may be a global solution, there may be a local or there may not be a global solution. But it is likely to have a it is going to have a local solution that local solution may or may not be unique.

(Refer Slide Time: 11:08)



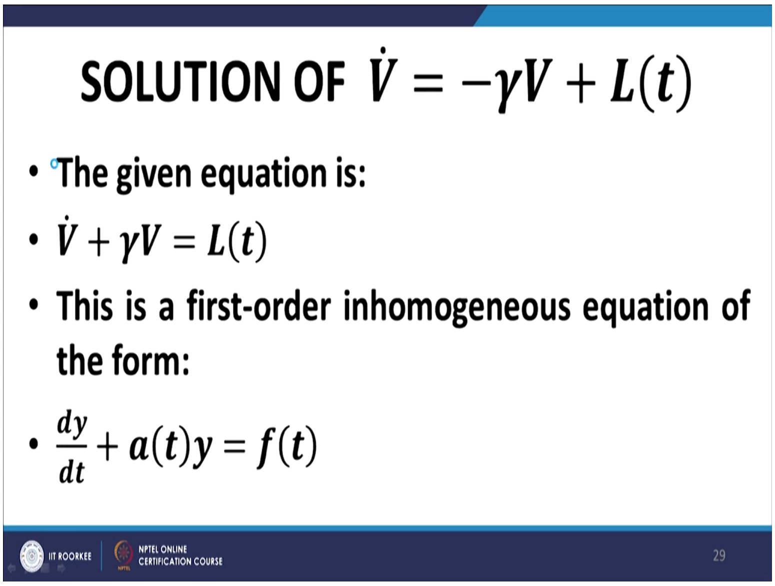
(Refer Slide Time: 11:10)



So, let us now look at how we move further towards that solution right. So, now we have first of all let us arrive at this solution here we have, V t is equal to V 0 exponential minus gamma t; in this situation; in this situation when the driving force is absent, only the damping forces there then V t is equal to V 0 exponential minus gamma t there is the velocity gradually dies down.

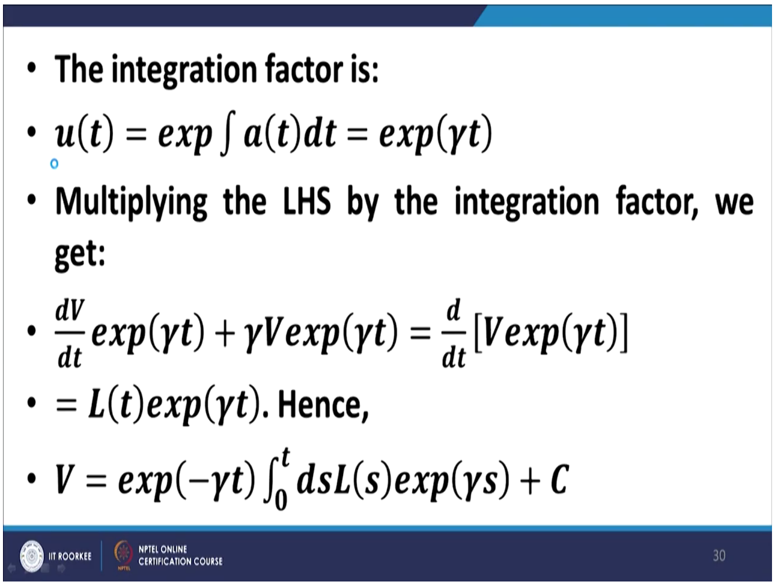
Now therefore, the solution of this expression V dot t I am sorry, V dot is equal to minus gamma V plus L t will be V t is equal to V 0 exponential minus gamma t plus integral dsL s exponential minus gamma t minus s. Let us see how we arrive at the solution to start with.

(Refer Slide Time: 12:04)



Now, this is a first order inhomogeneous differential equation. The Langevin equation that is given to us is a first order in a inhomogeneous differential equation of the form dy by dt plus a t y is equal to f t. So, it can be solved simply by using the integrating factor approach.

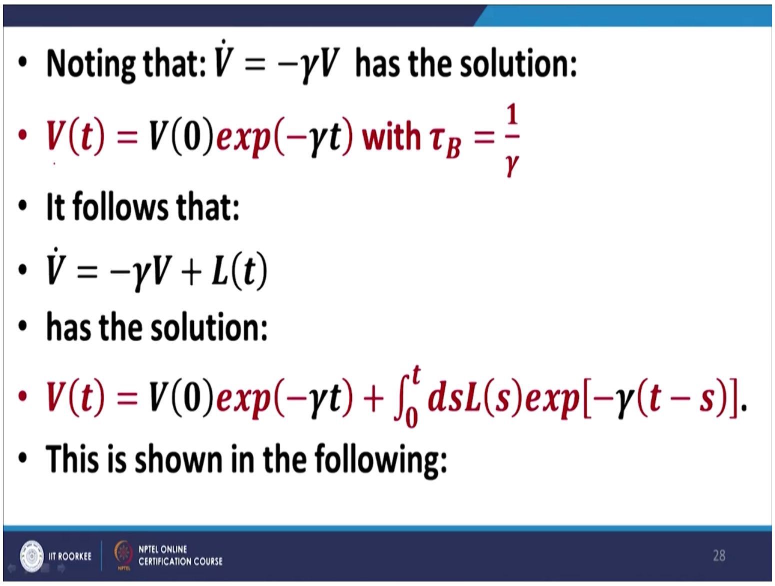
(Refer Slide Time: 12:31)



We work out the integrating factor, the integrating factor works out to exponential gamma t we multiply both sides of the equation by exponential gamma t. When we multiply both sides by exponential gamma t we find that the left hand side turns out to be a completely derivative of V exponential gamma t and therefore, the solution that we get is V is equal to exponential minus gamma t; now, this exponential gamma t on the going on the right hand side we give you an exponential minus gamma t.

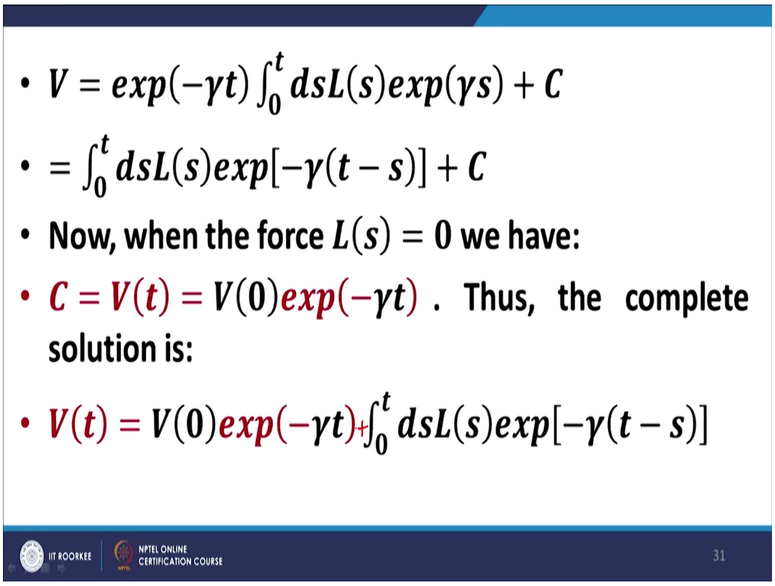
And the integral has to operate over the function on the right hand side which is 0 to t ds, s is the integration variable, dsL s exponential gamma s plus C, C is a constant. Now, how do we determine C? We determine C by the fact that when the force is absent our solution is what? Our solution is given our solution is this expression V t is equal to V 0 exponential minus gamma t.

(Refer Slide Time: 13:31)



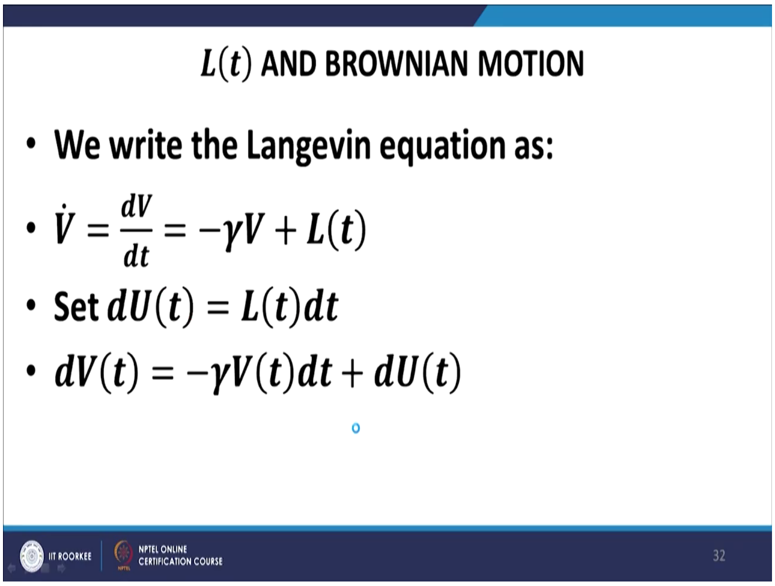
So, this enables us to find out the value of C because, when L t is equal to 0, our solution must turn out to be this expression.

(Refer Slide Time: 13:46)



And it follows, that the complete solution is now exponential V t is equal to V 0 exponential minus gamma t integral this expression exponential minus the thing, this should be plus here plus integral ds L s exponential minus gamma t minus s, this should be the complete solution of the Langevin equation.

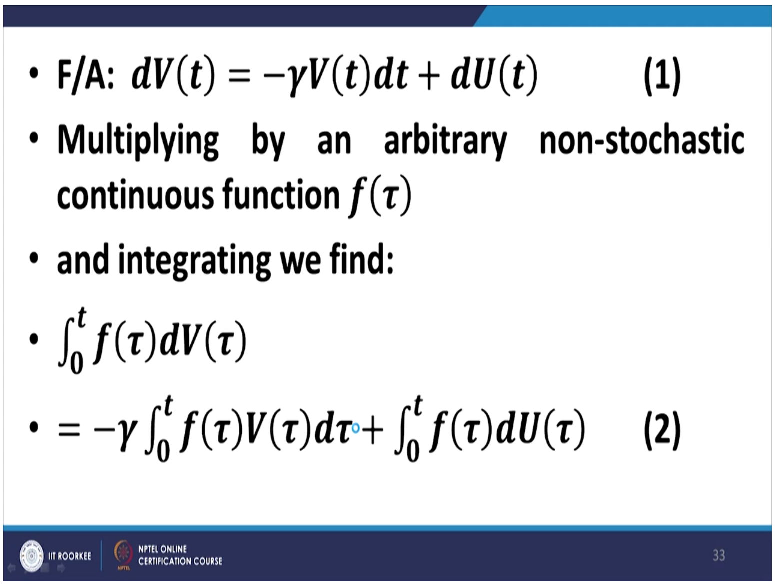
(Refer Slide Time: 14:13)



Now, let us investigate if the relationship between this force L t and the concept of Brownian motion. As I had mentioned earlier, this Brownian motion is the cornerstone of all stochastic processes. And that being the case it makes us presence felt almost everywhere and anywhere whenever we have stochastic analysis to do.

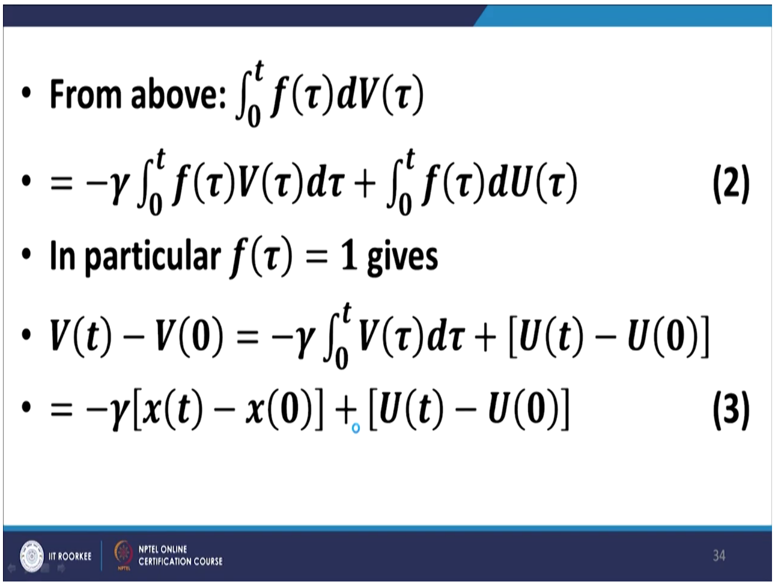
So, let us find out what L t has to do with Brownian motion, for that purpose we said d of U t equal to L t dt putting this making this substitution we can write the Langevin equation as dV t is equal to minus gamma V t dt plus dU t because, L t dt is equal to d of U t. Multiply by an arbitrary non-stochastic continuous function f tau and integrate over between 0 and t, we get the equation which equation number 2.

(Refer Slide Time: 15:10)



Integral of 0 to f tau dV tau that is the left hand side between 0 to t and the right hand side is this expression. Now, you can see that the integrations become viable.

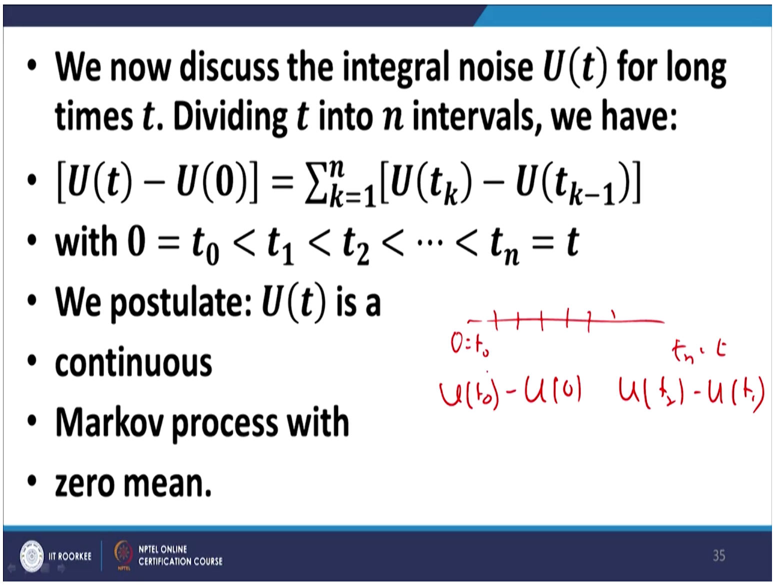
(Refer Slide Time: 15:42)



We when we do the integration, what do we get? We get V t minus V 0. Suppose, we take before that suppose we take V tau equal to 1, f tau equal to 1 I am sorry, suppose we take f tau equal to 1 and then do the integration what do we get? We get integral 0 to t d of V tau and that becomes V t minus V 0. And on the right hand side what do I have? Minus gamma 0 to t f tau is equal to 1 Recall, V tau dt d tau; what is V tau? V tau is the velocity remember.

So, it is dx tau upon d tau into d tau that is dx tau integrated between limits of 0 to t in terms of tau and that gives us x t minus x 0. And the last term when you have f tau equal to 1 and you integrate du tau between 0 and t you get, U t minus U 0. So, this solution is what we get when we have f tau equal to 1 and we do the integration.

(Refer Slide Time: 16:53)



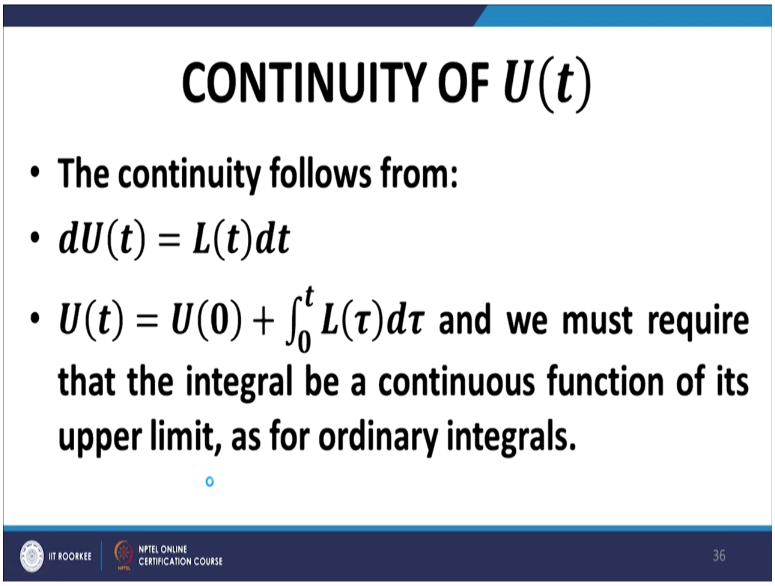
U is a function of time right, U t has a particular value, U 0 has a particular value. Now, we adopt the process of time slicing; we slice the time interval 0 to t into infinitesimal or very small intervals. Let us say we slice them into n time intervals, n time non overlapping intervals; non overlapping partitions and we write them as 0, t 0 to t 1, t 1 to t 2, t 2 to t 3 and so on up to t n.

So, naturally t 0 will be equal to 0; t n will be equal to t and the various points along the interval will be this will be say this is 0, this is taken as t 0, this is t n, this is taken as t and you have different points along this representing the partitions of the timeline.

Now, partitions of the timeline and you take the values of the function U t at various points along this timeline and form this particular sequence, U k t k minus U t k minus 1 plus its says u 0 u 1 u t 1 minus u 0; u t 1 minus u0 and then you have u t 2 minus u t 1 and so on.

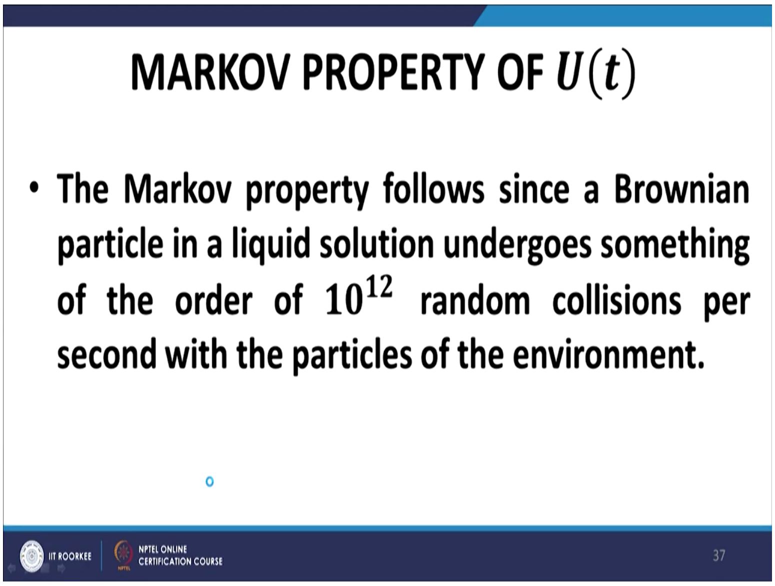
You form this particular expression; obviously, that some of this expressions if you do it all the intermediate terms will cancel out and you will get the left hand side here. Now, we postulate that U t is continuous. It is a Markov process and it is zero-mean. Let us see what is the justification of these three things.

(Refer Slide Time: 18:54)



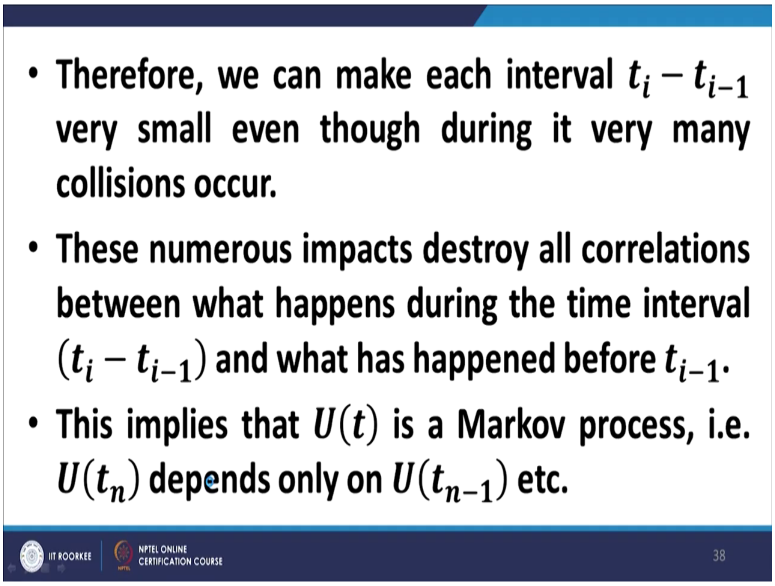
Now, we have d U t by definition d U t is equal to L t dt therefore, U t is equal to when you integrate both sides, you have U t is equal to U 0 plus integral 0 to t L tau d tau. Now, this integral has to be a continuous function of the upper limit and therefore, we find that L t is a continuous function and therefore, U t also has to be a continuous function.

(Refer Slide Time: 19:25)



Now, Markov property I had alluded to a little bit earlier, Markov property arises because the frequency of collisions. The number of collisions of the Brownian particle with the molecules is very very large, something of the order of 10 to the power 10 to 10 to the power 12 collisions per unit time per second right.

(Refer Slide Time: 19:48)

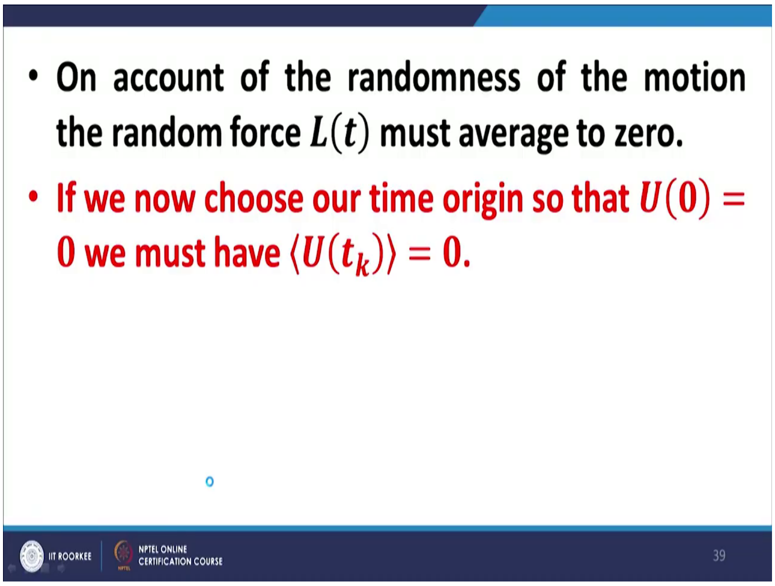


So, what happens is, even if you make t this time slicing; even if you make this time slicing very small, there would still be a number of collisions taking place within a particular time slice; and as a result of which for example, suppose you pick up the time slice t i minus t i minus 1.

The number of collisions that take place within this time slice t i minus t i minus 1 would be very large. And as a result of which, the influence of what had happened earlier is very rapidly wiped out from the memory of the Brownian particle. And therefore, one can say that howsoever small I make t i minus t i minus 1 howsoever small I make this time chunk or time slice.

This the whatever happens here would not be dependent on anything beyond t i minus 1 anything before t i minus 1 because the frequency of collisions is so fast, that the speed of the rate at which memory is lost by the Brownian particle is very rapid. And therefore, for all practical purposes we can say that whatever happens in this interval is at most dependent upon what happens at t i minus 1 and nothing before that and that is what is the characteristic of a Markov process.

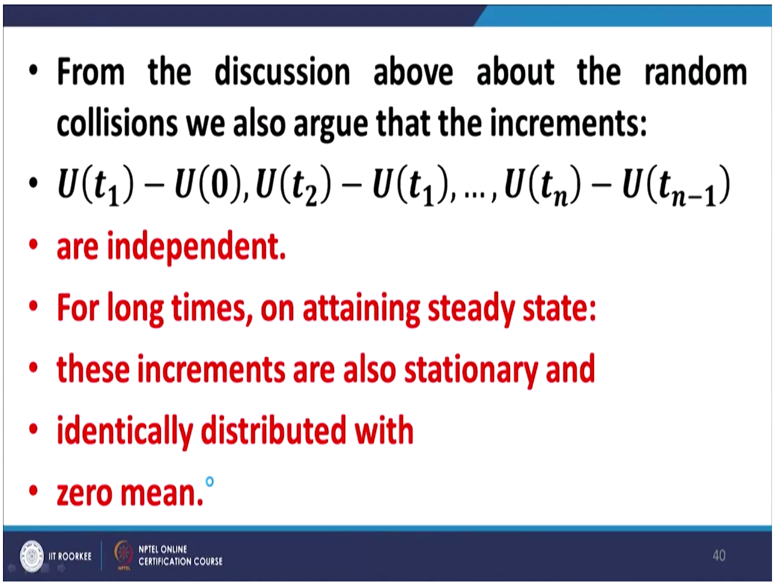
(Refer Slide Time: 21:29)



So, we can say that U t is a Markov process that is; U t depends on U t minus 1 and nothing earlier than that. Then, we have because of the randomness of the emotion; because of the randomness of the motion as I discussed earlier in fact, if there is some kind of a drift that drift can be incorporated in the damping term, and the net result is that we can for all practical purpose, purposes we can assume that the mean of or the average of L t should be approaching 0 should be 0.

And if we now select the time origin such that U 0 is equal to 0, we must then have the average of U t k has 0. Whatever the value of t k is whatever the time pointed with the time slices; the average value must be 0 provided you take the origin U 0; U at t equal to 0 has 0, then because the initial point is 0 and every subsequent interval is has the mean of 0 therefore, we have U t k equal to 0.

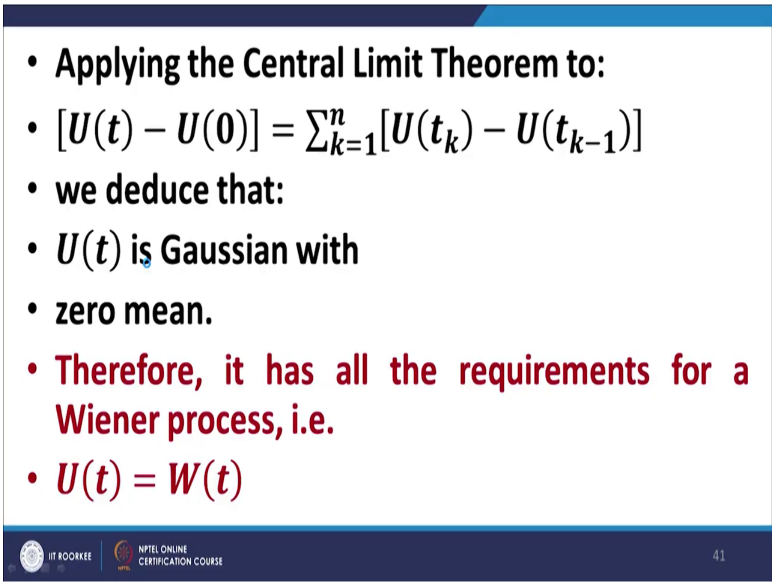
(Refer Slide Time: 22:33)



So, what is the outcome? The outcome is that outcome of all the earlier discussions is very interesting; what do we have? We have U t 1 minus U 0, U t 2 minus U t 1, U t n minus U t n minus 1; all these remember these are time slices and U t 1, U t 2 are basically related to the driving force or the random driving force and what do we find? The random driving force impact are independent in each of these time slices.

They over after the steady state is reached after a point in time of evolution; these increments are also stationary and they are identically distributed with a mean of 0. So, these 3 fundamental properties emanate from the behavior of the applied force that we have the fundamental thing is that it is firstly, it acts for a very small period of time, infinitesimal period of time and secondly, there are a huge number of collisions taking place for unit time.

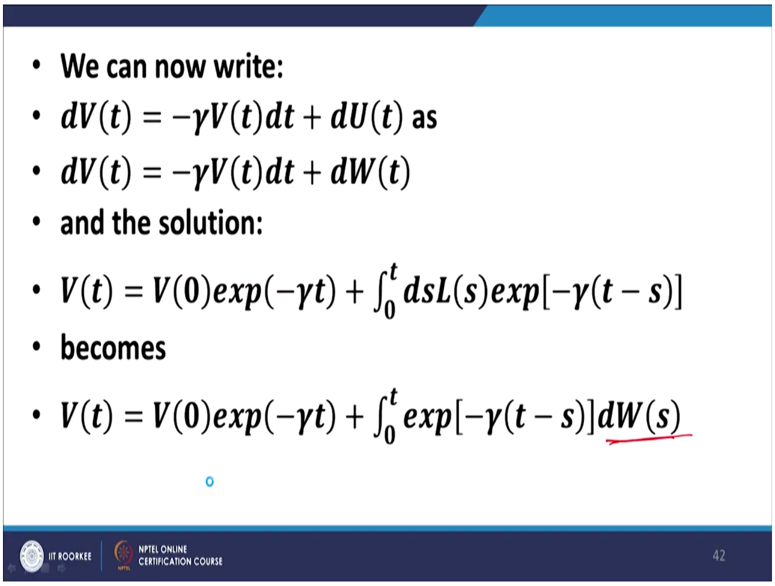
(Refer Slide Time: 23:45)



So, that being the case what do we have? We have because of the central limit theorem if you apply the central limit theorem to the expression on the right hand side what do we get? You get that this expression becomes a Gaussian and it has a zero mean and it has it meets all the requirements of a standard Brownian motion.

So, U t is nothing but this standard Brownian motion. That is what we conclude after this slightly elaborate extended analysis that U t what was U t? U t was L t dt d U t was L t dt; so it is an integral of L t dt L t is basically some kind of a noise and L t dt integrated over gives me U t and that turns out to be Brownian motion.

(Refer Slide Time: 24:36)



Therefore, making use of this particular property now, I can write the solution to my original Langevin equation as V t is equal to V 0 exponential minus gamma t plus I can replace this L s ds by dW s; and now this becomes an integral or over infinitesimal Brownian ah path Brownian motion V t is equal to V 0 exponential minus gamma t plus integral exponential minus gamma t minus dW s.

This is where the infinitesimal increment Brownian increment makes its appearance due to due to what? Due to the equivalence between U t and W t. So, just to recap; just to recap this part of the discussion, the continuity we establish because this integral needs to be continuous in units so, upper limit and therefore, U t has to be continuous so, that was not a major issue.

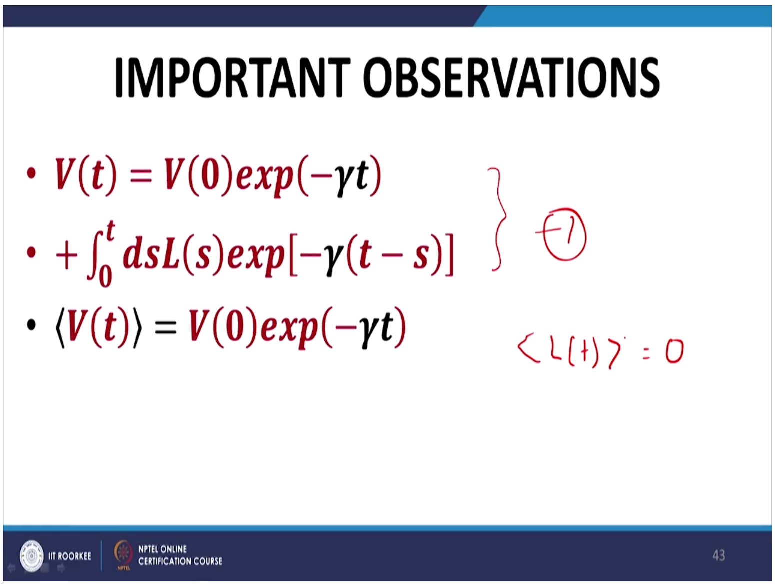
The Markov property arose because the frequency of collisions is so large. Frequency of collisions mean so large; that if you take any small interval even then the number of collisions is so much; so many that the memory carried by the Brownian particle is very short lived, very very short lived.

And for all practical purposes we can say that what happens in this small interval t i minus t i minus 1 is independent or does not depend on the history of the particle; beyond what it was at t i minus 1 so that is what is the property of a Markov process; so therefore, U t was a Markov process because of this, and then we also agreed that because if we assumed that we started the origin that is U 0 is equal to 0 and because each of these increments as a zero mean on the average because if there were any drift that drift would be captured by the damping term and we could have a zero mean process being represented by L t; and that is what was wanted.

And so, the every increment has a zero mean the initial point is 0 and therefore, on the average at any point you take the average at any point in time the mean would be 0. So, the mean is 0 here. And so what have what we have? We have U t minus U t 1 minus U 0, U t 2 minus U t 1; all these are independent of course, there independent right because they represent independent impulses and attending a steady state this increments also become stationary and they become identically distributed and they have zero mean.

So, all these properties we will established and because of these properties if you apply the central limit theorem to this then you end up with U t being a Gaussian variable are a normal variable with a zero mean and therefore, it has all the requirements of a Brownian motion and I can write U t as W t, right.

(Refer Slide Time: 28:12)



Now, we come to some important observations of this model of the revised model. We have talked about the earlier model; we talked about the flow in the earlier model the average of V t square would tended to get unbounded and now we look at what happens in the revised situation, when we introduce a damping term into the model.

Now, when you introduced the damping term into the model the solution as we have seen and becomes our in this equations which is nothing but equation 1. So, clearly if I take the average; if I take the average of V t the second time again goes away because the averaging over L t would be 0; we have already assume throughout that average of L t is equal to 0; average of L t is equal to 0 we have assumed throughout.

So, when we do the averaging of V t we find that the first term remains; the second term becomes 0 and the averaging is V 0 exponential minus gamma t it is quite clear that as time passes a passes V 0 this average tends to decrease because of the damping factor and gradually the velocity will fall.

(Refer Slide Time: 29:34)

$$V(t)=V(0)\exp(-\gamma t)+\int ds L(s)\exp[-\gamma(t-s)]$$

$$\langle V^2(t)\rangle = V^2(0)\exp(-2\gamma t)$$

$$+\left\langle \iint \begin{matrix} dt_1 dt_2 L(t_1)L(t_2)\exp[-\gamma(t-t_1)] \\ \exp[-\gamma(t-t_2)] \end{matrix} \right\rangle + \langle CROSS \rangle$$

$$= V^2(0)\exp(-2\gamma t)+\iint \begin{matrix} dt_1 dt_2 \langle L(t_1)L(t_2)\rangle \exp[-\gamma(t-t_1)] \\ \exp[-\gamma(t-t_2)] \end{matrix}$$

Now, let us look at the average of V squared t when we do the average of V square t again the cross terms will again vanish, for the same reason as I explained earlier the reason was that, the average of L t is 0 and because the average of L t is 0 the cross term which has one term in L t and one term in V 0, the L t average being 0 that cross term vanishes and we are left with these 2 terms.

The first term is V square 0 exponential minus 2 gamma t that is this is deterministic there is no problem with this. The second term involves a double integral; double integral let us call it dt 1 dt 2 L t 1 L t 2 exponential minus gamma t minus t 1 exponential minus gamma t minus t 2; remember this integration has to be done over the limits 0 to t.

(Refer Slide Time: 30:47)

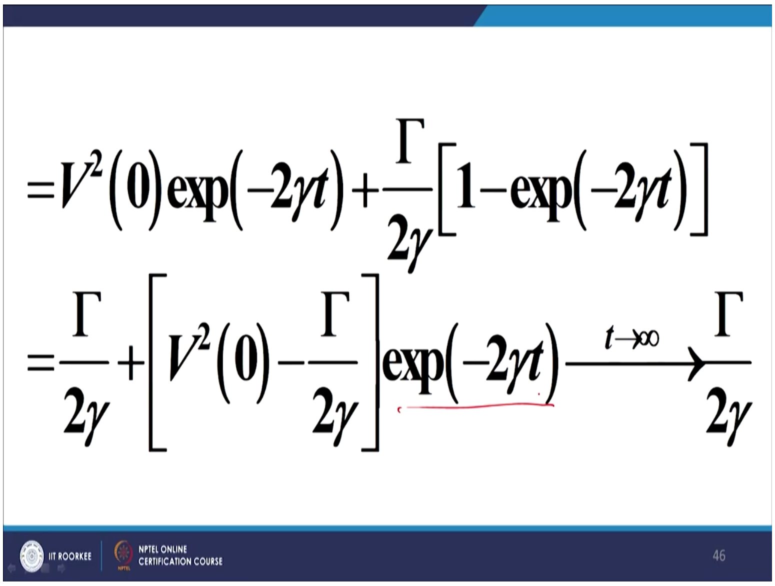
$$
\begin{aligned}
&= V^2(0)\exp(-2\gamma t) + \Gamma \iint dt_1 dt_2\, \delta(t_1 - t_2) \exp\left[-\gamma(t - t_1)\right] \exp\left[-\gamma(t - t_2)\right] \\
&= V^2(0)\exp(-2\gamma t) + \Gamma \int dt_1 \exp\left[-2\gamma(t - t_1)\right] \\
&= V^2(0)\exp(-2\gamma t) + \Gamma \exp(-2\gamma t) \int_0^t dt_1 \exp\left[2\gamma t_1\right] \\
&= V^2(0)\exp(-2\gamma t) + \frac{\Gamma}{2\gamma}\exp(-2\gamma t)\left[\exp(2\gamma t) - 1\right] \quad --(2)
\end{aligned}
$$

So, we since this is an averaging the angular brackets can be replaced with angular brackets within the integral because as I mentioned integral is the sum and now the angler brackets between L t 1 and L t 2 are nothing but the auto correlation functions and they are given by gamma delta t 1 minus t 2; so, we have substituted gamma delta t 1 minus t 2. Now, we perform let us say we perform the t 2 integration and then, what we end up with is that all t 2's will be replaced by t 1.

When you replace all t 2's by t 1 exponentially both the exponential terms become the same and I get exponential minus 2 gamma t minus t 1 integral dt 1; if you do this integration it become it gives me the expression which is given at the bottom of this slide let us call it equation 2.

Simple straightforward integration of the exponential in this 2 gamma in the exponential will manufactured itself in the denominator as 2 gamma, and the rest when you put the limits you get this expression 2 gamma t minus 1.

(Refer Slide Time: 32:01)

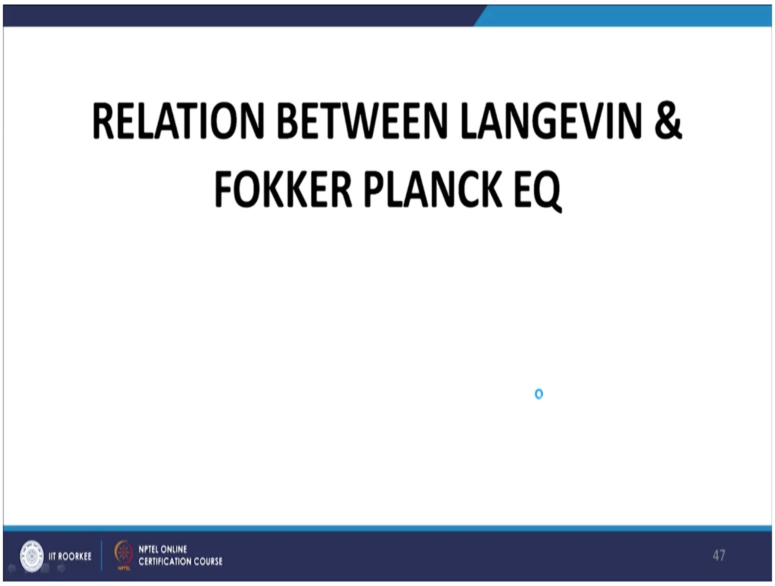


You simplify this a bit further; you simplify this a bit further and what I end up with this gamma upon 2 small gamma plus remember small gamma is the small gamma is a damping coefficient and capital gamma is the strength of the applied random force the stochastic force. So, gamma upon 2 gamma plus V square 0 minus gamma upon 2 gamma exponential minus 2 gamma t.

Now, as t tends to infinity a very interesting thing happens now and the interesting thing happening is that this term tends to 0, as t tends to infinity the exponential minus 2 gamma t time tends to 0; and that being the case; and that being the case this whole expression vanishes and I get a constant steady state value which is given by capital gamma upon 2 small gamma. Remember this is for a particle of unit mass right, for a Brownian particle of unit mass.

So, in this case the problem that we had in the earlier model has been taken care of if we end up with a finite value of the average of V square t and we also have a finite value of V c V t and V square t; so, the first and second moments both have finite values and to that extent this models in physically viable right.

(Refer Slide Time: 33:37)



So, we conclude here for today. We will continue in the next class first discussing the Langevin equation and the Fokker Planck equation and then deriving a path integral solution for the Langevin equation.

Thank you.