Path Integral Methods in Physics & Finance
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Lecture - 11
Diffusion Equation Path Integral (1)

Welcome, in the last lecture let us speak up a bit I introduced the concept of Brownian motion. Brownian motion as I mentioned forms the cornerstone of the theory of stochastic processes. So, let us review it once again very briefly.

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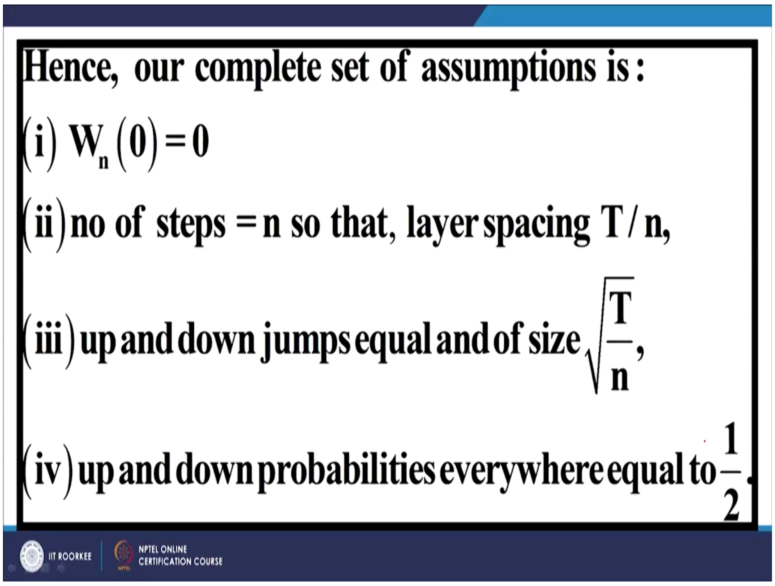

We started with the concept of a random walk which is a sequence of discrete random variables, binary random variables that can take only two values; let us say heads and tails and we had a sequence of those random variables representing a stochastic processes and then we extended that sequence to a continuous framework.

So, we found that in the case of the discreet set of random variables if we have n random variables the mean; obviously, turns out to be 0 and the variance turns out to be equal to n. Then we said that no because we want to model the stochastic process as a continuous process in time by reducing the step size, the time step size or increasing the number of steps towards infinity and that will blow up the variance, that will curtail its utility as a modeling tool.

And therefore, we changed the jumped dimensions we started with a, jump dimension of plus minus 1 we change the jump dimension to plus minus under root t upon n. So, let us recap the entire set of assumptions that we had at that point in time we started the process at the origin. So, we wrote W n 0 is equal to 0. Incidentally, W n represent the spectrum of values at time T which is given by the argument of W n, n represents the number of steps.

So, number of steps was n therefore, the layer spacing or the time step was T upon n and the upper up and down jumps were under root T upon n plus an under root T upon n minus. Probabilities, because we were considering a unbiased random walk the probability in either case was equal to 1 by 2.

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Hence, we have IIDs X and Y such that:

$$Y_i = \sqrt{\frac{T}{n}} X_i \text{ where } X_i \text{ are IIDs defined by}$$

$$X_i = \begin{cases} +1 \text{ with } p(X_i = +1) = 1/2 \\ -1 \text{ with } p(X_i = -1) = 1/2 \end{cases}$$

$$Y_i = \begin{cases} +\sqrt{\frac{T}{n}} \text{ with } p(Y_i = +1) = 1/2 \\ -\sqrt{\frac{T}{n}} \text{ with } p(Y_i = -1) = 1/2 \end{cases}$$

Then this new random variable which we termed as Y i and we had a sequence of these random variables for which you proceeded to take the limit by using or by invoking the central limit theorem.

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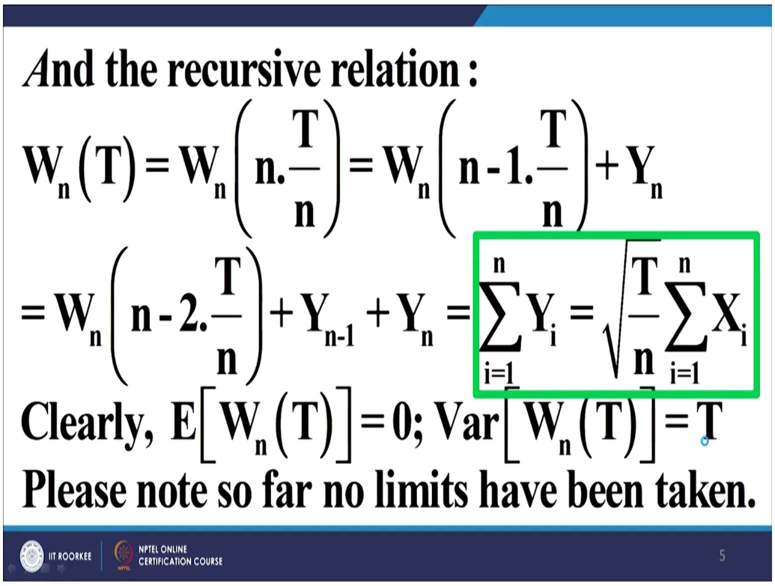

The entire sequence of random variables is captured by the displacements at any point in time is captured by the sum of these random variables as you can see here W n of capital T is equal to summation of all the random variables representing the fluctuations of the system or the transitions of the system at each time step.

And then we also worked out that for an this is for the total evolution that I have just discussed. For the case of an arbitrary time t small t that is in between 0 and capital T the in between any point or observation point within the total or the overall time of the evolution of the system, we find that the number of steps turns out turn out to be n small t upon capital T.

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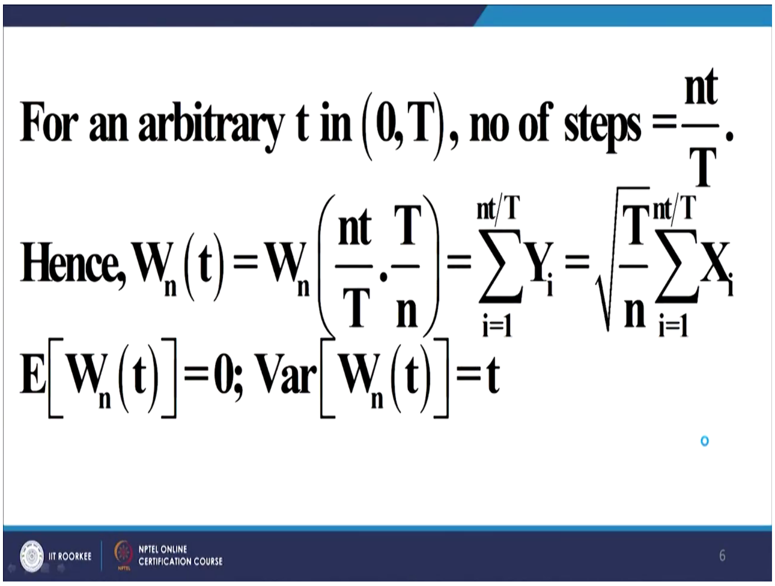

And accordingly the results get modified, the variance gets modified to small t and the mean of course, remains the 0.

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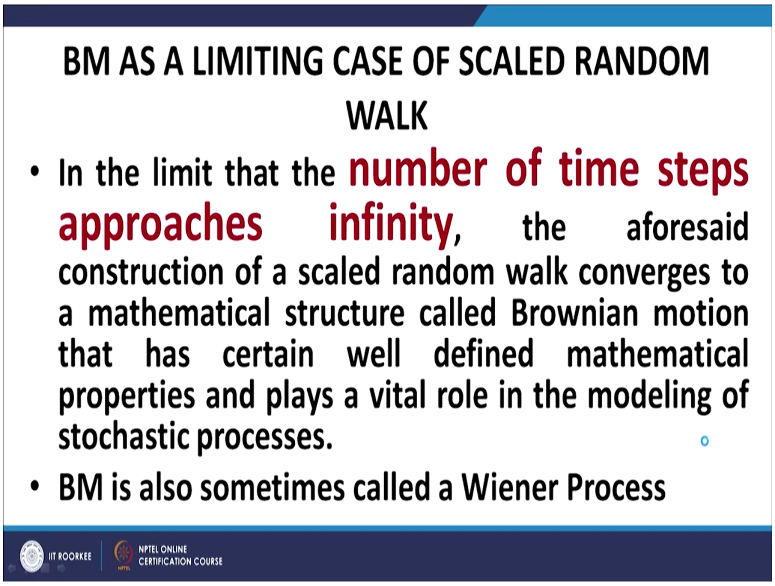

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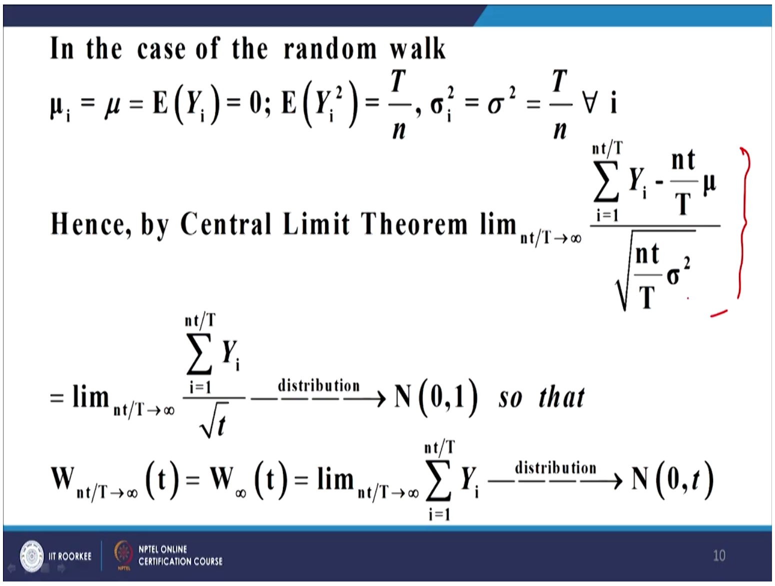

Then we did the limit taking. When we did the limit taking we found that, because of the central limit theorem this expression that I have here becomes normally distributed as a standard normal variable and because it becomes normally distributed as a standard normal variable and mu in our case is equal to 0, sigma in our sigma square in our case turns out to be T upon n therefore, we find that W n or now of course, I will use the term w infinity and ignore infinity so we will call it W t.

Wt is normally distributed with a mean of 0 and a variance of small t. This is a very fundamental result, this is the defining property of Brownian motion. The possible value the spectrum of values that the process, the Brownian process can take at any point in time is normally distributed with a mean of 0 and with a variance equal to the time elapsed since the origin of the Brownian motion.

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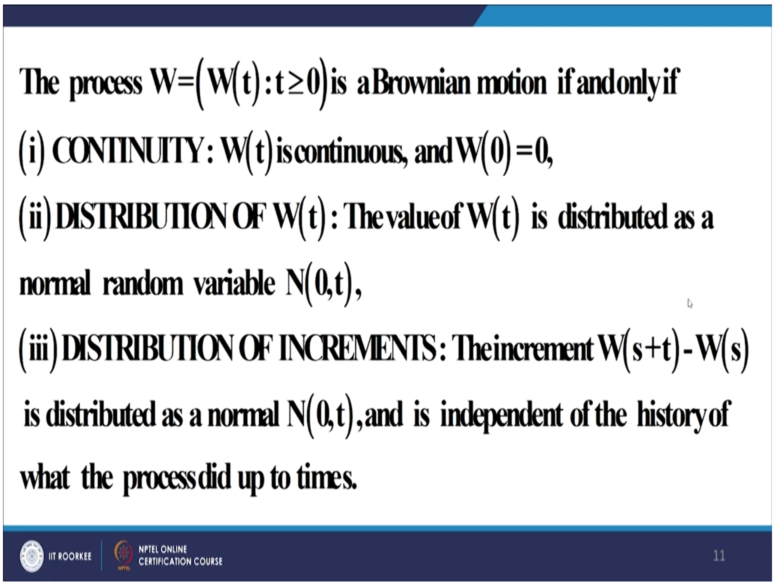

So, a quick recap on the properties of the Brownian motion. Brownian step, Brownian paths are continuous although, they are not differentiable at any point and they are nowhere differentiable to be more precise.

The Brownian motion is normally distributed or W t is normally distributed with a mean of 0 and a variance of t. In fact, the increment, the increment W s plus t minus W s is also normally distributed with a mean of 0 and a variance of t

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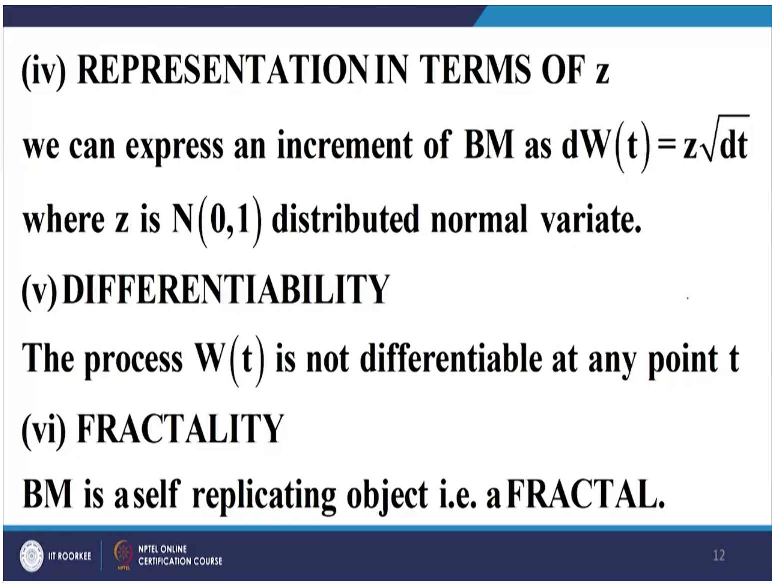

We can represent a Brownian motion or in infinitesimal increment of Brownian motion d W t in terms of the standard normal variable z and under root d t it can be verified very easily that the mean and variance of both this has turned out be the same and because in the case of a normal distribution it can be completely specified by the specification of the mean and the variance, the two terms on the equality holds.

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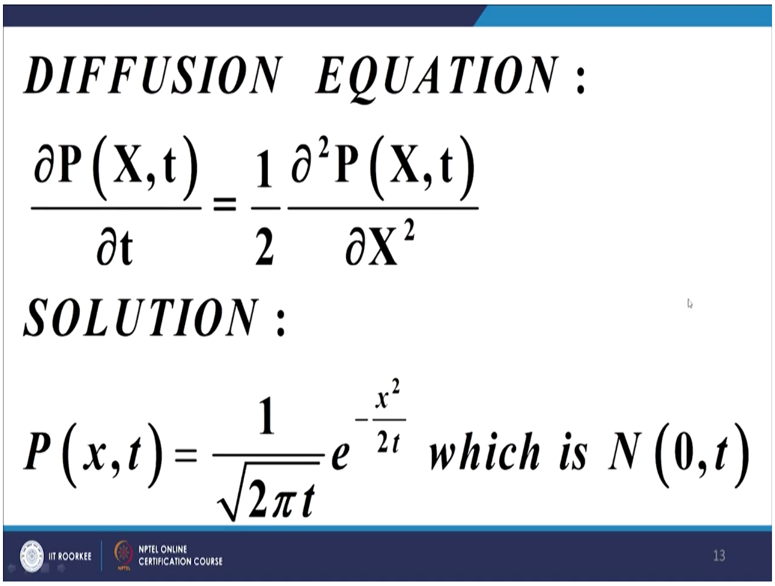

Brownian motion is a fractal as I mentioned earlier. Then we talked about the diffusion equation and we found that when we saw when we start with certain fundamental principles like the collisions of a Brownian particle, random motion of a Brownian particle due to collisions that it encounters when immersed in a fluid, due to the molecules of the fluid colliding with the Brownian particle can be represented by a diffusion equation.

And indeed when we solve this diffusion equation, we arrive precisely at the same result as in the case of Brownian motion which in the probability distribution clearly shows, the PDF clearly shows that it is a normal distribution with a mean of 0 and a variance of t.

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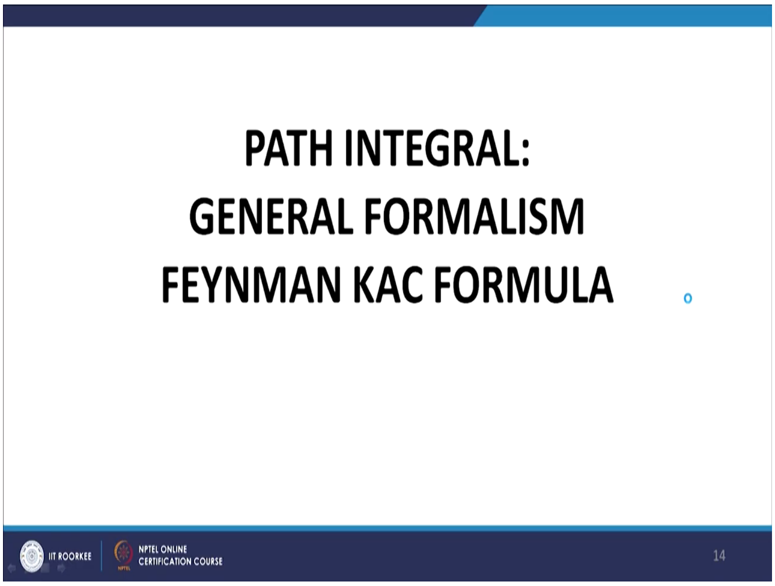

So, that is where we concluded the topic of Brownian motion. Today, we will extend concept of Brownian motion and use it to define a path integral arising out of the solution of the diffusion equation.

So, let us start with 1 dimensional diffusion equation, del phi upon del t is equal to u phi plus d upon 2 d is a diffusion constant. It is a constant that is the most important part and del square phi upon d x square.

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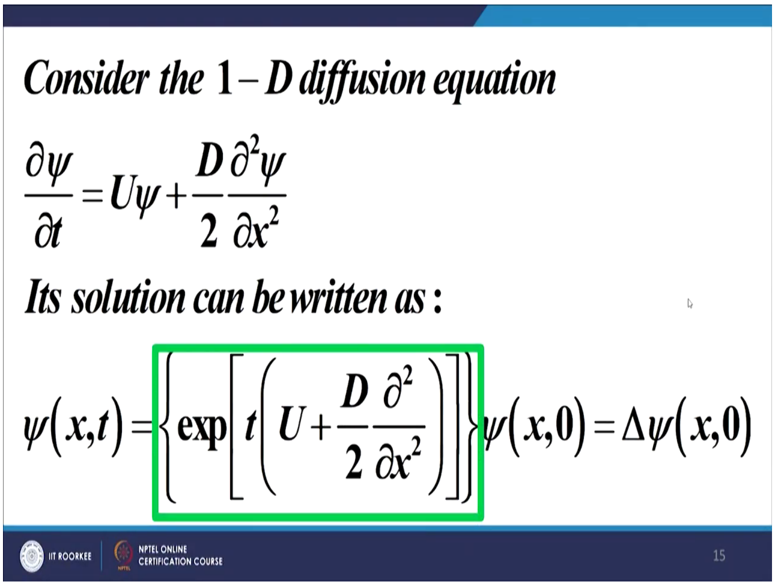

The solution can be written as it is quite simple a solution you can write in the form of phi of x t is equal to exponential of this expression and phi of x 0.

You can see that this expression that I have enclosed in the green box represents a kind of an operator, which operating on phi x 0 transform is transformed this to phi x t. This is the function of what we call an a propagator.

We shall be talking much more about this concept of propagator, when we talk about field theory, but for the moment its job is to transform a state, given state from one point in time to another point in time and that is precisely, which is explicitly is shown by this equation here. We represent it by delta.

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$$F/A: \psi(x,t) = \exp\left[t\left(U + \frac{D}{2}\frac{\partial^2}{\partial x^2}\right)\right]\psi(0) = \Delta\psi(x,0)$$

$Set: A = U; B = \frac{D}{2}\frac{\partial^2}{\partial x^2}$. *By Trotter's formula, we have*

$$\exp\left[t(A+B)\right] = \lim_{n\to\infty}\left[\exp(tA/n)\exp(tB/n)\right]^n \text{ so that}$$

$$\Delta = \exp\left[t\left(U + \frac{D}{2}\frac{\partial^2}{\partial x^2}\right)\right] = \lim_{n\to\infty}\left[\exp\left(\frac{t}{n}U\right)\exp\left(\frac{t}{n}\frac{D}{2}\frac{\partial^2}{\partial x^2}\right)\right]^n$$

Now, this expression the propagator the expression for the propagator, the exponential term we use Trotter's formula to simplify it further. What is Trotter's formula? Trotter's formula is given here; exponential of t into A plus B is equal to exponential t A upon n exponential t B upon n to the power n by with the limit moving towards infinity.

Limit n tending to infinity exponential t A upon n exponential t B upon n. To make this ah result applicable we make the substitution A is equal to U B is equal to D upon 2 del square upon del x square making these two substitutions we have our given expression the for the propagator delta is equal to limit n tending to infinity exponential t upon n into U exponential t upon n D upon 2 del square upon del x square to the entire thing to the power n and then with the limit n tending to infinity. So, our job is to simplify this expression to the extent we can do.

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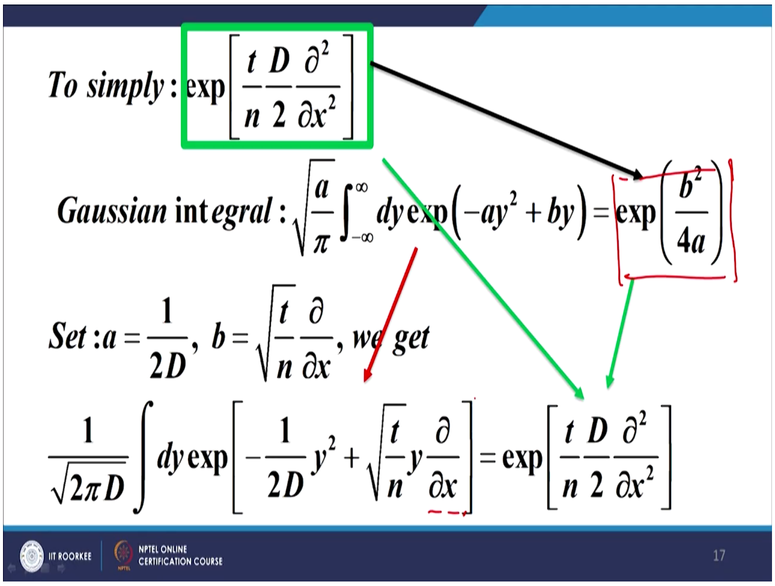

We shall be focusing on the second factor to start with exponential t upon n D upon 2 del square upon del x square let us start with this expression. Now, to simplify this expression let us recall a result in Gaussian distribution.

If you look at this integral under root a upon pi integral minus infinity to infinity d y exponential minus a y square plus by, this can be simplified by converting it to a perfect square and then using the Gaussian integral and we find that this the result that we get is exponential b square upon 4 a.

Let us make the substitution a is equal to 1 upon 2 D that is small a, a is equal to 1 upon 2 D b is equal to under root t upon n del by del x. We make the substitution in this particular expression.

When we make this particular substitution in this expression, we get the expression given in the top green box that is what we wanted to simplify exponential t upon n D by 2 del square upon del x square, this is the expression that we get. If I put a is equal to 1 upon 2 D s can be easily verified straight away, a is equal to 1 upon 2 D b is equal to under root t upon n del by del x.

So, b square upon 4 a comes out to be this expression precisely. Therefore, making the same substitution on the left hand side; making the same substitution on the left hand side I get this expression that is 1 upon under root 2 pi D integral d y exponential minus 1 upon 2 D y square plus under root t upon n y del by del x.

In a nutshell what has happened is the expression that we started with has now been converted to an integral given on the left hand side of the last equation on the slide. The important thing to note here is number 1 that this as the second order derivative appearing in the exponential has now been converted to a first order derivative as you can see on the left hand side this expression.

You can see that it is a first order derivative, the second order derivative has been converted to a first order derivative number 1 and number 2 which we will come to in just a moment that this gives the feel that, because when we have the first order derivative here and we have the exponential here, the relevance of this expression we come into the in the following slides.

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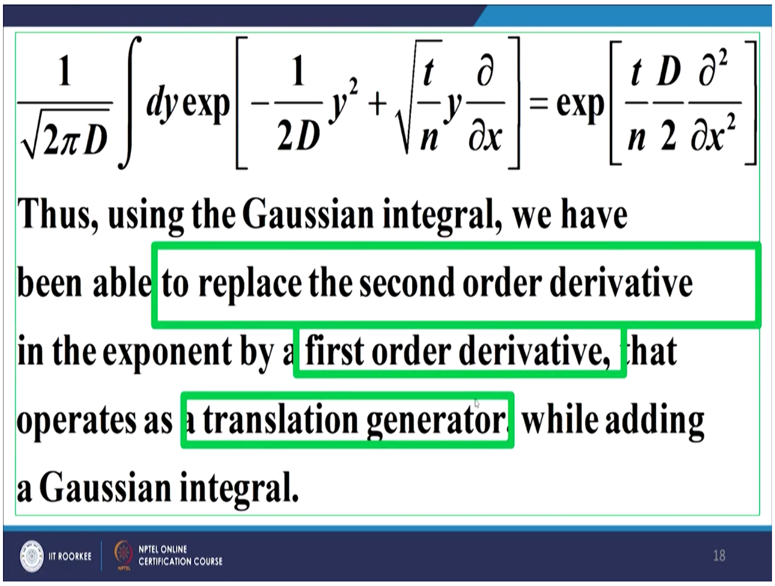

So, this is what we have to start with. Now, as I mentioned earlier, what we have been able to do is we have been able to replace the second order derivative with the first order derivative in the exponential that rings a bell. What bell it rings? Let us come to that.

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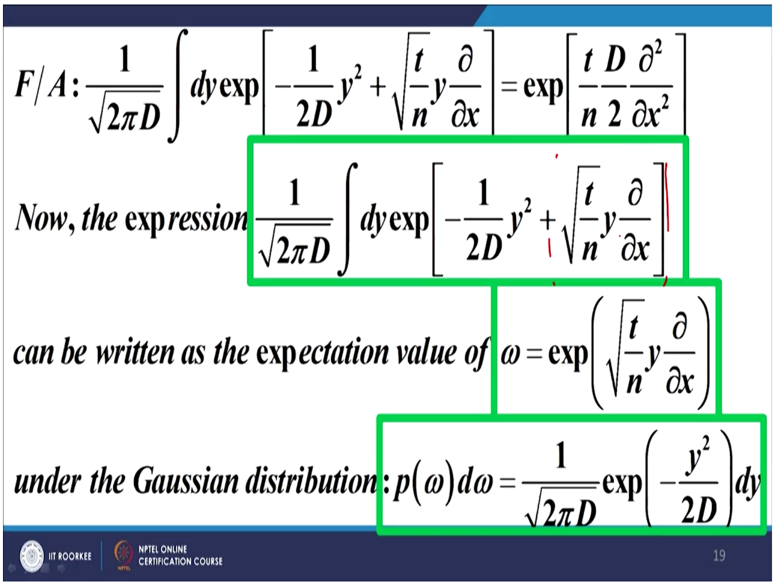

Now, this expression 1 upon under root 2 pi D exponential this whole thing. If you look at it carefully it can be segregated into two parts; the first part exponential minus 1 upon 2 D into y square and the second part exponential under root t upon n y del by del x.

You can look at it this way that we this is a random variable under root t upon n y del by del x is the random variable, you can look at that. Indeed, if you go back to the slide; if we go back to this particular slide where we have used the Gaussian distribution, we clearly find that y is actually a random variable.

So, this, because y is a random variable under root t upon n y del by del x is also random and we can look at this whole expression as a expectation value. Expectation value of what? Expectation value of omega exponential under root t by n y del by del x, this in this second part with respect to the probability defined by first term minus 1 upon 2 D y square you can see that this is Gaussian, exponential minus 1 upon 2 D y square and you have a factor of 1 upon under root 2 pi D also, outside the integral this is clearly a Gaussian probability distribution.

So, we have a Gaussian probability distribution and we have a Gaussian variable therefore, the whole integral can now be represented as an expectation.

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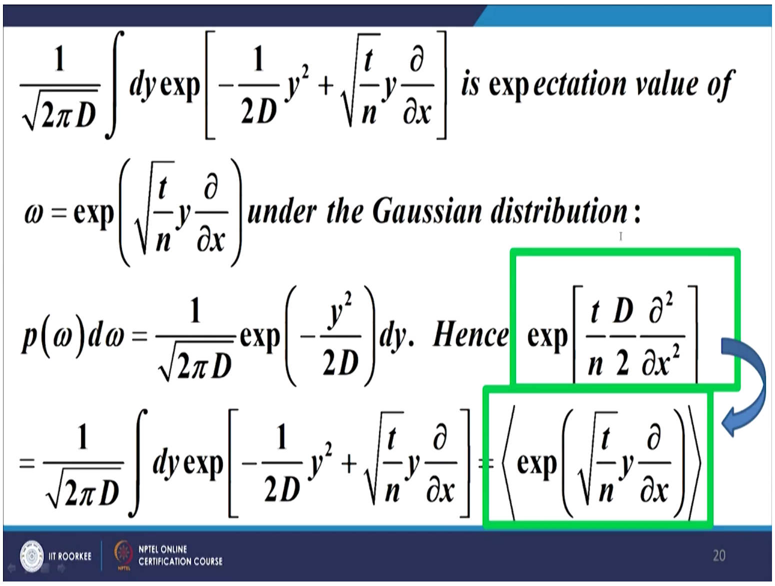

And that is precisely what we are doing here, we are writing this as the expectation value exponential under root t upon n y del by del x. This brackets represent the expectation value.

I have been using e factor as e operator as the expectation so far, but this is also the same thing precisely the same thing whether I use this angle brackets or I use the e e operator, it amounts to like working out the expectation of the quantity within and this is precisely what to repeat this.

In fact, if you work out the other way around it becomes more clear. Now, if you work out the expectation of this expression with which is within the angular brackets, this is my random variable and the probability distribution is given by exponential minus 1 by 2 Dy square 1 upon under root 2 pi D which is Gaussian and you are integrating over all of value of the random variable y and therefore, this whole thing becomes the expectation value that we capture in this angular brackets right.

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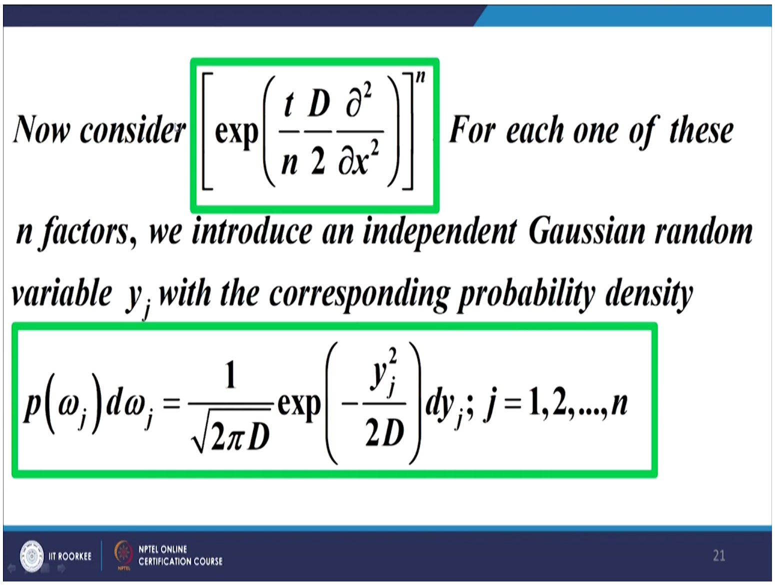

So, now let us look at now, you see we have simplified exponential t upon n D by 2 del square upon d x square and what have been shown it to be? We have shown it to be the expectation exponential under root t upon n y del by del x.

Please note this very carefully that we have a first order derivative within the expectation whereas, the term that we started with was the second order derivative in x. Now, but we have to do more, we have to work out the nth power of this expression in order to work out nth power of this expression.

In order to simplify the nth power of this expression, we introduced n such variables or n such expectations and for that purpose we introduce n such variables y 1 y 2 y 3 y j and with respect to each such y variable, we take the expectation of one factor within this total of n factors.

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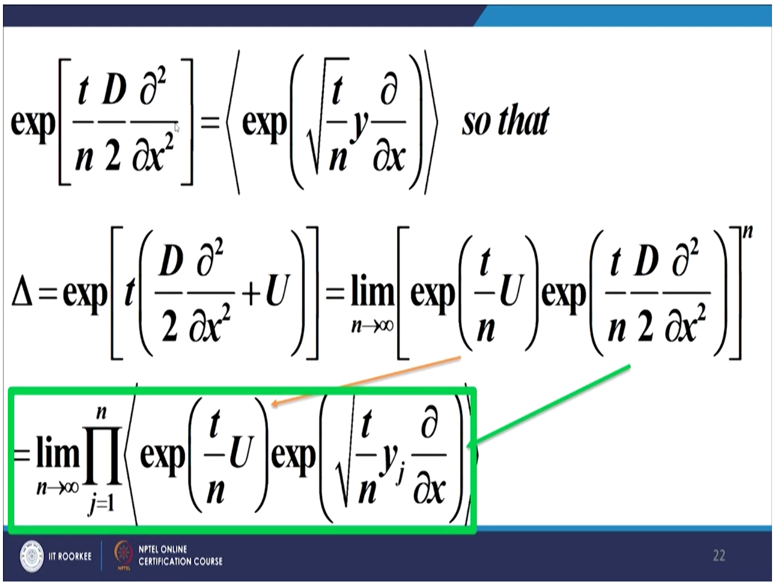

So, when we do that what do we get? We get the this expression in the green box right at the bottom. The limit n tending to infinity still to be taken so that remains outside. Product j equal to 1 to n exponential t upon n U you will recall that we had this term to start with.

So, no change in this particular term and the t upon n D by 2 del square upon del x square to the power n is written as a product of n random variables y 1 y 2 y 3 together with the expression under root t upon n d by dx or del by del x.

So, this expectation, this expectation over n random variable so each of which is Gaussian, each of which is Gaussian, each of which is independent of the other is represented by this limit n tending to infinity product of the expectations and now, because they are all independent so, e x e y is equal to e x y.

So, you can write it in this form or you can write it as expectation in terms of taking a product of expectation term or expectation of product terms it comes to the same, because they are independent of each other. The y i are independent y 1 is independent of y 2 is independent of y 3 and so on right.

So, some important observations with respect to the last expression that we had on the slide, the green box expression let us go through those observations. The expectation brackets stands for all the n expectation values of Gaussian integrals. So, all the expectations values are covered within the product and the angular brackets.

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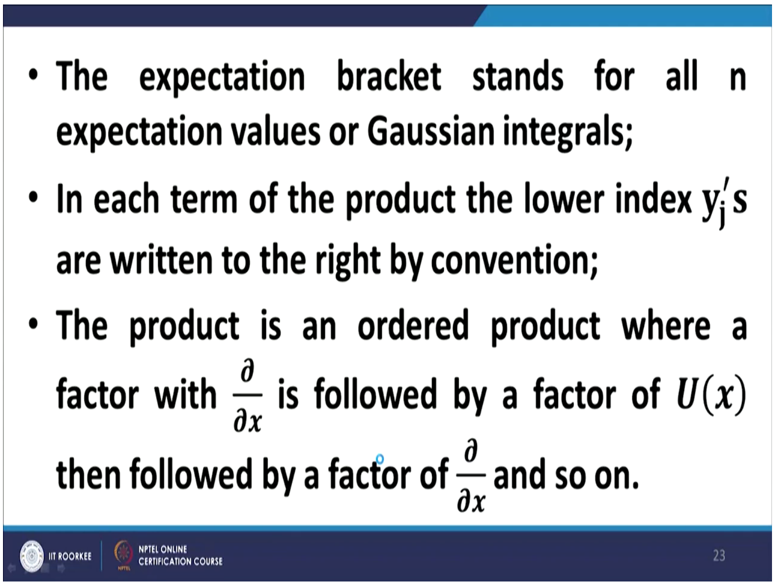

In each term the product with a lower index y i that is; y 1 will come first to the right, then y 2, then y 3. So, the order is increasing from the right to the left, this is by convention. The lowest y i that is y i y 1 is represented at the first term and the y 2 at the second term and so on the order is taken in the ascending order right.

Then the you can look at this, as I mentioned you can have all the x terms within this expectation or you can have it as a product of expectations if you have it within the expectation then what will I have? I will have a term in y 1, then I will a term in U, then I will have a term in y 2, I have a term in U, I have a term y 3, and I have a term in u and so on. So, they keep on alternating, y and U keep on alternating with each other.

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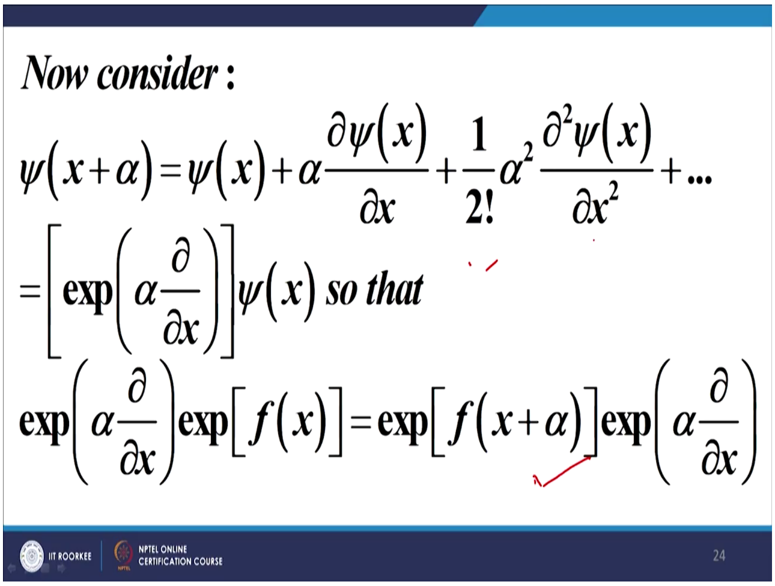

Now, we come to the point that I mentioned earlier that when you have a first order derivative in the exponential it becomes very interesting. It becomes interesting because this acts as a translation generator.

Now, translation generator means this equation holds. This equation is a variant of the first equation; the first equation clearly exemplifies the translation property of exponential alpha del by del x. When exponential alpha del by del x operates on any variable psi of x it simply gives you the variable psi x plus a. That means, it shifts, it makes a translation shift from x to x plus a. The operation by this variable exponential alpha del by del x simply shift the underlying variable x by an amount of a and we get the new the new function at the new point phi of x plus a.

On this basis we get the second equation that we are going to use exponential alpha del by del x exponential f x is equal to exponential f x plus a, exponential alpha del by del x. This is simply an extension of the first equation that we had here that I just explained.

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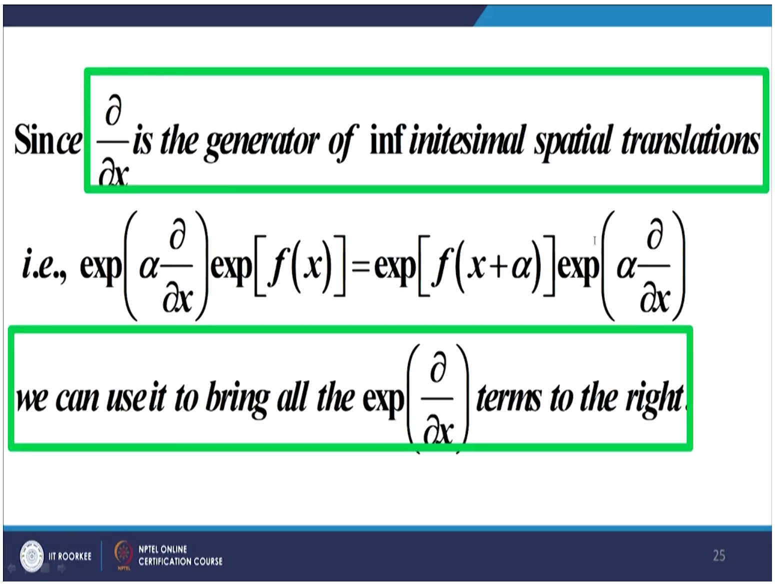

So, del by del x acts as a generator of infinitesimal special translations and exponential del by del x, it has a finite translation with cumulation of infinitesimal translations.

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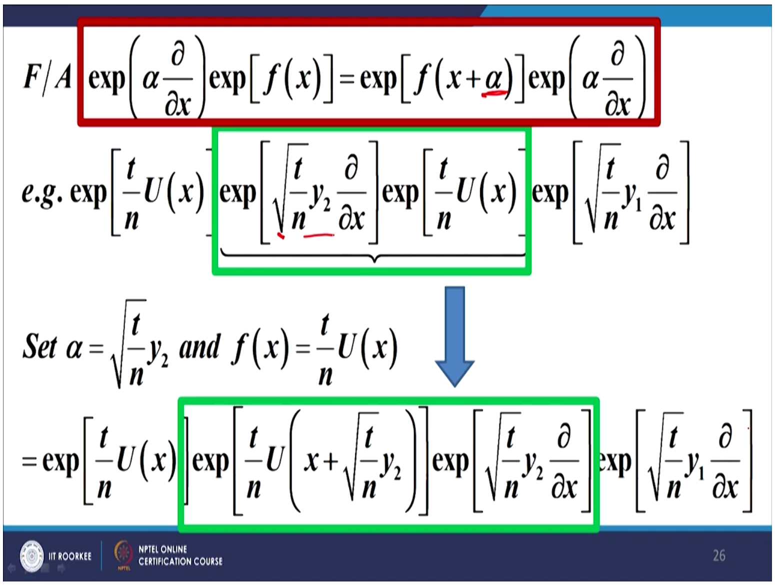

Why do we need it? Let us look at this. What we can do by this is. As i mentioned in a previous slide, when you look at this you have alternating terms of y and U, y and U, y and U.

We can bring all the y terms to the right hand side and all the U terms to the left hand side by making use of this property of being a translation generator. So, that is where the role of this translation generator comes into play, let us see this explicitly. So, we have in this expression, this is an example how it operates; exponential t upon n into U x, exponential under root t upon n y 2 del by del x, exponential t upon n U x, exponential t upon n y 1 del by and del x.

Now, I have combined the terms two middle terms. And the two middle terms means exponential under root t upon n y 2 del by dx exponential t upon n U x. These pair of terms I will change their order; I will change their order by making the use of the x 1 of the property, the translation radiant property of exponential t upon n U x.

By making this use of this property the translation generator property of exponential t upon n U x I will reverse the order of the appearing of these two quantities. Let us see how we do it. The property as I explained earlier gives us exponential alpha del by del x exponential f x is equal to exponential x plus a f x plus a exponential alpha del by del x. In our case what happens? We write alpha equal to under root t upon n y 2; alpha is equal to under root t upon n y 2, this particular portion, and we write f x is equal to t upon n U x.

By making these substitutions and making use of the first equation the red box equation we simply arrive at a reversal of their order; the del by del x term and together with the y 2 term now shift to the right hand side and the U x term shifts to the left hand side.

So, what do we have? We this t upon n U x is as earlier no change in that, but when this U x term shifts to the left hand side it will have an additional factor of alpha as you can see in the first equation in the red box equation. Therefore, instead of U x we will now have U x plus under root upon n y 2.

That represents this particular alpha here. As you can see we have used x plus under root t upon n y 2, and under root t upon n y 2 is our alpha. So, f of x plus alpha becomes exponential t upon n U x plus this particular expression which corresponds to alpha.

And now this derivative term moves to the right hand. On the right we have the two derivatives terms together, we have under root t upon n y 2 del by del x and under root t upon n y 1 del by del x both with exponentials of course,.

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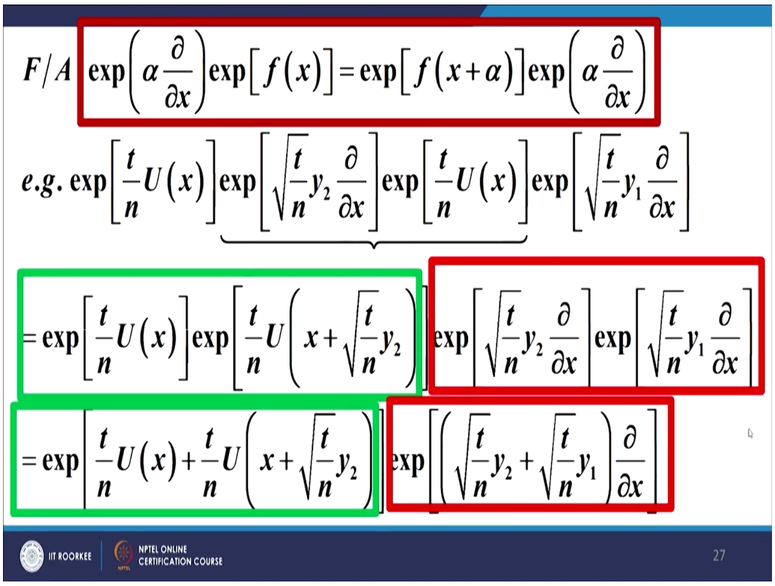

Now, little bit more simplification simply adding the exponents we get this particular expression. When you add the exponents you get t upon n U x plus t upon n U x plus under root t upon n y 2 e e to the power x e to the power y is equal to e to the power x plus y.

And similarly, here we use the same thing e to the power x plus e to the power x; e to the power y is equal e to the product x plus y right.

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$$\exp\left[\frac{t}{n}U(x)\right]\exp\left[\sqrt{\frac{t}{n}}y_3\frac{\partial}{\partial x}\right]\exp\left[\frac{t}{n}U(x)\right]\exp\left[\sqrt{\frac{t}{n}}y_2\frac{\partial}{\partial x}\right]\times$$

$$\exp\left[\frac{t}{n}U(x)\right]\exp\left[\sqrt{\frac{t}{n}}y_1\frac{\partial}{\partial x}\right]$$

$$=\exp\left[\frac{t}{n}U(x)\right]\exp\left[\frac{t}{n}U\left(x+\sqrt{\frac{t}{n}}y_3\right)\right]\exp\left[\frac{t}{n}U\left(x+\sqrt{\frac{t}{n}}y_2+\sqrt{\frac{t}{n}}y_3\right)\right]\times$$

$$\exp\left[\sqrt{\frac{t}{n}}y_3\frac{\partial}{\partial x}\right]\exp\left[\sqrt{\frac{t}{n}}y_2\frac{\partial}{\partial x}\right]\exp\left[\sqrt{\frac{t}{n}}y_1\frac{\partial}{\partial x}\right]$$

So, now we bring another pair of terms into the picture. We bring another pair of terms in the in the picture what are they? t upon n U x exponential under root t upon n y 3 del by del x and whatever is there earlier is retained.

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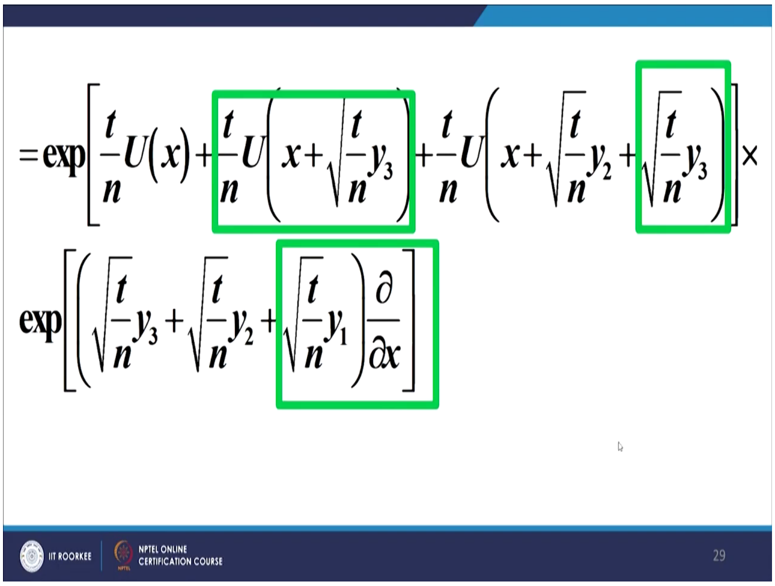

When you again apply the translation generator what we end up with is this, slightly extended equation we have exponential t upon n U x t plus t upon n U x plus under root t upon n y 3 plus t upon n U U x plus t upon n y 2 plus under root t upon n y 3 and this is. And now, again and now again all the d y by d x terms come to the right hand side and we have exponential under root t upon n y 3 plus under root t upon n exponential. Sorry y 2 plus under root t upon n y 1 with d by d x or del by del x common throughout.

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$$\Delta = \lim_{n\to\infty} \prod_{j=1}^{n} \left\langle \exp\left(\frac{t}{n}U\right) \exp\left(\sqrt{\frac{t}{n}}\, y_j \frac{\partial}{\partial x}\right) \right\rangle$$

$$= \lim_{n\to\infty} \left\langle \exp\left(\frac{t}{n}\sum_{k=1}^{n} U\left(x + \sqrt{\frac{t}{n}}\sum_{j=k}^{n} y_{j+1}\right)\right) \exp\left(\sqrt{\frac{t}{n}}\sum_{k=1}^{n} y_k \frac{\partial}{\partial x}\right) \right\rangle$$

$$\textit{with } y_{n+1} = 0.$$

So, the net result is that by playing with or by making use of the property of the translation generators or using exponential t upon n U as a translation generator we are able to transform all the derivative terms to one side and all the U terms to the other side; that is the most important thing that is precisely what we have done and we get this ultimate result.

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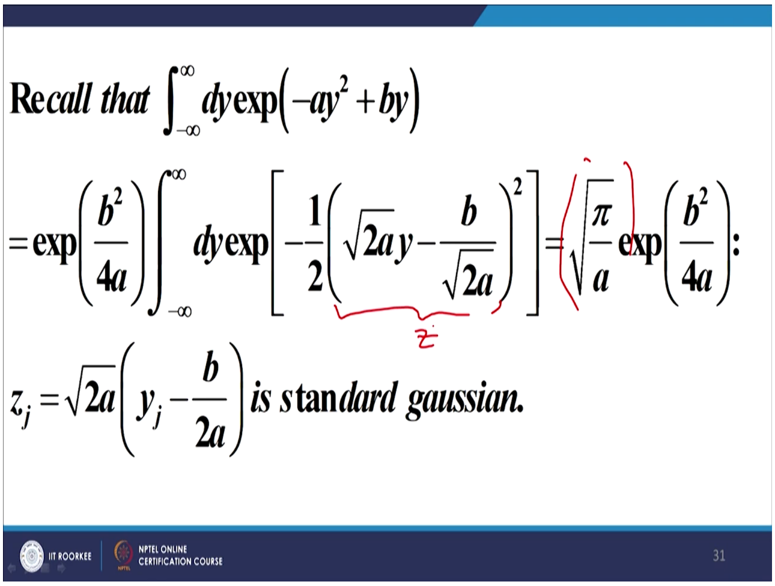

Now, let us go back to our Gaussian Integral, we had this Gaussian integral d y exponential minus a y square plus b y. When you simplify this we by completing the square we get exponential b square upon 4 a and d y exponential minus 1 by 2 under root 2 a y minus b upon under root 2 a whole square and that gives us this pre factor of under root t upon a.

But the important observation from this is that this particular expression that within the round bracket under root 2 a y minus b upon under root 2 a is a standard Gaussian or a standard normal variable and it is distributed and normally with a mean of 0 and a variance of 1. This expression if I call this particular variable as z, then z is normally distributed with a mean of 0 and a variance of 1 right.

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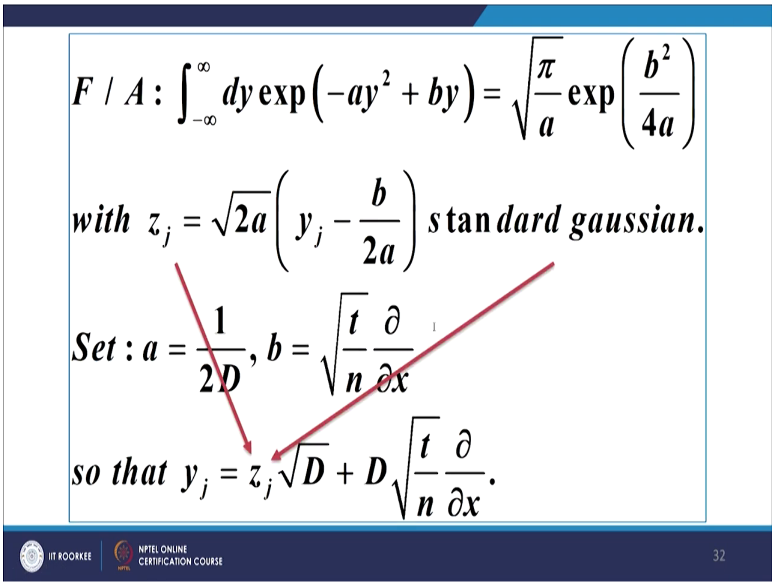

So, just to recall z is normally distributed with a mean of 0 and a variance of 1, it is standard Gaussian and put putting a equal to 1 upon 2 D b equal to under root 2 upon under root t upon n del by del x.

As we did earlier we get y j is equal to z j under root D plus D under root t upon n dell by del x. This is simply we are making the substitutions a equal to one upon 2 D, b equal to under root t upon n del by del x in this expression for z j and then expressing y j in terms of z j.

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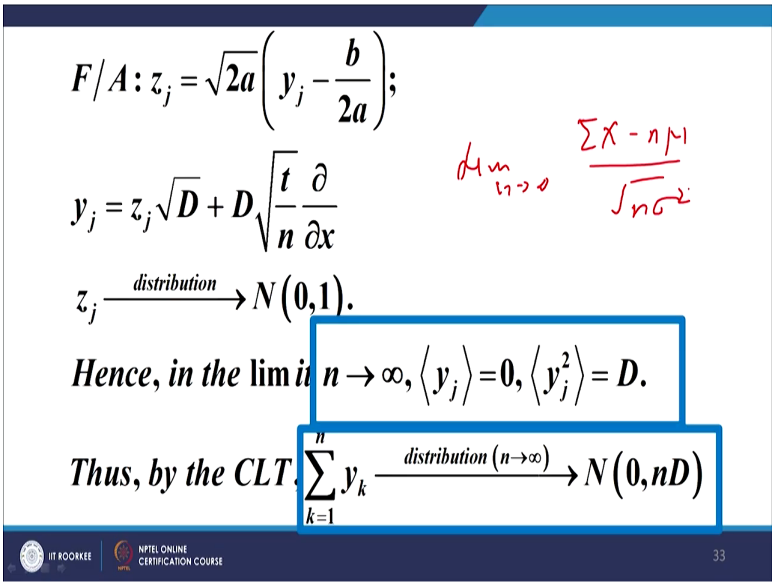

So, y j is equal to z j root D plus D root t upon n del by del x. Now, recall z j is standard Gaussian z j is standard Gaussian. So, let us look at the properties of y j, first as n teds to infinity what happens? Before I take the limits let us look at the properties of each y j. Each y j now the expectation of y j is clearly 0 when n tends to infinity. When n becomes very large the second term tends to 0, the mean of the first term the mean of z is equal to 0.

Therefore in the limit that n tends to infinity in the limit that n tends to infinity y j has the expectation value of 0, because the second term vanishes when because it has N in the denominator and as n tends to infinity, the second term vanishes and as far as the first term is concerned the mean of z j is 0 and therefore, the mean of y j is 0.

What about y j square? When you look at y j square there is one term z j square D, now the mean of z j square is equal to 1. And what about the other terms? The other terms if you look at it when you square this expression y j y j square, the other terms will contain a factor of n in the denominator either root n or n in the denominator. Therefore, all those terms will vanish in the limit n tending to infinity.

And hence when we have these two values the mean of y j is equal to 0, the mean of y j square is equal to D. So, this is important hence by the central limit theorem what do we get, recall central limit theorem says limit n tending to infinity limit n tending to infinity sigma X minus n mu upon under root n sigma square is standard Gaussian.

In our case what happens? mu is equal to 0; sigma square is equal to D. Therefore, our sigma y j; sigma y j or summation over the some of the sequence of ys. In the limit n tending to infinity some of the sequence of y is normally distributed with a mean of 0 and with a variance of n D, because sigma square in our case is equal to D. Each y has a mean of 0 has a variance of D; therefore, summation y j is normally distributed with a mean of 0 and a variance of n D. We will continue from here after the break.

Thank you.