Path Integral Methods in Physics & Finance
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Lecture - 10
Diffusion Equation

So, this is the defining property of Brownian motion; that the mean of the process is 0 and the variance of the process measured at any particular time is equal to the length of time elapsed since the origin of the process.

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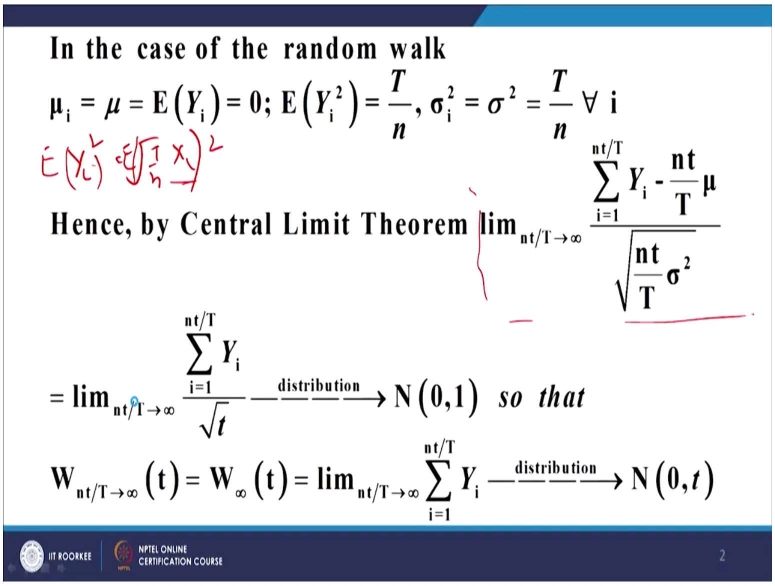

In other words, the spectrum of various possible values that the Brownian motion process can take is normally distributed with the mean of 0 and a variance equal to the length of time from the origin of the process to the point under review right.

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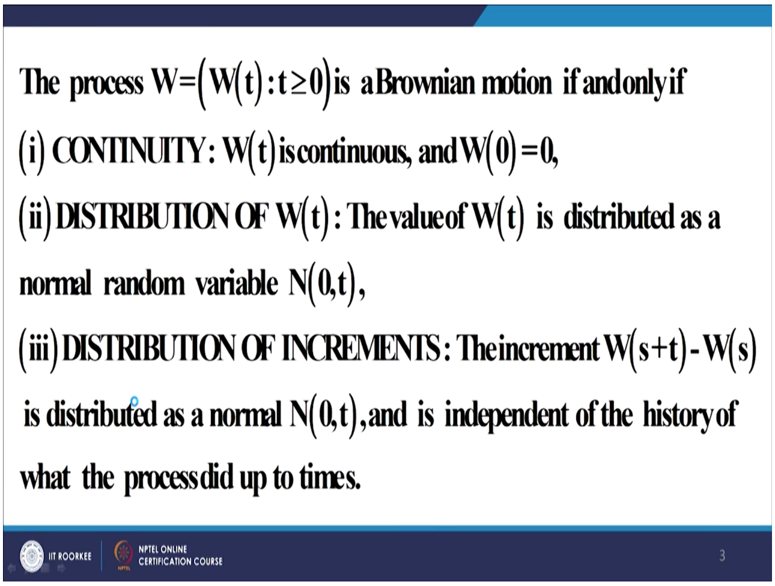

So, let me summarize now the various the process the properties of Brownian motion. The first is that the process starts at the origin. Brownian motion is always deemed to start at the origin W 0 is equal to 0. Continuity the Brownian paths as you will see from the diagram itself are continuous.

Although the process the paths are no were differentiable; because the zigzagging is so much, zigzagging of the paths is so large that if you take pick up any point in time and try to differentiate the process at that point and time that will not give a finite result and so differentiable differentiability is lost, but continuity is retained.

The distribution of W t I have already mentioned, I emphasize once again the value of W t is normally distributed with a mean of 0 and a variance of t. Another very important feature, very interesting feature about Brownian motion is even the increments you take two points: take a point t equal to s and take a point t equal to s plus t. Then the increment is also normally distributed with a mean of 0 and a variance equal to t.

In other words, if you have this increment W s plus t minus W s, this increment is normally distributed with a mean of 0 and a variance of t and above all, it is independent of the history of the process up to the point s; the history or the memory of the process up to the point s is irrelevant to the future evolution of the process. Therefore, Brownian motion is a Markov process.

Brownian motion is a Markov process, it is also a martingale which we shall talk about later, but basically what it means is that the expected value of the Brownian motion at any point and time is equal to its value at that point.

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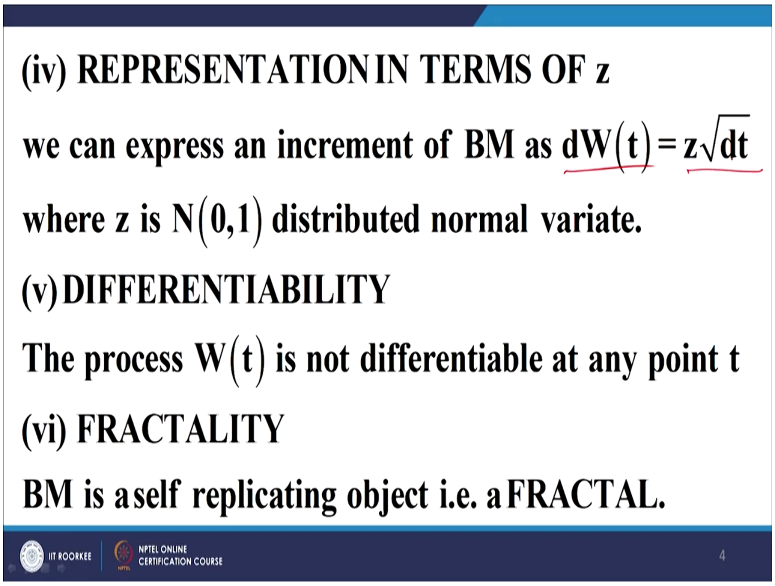

Now, because the increment d W t the which we discussed just now. The infinitesimal increment d W t is normally distributed with a mean of 0 and a variance of dt, we can write it as d W t is equal to z under root dt.

If you try to check up the mean and variance of both its both the sides is quite obvious the mean is 0 because the mean of z is 0, z is the standard normal variate remember. So, the mean of d W t is 0 because and on the right-hand side, the mean of z under root dt is also 0 because the mean of z is 0.

The variance of d W t according to our requirement should be dt and the variance of z root dt is also dt because variance of z square is the expected value of z square is equal to 1. And indeed, the variance of z square is equal to 1 and root dt or dt for that matter is not stochastic. So, multiplying z by root dt amounts to multiplying the variance by dt. So, dt into 1 gives you dt.

So, the mean and variance both coincide, both are normally distributed and we also know that in the case of a normal distribution. The distribution is completely specified by its mean and variance and putting all these pieces together, we find that this in infinitesimal increment of a Brownian motion can be represented in this form.

Differentiability: I have just mentioned it is not differentiable at any point and fractality just like a straight line, if you zoom in on a straight line as much as you like, you still get a straight line. Similarly, if you zoom in on a Brownian motion, you may use as much magnification as you like, but you still end up with the same zigzag structure that is what it is called in mathematics a fractal right.

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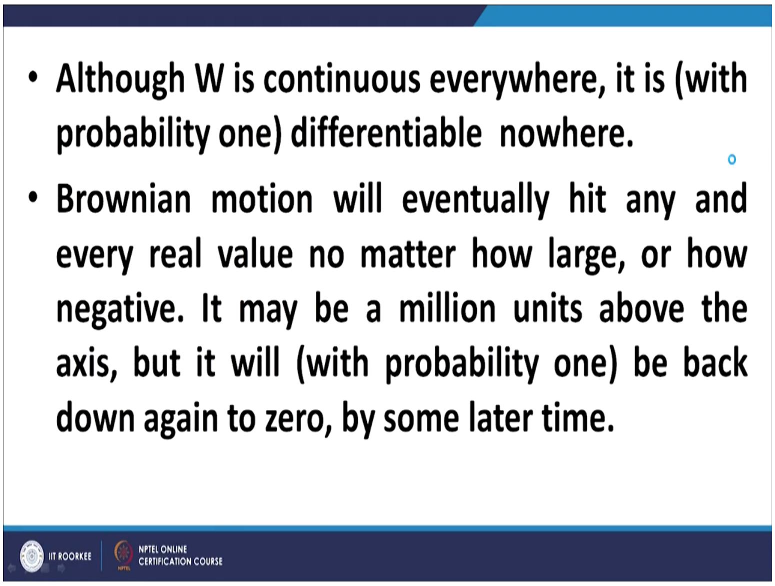

So, Brownian motion is continuous. It is differentiable nowhere. Brownian motion will eventually hit any and every real value no matter how large or how negative and why it will yeah hit every point infinitely often.

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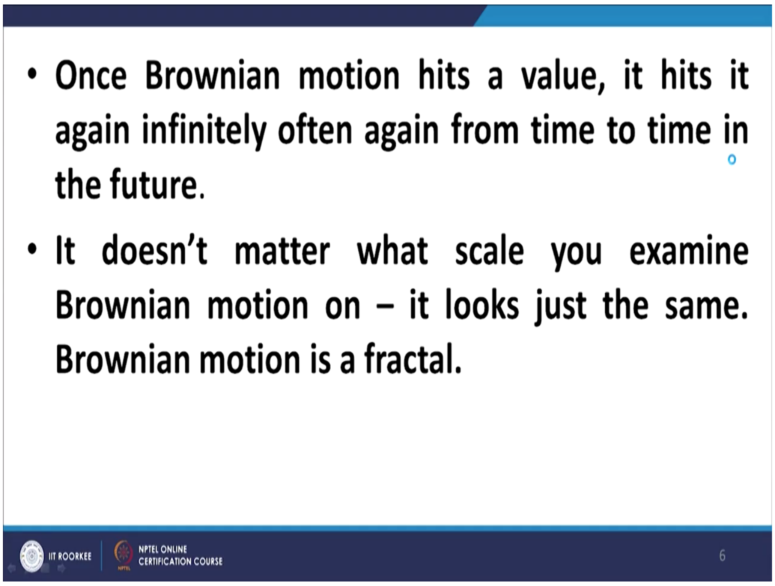

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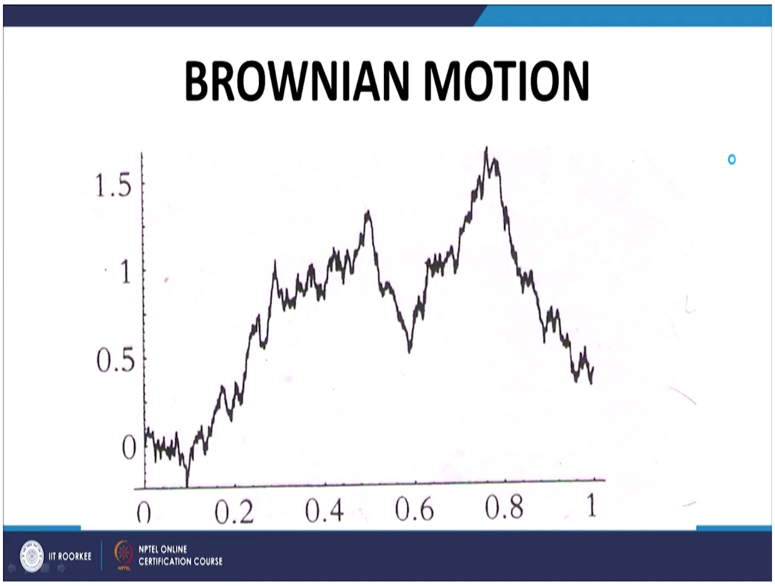

And then, as I told you it is a fractal also. This is the typical diagram of a Brownian motion. Just look at the amount of zigzagging that is there and because of the zigzagging, the motion the process or the curve loses its differentiability property and notwithstanding the fact, that it retains its continuity.

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RANDOM WALKS	TIME LENGTH /LAYER SPACING	JUMP SIZE	EXP VALUE	VARIANCE
SINGLE STEP RW	T	±1	0	1
TWO STEP RW	T/2	±1	0	2
n-STEP RW	T/n	±1	0	n
n-STEP SCALED RW	T/n	$\pm\sqrt{(T/n)}$	0	T
n-STEP SCALED RW	T/n	$\pm\sigma\sqrt{(T/n)}$	0	σ^2T
n-STEP SCALED RW WITH DRIFT	T/n	$(\mu T/n)\pm\sigma\sqrt{(T/n)}$	μT	σ^2T
Brownian Motion	->0	->0	0	T
Scaled BM	->0	->0	0	σ^2T

Now, this Brownian motion see why is it so important? Why is it so relevant? Let us devote a few minutes to this. Just as when we talk about deterministic curves, when we are talk about Newtonian differentiation, Newtonian calculus; we assume that when we differentiate a function, we assume that x and x plus delta x are very closely spaced points.

And then, we assume that the points are so closely spaced that they are and they the curve between them, the distance between them, the curvature between them can be approximated, can be ignored in fact. And they can the point x f x and f x plus delta x are so close that the curve joining them and can be approximated by a straight line and then, we do various calculus operations on them.

The fundamental assumption is that to point there are two points are so close, so very close that at the infinitesimal level, they can be approximated by a straight line. In other words, we can put it in another way that the entire curve can be considered as an assortment of infinitesimally small straight lines. Similarly, stochastic processes, random processes that evolve in time can be assumed to be functions of Brownian motion in some sense.

And this Brownian motion can be considered as building blocks of these stochastic or random processes that evolve in time in an unpredictable manner. So, that is why this Brownian motion is so important and the calculus relating to Brownian motion that we will discuss in a little bit of detail is becomes so very important right.

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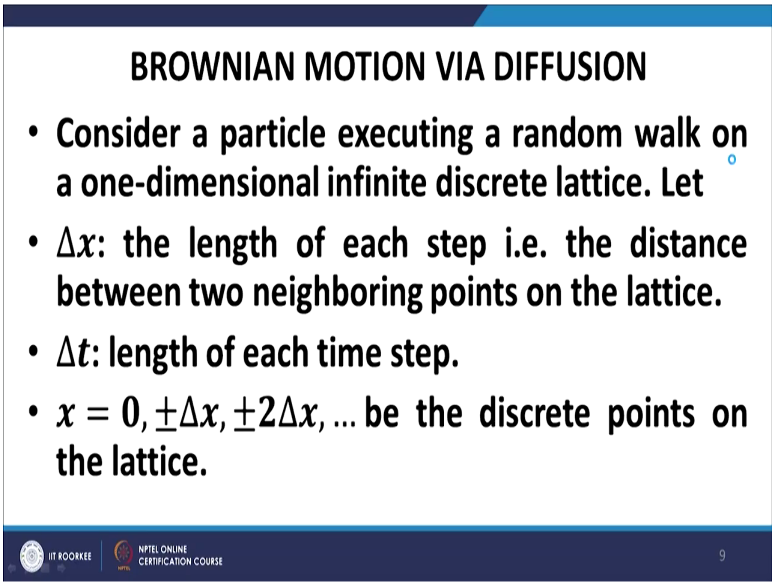

So, now, as I mentioned at the beginning of today's lecture, there are various approaches to reaching this concept of Brownian motion to arriving at Brownian motion from ab initio random walk or a discrete random walk. One of the approaches which uses the central limit theorem I have already elucidated. Now, I want to discuss the diffusion approach or the approach to Brownian motion via diffusion.

In fact, this approach is important historically as well because Robert Brown in in whose memory this Brownian motion term is coined has studied this structure of Brownian motion in context of the diffusion of pollen grains when immersed in fluid, in immersed in a liquid. The patterns that evolved as this pollen grains were hit on various randomly by the molecules of the fluid was what was later termed on to be Brownian motion.

So, we now take up this case of Brownian motion through diffusion. In one-dimension and we assume that a particle is executing random motion on that. We assume that delta x is the length of each step then that is the distance between two neighboring lattice sites which we assume uniform throughout and delta t is the length of each time step x is the time, x axis is the time step, y axis represents the delta and the motion of the particle.

And the coordinates are number and name numbered as 0, the origin plus minus delta x plus minus 2 delta x and so on. These are these discrete points on the lattice which come in which the particle is executing random motion.

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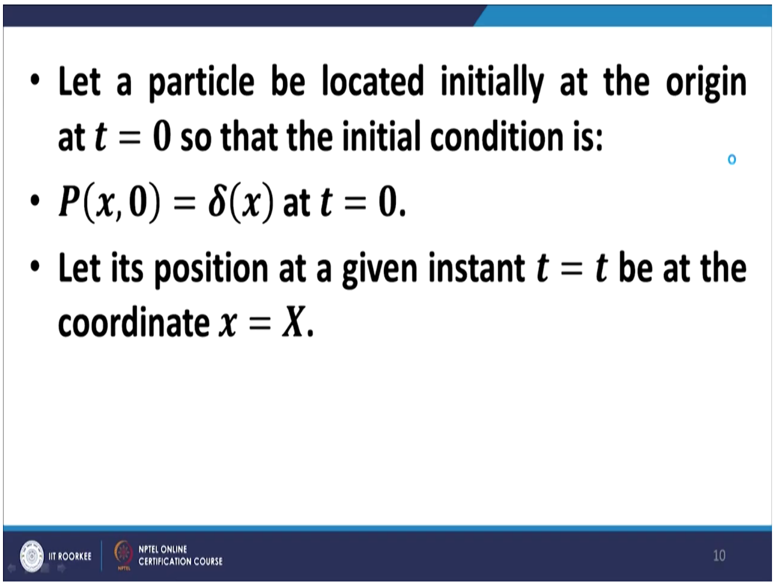

Initially, the we assume that the particle is located at the origin at t equal to 0. So that, the initial condition can be represented as the delta function P x, 0 is equal to delta x at t equal to 0 and let its position at any other part, in any arbitrary point in time t be given by capital X, x equal to capital X.

That means x equal to capital X is a arbitrary position at an arbitrary time at capital T equal to small t and the particle initially is assumed to lie at the origin.

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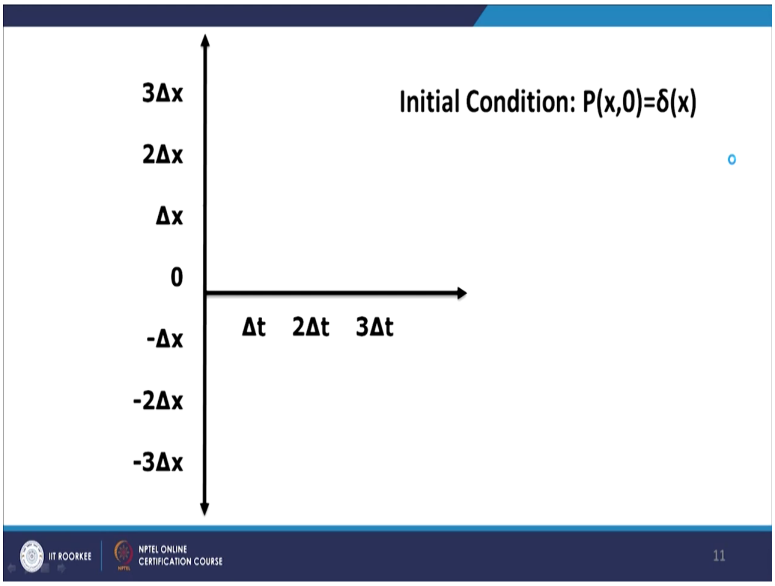

This is the sketch, we have the time along the horizontal axis and we have the displacements along the vertical axis and the particle initially is at the origin.

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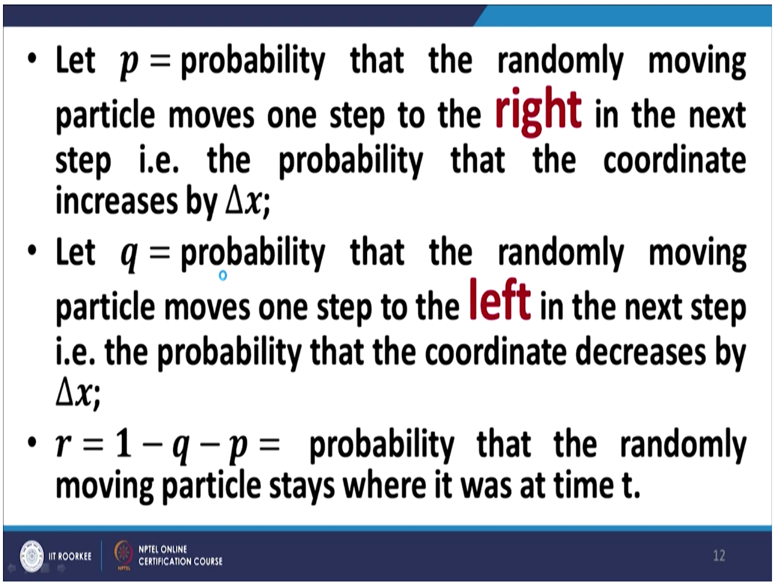

So, we assume that p is the probability; small p is the probability that the randomly moving particle moves one step to the right in the instant of time delta t. Let us say, it is at one particular point and then, it moves one step to the right at time t equal to delta t that is the coordinate increases by 1 unit this is probability. Probability of this is given by p.

q is the probability that the particle moves to the left, one step at time delta t, t equal to delta t and in other words, we can say that q is the probability that delta x decreases by 1 unit or the coordinate decreases by delta x in the time delta t; the coordinate decreases by delta x in the time delta t.

r equal to 1 minus q minus p the residual probability is that the randomly moving particle stays where it is at time t.

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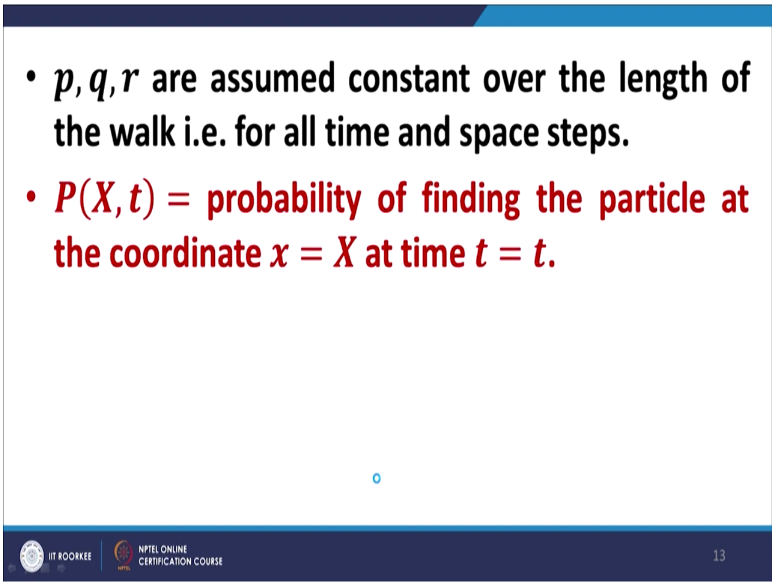

Now, p, q and r they are assume constant over the length of the walk for all time and space steps and P X, X is remember X is an arbitrary position of the particle at time t. And this is the probability of finding the particle at this point at point x equal to capital X at time t equal to small t.

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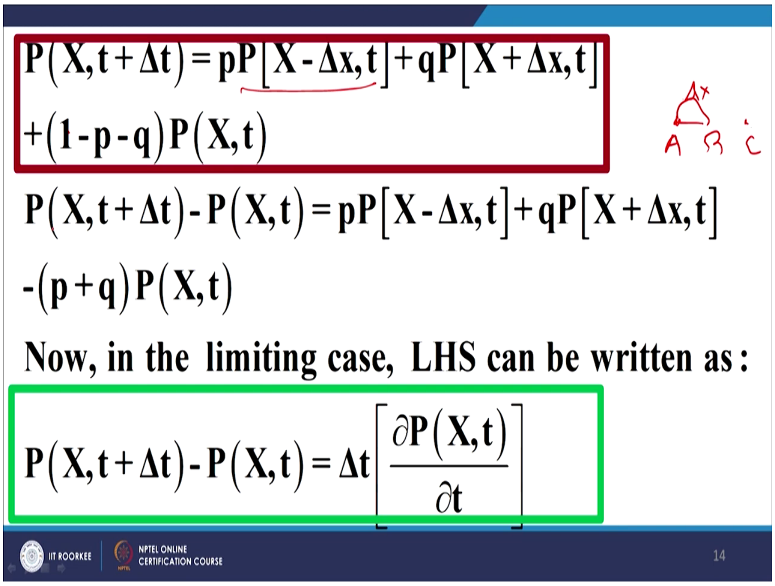

Now comes the very important stuff. The probability that the particle is at the point capital X at time t plus delta t is equal to what? Is equal to the probability that the particle was at X minus delta x that is one step to the left at time t and then, it makes a jump to the right which is of length in delta x in the time delta t. So, which is given by small p.

In other words, the particle is initially at X minus delta x, initially at X minus delta x and then it makes a jump from there to the right in the time delta t by one step which is of length delta x and the probability of this jump is equal to small p and the probability that the particle is initially at this point at the left point is P X minus delta x into t.

So, the probability that the particle is at this point A and then, it reaches the point B, at this point A at time t and then, it reaches the point B at time t plus delta t because that is what we want is given by probability that it is at the point A which is this this quantity and then, it makes a jump which is given by the small p factor.

Similarly, it could also happen that the particle was at the point C one step to the right; because it is executing random motion. So, it could have been one step to the right and the probability of which is P X plus delta x at time t and then, it makes a jump to the left in time delta t which is given by q. So, the probability of this event is given into q into P X plus delta x into t or it could so happen that the particle is already there at the point B and it remains there. The probability of which is 1 minus p minus q into P X, t this is the defining equation. This is the most important equation that the rest of it is more or less algebra. The this is the fundamental equation.

The probability I repeat, the probability that you find, the particle at the point X at time t plus delta t is equal to number 1; the probability that it is one step to the left and it makes a jump during delta t of delta x or it is one step to the right at time t and it makes a jump to the right to the left I am sorry to the left of delta x in time delta t or it was at point X and it remains at point X does not move during that time delta t. So, these three probabilities are added.

Now, we simplify this; we simplify this, we take this one factor with this P X, t to the left-hand side and we the rest of the things, we rearranged a little bit. Now in the limiting case, this left-hand side can be written in the differential form as d of P X, t upon dt into delta t this is the left hand side mind you.

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For the RHS, $pP[X-\Delta x,t]+qP[X+\Delta x,t]$

$-(p+q)P(X,t)$

$$=\frac{1}{2}(p+q)\{P[X-\Delta x,t]-2P(X,t)+P[X+\Delta x,t]\} \quad (B)$$

$$+\frac{1}{2}(p-q)\{P[X-\Delta x,t]-P[X+\Delta x,t]\} \quad (A)$$

Now, let us look at the right-hand side. The right-hand side is given by this expression. We write it in a slightly rearranged form which is given by this expression. If you look at we let us call this expression A and let us call this expression B.

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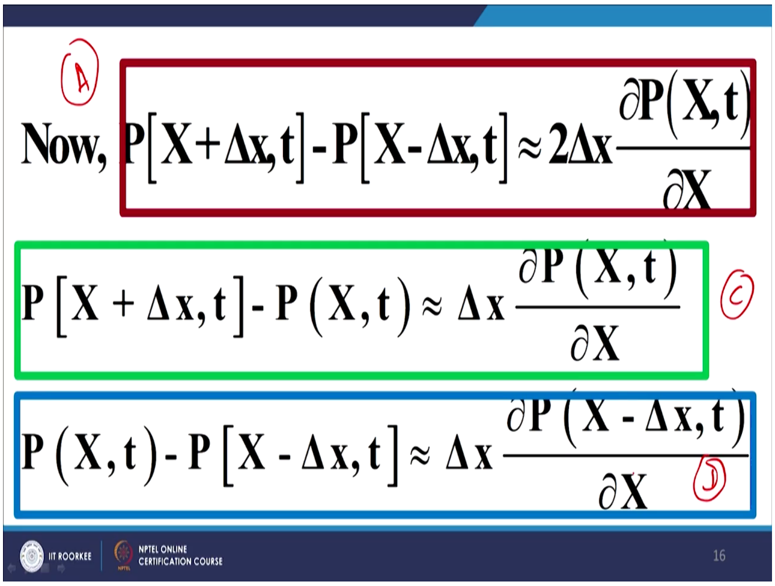

Let us look at expression A first. Expression A is quite straightforward P X plus delta x, t minus P X minus delta x, t this is nothing but d P by d X into 2 delta x. Because the difference in space coordinates is 2 delta x; so, we simply multiply the gradient by 2 delta x and we arrive at this difference. So, this is as far as equation A is concerned.

Equation B is slightly more involved. To work out equation B, we will need equations this equation and in this equation let us call the C and let us call this D. What we do is we subtract this C from D.

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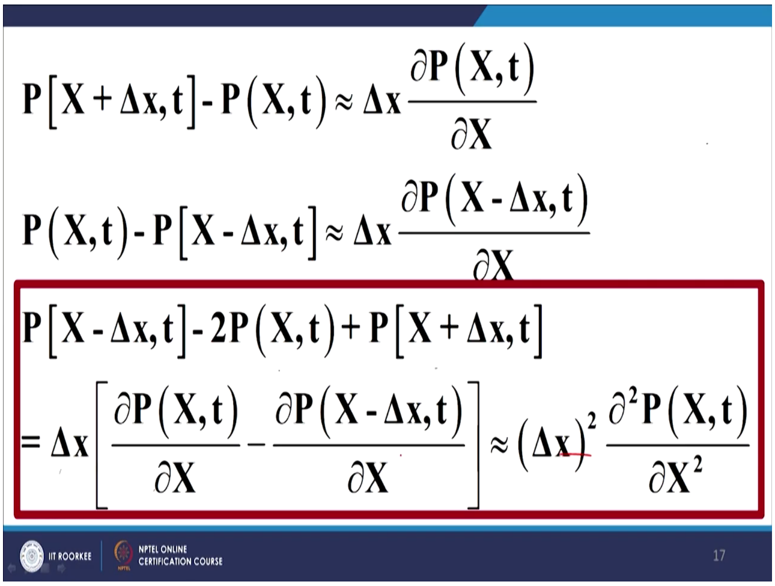

When we subtract C from D, what do we get? We get this expression this and this was my C, the first one was my C, the second one was my D. So, when I subtract C from D, I get this expression. Now, if you look at this; if you look at this carefully, look at these two, look at the right-hand sides, the left-hand side is what we wanted.

So, we have got left-hand side. The right-hand side is a different of two difference of two derivatives d P by d X at X and d P by d X at X minus delta x obviously, we can this difference of two derivatives gives the field that we can use the second derivative to represent that, to represent the difference of two derivative.

Just like we use this expression X plus delta x, P X plus delta x minus P X and we introduce the first derivative. Similarly, when we are having difference of two derivatives, we can use this second derivative. So, precisely that is what I have done and we have used the second derivative to represent in this difference of two derivatives.

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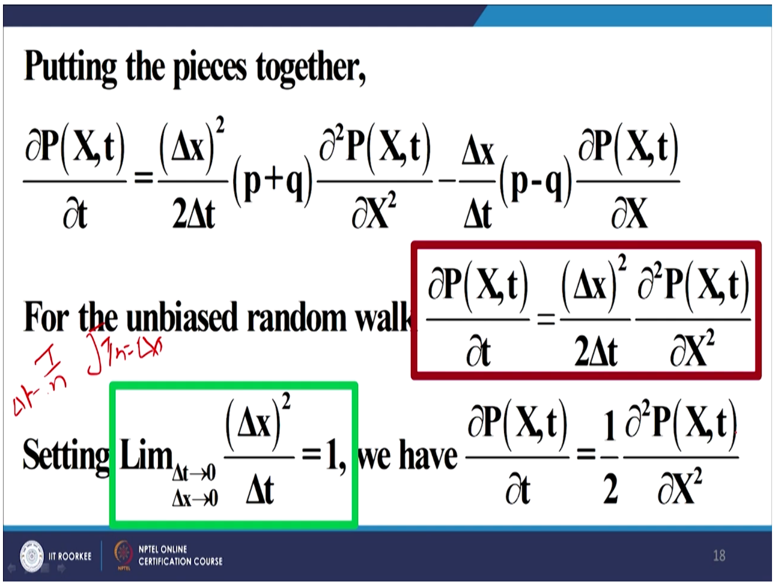

Now we put, rearranged put all these expressions back to where they were in terms of theory while in terms of the derivatives and the limiting quantities and we get this expression.

For an unbiased random walk what happens? p plus q become equal to 1 and p minus q become equal to 0. So, what are we left with? We are left with d P by dt is equal to delta x square upon 2 delta t d 2 P upon d x squared now comes a very interesting thing.

Now, we do this limiting limit delta t tending to 0, delta x tending to 0 delta x square upon delta t is equal to 1. Now this is very interesting. This if you look, if you recall this can be related to the limiting that we done, we had done when we talked about the random walk and so on.

If you recall our time length, our time step was T by n. When we talked about the discrete random walk, the time length of the time step length was T upon n and what was the jump size? The jump size was under root T upon n. So, if this is my delta x this is the jump size which is delta x and this is delta t, then clearly, we have delta x square upon delta t is equal to 1.

So, the rational for this could be could well be looked at or could well have been built into the random walk process that we analyzed earlier. So, that being the case, this expression gives you 1 and we end up with this equation which is a diffusion equation d P by dt is equal to 1 by 2 d 2 P upon dx square the second derivative P with respect to x squared.

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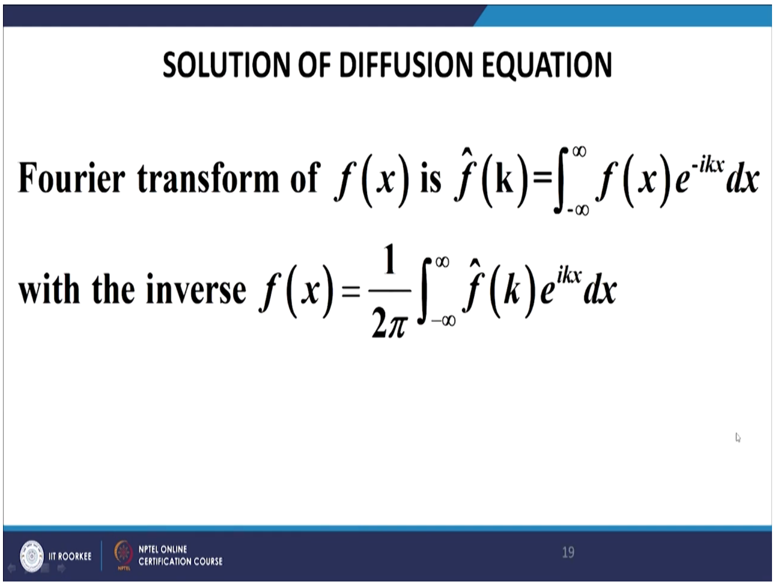

To arrive at the explicit result that Brownian motion is present in this particular case, let us solve the diffusion equation, it is quite simple, we simply use the Fourier transform method and there are new a number of methods of course. Fourier transform happens to be one of the simplest ones in this case and we use that.

So, we use f x the Fourier transform of f x is given by f hat k which is given by this and f x is recovered by taking the inverse Fourier transform where we have the factor of 1 upon 2 pi and we use this expression for the inverse Fourier transform.

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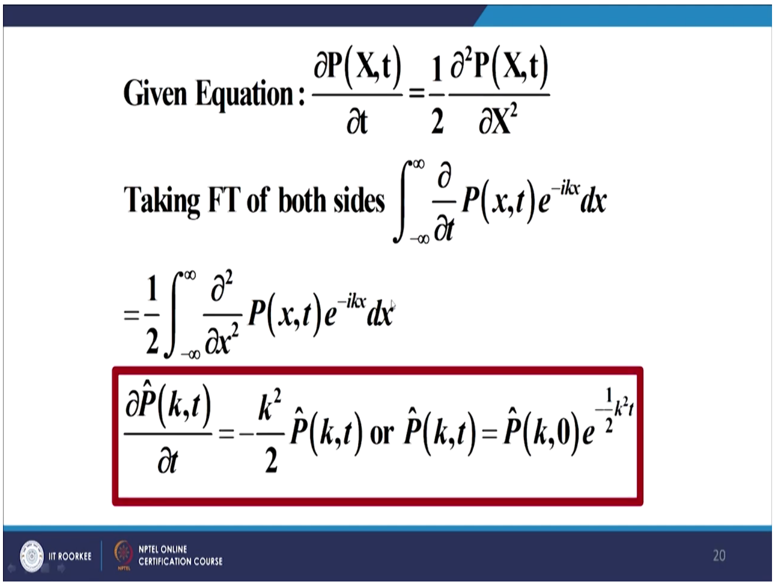

So, the given diffusion equation is this one. You take the Fourier transform of both the sides, integrate them by after multiplying by exponential minus ikx dx what do I get on the left-hand side? On the left-hand side, I get the Fourier d by dt I take, I can take outside the integral and the rest is nothing but the Fourier transform of P x, t so, I get d P by d P hat upon dt d P by d d by dt is taken outside the integral and P e minus ikx constitutes the Fourier transform of P x, t which is nothing but P hat k, t so, we have d P hat k, t upon dt.

Now, let us look at the right-hand side. when I take the Fourier transform of right-hand side, this d square upon dx square operates on e to the power minus ikx and I get this expression minus k square upon 2 and the rest what remains is simply the Fourier transform of P x, t.

I repeat, the this differentiation operator operates on e to the power minus as xk twice and we recover minus ik two times which gives us k square upon 2 in this 1 upon 2 is already there. And so, a minus k squared upon 2 which I can take outside the integration and inside the integration, I am left with P x, t e to the power minus ikx dx which is nothing but the Fourier transform of P x, t which is P hat k, t.

So, this is my right-hand side. Clearly the solution is very simple, straightforward solution it is P P hat k, t is equal to P hat k, 0 e to the power minus 1 by 2 k square t.

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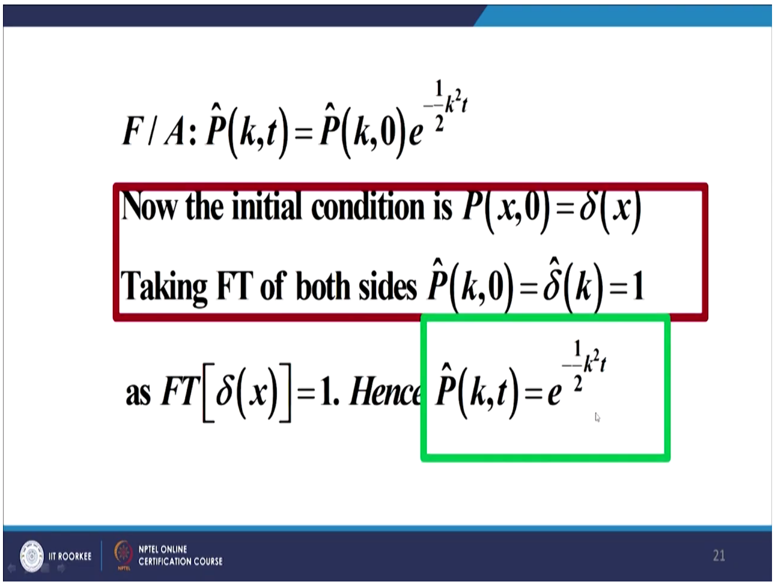

Let us look at the initial condition now. What was the initial condition? The initial condition was P x of 0 is equal to delta x. Let us take the Fourier transform of both sides because the Fourier transform of delta x is equal to 1, we get P hat of k, 0 is equal to 1 and this equation simplifies even more we get P hat of k, t is equal to exponential minus 1 by 2 k square t.

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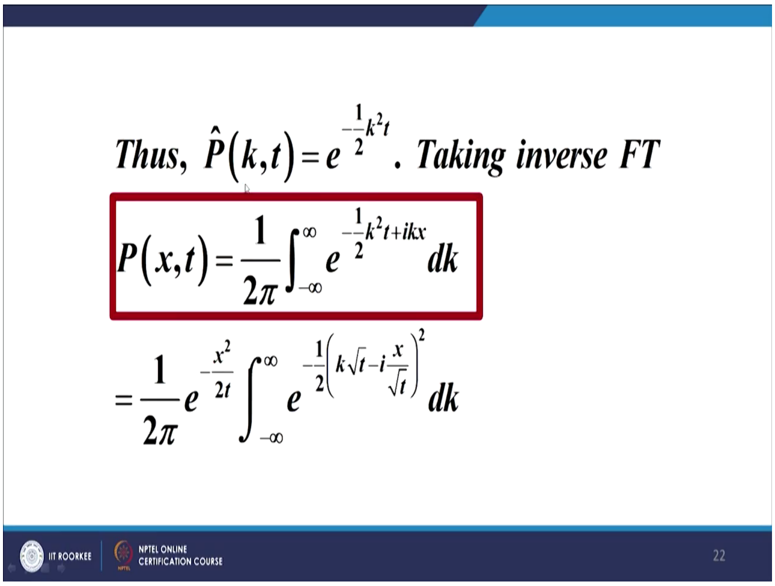

Now, we take the inverse Fourier transform to recover P x, t. When we take the inverse Fourier transform, we have a factor of 1 upon 2 pi here and also have a factor of exponential ikx dk and in the integration obviously, with respect to d with respect to k because we are having the inverse Fourier transform and this is clearly a Gaussian integral, simple Gaussian integral. We do the Gaussian integration.

And what do we end up with? We end up with in this expression where we make a simple substitution y is equal to k root t and once you substitute y equal to k root t, you get a factor of root t here in the denominator and this is a whole integral gives you root 2 y root 2 pi I am sorry and in the net result is in this expression, 1 upon under root 2 pi t e to the power minus x square upon 2 t.

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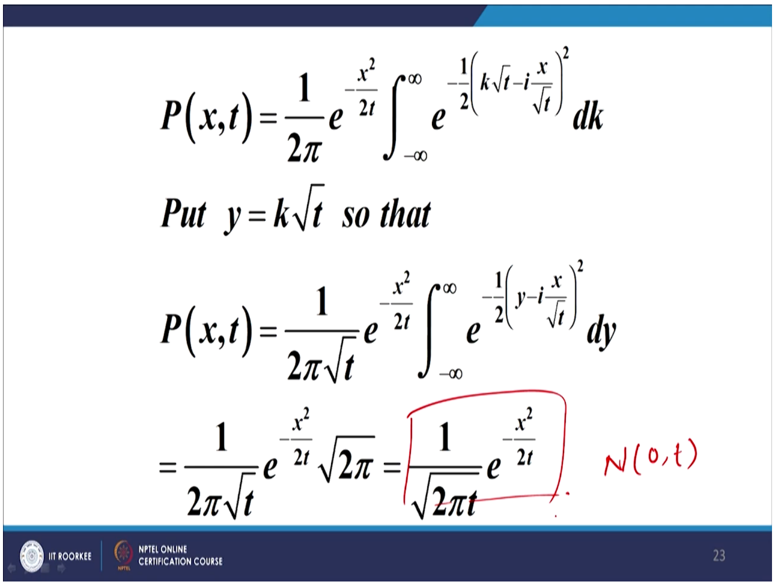

Now, clearly this is the probability density function; this is the probability density function of a normal distribution with a mean of 0 and a variance of t. You can compare it with the standard form of the probability density function of a normal distribution; you find that this is the probability density function of a normal distribution with a mean of 0 n, mean of 0 variance of t.

So, that establishes the connection between the standard normal distribution or the normal distribution and Brownian motion and diffusion. Diffusion, Brownian motion and normal distribution, there is a the clear linkage between them. The solution of the diffusion equation gives us the Brownian motion which obviously, Brownian motion is defined in terms of the Gaussian or the normal distribution.

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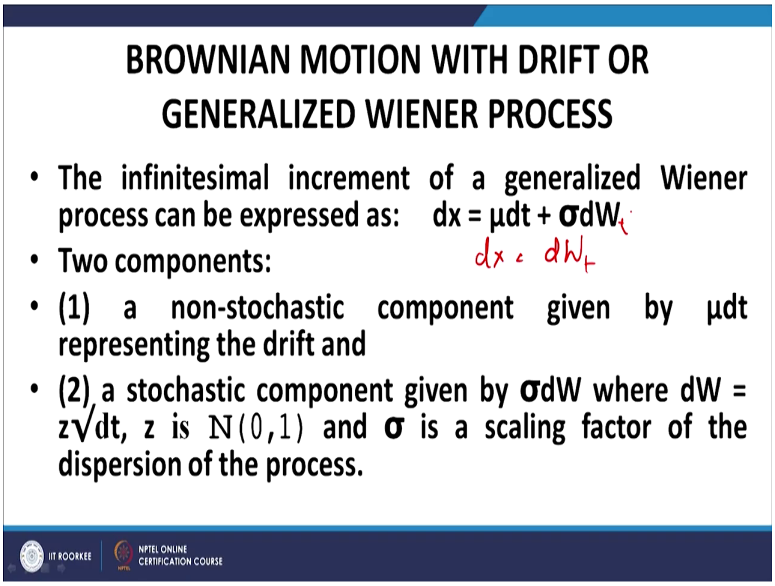

Now, we have talked about Brownian motion or the standard Brownian motion as being a 0 mean process, a Brownian motion that has a mean of 0 and a variance of dt and that is usually expressed in the form dx is equal to d W or you can have a t here if you want to explicitly show that it is a function of time that. There is an extension of this or there is a generalization of this of the form dx is equal to mu dt plus sigma d W t.

Now, in this case, what is happening is an underlying drift is being is superposed on the standard Brownian motion. In other words, we are having an underlying drift or underlying incline either upwards or downwards in a straight line and over that, we are super opposing the zigzag of the Brownian motion.

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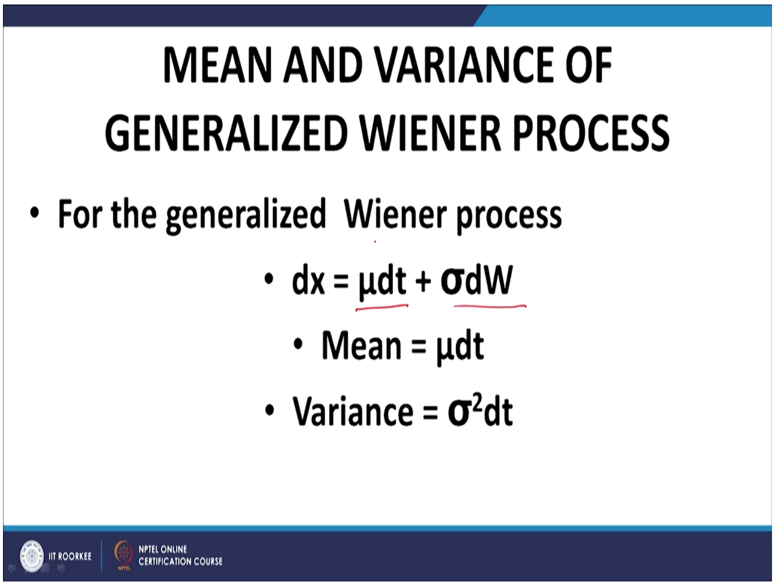

And it is quite easy to see here that the mean of this to this process is mu dt it is not 0 because the both these terms are not stochastic. So, the mean of this expression is mu dt and the variance is sigma square dt because of this term. The first term contributes to the mean does not contribute to the variance and the second term, contributes to the variance does not contribute to the mean because this is d W is a 0 mean process and mu dt is a 0 variance process.

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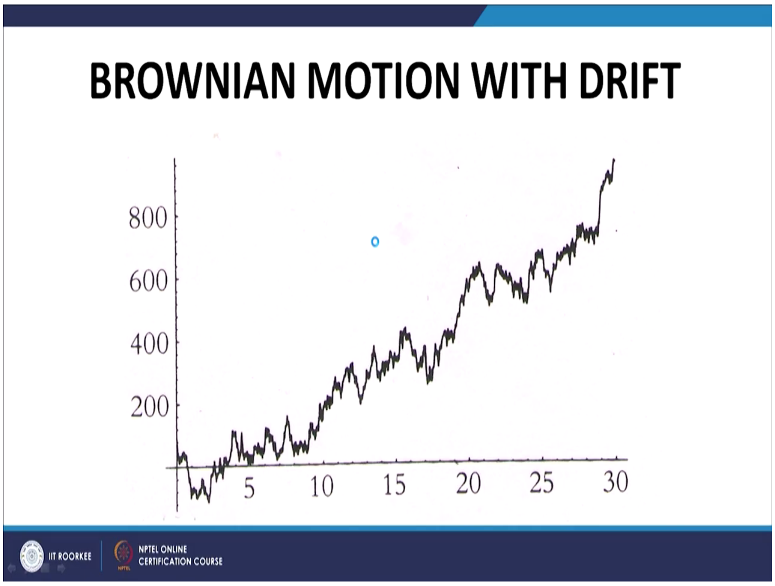

So, the generalized wiener process let us have a look at this. This is what is a Brownian motion with drift. There is an underlying pattern, on that pattern; a superposition of a Brownian motion is taking place.

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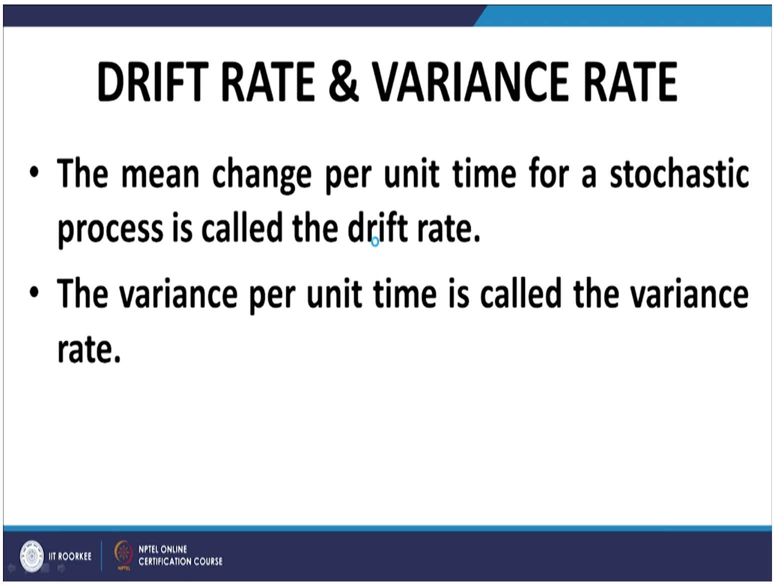

The mean change per unit time of a stochastic process is the drift rate, the drift rate and the variance rate. The variance per unit time is the variance rate of the process. The average change per unit time is the drift rate of the process. This is an example of exponential Brownian motion which you we shall encounter when we do modeling of stock prices and we talk about derivatives of stock prices like call options and futures and forwards.

But, this is the what is exponential Brownian motion which is generally used for the modeling of stock prices and which forms the premise of the Black-Scholes model that that is used in stock prices for modeling of options.

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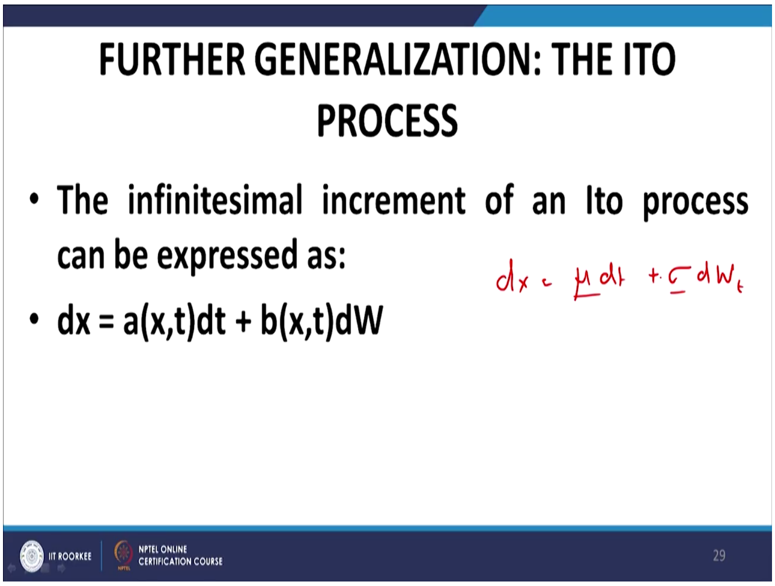

Then, we have further generalizations of this Brownian motion concept. We have the concept where the coefficients so far, in the generalized wiener process, mu and sigma were treated at constants.

We talked about the process dx is equal to mu dt plus sigma d W t and mu and sigma were treated as constants. But, there can be processes, there are processes where this mu and sigma both can be functions of x and time and they can be explicit functions of time and these processes are known as Itô processes. They are extensions; they are further generalizations of the concept of Brownian motion right.

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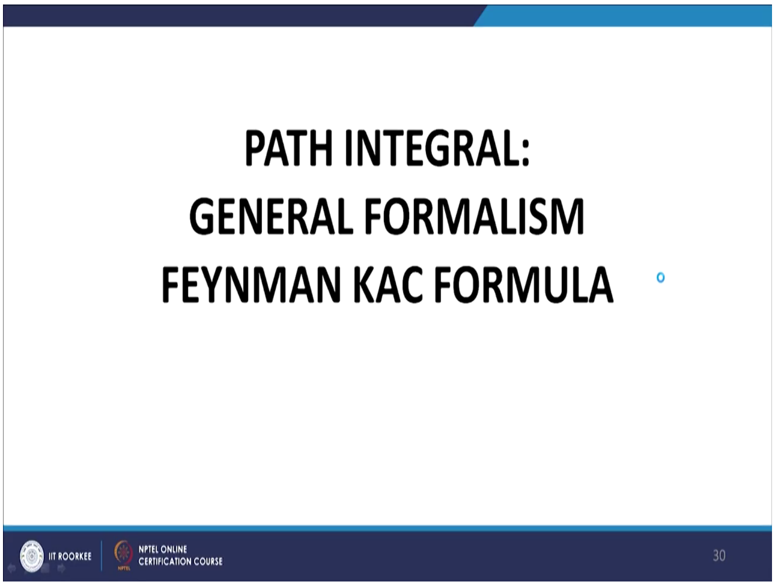

Now, we come to the concept of path integral. This is the first real encounter with the path integral formalism. In this, what we will do is we start with the diffusion equation and we show that the path integral forms the complete solution of the diffusion equation. I will take it up in the next lecture.

Thank you.