Financial Derivatives and Risk Management
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Forward and Future Prices, Exposure
Lecture 7

Forward prices vs futures prices

Consider a futures contract which lasts for N days and the price at the end of day n is F_n ($0 \leq n \leq N$) per unit of underlying. Let the lot size of the futures be z i.e. one contract entails delivery of z units of the underlying. Define d as the risk-free rate of interest per day, compounded daily. We assume that d is constant. Consider the following strategy:

1 At the end of day 0 (start of contract), go long $(1+d)$ contracts.
2 At the end of day 1, increase long position to $(1+d)^2$ contracts.
3 At the end of day 2, increase the long position to $(1+d)^3$ contracts …. and so on.

At the end of day *n-1*, the long position will be $(1+d)^n$ contracts. This will also be the position beginning of day n i.e. the long position at the beginning of day n will be $(1+d)^n$ contracts. At the end of the day this position will lead to a profit (possibly negative) given by: $z(F_n - F_{n-1})(1+d)^n$

Suppose that this profit is compounded daily at the rate d per day till the day N. Its value at the end of day N is: $z(F_n - F_{n-1})(1+d)^n (1+d)^{N-n} = z(F_n - F_{n-1})(1+d)^N$.

Thus at the end of day N, the entire investment will have accumulated to: $\sum_{n=1}^{N} z\left(F_n - F_{n-1}\right)\left(1+d\right)^N = z\left(F_N - F_0\right)\left(1+d\right)^N$.

However, because of convergence between spot and futures prices as on the date of maturity of the futures, the final settlement price of the futures is marked to the spot price of the underlying on maturity so that $F_N=S_N$. Thus, the above strategy yields a payoff of $z(S_N-F_0)(1+d)^N$.

Therefore, the above strategy plus an investment of zF_0 in risk-free bond at the end of day 0, will give a terminal value of: $zF_0(1+d)^N + z(S_N - F_0)(1+d)^N = zS_N(1+d)^N$

This combined strategy requires an investment of zF_0 at the end of day 0 and no further investment. It pays off $zS_N(1+d)^N$ on maturity at time N.

Now suppose a forward contracts are available on the same asset. Let the forward price per unit of underlying at the end of day 0 be G_0. Consider the following strategy:

Invest zG_0 in a risk-free bond and buy $z(1+d)^N$ units of underlying forward with the same maturity as the futures contract i.e. N. .
At the end of day N, the investment will have matured to $zG_0(1+d)^N$. This is used to pay for the delivery on the forward purchases of $z(1+d)^N$ units @ G_0 per unit.

The investor will receive $z(1+d)^N$ units of the underlying under the forward contract. These will be worth $zS_N(1+d)^N$ at t=N.

Thus the combined risk-free bond of value zF_0 plus futures strategy and the risk-free bond of value zG_0 plus forward strategy have identical terminal payoffs. Further, neither the futures strategy nor the forward strategy entails any initial investment. All the intermediate cash flows arising from MTM in futures is reinvested to yield the terminal payoff and there is no other intermediate cash flow. To prevent arbitrage, the initial investment requirements must also be identical. Hence $F_0 = G_0$ i.e. futures and forward prices are identical.

Thus, the price of a particular underlying with a particular maturity in the futures market and the forwards market should be very close to each other. In the above arbitraging, we have made a cardinal assumption that the rate of interest remains constant over the life of the futures contract.

Important Observations

(i) No arbitrage considerations create a direct relationship between forward and spot prices e.g. $F(0,T)=S(0)*e^{rT}$.

(ii) No arbitrage considerations also mandate convergence between forward and futures prices.

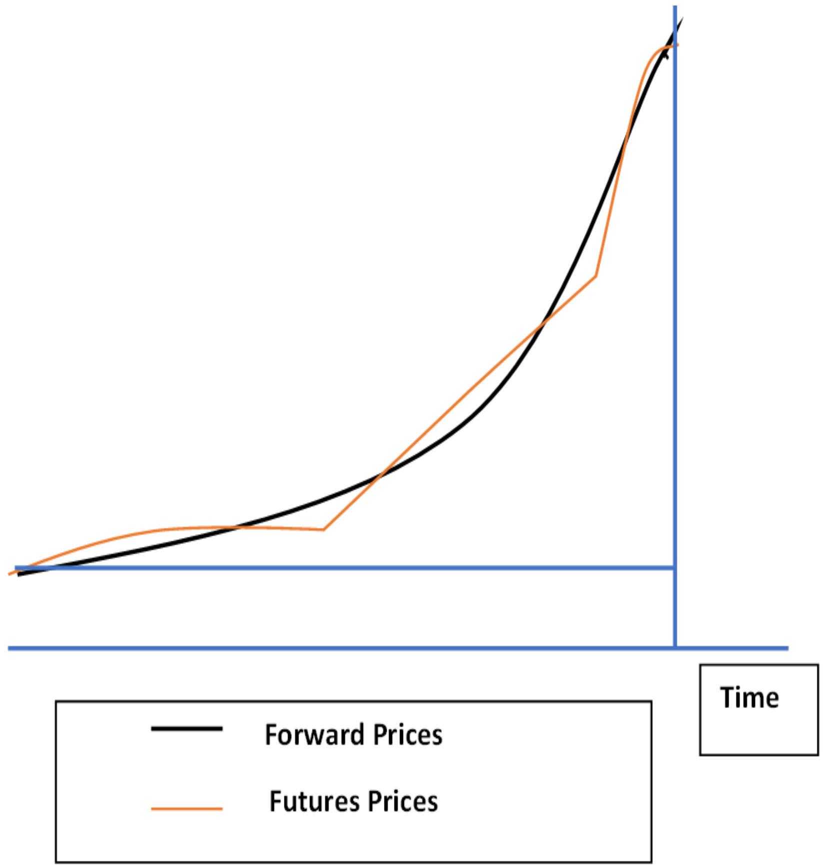
(iii) Thus, futures prices should also move in tandem with the spot prices.

(iv) Nevertheless, since futures are freely tradeable, their prices, ultimately (at the macro level), are determined by the interaction of demand and supply at the marketplace.

(iv) The net result is that futures prices hover in close vicinity of forward prices although occasional divergence would subsist due to heterogeneous risk-return preferences, market asymmetries and anomalies.

(v) However, the demand and supply, themselves, are functions of fundamental variables. These fundamental variables also manifest themselves as determinants of the inputs that go into the intrinsic value e.g. the cash flow projections, its risk profile etc.

We have seen that the no-arbitrage forward prices for a given maturity vary in direct proportion to the spot prices. No-arbitrage also requires that futures prices and forwards prices should align with each other. It follows that futures price should also move in proportion to the spot prices. This is what the no-arbitrage mandate seems to imply.



This diagram depicts how the forward price would move, given a particular spot price, as the maturity of the contract increases. The forward price will also increase exponentially as it is given by $F_0=S_0e^{rT}$. As time increases the forward price would also increase. Futures price should also increase in a similar pattern because futures contracts are akin to forward contracts in terms of the net price that is paid by the long party for acquiring the asset.

However, there are caveats. Futures prices are determined by demand and supply of the underlying in the futures markets. Thus, these prices also embed the impact of many other factors like the heterogeneity of the market participants in terms of the risk return trade-offs, asymmetry of information flow in the market, finite time of information dissemination etc. All these factors contribute to the lack of perfect market efficiency. This market imperfection manifests itself in some divergences between spot and future prices.

Futures contract being traded on the exchange is subject to forces of demand and supply. Demand and supply do not necessarily move in tandem with the forward price curve. The interaction of demand and supply which would determine the equilibrium market price of the futures contract does not necessarily move absolutely in line with the forward price curve. Lack of absolute perfection in market efficiency contributes to futures price being a slight variance with the forward price.

The futures prices would hover around the forward prices but they would not necessarily coincide with the forward prices at all points. However, at the same time, massive divergence of the form which is shown in the figure is also extremely unlikely. Although such divergences cannot entirely be ruled out because future prices are random variables, the probability of such occurrences would be extremely small. They would normally move along with this curve but with slight variations up and down, around this particular curve.

The demand and supply of any asset (including derivatives) is determined by the fundamental attributes of the asset i.e. its future earning capacity which is measured by its intrinsic value by various market players.

A comparison of this intrinsic value with the prevailing market prices creates the perception of assets being overpriced or under-priced, which result in market trades.

Exposure, risk & hedging

Before we take up hedging, it is necessary to understand the concept of risk. But risk is only relevant if there is an exposure, if there is no exposure there is no risk. So we explain the concept of exposure. But exposure is usually measured in terms of regression coefficients. A primer on regression is thus in order.

Appendix

A primer on regression

Suppose a marketing analyst is trying to predict the relationship between the sales of umbrellas (Y) and the average daily level of rainfall (X) over the next rainy season. He identifies several factors that influence Y e.g. average daily rainfall, income levels, outdoor employment, sales of substitutes like raincoats etc. Regression analysis enables the analyst to mathematically study the impact of the factors that he chooses to study on the dependent variables.

In regression analysis, these independent and dependent factors are called variables. The dependent variable is the factor on whom the impact of various independent variables is attempted to be studied. It is important to emphasize here that the impact of not all factors may be studied simultaneously. We may study:

(i) the impact of one factor at a time holding the others constant; or
(ii) some factors at a time while holding some others constant; or
(iii) we may assume that of the many factors identified, there exists one single factor that explains a significant proportion of the relationship so that the rest may be modelled collectively as a random term.

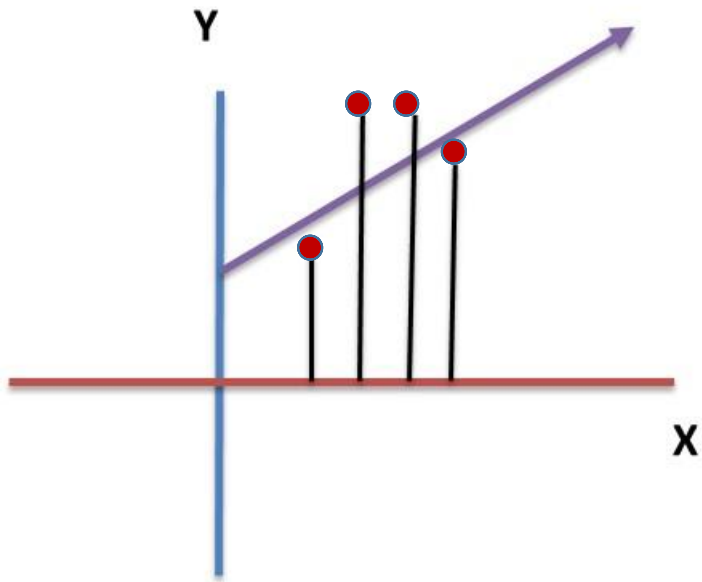
In order to conduct a regression analysis, the first step is, obviously to identify the dependent variable (Y) and the independent variable (X) whose impact on Y one has decided to investigate. The next step is to gather data, usually historical data, on the variables X,Y. For example, while studying the impact on the sale of umbrellas of the average daily rainfall in the 3-month rainy season, one can select several historical rainy seasons of each of 3 months and collect data on the sale of umbrellas over these intervals.

The Y-axis is the sale of umbrellas (the dependent variable, the thing you're interested in, is always on the Y-axis) and the X-axis is the average daily rainfall. The collected data is then plotted on the two-dimensional XY plane to yield the various data points (x_i,y_i), $i=1,2,\ldots,n$. This plot is called the scatter diagram. Each point (x_i,y_i) on this two-dimensional plot represents a data=point, an observation.

Now imagine drawing a line through the chart above, one that runs roughly through the middle of all the data points. This line will help the analyst to answer, with some degree of certainty, how many umbrellas can be sold in the next rainy season corresponding to a given average daily rainfall level. Regression aims to identify the parameters of the straight line that best fits the given data points based on a certain criterion.

Let us assume that the equation of this "best fit" line is $Y=mx+c$. (1)

Consider a data-point on the given XY plane, say (x_i,y_i). The ordinate of the point on the line (1) corresponding to $X=x_i$ is $Y_i=mx_i+c$.



Therefore, the error that would have been made if we had used the line (1) for representing the given data instead of the actual observed data is: $e_i=y_i-Y_i = y_i-mx_i-c$ or $y_i=mx_i+c+e_i$ (2)

Both x and e are random variables, each has a probability distribution, mean and variance.

Now, the criterion that we adopt for identifying the "best fit" straight line (1) is that $Z=\sum_{i=1}^{n} e_i^2$ is a minimum, the conditions for which are:

$$0=\frac{\partial Z}{\partial m}=2\sum_{i=1}^{n}\left(y_i-mx_i-c\right)\left(-x_i\right)=\sum_{i=1}^{n}x_iy_i-m\sum_{i=1}^{n}x_i^2-c\sum_{i=1}^{n}x_i \qquad (3)$$

$$0=\frac{\partial Z}{\partial c}=2\sum_{i=1}^{n}\left(y_i-mx_i-c\right)\left(-1\right)=\sum_{i=1}^{n}y_i-m\sum_{i=1}^{n}x_i-nc \qquad (4)$$

Eqs (3) & (4) yield:

$$m=\frac{\sum_{i=1}^{n}x_iy_i-\sum_{i=1}^{n}x_i\sum_{i=1}^{n}y_i}{\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}x_i\sum_{i=1}^{n}x_i} \qquad (5)$$

and

$$c=\bar{y}-m\bar{x} \qquad (6)$$

Also from eq (3), we have

$$\sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i \left(y_i - Y_i \right) = \sum_{i=1}^{n} x_i \left(y_i - m x_i - c \right) = 0 \qquad \text{whence Cov(x,e)=0} \tag{7}$$

and from eq (4) $\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} \left(y_i - Y_i \right) = \sum_{i=1}^{n} \left(y_i - m x_i - c \right) = 0$ whence E(e)=0 (8)

An important outcome of eq (7) is that regression analysis guarantees the that x and e are uncorrelated. This property ensures that how well the regression equation describes is independent of the values that x can take. Cov(*e*,x)=E[(e-0),(x-E(x))]=0 implies that the partition of y into Y and e, as above, is orthogonal.

The mathematics of regression can be put in an elegant form with matrix notation:

$$\textit{We have} \begin{pmatrix} y_1 \\ . \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ . & . \\ 1 & x_n \end{pmatrix} \begin{pmatrix} \beta_1 = \alpha \\ \beta_2 = \beta \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ . \\ \varepsilon_n \end{pmatrix} \quad \textit{or} \ \mathbf{Y = X\beta + \varepsilon}$$

$$S(\boldsymbol{\beta}) = \boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})^T (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})$$

$$= \mathbf{Y}^T\mathbf{Y} - \mathbf{Y}^T\mathbf{X}\boldsymbol{\beta} - \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{Y} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}$$

$$= \mathbf{Y}^T\mathbf{Y} - 2\mathbf{Y}^T\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} \quad \begin{pmatrix} \sin ce \ \ \boldsymbol{\beta}^T{}_{1\times 2}\mathbf{X}^T{}_{2\times n}\mathbf{Y}_{n\times 1} \ \ \textit{is} \ 1\times 1 \ \textit{matrix} \\ \textit{so that} \ \left(\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{Y}\right)^T = \mathbf{Y}^T\mathbf{X}\boldsymbol{\beta} = \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{Y} \end{pmatrix}$$

$$\frac{\partial S}{\partial \beta} = -2\mathbf{Y}^T\mathbf{X} + 2\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X} = 0 \Rightarrow \mathbf{Y}^T\mathbf{X} = \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X} \Rightarrow \mathbf{X}^T\mathbf{Y} = \mathbf{X}^T\mathbf{X}\boldsymbol{\beta}$$

$$\Rightarrow 0 = \mathbf{X}^T (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) = \mathbf{X}^T\boldsymbol{\varepsilon} \ (\textit{Orthogonality of } \mathbf{X} \ \textit{and} \ \boldsymbol{\varepsilon}.$$

$$\textit{Also} \ 0 = \mathbf{X}^T\boldsymbol{\varepsilon} = \begin{pmatrix} 1 & x_1 \\ . & . \\ 1 & x_n \end{pmatrix}^T \begin{pmatrix} \varepsilon_1 \\ . \\ \varepsilon_n \end{pmatrix} = \begin{pmatrix} 1 & . & 1 \\ x_1 & . & x_1 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ . \\ \varepsilon_n \end{pmatrix} = \begin{pmatrix} \sum \varepsilon_i \\ \sum x_i \varepsilon_i \end{pmatrix}$$

$$\textit{whence} \sum \varepsilon_i = \sum x_i \varepsilon_i = 0$$

$$\textit{EVALUATION OF } \frac{\partial \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}}{\partial \boldsymbol{\beta}}$$

$$\textit{Let } \mathbf{A}_{2\times 2} = \mathbf{X}^T{}_{2\times n}\mathbf{X}_{n\times 2} \quad \textit{then } \mathbf{A}^T = \mathbf{A} \ (\mathbf{A} \ \textit{is symmetric})$$

$$\textit{Let } \boldsymbol{\Sigma} = \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} = \boldsymbol{\beta}^T\mathbf{A}\boldsymbol{\beta} = \sum_i \sum_j a_{ij}\beta_i\beta_j$$

$$\frac{\partial \boldsymbol{\Sigma}}{\partial \beta_k} = \frac{\partial}{\partial \beta_k}\left(\sum_i \sum_j a_{ij}\beta_i\beta_j \right) = \sum_j a_{kj}\beta_j + \sum_i a_{ik}\beta_i$$

$$\frac{\partial \boldsymbol{\Sigma}}{\partial \boldsymbol{\beta}} = \boldsymbol{\beta}^T\mathbf{A}^T + \boldsymbol{\beta}^T\mathbf{A} = \boldsymbol{\beta}^T\left(\mathbf{A}^T + \mathbf{A}\right) = 2\boldsymbol{\beta}^T\mathbf{A} = 2\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}$$

For example, we have:

$$\boldsymbol{\beta}^{\mathbf{T}}\mathbf{A}\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

$$\frac{\partial \boldsymbol{\beta}^{\mathbf{T}}\mathbf{A}\boldsymbol{\beta}}{\partial \beta_1} = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} +$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

$$= \left(a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 + a_{41}\beta_4 \right) + \left(a_{11}\beta_1 + a_{12}\beta_2 + a_{13}\beta_3 + a_{14}\beta_4 \right)$$

$$\frac{\partial \boldsymbol{\beta}^{T}\mathbf{A}\boldsymbol{\beta}}{\partial \boldsymbol{\beta}} = \boldsymbol{\beta}^{T}\mathbf{A} + \boldsymbol{\beta}^{T}\mathbf{A}^{T}$$

In the above, we assume that the dependent variable Y is largely explained or influenced by the independent variable X. There may, of course, be “other” factors also affecting Y, but their effect is either not significant or is not the ubject matter of the study. Since we have no information about the identity of these “other” factors and/or the nature of their impact on Y, we assume that these “other” factors collectively make a random contribution to Y represented by the term (c+e).

Hence, we can write: $y_i=mx_i+c+e_i$ where index i indicates observation sequence. mx_i is the value of dependent variable Y due to the variable X while c+e is the random cumulative contribution to Y due to "other" factors.

We can write $a_i=c+e_i$ with $E(a)=c$ and $E(e)=0$ whence $y_i=mx_i+a_i$

Systematic relationship

A systematic relationship exists when two variables move in a predicted manner "on the average".

Given the relationship equation, $y_i=mx_i+(c+e_i)$, the first term captures the systematic relationship between the independent variable x and the dependent variable Y.

Since, x and *e* are random variables, we shall not always get a straight line fit with all observations. It is common practice to run a least squares regression that gives us best fit values for m and c.