Financial Derivatives and Risk Management
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Lecture 60
Value at Risk: Computation for Bond & Derivative Portfolios

VaR of portfolio of many assets

Consider a portfolio Π comprising of investments of amounts S_i, i=1,2,…,n in i=1,2,…,n assets each with standard deviation of returns σ_I, i=1,2,…,n and mutual correlation coefficients ρ_{ij}, I,j =1,2,…,n. The variance of the portfolio is, then, given by $\sigma_\Pi^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}S_iS_j\sigma_i\sigma_j$.

Assuming that the differential Black Scholes model holds for each asset i, i=1,2,…,n, and that the expected return $\mu_{i,day}$ over a 1-day period is insignificant relative to the 1-day volatility (or 1-day SD of price changes), we can write: $dS_{i,day}=Z_i\sigma_{i,day}S_i$ whence $SD(dS_{i,day})=\sigma_{i,day}S_i$. The variance of the 1-day value changes of the portfolio Π is, then, $Var\left(d\Pi_{day}\right)=Var\left(\sum_{i=1}^{n}dS_{i,day}\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\left[SD\left(dS_{i,day}\right)\right]\left[SD\left(dS_{j,day}\right)\right]=\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}S_iS_j\sigma_{i,day}\sigma_{j,day}$.

Alternatively,

$$Var\left(d\Pi_{day}\right)=Var\left(\sum_{i=1}^{n}dS_{i,day}\right)=Var\left(\sum_{i=1}^{n}Z_iS_i\sigma_{i,day}\right)=E\sum_{i=1}^{n}\sum_{j=1}^{n}\left[Z_iS_i\sigma_{i,day}\right]\left[Z_jS_j\sigma_{j,day}\right]$$
$$=E\sum_{i=1}^{n}\sum_{j=1}^{n}\left[Z_iZ_j\right]\left[S_iS_j\sigma_{i,day}\sigma_{j,day}\right]=\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}S_iS_j\sigma_{i,day}\sigma_{j,day}\ .$$

Please note that $E\left(\sum_{i=1}^{n}Z_iS_i\sigma_{i,day}\right)=\sum_{i=1}^{n}E\left(Z_i\right)S_i\sigma_{i,day}=0$. Knowing the standard deviation of the 1-day changes in value of the portfolio Π, we can work out the VaR for any confidence level by taking the corresponding z-value from the standard normal tables and for any time horizon by invoking the square root rule i.e. VaR (Π, α%,N days)= $SD(d\Pi_{day})*|z_\alpha|*\sqrt{N}$.

VaR for bond portfolios

Let us now move to bond portfolios. Now for bond portfolios there are three standard approaches for working out the value at risk. We have the (i) duration based approach; (ii) cash flow mapping; and (iii) principal component analysis.

Duration based approach

Consider a bond portfolio Π comprising of investments of amounts S_i, i=1,2,…,n in bonds i=1,2,…,n each with duration D_i, i=1,2,…,n. Then, we have

$$S_i = \sum_{t=1}^{T} C_t e^{-ty} \rightarrow \frac{dS_i}{dy} = -\sum_{t=1}^{T} tC_t e^{-ty} \rightarrow \frac{dS_i}{S_i} = \frac{-\sum_{t=1}^{T} tC_t e^{-ty}}{S_i} dy = \text{-}D_i dy$$

$$\Delta\Pi = \sum_{i=1}^{n} \Delta S_i = \sum_{i=1}^{n} \frac{\Delta S_i}{S_i} S_i = \text{-}\sum_{i=1}^{n} D_i S_i \Delta y = \text{-}\Pi\Delta y \sum_{i=1}^{n} D_i \frac{S_i}{\Pi} = \text{-}\Pi\Delta y \sum_{i=1}^{n} D_i w_i = \text{-}D_\Pi \Pi \Delta y$$ whence

$$\sigma_\Pi = D_\Pi \Pi \sigma_y .$$

Please note that in the above, the final cash-flow C_T would also include the principal. In fact, if the principal is distributed and redeemable in installments, the relevant cash-flows will include both coupon and principal redemptions. The C_t's represent the total cash-flows from the instrument, whatever the source of the cash flow may be, whether it is coupon, principal or both.

Now, before I move to an example there is a very important assumption in this model that needs to be highlighted. Let us revisit the step: $\Delta\Pi = \text{-}\sum_{i=1}^{n} D_i S_i \Delta y = \text{-}\Pi\Delta y \sum_{i=1}^{n} D_i \frac{S_i}{\Pi}$ $= \text{-}\Pi\Delta y \sum_{i=1}^{n} D_i w_i = \text{-}D_\Pi \Pi \Delta y$. In this step we have assumed Δy to be independent of the summation index i, and accordingly taken it outside the summation. The quantity within the summation then works out to be the portfolio duration D_Π. We need to understand the implications of this assumption. Now, the portfolio Π consists of a number of bonds, these bonds will more likely than not have different maturities and therefore be influenced by spot rates of different maturities. Now, usually spot rates of different maturities are different, in line with the term structure of interest rates. Indeed, it is also true that if and when a stimulus inducing shifts in interest rates operates, it influences the different maturity spot rates to varying extents and consequently, the shifts in these different maturity spot rates are by different magnitudes. Stated otherwise, the shifts in the yield curve consequent to a stimulus is not parallel to itself. The shift in a particular spot rate will relate to the demand-supply dynamics & equilibrium for money of the same maturity. It is not necessary that interest rates for all maturities change by this same magnitude e.g. the 1-year interest rate changes by 0.5 percent, the 2-year interest rate changes by 0.5 percent, the 10-year interest rate changes by 0.5 percent and so on. The amount of change depends on the demand and supply for money corresponding to those maturities.

Therefore, in view of the above, by taking Δy as the same for all the relevant spot rates, we are introducing a strong approximation. In other words, I am assuming a parallel shift in the interest rate curve representing the term structure. This happens to be the biggest drawback of the duration model and forms the fundamental limitation of this model. That is what restricts its application spectrum.

<u>Example</u>

A company has a position in bonds worth USD 6 million. The duration of the portfolio is 5.2 years. Assume that only parallel shifts in the yield curve can take place and that the standard deviation of the daily yield change (when yield is measured in percent) is 0.09. Use the duration model to estimate the 20-day 90% VaR for the portfolio (z=1.28).

Solution

Portfolio value Π: USD 6,000,000; Portfolio Duration: 5.2 years; σ_y =0.09%=0.0009
σ_Π =5.2*6,000,000*0.0009=28,080; 20 day 90% VaR= 1.28*28,080*√20=160,739

Cash flow mapping

Data in respect of bonds of certain standard maturities is tracked and reported on electronic and print media on real-time basis by several reputed financial monitoring agencies like Reuters, Bloomberg etc. These standard maturities are 1 month, 3 months, 6 months, 1 year, 2 years, 5 years, 7 years, 10 years, and 30 years. Prices & volatilities of zero coupon bonds with aforesaid maturities are, therefore, accessible on real-time basis by the investor.

Therefore, for bond portfolios, the procedure usually followed for VaR calculation is to choose as market variables, the prices of these zero-coupon bonds with standard maturities. The cash flows from instruments in the portfolio are mapped into cash flows occurring on the standard maturity dates. This is called cash flow mapping.

A bond portfolio will usually consist of bonds of maturities different from the above standard maturities. Even standard maturity coupon bonds will have coupon flows at non-standard maturities e.g. a 2 year 6-monthly coupon bond will have a coupon payment at 1.50 years which is not a standard maturity.

Now the data in respect of these non-standard maturity cash flows will, obviously, not be readily available. Cash flow mapping is one approach which is very useful and intuitive. We bifurcate each and every cash-flow emanating from the bond (coupon, principal or both) into two parts, one relating to the immediately preceding standard maturity bond and to the immediately succeeding standard maturity bond.

For example, a 0.30-year bond would be mapped into a 3-month bond and a 6-month bond. A 6-monthly coupon bond with a maturity of one year and two months would be mapped as follows: The next coupon will be at t=2 months from now will be mapped to the 1-month and the 3-month zero coupon bonds. The coupon at t=8 months will be mapped to 6-month and 1-year zeros while the final cash flow (coupon principal) will be mapped to 1-year and 2-year zeros.

So this is how the scheme operates, each and every cash flow emanating from the bond portfolio are split into the relevant standard maturities and then the cashflows for each standard maturity aggregated. These form the basis of our VaR calculations.

Example

A 0.3-year zero-coupon bond with a principal value of $50,000 is mapped into positions in a three-month bond and a six-month bond. The 0.25 and 0.50 years spot rates are 5.50% and 6.00% respectively. Daily volatilities of the prices of the 0.25 year and 0.50 year zero coupon bonds are 0.06% and 0.1% with a correlation of 0.9. Calculate the values of the 0.3-year bond mapped to each of the standard bonds.

Solution

The bond entails only one cash-flow of USD 50,000 at the end of 0.30 years from now.

The 0.3-year cash flow is mapped into a 3-month zero coupon bond and a 6-month zero coupon bond.

The 3-month and 6-month spot rates are 5.50% and 6.00%. Linear interpolation gives the 0.30-year spot rate as 5.60%. Therefore, present value of 50,000 to be received in 0.30 years is: $50{,}000/[(1.0560)^{0.30}] = 49{,}189.32$. This is the amount to be mapped to the 3-month & 6-month zero.

The 1-day volatility of the 3-month and the 6-month bonds are 0.06% and 0.10%. The interpolated volatility of the 0.30-year bond is, therefore, 0.068% per day.

Assume that α of the value of the 0.30 year bond gets mapped to the 3-month bond and $(1-\alpha)$ to the 6-month bond. To match variances, we must have

$0.00068^2 = 0.0006^2\alpha^2 + 0.001^2(1-\alpha)^2 + 2\times 0.9\times 0.0006\times 0.001\times\alpha\times(1-\alpha)$ or

$0.28\alpha^2 - 0.92\alpha + 0.5276 = 0$ whence $\alpha=0.760259$ so that the amount allocated to the 3 month bond is 37,397 and that to the 6-month bond is 11,793.

Please note that the bifurcation envisaged by the cash-flow mapping is a notional bifurcation. It is not a physical bifurcation. We are not physically converting our bond to the 3-month & 6-month zeros, we are simply doing a notional split in order to be able to work out the value at risk.

Principal component analysis

One approach to handling the risk arising from groups of highly correlated market variables is principal components analysis. This takes historical data on movements in the market variables and attempts to define a set of components or factors that explain the movements.

The principal component analysis is usually carried out through a software package. We start by identifying the factors that influence our bond portfolio. There usually turn out to be the spot interest rates for various maturities. However, there may be a large number of such rates, far too many for the analyst to manipulate further. Because there could be a large spectrum of interest rates which would influence our portfolio, we try to reduce the dimensions of the problem. In fact, principal component analysis is a technique of dimensional reduction. Given a certain amount of data having a large number of dimensions, we try to extract certain factors which are good enough to explain substantively the given data and which are mutually uncorrelated. The procedure attempts to do a dimensional reduction i.e. to reduce to the number of factors required for explaining the data. Obviously, as a consequence of such reduction, the complete explanatory power of the set of independent variables is lost. Nevertheless, the algorithm ensures that this loss of explanatory power is kept to a minimum.

Further, the variables that are identified through the principal component analysis are linearly independent and uncorrelated.

Thus, in the above problem, the spectrum of interest rates would be reduced yet these rates explain a significant proportion of the variance of the data that is provided to us. As a result, we do not lose much information by using a lesser number of variables to represent the same problem. In addition, we have a massive advantage of using uncorrelated variables.

Example

Suppose that the daily change in the value of a portfolio is, to a good approximation, linearly dependent on two factors, calculated from a principal components analysis. The delta of a portfolio with respect to the first factor is 6 and the delta with respect to the second factor is -4. The standard deviations of the factors are 20 and 8 respectively. What is the 5-day 90% VaR?

Solution

$$\textit{Let the portfolio value be represented as}: \Pi = \alpha_1 F_1 + \alpha_2 F_2 \ \textit{so that}$$
$$\Delta_1 = \frac{\partial \Pi}{\partial F_1} = \alpha_1 = 6;\ \Delta_2 = \frac{\partial \Pi}{\partial F_2} = \alpha_2 = -4.\ \textit{Hence},\ \Pi = 6F_1 - 4F_2;\ \Delta\Pi = 6\Delta F_1 - 4\Delta F_2$$
$$\textit{and}\ \sigma_\Pi^2 = 36\sigma_1^2 + 16\sigma_2^2 = 36\times 20^2 + 16\times 8^2 = 15424;\ \sigma_P = 124.19\ \textit{as as}\ \rho_{12} = 0$$
$$5\ \textit{day}\ 90\%\ \textit{VaR} = 1.28\times 124.19\times\sqrt{5} = 355.46$$

VaR of a portfolio of derivatives

Now we come to a portfolio comprising of derivatives. In the case of a portfolio which involves derivatives, again the process is more or less absolutely parallel to what we had for the cash assets. We have, given a derivative $\pi \equiv \pi(S)$

$$\delta = \frac{\Delta\pi}{\Delta S} \rightarrow \Delta\pi = \delta\Delta S = S\delta\frac{\Delta S}{S} = S\delta\left(\mu_S dt + \sigma_S Z\sqrt{dt}\right)$$ so that for dt=1 day

$$\Delta\pi_{day} = \delta\Delta S_{day} = S\delta\frac{\Delta S_{day}}{S} = S\delta\sigma_{S,day}Z$$ whence $\sigma_{\pi,day} = S\delta\sigma_{S,day}$ when we neglect the daily return $\mu_{S,day}$.

For a multi-derivative portfolio, $\Delta\Pi_{day} = \sum_{i=1}^{n}\Delta\pi_{i,day} = \sum_{i=1}^{n}\frac{\Delta\pi_{i,day}}{\Delta S_{i,day}}\frac{\Delta S_{i,day}}{S_i}S_i = \sum_{i=1}^{n}\delta_i S_i \sigma_{S_i,day}Z_i$

whence $\sigma_{\Pi,day}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i S_i \delta_j S_j \rho_{ij}\sigma_{S_i,day}\sigma_{S_j,day}$.

So, if we compare the above expression with what we had for the cash asset, we find that it is absolutely the same but for the additional δ term. This delta is the rate of change of the portfolio value with respect to the price of the underlying.

Why this term is here? To understand consider the following: $df\left(x\right)=\frac{df\left(x\right)}{dx}dx=\delta dx \text{ where } \delta=\frac{df\left(x\right)}{dx}$. So when we are talking about a function of x and we want to work out the change in value of the function, we have to incorporate the rate of change of the function with respect to x and then multiply by change in x i.e. dx. A derivative's value is a function of the value of underlying asset. Therefore, when we work out the price change of the derivative, we have to incorporate the rate of change of value of the derivative with respect to the price of underlying asset and then multiply it by the change in price of underlying asset. So that is precisely what is happening, this δ is the rate change of derivative value with respect to underlying.

Example

A portfolio consists of options on Microsoft and AT&T. The options on Microsoft have a delta of 1,000, and the options on AT&T have a delta of 20,000. The Microsoft share price is $120, and the AT&T share price is $30. Assuming that the daily volatility of Microsoft is 2% and the daily volatility of AT&T is 1% and the correlation between the daily changes is 0.3, what is the standard deviation of the portfolio price (in thousands of dollars)?

Solution

We have, $\sigma_{\Pi,day}^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i S_i \delta_j S_j \rho_{ij}\sigma_{S_i,day}\sigma_{S_j,day}$. For a two asset portfolio,

$=\left(\delta_1 S_1 \sigma_{S_1,day}\right)^2+\left(\delta_2 S_2 \sigma_{S_2,day}\right)^2+2\rho_{12}\left(\delta_1 S_1 \sigma_{S_1,day}\right)\left(\delta_2 S_2 \sigma_{S_2,day}\right)$. In the given problem:

$$\delta_M=1000;\ \delta_A=20000;\ P_M=120;\ P_A=30;\ \sigma_M=0.02;\ \sigma_A=0.01;\ \rho_{MA}=0.30$$

$$\sigma_\Pi^2=\left(1000\times120\times0.02\right)^2+\left(20000\times30\times0.01\right)^2+2\times0.30\times\left(1000\times120\times0.02\right)\times\left(20000\times30\times0.01\right)$$

$$=5760000+36000000+8640000=50400000,\ \sigma_P=7099.30$$

VaR of a forward contract

Some time ago a company has entered into a forward contract to buy ₤1 million for $1.5 million. The contract now has six months to maturity. The daily volatility of a six-month zero – coupon sterling bond (when its price is translated to dollars) is 0.06% and the daily volatility of a six-month zero-coupon dollar bond is 0.05%.

The correlation between returns from the two bonds is 0.8. The current exchange rate is 1.53. Calculate the standard deviation of the change in the dollar value of the forward contract in one day. What is the 10-day 99%VaR? Assume that the six-month interest rate in both sterling and dollars is 5% per annum with continuous compounding.

Solution

The contract is equivalent to a long position in a sterling bond and a short position in a dollar bond.
USD value of the sterling bond: $1.53e^{-0.05x0.50}$=1.492 m
USD value of dollar bond: $1.5e^{-0.05x0.50}$=1.463 m
The variance of the change in the value of the contract in one day is calculated as follows:

$$\sigma^2 = 1.492^2 \times 0.0006^2 + 1.463^2 \times 0.0005^2 - 2 \times 0.8 \times 1.492 \times 0.0006 \times 1.463 \times 0.0005$$
$$= 0.000000288 \textit{ so that } \sigma = 0.000537 \textit{ and } VaR = 0.000537 \times 2.33 \times \sqrt{10} = 0.00396$$

Since the valuer has committed to buy GBP i.e. it will receive GBP, it is long in GBP bond and short in USD bond. The USD value of the two bonds will be calculated at the current exchange rate i.e. GBP 1.00=USD 1.53. Thus, the value of the GBP bond is the PV of GBP 1.00 million calculated at the GBP spot rate of 5% p.a. and this PV is then converted to USD at the current exchange rate of 1.53 to get the USD value of GBP bond. We get USD 1.492 million

The current USD value of the USD bond is simply the PV of 1.50 million USD calculated at the USD spot rate of 5% p.a. That comes to USD 1.463 million.

The variance of the combination of the two bonds (long GBP + short USD) is calculated using the standard portfolio formula. The negative sign arises due to the long/short positions.

Appendix

Principal component analysis

Let us assume that out portfolio Π is influenced by three variables Y_i, i=1,2,3 so that its value can be expressed as:

$\Pi = \beta_1 Y_1 + \beta_2 Y_2 + \beta_3 Y$ whence $\Delta\Pi = \beta_1 \Delta Y_1 + \beta_2 \Delta Y_2 + \beta_3 \Delta Y_3$.

Let us also assume that these each of these market variables Y_i can be explained sufficiently precisely by two uncorrelated factors X_1 and X_2 with factor loadings α_{i1} and α_{i2} so that:

$Y_i=\alpha_{i1}X_1+\alpha_{i2}X_2$ and $\Delta Y_i=\alpha_{i1}\Delta X_1+\alpha_{i2}\Delta X_2$

These factors X_1, X_2 and factor loadings α_{i1} and α_{i2} are usually obtained through software by best fitting of historical data on the moments of these variables. The software ensures that the factors X_1 and X_2 are uncorrelated. The quantity of a particular factor i.e. coefficient of X_j in a particular observation of the market variable Y_i is called the factor score of the observation.

Importance of a particular factor is proportional to the SD of is factor score σ_j.

Change in variable Y_i due to one SD change in factor $X_j=\alpha_{ij}\sigma_j$.

The factor X_j accounts for $\dfrac{\sigma_j^2}{\sum_{k=1}^{2}\sigma_k^2}$ of the total variance. There are no cross terms since the factors are uncorrelated. From above,

$\Delta\Pi = \sum_{i=1}^{3}\beta_i \Delta Y_i = \sum_{i=1}^{3}\beta_i\left(\alpha_{i1}\Delta X_1 + \alpha_{i2}\Delta X_2\right) = \Delta X_1 \sum_{i=1}^{3}\beta_i\alpha_{i1} + \Delta X_2 \sum_{i=1}^{3}\beta_i\alpha_{i2}$ so that

$$\sigma_\Pi^2 = \left(\sum_{i=1}^{3}\beta_i\alpha_{i1}\right)^2 \sigma_1^2 + \left(\sum_{i=1}^{3}\beta_i\alpha_{i2}\right)^2 \sigma_2^2$$