Financial Derivatives & Risk Management
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Lecture 57
Swaps: Valuation of Interest Rate Swaps

Valuing a floating rate bond

The first point to note is that the value of a newly issued floating-rate bond that pays 6-month LIBOR is always equal to its principal value (or par value) when the LIBOR/swap zero curve is used for discounting. The reason is that the bond provides a rate of interest of LIBOR, and LIBOR is the discount rate.

Now, what is a floating rate bond? A floating rate bond is a bond on which the interest rate is fixed at the beginning of each period to which it relates, that period may be at 6 monthly period or annual, it is usually a 6-monthly period. At the beginning of every 6-month period, the interest rate is fixed on the basis of the market rate that is prevailing at that point in time, usually the LIBOR or T-bill rate or prime rate or any other benchmark specified in the issue contract. This interest rate applies to the immediately following 6-month period, whereafter the interest rate is again reset on the basis of the updated value of the benchmark rate at this new reset date. However, interest is actually due for payment at the end of the six-month period to which it relates as per convention.

The value of a floating rate bond invariably equals its par value on the date of issue since it pays coupons equal to the market rate over its entire life span. As an illustration, consider a floating rate bond, indexed to 12-month LIBOR issued at t=0 with 1-year maturity and face value of 100. Let the t=0 LIBOR be 5% p.a. Clearly, then, the bond's coupon will be tagged at 5% for the one-year life-span. Hence, its cash flow at t=1 will be 105 (including the principal of 100). Its intrinsic value will be 105/(1+0.05)=100 = Face value.

Thus, the inference is that if the coupon rate is tagged with a benchmark rate and the discounting is also done at that benchmark rate, the intrinsic value of the bond equals its face value. This is also logical, in the sense that if the return provided by the instrument (coupon rate) equals the required return (discount rate), the bond should be worth its par value.

Let us consider a second example. Consider a floating rate bond that has 2 years to go to maturity from now (t=0). Let the bond pay annual LIBOR with the last interest payment having just been made. Further, let the face value of the bond be 100 and the LIBOR reset for the next 1 year be 5.00%. Further let the 1-year and 2-year zero rates 5.00% and 6.00%, all rates with annual compounding. We need to compute the current value of the bond.

The implied LIBOR for the second year can be obtained from the principle of no arbitrage and will equal the forward rate for that period. We have, by no arbitrage:

$(1+S_{02})^2 = (1+S_{01})\ (1+f_{12}) = (1+0.06)^2 = (1+0.05)(1+f_{12})$ whence $f_{12} = 7.01\%$.

Hence, the value of the bond is: $\frac{5}{1.05}+\frac{107.01}{1.06^2}=4.7619+95.2385=100.00$ i.e. the face value of the bond. This is no coincidence.

The present bond has a maturity of 2 years with annual coupons. Coupon at the end of year 1 will be paid at the rate fixed at t=0 in accordance with the prevailing LIBOR at that time. Coupon at the end of year 2 will be paid at the rate fixed at the end of year 1 in accordance with the LIBOR prevailing at that time i.e. t=1.

We, obviously know the LIBOR at t=0 so that the coupon rate for the first year can be fixed unambiguously and poses no problem. It is given at 5.00% p.a. Hence, the cash flow at t=1 (coupon only) will be 5.00. This cash flow will be discounted at the spot rate for 1-year maturity, which is also 5.00%.

However, the coupon rate for the second year poses a problem because we do not know the LIBOR at t=1. In fact, this is a random variable and will carry a certain probability distribution. In fact, this is the reason why this bond is called a floating rate bond i.e. because the interest rate for future years would depend on the market rates that would prevail at the respective points in time at which the rates are to be determined.

However, we can have a proxy for these floating (undetermined) rates. For this purpose, we invoke the principle of no-arbitrage. Using the spot rates at t=0 for various maturities, we can work out the forward rates that should prevail at the various time points in the future, on the premise that the market efficiency will not allow any arbitrage opportunities to sustain. The forward rates so obtained can then be used to serve as proxies for the unknown undermined floating rates.

Thus, knowing the spot LIBOR for 1 year and 2year maturities, we can work out the forward rate for the second year using the principle of no arbitrage. It turns out to be:

$(1+S_{02})^2 = (1+S_{01})\ (1+f_{12}) = (1+0.06)^2 = (1+0.05)(1+f_{12})$ whence $f_{12} = 7.01\%$.

In other words, because the LIBOR at t=1 is unknown, undetermined and random and, obviously we do not know it will precision at t=0, we simply use a proxy for it which is the forward rate for the relevant period determined on the premise of no-arbitrage. Obviously, as time passes this forward rate may also change due to changes in spot rates, but this rate is fixed and unambiguous at a given instant since it is obtained via an equality connecting various spot rates prevailing at that instant (which are obviously known with precision).

Given the data in the problem, the forward rate works out to 7.01%. Hence, the coupon rate for the second year will be 7.01% since we are proxying the LIBOR at t=1 by the forward rate at the instant. The cash flow at the end of the second year will be 107.10 since the principal of 100 will also be repaid. This will be discounted at the 2-year spot rate i.e. 6%.

The value of the bond, thus, works out to: $\frac{5}{1.05}+\frac{107.01}{1.06^2}=4.7619+95.2385=100.00$.

Therefore, if the LIBOR at t=1 actually turns out to be the same as the forward rate worked out as above, i.e. the forward rate that we have worked out is actually realized, then the intrinsic value of the bond will work out to its face value once again.

Let us establish the general case. We have:

$$P_0 = \sum_{i=1}^{T} \frac{f_{i-1,i}F}{\left(1+S_{0i}\right)^i} + \frac{F}{\left(1+S_{0T}\right)^T} = \sum_{i=1}^{T-1} \frac{f_{i-1,i}F}{\left(1+S_{0i}\right)^i} + \frac{f_{T-1,T}F}{\left(1+S_{0i}\right)^T} + \frac{F}{\left(1+S_{0T}\right)^T}$$

$$= \sum_{i=1}^{T-1} \frac{f_{i-1,i}F}{\left(1+S_{0i}\right)^i} + \frac{\left(1+f_{T-1,T}\right)F}{\left(1+S_{0i}\right)^T} = \sum_{i=1}^{T-1} \frac{f_{i-1,i}F}{\left(1+S_{0i}\right)^i} + \frac{\left(1+f_{T-1,T}\right)F}{\prod_{j=1}^{T}\left(1+f_{j-1,j}\right)} = \sum_{i=1}^{T-1} \frac{f_{i-1,i}F}{\left(1+S_{0i}\right)^i} + \frac{F}{\prod_{j=1}^{T-1}\left(1+f_{j-1,j}\right)}$$

$$= \sum_{i=1}^{T-2} \frac{f_{i-1,i}F}{\left(1+S_{0i}\right)^i} + \frac{f_{T-2,T-1}F}{\left(1+S_{0i}\right)^{T-1}} + \frac{F}{\prod_{j=1}^{T-1}\left(1+f_{j-1,j}\right)} = \sum_{i=1}^{T-2} \frac{f_{i-1,i}F}{\left(1+S_{0i}\right)^i} + \frac{\left(1+f_{T-2,T-1}\right)F}{\prod_{j=1}^{T-1}\left(1+f_{j-1,j}\right)}$$

$$= \sum_{i=1}^{T-2} \frac{f_{i-1,i}F}{\left(1+S_{0i}\right)^i} + \frac{F}{\prod_{j=1}^{T-2}\left(1+f_{j-1,j}\right)} = \ldots = \frac{f_{0,1}F}{\left(1+S_{01}\right)^1} + \frac{F}{\left(1+f_{0,1}\right)} = \frac{f_{0,1}F}{\left(1+f_{0,1}\right)} + \frac{F}{\left(1+f_{0,1}\right)} = F$$

The conclusion is that, given a floating rate bond, its value on any interest payment date, immediately after the payment of the interest due on that date, is its par value. This valuation is based on the assumption that the expected floating rates over the life of the bond are captured by the forward rates prevailing on the date of valuation i.e. that the forward rates will be realized.

What happens if the bond pays a floating rate e.g. LIBOR plus a fixed rate? In this case, we simply value the bond as if it consisted of two separate instruments A, a floating rate bond and B, a fixed rate bond and aggregate the values so obtained.

<u>Example 1</u>

A floating rate bond has a principal value of USD 400 million and a remaining life of 1.25 years. It pays 6 month LIBOR semi-annually (with semi-annual compounding). The last reset of the 6-month LIBOR was 8% (with s.a. compounding). The LIBOR rate for 3-month maturity is 18% continuously compounded. Calculate its intrinsic value (in million USD).

<u>Solution</u>

PRINCIPAL VALUE		**400**
REMAINING LIFE		**1.25**
3 MONTH LIBOR		**0.18**
LAST RESET LIBOR 3 MONTHS AGO		**0.08**
AT THE NEXT PAYMENT DATE THE BOND WILL BE WORTH		
ITS PAR VALUE AFTER THE INTEREST PAYMENT		
INTEREST PAYMENT ON NEXT PAYMENT DATE		
3 MONTHS FROM NOW		**16**
PAR VALUE		**400**
VALUE OF BOND 3 MONTHS FROM NOW		**416**
DISCOUNT FACTOR		**0.955997482**
PRESENT VALUE OF BOND		**397.6949524**

Example 2

Consider a floating rate bond of face value of 1,000 that pays 6-monthly floating rate interest indexed to 6-month LIBOR. The interest rate is set at the beginning for the 6-month period to which it relates. The bond was issued 9 months ago and has a remaining life of 4.25 years. The last LIBOR re-set was made 3 months ago at 12% p.a. compounded half yearly. The current 3-month LIBOR is 9% p.a. with continuous compounding. What is the current intrinsic value of the bond?

Solution

FACE VALUE	**1000**
LAST LIBOR RESET	**0.12**
LAST LIBOR RESET AT	**-0.25**
NEXT INTEREST PAYMENT	
ON	**0.25**
AMOUNT	**60**
VALUE IMMEDIATELY AFTER INTT	**1000**
TOTAL VALUE AT T=0.25 YEARS	**1060**
DISCOUNT RATE	**0.09**
PRESENT VALUE	**1036.416311**

Now, in this problem, as is usually the case, interest rate is set at the beginning of every 6-month period to which it relates, the actual payment takes place at the end of the period. The bond was issued 9 months ago and it has a remaining life of 4.25 years. Thus, the life of the bond is 5 years.

The last LIBOR was re-set at 3 months ago from today(t=0) i.e. at t = (-)3 months = (-)0.25 years. Since, the interest rate is fixed at 6-monthly intervals, the rate fixed at t = (-)3 months shall operate for the period from t = (-) 3 months to t = (+) 3 months. This rate was 12.00% compounded s.a. i.e. 6% for this half year period. Therefore, the interest payment that will take place at t=3 months on this bond (Face value=1,000) will be 6% of 1,000=60.

Immediately after this bond is stripped off this interest payment i.e. after this interest payment has been made, the remaining bond is worth its face value (1,000). Immediately after paying out this interest, the bond becomes a freshly issued floating rate bond. Therefore, the value of the bond after payment of this interest, will be equal to its face value. Thus, the total value of this bond as on the date of this interest payment i.e. at t= 3 months (including the interest) is 1,060

The current (t=0) 3-month LIBOR is given at 9.00% with continuous compounding.

Therefore, the value of this bond at t=0 =PV of 1,060 at t= 3 months @ 9.00 p.a. cc =$1,060e^{-0.0225}$ = 1,036.42.

<u>Valuation of a fixed rate bond</u>

We value a fixed rate bond simply by calculating the present value of all future cash flows that are expected to emanate from the bond by discounting them at the appropriate risk adjusted spot rates of maturities corresponding to the dates of payment of interest and principal.

<u>Valuation of an interest rate swap</u>

As explained earlier, an interest rate swap can be viewed as equivalent to an exchange of a fixed rate and a floating rate bond. Therefore, the value of the swap will equal the difference between the value of the fixed rate bond and that of the floating rate bond.

An alternative approach valuation of an IRS is to consider it as a string of FRAs and to proceed to value each of these FRAs. The cumulative value of these FRAs will give the value of the IRS.

Further, if the value of a swap to one party is, say X, then its value from the perspective of the other party will be (-)X since it is a zero-sum game.

<u>Example 3</u>

Suppose that a financial institution has agreed to pay 6-month LIBOR and receive 8% per annum (with semi-annual compounding) on a notional principal of USD 100 million. The swap has a remaining life of 1.25 years. The LIBOR rates with continuous compounding for 3-month, 9-month, and 15-month maturities are 10%, 10.5%, and 11%, respectively. The 6-month LIBOR rate at the last payment date was 10.2% (with semi-annual compounding). Calculate the swap’s value.

Solution

Method 1: As the difference of the value of a fixed rate bond and a floating rate bond

The financial institution (FI) is the fixed rate receiver. Therefore, so from the perspective of the financial institution it is receiving fixed rate, it is paying the floating rate and as such it will ascribe a positive value to the fixed leg and a negative value to the floating leg. The value of the swap is the value of the fixed leg that it is receiving less the value of the floating leg that it is paying. Hence, the value of the swap from its perspective $V=V_{fixed}-V_{floating}$.

For the fixed leg, FI is receiving 8.00% semi-annually with semi-annual compounding, so this is equivalent to 4.00% for every 6-month period. The notional principal is USD 100 million. The swap has a remaining life of 1.25 years i.e. 15 months and interest is payable every 6-months. It follows that the next interest payment date will be after 3 months from now (t=0) and thereafter after 9 months and 15 months respectively from now. Please note this, bond has a remaining life of 1.25 years. Hence, the cash flows will be 4 million at t=3 months, 4 million at t=9 months and 104 million (including principal of 100 million) at t=15 months. Discounting them at the respective spot rates, we have:

TIME	CASHFLOW	DISC FACTOR	DCF
0.25	4	0.975309912	3.901239648
0.75	4	0.924270963	3.697083853
1.25	104	0.87153435	90.6395724
			98.2378959

Now, we come to the floating leg. Since the next interest payment is 3 months after now i.e. at t=+3months and interest rate is reset every 6-months, the rate for this interest payment would have been set 3 months ago i.e. at t = (-)3 months. This rate has been set at 10.20% with semi-annual compounding i.e. 5.10% per the 6-month period from t=(-) 3 months to t=+3 months.

Thus, the floating leg will pay out a cash flow on account of interest at t=+3 months of 5.10% on the principal of USD 100 million i.e. USD 5.10 million. Since, it is a floating rate bond, immediately after this interest payment, the bond will be worth its face value of 100 million. It follows that the total value of the bond at t=+3 months is 5.10+100=105.10 million.

But this is the value of the floating leg at t=+3 months. We need to discount it to t=0 for getting its value at t=0. The 3-month spot rate for this purpose is given as 10.00% continuously compounding. Hence, the value of the floating leg at t=0 is $V_{floating}=105.10e^{-0.0250}=102.5051$.

Thus, the value of the swap from the perspective of the FI = $V_{fixed}-V_{floating}$ = 98.2379-102.5051 = (-)4.2672.

Method 2: As a series of FRAs

Step 1: Work out the interest reset and payment dates

The swap has a remaining life of 1.25 years i.e. 15 months and interest is payable every 6-months. It follows that the next interest payment date will be after 3 months from now (t=0) and thereafter after 9 months and 15 months respectively from now.

Further since reset of floating rate is also done at every 6-monthly intervals, the last reset would have been at t=(-) 3 months with subsequent resets at t=+3, +9 months corresponding to interest payments at t=+3, +9 and +15 months respectively.

Step 2: Work out the forward rates at each reset date

t=+3 months

We have, $\exp\left(S_{0,3m}\times\frac{3}{12}\right)\times\exp\left(f_{3m,9m}\times\frac{6}{12}\right)=\exp\left(S_{0,9m}\times\frac{9}{12}\right)$ so that

$\left(f_{3m,9m}\times\frac{6}{12}\right)=\left(S_{0,9m}\times\frac{9}{12}\right)-\left(S_{0,3m}\times\frac{3}{12}\right)$ whence

$$f_{3m,9m}=\frac{\left(S_{0,9m}\times\frac{9}{12}\right)-\left(S_{0,3m}\times\frac{3}{12}\right)}{\frac{6}{12}}=\frac{\left(10.5\times\frac{9}{12}\right)-\left(10\times\frac{3}{12}\right)}{\frac{6}{12}}=10.75$$

Hence, the forward rate for the period t=+3 months to t=+9 months is 10.75%. But this rate is continuously compounded rate. However, the swap operates with semi-annually compounded rates. Hence, we need to convert this continuously compounded rate to semi-annually compounded rate. We have, $\exp(r_{cc})=\left(1+\frac{r_{sa}}{2}\right)^2$. For r_{cc}=10.75%, we get r_{sa}=11.044%.

t=+9 months

We have

$$f_{9m,15m}=\frac{\left(S_{0,15m}\times\frac{15}{12}\right)-\left(S_{0,9m}\times\frac{9}{12}\right)}{\frac{6}{12}}=\frac{\left(11.0\times\frac{15}{12}\right)-\left(10.5\times\frac{9}{12}\right)}{\frac{6}{12}}=11.75$$

Hence, the forward rate for the period t=+9 months to t=+15 months is 11.75%. But this rate is continuously compounded rate. We have, $\exp(r_{cc})=\left(1+\frac{r_{sa}}{2}\right)^2$. For r_{cc}=11.75%, we get r_{sa}=12.102%.

Step 3: Work out the cash flows at the fixed and forward rates at each interest payment date

Time	Fixed Rate (%)	Floating Rate (%)	Fixed Leg Cash Flow	Floating Leg Cash Flow	Difference
3m	8.00	5.100	4.00	5.100	-1.10
9m	8.00	11.044	4.00	5.522	-1.522
15m	8.00	12.102	4.00	6.051	-2.051

Step 4: Calculate the aggregate present value of the difference at the spot rates of appropriate maturity

Time	Fixed Leg Cash Flow	Floating Leg Cash Flow	Difference	Discount Rate (%) cc	Present Value
3m	4.00	5.100	-1.10	10.00	-1.0728
9m	4.00	5.522	-1.522	10.50	-1.4068
15m	4.00	6.051	-2.051	11.00	-1.7875
					-4.2672

We assume, in this approach that the forward rates that are fixed at t=0 by no arbitrage actually realize and turn out to be the LIBOR rates at the various reset dates. Thus, the forward rate of 11.044% worked out for t=3 months to t=9 months is assumed to be the what LIBOR will evolve to be at t = 3 months when we do the rest of the floating rate. On this premise, we work out the floating rate interest for the period t=3 months to t=9 months at this rate (11.044%).

So, these are the two approaches that can be used for calculating the value of an interest rate swap. In the first method, we follow the prescription that interest rate swaps can be considered as an exchange of fixed rate bond and a floating rate bond. We value them separately and calculate the difference in their value. In the second approach, we assume that the forward rates that are implied by the spectrum of spot rates prevailing as on the date on which the valuation is carried out are actually realized and LIBOR or the floating rate is fixed in accordance with this forward rate.

Example 4

A financial institution has entered into an interest rate swap with company X. Under the terms of the swap, it receives 10% per annum and pays 6-month LIBOR on a principal of USD 10 million for 5 years. Payments are made every 6 months. Suppose that company X defaults on the sixth payment date (at the end of year 3) when the interest rate (with semi-annual compounding) is 8% per annum for all maturities. What is the loss to the financial institution? Assume that 6-month LIBOR was 9% per annum halfway through year 3.

FINANCIAL INSTITUTION RECEIVES				0.1	FIXED
PAYS				LIBOR	
PRINCIPAL				10	MILLION
LIFE				5	YEARS
DEFAULT				3	YEARS
DISCOUNT RATE				0.08	
CASH FLOWS REMAINING ON THE SWAP					
TIME	FIXED RATE	AMOUNT	DISC FACTOR	DCF	
3	0.05	0.5	1	0.5	
3.5	0.05	0.5	0.961538462	0.480769231	
4	0.05	0.5	0.924556213	0.462278107	
4.5	0.05	0.5	0.888996359	0.444498179	
5	0.05	10.5	0.854804191	8.975444006	
				10.86298952	
VALUE OF FLOATING LEG			0.09	10.45	
NET VALUE OF SWAP AT T=3				0.412989522	