**Financial Derivatives and Risk Management**
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**Lecture 52**
**Option Greeks: Further Properties, Role in Trading Strategies**

## Delta & Gamma neutral portfolio

Let us first address the issue: Why do we need a delta & gamma neutral portfolio?

$\Delta$ of a portfolio represents the rate of change in its value corresponding to an infinitesimal change in the stock price around a given value of the stock price. It follows, then that that if a portfolio is $\Delta$ neutral at the current stock price, then its value does not change in response to small changes in the stock price in the infinitesimal neighbourhood of the current value of the stock price.

The problem, however, is that this immunity is confined to infinitesimal changes i.e. very small changes in the stock price. In other words, if the jump in the stock price from the current value (at which the portfolio has been made $\Delta$ neutral) is large, this immunity breaks down and the portfolio value does undergo a change.

This is because, the call price curve is curved and hence, the value of $\Delta$ itself changes from point to point along the curve. If a portfolio is $\Delta$ neutral at a given stock price it means that the composition of the portfolio has been made $\Delta$ neutral with respect to the $\Delta$ value at that price. Now, when the stock price makes a jump, we have a new point on the portfolio (call) price curve. At this point we have a different $\Delta$ from the earlier $\Delta$. Thus, the portfolio which was $\Delta$ neutral at the earlier point no longer remains $\Delta$ neutral at this point. Because, the portfolio is, now, not $\Delta$ neutral, it follows that the immunity is lost.

If we want to regain $\Delta$ neutrality, we need to adjust the portfolio $\Delta$ to its new value i.e. its value at the new stock price. This process is called rebalancing. Ideally, rebalancing needs to be done continuously i.e. after every move of the stock price. However, in reality the frequency of rebalancing depends on the rapidity with which delta changes corresponding to change in stock price i.e. the curvature of the portfolio price curve. Greater the curvature, greater the responsiveness of the slope ($\Delta$) to changes in stock price and therefore, higher the need for frequent rebalancing. On the other hand, if the curve is flat, $\Delta$ changes very slowly with stock price and hence, we need rebalancing of the portfolio at infrequent intervals.

The rate of change of $\Delta$ with stock price and hence, the curvature of the price curve is measured by $\Gamma = \frac{\partial \Delta}{\partial S}$. If $\Gamma$ is small, the portfolio price curve is flat, $\Delta$ does not change rapidly and rebalancing need is infrequent and vice versa.

Thus, if we have a $\Delta$ neutral, $\Gamma$ neutral portfolio, it means that $\Delta$ is not changing over a significant range of price movements. Thus, for such portfolios immunization is robust and would be preserved over significant stock price moves. The portfolio value will remain protected against relatively large stock price moves. $\Delta$ neutrality of the portfolio would be preserved over a relatively larger movement of the stock price and we need to rebalance it at a lower frequency.

We, now, discuss the construction of a Δ neutral & Γ neutral portfolio. At the outset, we note that the Δ and Γ of stock are necessarily +1 and 0 respectively. Keeping this in mind we,

(i) first create a gamma neutral portfolio using the options available without bothering about the delta.
(ii) then, neutralize the delta of the portfolio to zero by an appropriate position in stock. This will not disturb the gamma since the stock has zero gamma.

**<u>Example</u>**

A bank's position in options on a stock has a delta of 300 and a gamma of 140. An option X that is being traded has a delta of 0.60 and a gamma of 0.70. How to make the portfolio delta and gamma neutral?

We first make the portfolio gamma neutral by shorting 200 units of option X.
The delta of our portfolio is, now, 300-0.60*200=180
Now, it will convert delta to zero, by shorting 180 units of the stock.
Now, we have both delta and gamma neutrality.

**<u>Positive gamma & the investor</u>**

To understand this phenomena, we proceed along the following:

(i) Construct a riskfree portfolio Π at t=0, $S=S_0$ consisting of:
  (a) one unit of the derivative
  (b) $-\Delta_0 = -\Delta\big|_{S=S_0} = -\left.\dfrac{\partial C}{\partial S}\right|_{S=S_0}$ units of the stock.

  The value of this portfolio is $\Pi_0 = C_0 - \left.\dfrac{\partial C}{\partial S}\right|_0 S_0 = C_0 - \Delta_0 S_0$ and the portfolio $\Delta_{p0}=0$.
(ii) Let the stock price make a movement dS at this point to $S_1$. The change in the value of portfolio Π due to this stock price movement is $d\Pi_0 = dC_0 - \Delta_0 dS$ .
(iii) The portfolio Δ value will also change due to the change in the stock price. Let the new $\Delta_p$ be:

$$\Delta_{p1} = \Delta_{p0} + d\Delta = 0 + \left.\frac{\partial \Delta_p}{\partial S}\right|_0 dS = \Gamma_{p0} dS$$

(iv) To retain Δ neutrality of our portfolio Π at the new stock price $S_1$, we need to ***short*** a further $\Delta_{p1} = \Gamma_{p0}dS$ units of the stock. This shorting will be done at the current price of $S_1=S_0+dS$.
(v) If $\Gamma_0 > 0$, and if the stock price increases i.e. dS>0, this means increasing the short content of stock in Π i.e. selling stock. Thus, the net effect in that case would be that the investor would be selling stock after a price rise, thereby making a profit.
(vi) Similarly, if the stock price falls, i.e. dS<0, this means decreasing the short content of stock in Π i.e. buying stock. Thus, the net effect in that case would be that the investor would be buying stock after a price fall, thereby again, making a profit.

(vii) Let the stock price, now, make another movement -dS' at this point to $S_2=S_1$-dS'. The change in the value of portfolio $\Pi$ due to this stock price movement is $d\Pi_1 = dC_1 + \Delta_1 dS'$.

(viii) The portfolio $\Delta$ value will also change due to the change in the stock price. Let the new $\Delta$ be:

$$\Delta_{p2} = \Delta_1 + d\Delta' = 0 - \Gamma_{p0} dS' = -\Gamma_{p0} dS'$$

(ix) To retain $\Delta$ neutrality of our portfolio $\Pi$ at the new stock price $S_2$, we need to long $\Delta_{p2}$ $=\Gamma_{p0}$dS' units of the stock. This buying will be done at the current price of $S_2=S_1$-dS'.

(x) Thus, in totality we have shorted $\Delta_{p1} =\Gamma_{p0}$dS units of the stock at $S_1$ and bought $\Gamma_{p0}$dS' units of the stock at $S_2$. If dS=$S_1$-$S_0$>-dS'=$S_1$-$S_2$>0 then clearly $S_1>S_2>S_0$. The net sale proceeds are $\Gamma_{p0}$dS'$^*S_1$- $\Gamma_{p0}$dS'*$S_2$ = $\Gamma_{p0}$ dS'$^*$($S_1$-$S_2$). Thus, if $\Gamma_{p0}>0$, this amount is clearly positive because dS, -dS'>0.

**<u>Example</u>**

A bank's position in options on a stock has a delta of 0 (delta neutral) and a gamma of 800. The stock price is 90. After a short period of time, the price moves to 93. Estimate the new delta. What additional trade is necessary to keep the position delta neutral? The stock price again makes a move down to 91. The bank again enters into a trade to make the portfolio delta neutral. Has the bank gained or lost money from the stock price movements?

**<u>Solution</u>**

After the stock price goes up to 93, the portfolio will lose its delta neutrality. Its new delta will be: New Delta= Old Delta+Gamma$^*$Change in stock price: 0+800*3=2,400.

Hence, I short further 2,400 stock to maintain delta neutrality. Therefore, I have shorted 2,400 units of stock @ 93.

Now, stock price changes to 91. New delta is: 0-800*2=-1,600.

Hence, to maintain delta neutrality I will need to buy 1,600 stock @ 91.

Thus, my net profit booked: 1,600*(93-91)=3,200.

We are assuming that $\Gamma$ has not changed because of the small price movements i.e. it remains at 800. In actual fact, it would change marginally.

***Thus, an investor makes a profit from the above strategy of maintaining $\Delta$ neutrality (provided he holds a positive $\Gamma$ position) if the stock price fluctuates.***

**<u>Gamma positive strategies</u>**

(i) Long Call
(ii) Long Put
(iii) Long Straddle
(iv) Long Strangle
(v) Long Calendar Spread
(vi) Vertical Bull Call Spread

(vii) Vertical Bear Put Spread

**Gamma negative strategies**

(i) Short Call, Short Put
(ii) Short Straddle, Short Strangle
(iii) Vertical Credit Spread (Vertical Bear Call Spread, Vertical Bull Put Spread)
(iv) Covered Call Write, Covered Put Write
(v) Iron Condor
(vi) Butterfly

**Summary**

(i) Gamma measures the rate of change for delta with respect to the underlying asset's price.
(ii) All long options have positive gamma and all short options have negative gamma.
(iii) The gamma of a position tells us how much a $1.00 move in the underlying will change an option's delta.
(iv) In BSOPM, put and calls have the same gamma.
(v) The gamma is maximum for ATM calls and approaches 0 as the stock price decreases or increases. The gamma of both OTM and ITM calls approaches zero.
(vi) As expiration approaches, ATM call gamma increases very rapidly while gamma of OTM and ITM both approaches zero. If volatility increases,
(vii) When volatility is low, the gamma of At-The-Money options is high while the gamma for deeply into or out-of-the-money options approaches 0.
(viii) When volatility is high, gamma tends to be stable across all strike prices.
(ix) If gamma of a position is small, delta neutrality is robust and frequent rebalancing is not required.
(x) Positive gamma can be used to book profits.

**Theta**

Now, we come to Θ. Θ captures the impact of passage of time on the option price. The sensitivity of the derivative price to the passage of time is measured by Θ. Now, Θ can be measured with reference to the time elapsed since option inception or the time remaining till maturity. Usually the former is adopted. Thus, Θ of an option is the rate of change of its value with respect to the passage of time with all else remaining the same. Theta is sometimes referred to as the time decay of the portfolio.

The Θ in the BSOPM is given by: $\Theta_c = \frac{\partial c}{\partial t}$; $\Theta_p = \frac{\partial p}{\partial t}$; $\Theta_c = -\frac{S\mathbf{N}'(d_1)\sigma}{2\sqrt{(T-t)}} - rKe^{-r(T-t)}\mathbf{N}(d_2)$.

Θ is the partial derivative of the option value with respect to time expired. t is the time expired, T is the time to maturity and therefore T - t is the time to expiration.

Theta is usually negative for a long option. This is because, as time passes with all else remaining the same, the option tends to become less valuable. Generally expressed as a negative number, the theta of an option reflects the amount by which the option's value will decrease every day. Theta is measured in terms is $/year or any other money units per time unit.

Options usually derive their value from two sources viz the current moneyness and the moneyness that may arise at option maturity. The former is called intrinsic value and the latter the time value. In other words, time value is indicative of the chances of the option finishing in the money at maturity. It arises from the volatility of the stock price whereby there is a finite probability that options that are currently OTM can end up ATM/ITM due to favourable stock price movements. Options that are deep OTM have little chance of ending up ATM/ITM so that they have little time value. Similarly, deep ITM options are also likely to remain ITM and hence, have little time value. But the potential moneyness/payoff at maturity of ATM options fluctuates rapidly with the price change of the underlying, making ATM options carry high time value. In essence, time value of options arises due to the uncertainties of its payoff at maturity. Since current stock price and volatility encapsulate the projected behaviour of stocks, it is these attributes that determine the time value of options.

Further, it must necessarily be that the time value at option maturity be zero since there is no uncertainty about its payoff at that point. Hence, all the time value that subsists at a given point must be necessarily amortized over its remaining life. Equivalently, it may be said that as maturity approaches, the variation of stock prices tends to decrease so that the probability of the stock price lying in a prescribed interval increases i.e. stock prices become more predictable if prediction is over small period of time (Recall that BM that represents randomness in the lognormal model has variance directly proportional to time). Thus, time value falls as maturity approaches as uncertainty over moneyness at maturity reduces.

The time value of the option erodes with the passage of time since time value arises due to uncertainty in the stock price, as explained above. The value of the option is, therefore, likely to decrease with time elapsed. Therefore Θ is negative for a long option position and positive for a short option. The passage of time hurts the option holder as his holding becomes less valuable. The passage of time benefits the option writer.

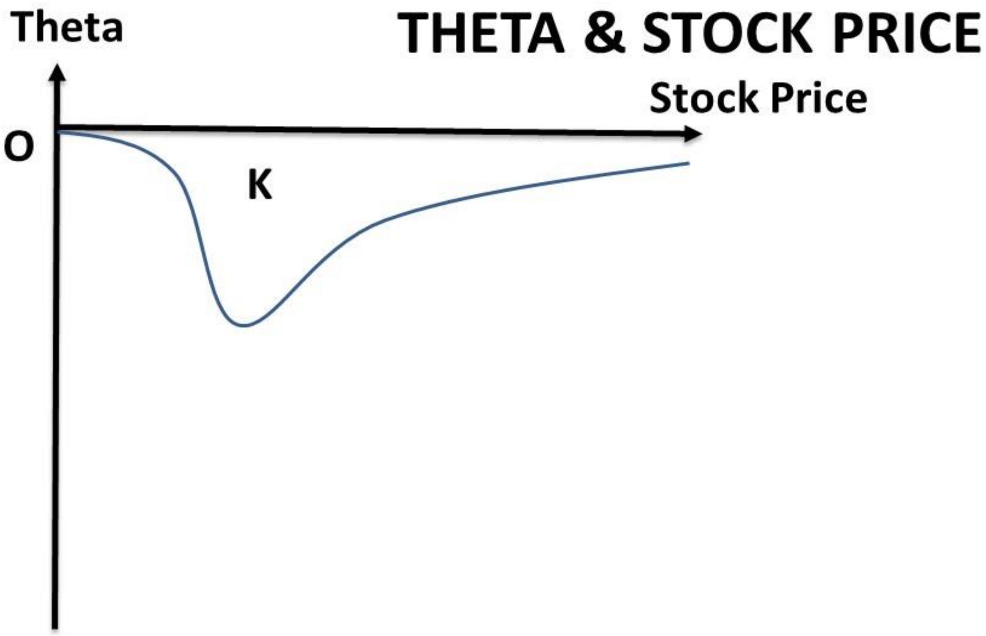
**Black Scholes Theta**

$$\Theta = \frac{\partial c}{\partial t} = S\mathbf{N}'(d_1)\frac{\partial d_1}{\partial t} - Ke^{-r(T-t)}\mathbf{N}'(d_2)\frac{\partial d_2}{\partial t} - rKe^{-r(T-t)}\mathbf{N}(d_2)$$

$$= -rKe^{-r(T-t)}\mathbf{N}(d_2) + S\mathbf{N}'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right) \text{since } S\mathbf{N}'(d_1) = Ke^{-r(T-t)}\mathbf{N}'(d_2)$$

$$= -rKe^{-r(T-t)}\mathbf{N}(d_2) + S\mathbf{N}'(d_1)\frac{\partial}{\partial t}\left(\sigma\sqrt{T-t}\right) = -rKe^{-r(T-t)}\mathbf{N}(d_2) - S\frac{\sigma}{2\sqrt{T-t}}\mathbf{N}'(d_1)$$

**Theta & stock price**




Clearly, from the above analytical expression for Θ viz $\Theta = -rKe^{-r(T-t)}\mathbf{N}(d_2) - S\dfrac{\sigma}{2\sqrt{T-t}}\mathbf{N}'(d_1)$, as S→0, ln (S/K)→-∞ so that $d_{1,2}$→-∞ whence both the terms and hence, Θ→0. This is natural, as S→0 make the call options deep OTM. Hence, they have insignificant time value. It, therefore, follows that the rate of erosion of time value (captured by Θ) is also negligible. Obviously, if the option is completely out of the money and there is very little chance of the option bouncing back into at the money then the option price is not going to be affected by passage of time, it is going to remain close to 0 and therefore in that scenario the value of Theta would also be close to 0.

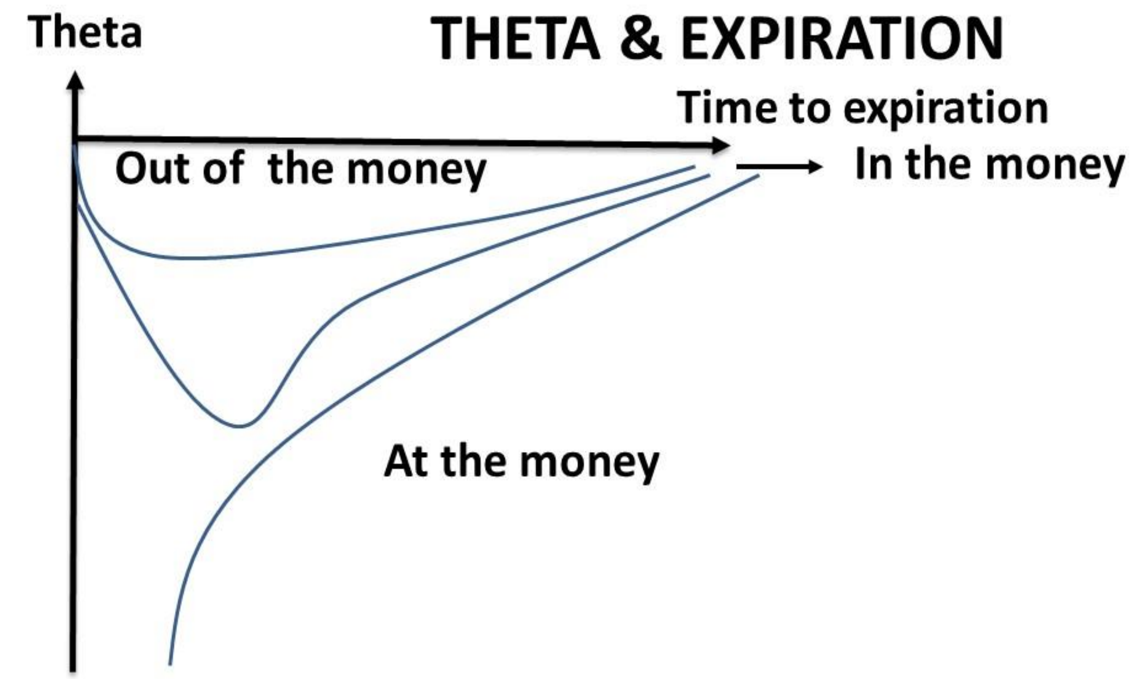
However, as S→∞, ln (S/K)→∞ so that $d_{1,2}$→∞ whence **N**($d_2$) →1. Also

$$S \to \infty \Rightarrow d_1 = \frac{\ln\frac{S}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \to \ln S \Rightarrow e^{\frac{1}{2}d_1^2} \to e^{(\ln S)^2} \to \left(e^{\ln S}\right)^{\ln S} = S^{\ln S}$$

$S\mathbf{N}'(d_1) = S\dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2} = S\dfrac{1}{\sqrt{2\pi}}\dfrac{1}{e^{\frac{1}{2}d_1^2}} \to S^{1-\ln S} \to 0$, Thus, when S→∞, Θ approaches the limiting value $\Theta = -rKe^{-r(T-t)}$. Thus, for deep ITM calls, Θ asymptotically approaches a small limiting finite negative value. This represents the erosion of a finite positive time value over the remaining maturity at a rate given by the riskfree interest on the present value exercise price. The limit S→∞ essentially epitomizes deep ITM calls. In their case, the maturity payoff is pretty much known with certainty (since they are very likely to be exercised and the stock price is also likely to be in a very high band). Hence, the only issue in such case relates to the time value in relation to exercise price. Since call exercise is a certainty, payment of exercise price is also certain. However, the option holder needs to borrow the present value of the exercise price for making the payment of the exercise price at maturity. As maturity approaches, this present value increases whence he has to borrow a higher amount entailing a higher interest cost per unit time. In other words, the interest cost of the option holder per unit time associated with the borrowing of present value of exercise price till maturity increases as option approaches maturity because the present value of exercise price increases. Due to no arbitrage requirements, the value of the option reduces by the same amount.

For an ATM call, $\Theta$ is large and negative. It is more negative than the limiting asymptotic value at $S \rightarrow \infty$. The minima of $\Theta$ is close to its exercise price i.e. for ATM calls.

**Theta & expiration**




Initially, OTM options have a faster rate of theta decay than ATM, but as expiration nears, the rate of theta option time decay for OTM options slows and the ATM options begin to experience theta decay at a faster rate.

Initially, OTM calls have a faster rate of theta decay than at the money options, but as expiration nears, the rate of theta option time decay for OTM options slows and the ATM options begin to experience theta decay at a faster rate. This is a function of theta being a much smaller component of an OTM option's price, the closer the option is to expiring.

Recall that short dated ATM calls tend to have high positive $\Gamma$. Similarly, as ATM calls approach maturity $\Theta$ decreases rapidly (Recall that $\Theta$ is negative for long calls). Thus, $\Theta$ increases rapidly in magnitude as ATM calls approach maturity. Now, ATM calls have significant time values. At maturity that time value has to be zero. Therefore, this significant time value must amortize over the remaining term to expiration. Hence, $\Theta$ have large negative value. Further, as expiration approaches, this rate of amortization of time value and, therefore, $\Theta$ also increases.

**Theta based strategies**

(i) Option sellers use theta to their advantage, collecting time decay every day.

(ii) Calendar spreads involve buying a longer-dated option (negative theta) and selling a nearer-dated option (positive theta), thereby creating a net positive $\Theta$ that operates to their advantage.

(iii) Selling options with close expiration will give higher positive theta per day and buying longer expiration calls will give lower negative theta, yielding positive net time decay per day. This holds if the underlying does not move.

(iv) However, selling short dated calls will also create larger negative gamma compared to buying longer dated calls. Thus, the spread will have significant negative gamma. That means that a sharp move of the underlying will cause much higher loss. However, if the underlying doesn't move, then theta will kick off and the strategy will just earn money with every passing day.

The $\Theta$ of short calls is positive so that as these calls approach maturity on a day-to-day basis, the short position is increasing in value. Thus, option sellers benefit by a large $\Theta$.

A long calendar spread usually consists of a short near dated call and a long far dated call with similar strike prices. Now, the near dated call, being close to expiration will have a large $\Theta$ and it will be positive because it is a short call. On the other hand, the long far dated call will have a smaller negative $\Theta$. The net effect of the spread is therefore to create a positive $\Theta$ strategy whence its value will increase (other things remaining constant) with every passing day. Other things remaining constant is very important, if stock price changes then it may very well happen that the changes in the portfolio value due to the change in stock price could overshadow the positive value growth due to positive net $\Theta$ and passage of time.

## <u>Gamma vs theta</u>

If one buys short-dated ATM calls, they will create a large positive $\Gamma$. However, long short-dated ATM calls also have negative $\Theta$ implying that their value will erode with the passage of time as maturity approaches. Thus, if a significant and quick stock price move occurs, such options will gain on account of the positive $\Gamma$ more than the loss in their value due to negative theta. But if the stock doesn't move, the negative theta will start to kick off much faster while the positive $\Gamma$ effect will be minimal.

On shorting short-dated ATM calls, negative $\Gamma$ portfolios are created. Further $\Gamma$ will increase significantly in magnitude as the options approach expiration. Thus, short call positions with close expiration will give higher positive theta per day but higher negative gamma. That means that a sharp move of the underlying will cause much higher loss because of negative gamma. But if the stock doesn't move, then theta will kick off and the portfolio will just earn money with every passing day. But if it does move, the loss will become very large very quickly.

Theta is often called a "silent killer" of option buyers. Buyers, by definition, have only limited risk in their strategies together with the potential for unlimited gains. While this might look good on paper, in practice it often turns out to be death by a thousand cuts. In other words, it is true you can only lose what you pay for an option. It is also true that there is no limit to how many times you can lose (by taking up long positions and paying premia again & again). And as any lottery player knows well, a little money spent each week can add up after not hitting the jackpot for a long time. For option buyers, therefore, the pain of slowly eroding your trading capital sours the experience.

When buying options, you can reduce the risk of negative theta by buying options with longer expiration. The trade-off is smaller positive gamma, which means that the gains will be smaller if the stock moves.