Financial Derivatives & Risk Management
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Lecture 51
Option Greeks: Further Properties

Black Scholes Delta

$$\Delta = \frac{\partial c}{\partial S} = \mathbf{N}(d_1) + S\mathbf{N}'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}\mathbf{N}'(d_2)\frac{\partial d_2}{\partial S}$$

$$= \mathbf{N}(d_1) + \left[S\mathbf{N}'(d_1) - Ke^{-r(T-t)}\mathbf{N}'(d_2)\right]\frac{\partial d_1}{\partial S_t} \text{ because } d_2 = d_1 - \sigma\sqrt{(T-t)}$$

$$\text{so that } \frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S};\ \Delta = \mathbf{N}(d_1) + \frac{1}{\sqrt{2\pi}}Se^{-\frac{1}{2}d_1^2}\left[1 - \frac{K}{S}e^{-r(T-t)+\frac{1}{2}\left(d_1^2-d_2^2\right)}\right]\frac{\partial d_1}{\partial S}$$

$$\text{From above}: \Delta = \mathbf{N}(d_1) + \frac{1}{\sqrt{2\pi}}Se^{-\frac{1}{2}d_1^2}\left[1 - \frac{K}{S}e^{-r(T-t)+\frac{1}{2}\left(d_1^2-d_2^2\right)}\right]\frac{\partial d_1}{\partial S}$$

$$= \mathbf{N}(d_1) + \frac{1}{\sqrt{2\pi}}Se^{-\frac{1}{2}d_1^2}\left[1 - \frac{K}{S}e^{-r(T-t)+\frac{1}{2}\left(2\sigma\sqrt{T-t}d_1-\sigma^2(T-t)\right)}\right]\frac{\partial d_1}{\partial S_t}$$

$$= \mathbf{N}(d_1) + \frac{1}{\sqrt{2\pi}}Se^{-\frac{1}{2}d_1^2}\left[1 - \frac{K}{S}e^{-r(T-t)-\frac{1}{2}\sigma^2(T-t)+\ln\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}\right]\frac{\partial d_1}{\partial S_t}$$

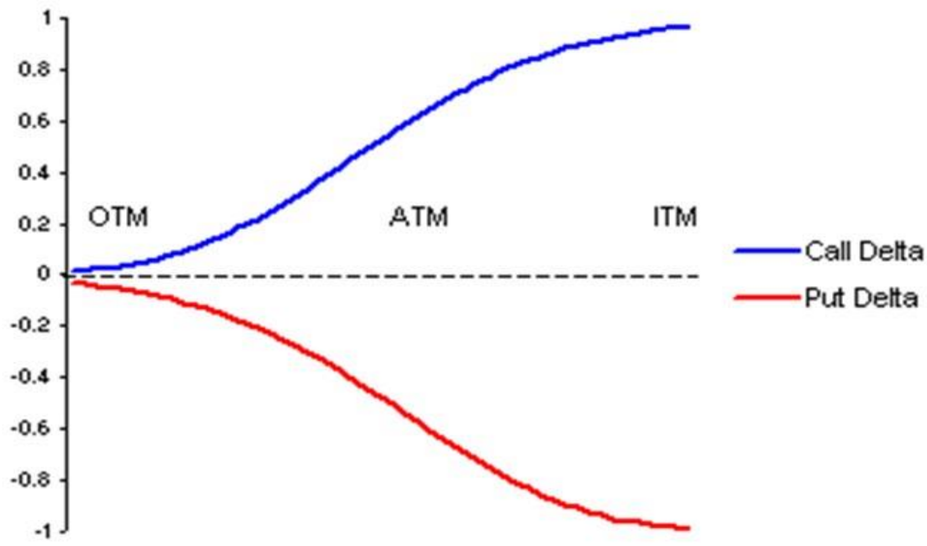
$$= \mathbf{N}(d_1) + \frac{1}{\sqrt{2\pi}}Se^{-\frac{1}{2}d_1^2}\left[1 - \frac{K}{S}e^{\ln\left(\frac{S}{K}\right)}\right]\frac{\partial d_1^t}{\partial S_t} = \mathbf{N}(d_1)$$

Delta & stock price

As you can see from the above diagram, the change in price of the call option has a certain curvature. The slope of the curve changes from point-to-point along the curve. The slope at a given point gives the value of Δ at that point. In fact, the curvature is captured by the second derivative Γ.

In the Black Scholes framework Δ=**N**(d_1) so that: $\Delta = \mathbf{N}(d_1);\ d_1 = \dfrac{\mathbf{ln}\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$.

But $\mathbf{ln}\left(\frac{S}{K}\right)$ is a monotonic one-one increasing function of S. d_1 is a monotonic increasing function of $\mathbf{ln}\left(\frac{S}{K}\right)$. **N**($d_1$) is a monotonic increasing function of d_1. It follows that Δ=**N**(d_1) is a monotonic increasing function of S i.e. the slope of the call price curve increases with increasing S so that $\Gamma = \frac{\partial \Delta}{\partial S} > 0$.



The Δ of OTM calls is close to 0 and that of ITM calls close to 1. It is immediate that $\Delta_{call}=\mathbf{N}(d_1) \in (0,1)$. Δ of OTM puts is close to 0 and that of ITM puts close to -1. $\Delta_{put}=\mathbf{N}(d_1)-1\in(-1,0)$.

As a general rule, of course, ITM options we will move more than OTM options because their Δ magnitude is higher indicating higher sensitivity to the price of the underlying.

(i) More a call is in the money, the greater is the probability of realizing its payoff.

(ii) Also, the payoff of an ITM call at maturity is S_T-K. This payoff is changes @ one unit for every unit change in stock price.

Thus, a deep ITM call literally l mimics the stock. Therefore, any change in stock price is mirrored by an equivalent change in the call price and we have Δ=1.

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Conversely, if the call is well out of the money, then the probability of it being exercised at maturity becomes very small and therefore the realizability of the payoff is very small. Thus, small changes in the stock price so not significantly affect the realizability of the payoff even though the projected amount of payoff may change. The net result is that the call price does not get impacted by small changes in the stock price. Thus, for OTM options $\Delta\to 0$.

<u>Delta & expiration</u>

We have:

$$\Delta = \mathbf{N}(d_1);\ d_1 = \frac{\mathbf{ln}\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad \left(T \text{ is time to } \mathbf{exp}\text{iration}\right)$$

$$\backslash \frac{\partial d_1}{\partial T} = -\frac{1}{2\sigma T^{3/2}}\mathbf{ln}\left(\frac{S}{K}\right)+\frac{1}{2\sigma\sqrt{T}}\left(r+\frac{1}{2}\sigma^2\right) = -\frac{1}{2\sigma\sqrt{T}}\left[\frac{1}{T}\mathbf{ln}\left(\frac{S}{K}\right)-\left(r+\frac{1}{2}\sigma^2\right)\right]$$

$$\frac{\partial \Delta}{\partial T} = -\mathbf{N}'(d_1)\frac{1}{2\sigma\sqrt{T}}\left[\frac{1}{T}\mathbf{ln}\left(\frac{S}{K}\right)-\left(r+\frac{1}{2}\sigma^2\right)\right]$$

$$Let\ T^*\ be\ time\ \mathbf{exp}ired,\ then\ \frac{\partial \Delta}{\partial T^*} = -\frac{\partial \Delta}{\partial T};\ \Phi'(d_1)\ being\ a\ pdf\ is\ always \geq 0$$

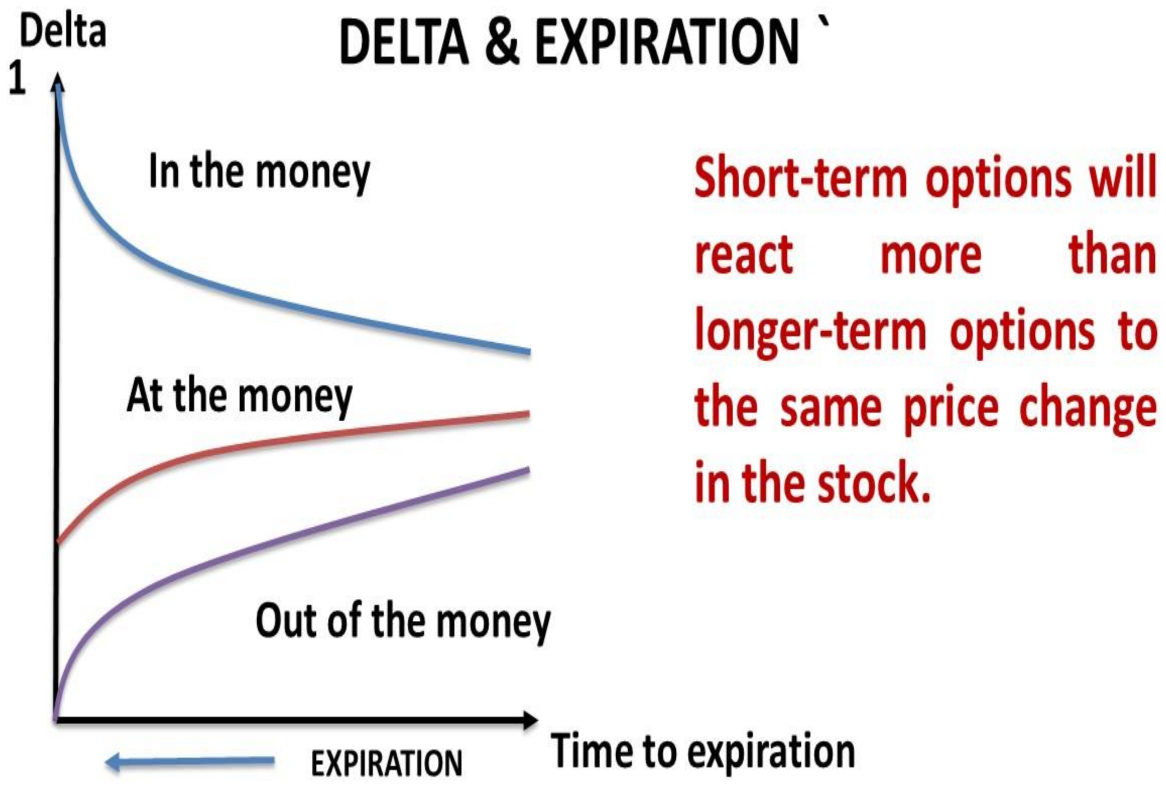
$$If\ S \to \infty,\ \frac{\partial \Delta}{\partial T^*} \to \infty\ i.e.\ for\ ITM\ calls\ as\ time\ \mathbf{exp}ired\ increases,\ \Delta\ increases\ rapidly.$$

$$If\ S \to 0,\ \frac{\partial \Delta}{\partial T^*} \to -\infty\ i.e.\ for\ OTM\ calls\ as\ time\ \mathbf{exp}ired\ increases,\ \Delta\ decreases\ rapidly.$$

We find that:

(i) if S $\to\infty$ i.e. the call is deep ITM, then Δ increases rapidly to its limiting value of unity as the option approaches maturity.

(ii) if S $\to 0$ i.e. the call is deep OTM, then Δ decreases rapidly to its limiting value of zero as the option approaches maturity.



The above relationship between Δ and expiration is also depicted in the diagram. Please note that because time to expiration decreases as time passes i.e. we are moving closer to expiration with passage of time, we need to interpret this diagram while moving towards the left-hand side i.e. towards the origin. The origin represents the expiration.

As seen in the diagram, when there is little time left to expiration, Δ moves very rapidly towards 1 for ITM calls and 0 for OTM calls. For ATM calls, it tends to decrease at an increasing rate to some intermediate value. For ATM calls, we have $\ln\left(\frac{S}{K}\right)=\ln(1)=0$ so that $d_1=\frac{\left(r-q+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}=\frac{\left(r-q+\frac{1}{2}\sigma^2\right)}{\sigma}\sqrt{T}\propto\sqrt{T}$. Thus, as maturity approaches T and hence, d_1 and so $\Delta=\mathbf{N}(d_1)$ also decreases.

Short-maturity options react very significantly to changes in stock price compared to long maturity options. Long-maturity options are much less sensitive to changes in stock price or time to expiration compared to those that are very close to maturity. The underlying logic is that the time value of the option gets eroded very fast as expiration approaches. Therefore, the option prices become more sensitive towards expiration.

Summary

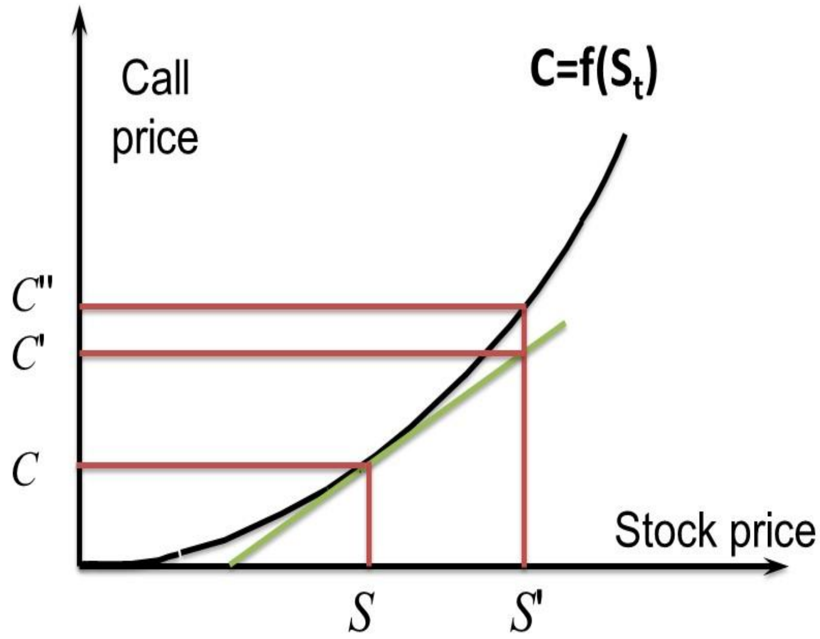
(i) Positions with positive delta increase in value if the underlying goes up.
(ii) Positions with negative delta increase in value if the underlying goes down.
(iii) $dc=\Delta.dS$ to first order. If Δ is positive, then positive dS will cause increase in call value and vice versa.
(iv) Call delta increases with increase in stock price from 0 for deep OTM calls to 1 for deep ITM calls.
(v) As expiration approaches, ITM call delta approaches 1 with increasing rapidity, OTM call delta approaches 0 with increasing rapidity. This is because time value erodes quickly as you approach maturity.
(vi) Delta hedging provides immunity against price changes, but only in an infinitesimal region. For perfect hedging continuous rebalancing of portfolio is required. This is because delta value changes with every move of the stock price.
(vii) However, if gamma is small, then the delta hedged portfolio is robust and rebalancing may be done at infrequent intervals.

Gamma

Gamma (Γ) is the first derivative of Δ with respect to the stock price. Equivalently, it is the second derivative of the call option with respect to the stock price S, $\Gamma=\frac{\partial\Delta}{\partial S}=\frac{\partial^2 c}{\partial S^2}$. So, it measures the rate of change of the slope of the call price curve i.e. the curvature of the call price curve. If Γ is small, the call price curve has small curvature, is relatively flat, and if Γ is large the curvature is significant. In this case, Δ changes very rapidly from point to point along the curve. Like the delta, the gamma is constantly changing along the call price curve, even with tiny movements of the underlying stock price.

Gamma & hedging error

GAMMA & CURVATURE **HEDGING ERROR = C”-C’**



Γ is a measure of the hedging error. Let us assume that at an arbitrary t, the stock price is at S and the corresponding call price is at C. At this point, let the stock price jump to a new value, say S' whence the call price moves to C" along the call price curve C=f(S). At t, before the jump in the stock price from S to S', suppose an investor creates a portfolio consisting of:

(i) one unit of the call C;
(ii) -Δ=-(C'-C)/(S'-S) units of the stock.

Then, after the stock price move to S', the value of the portfolio becomes:

Clearly the change in the call price is δC=C"-C per unit
The change in the stock price is δS=S'-S per unit
Thus, change in value of -Δ=-(C'-C)/(S'-S) units of stock = -(C'-C)

Now, if the call price curve had no curvature, this portfolio would have been perfectly hedged, since the point C" would have coincided with C' and the portfolio value would not have changed at all. The existence of curvature of the call price curve introduces a hedging error since the Δ value changes from point to point.

Therefore, change in value of the portfolio after the jump = Hedging error due to the curvature of the call price curve = (C"-C)-(C'-C) = C"-C'.

Now, $C''-C'=(C''-C)-(C'-C)=\dfrac{\dfrac{(C''-C)}{(S'-S)}-\dfrac{(C'-C)}{(S'-S)}}{(S'-S)}\times(S'-S)$

$$=\frac{\textit{Change in }\Delta}{\textit{Change in stock price}}\times(S'-S)=\Gamma\times(S'-S)$$

Thus, Γ is a measure of the hedging error introduced in delta hedging due to curvature of the call price curve.

Expression for Black-Scholes gamma

$$\Gamma=\frac{\partial\Delta}{\partial S}=\mathbf{N}'(d_1)\frac{\partial d_1}{\partial S}=\frac{1}{S\sigma\sqrt{T-t}}\mathbf{N}'(d_1)\text{ since }\Delta=\mathbf{N}(d_1),\ \frac{\partial\Delta}{\partial d_1}=\mathbf{N}'(d_1)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}$$

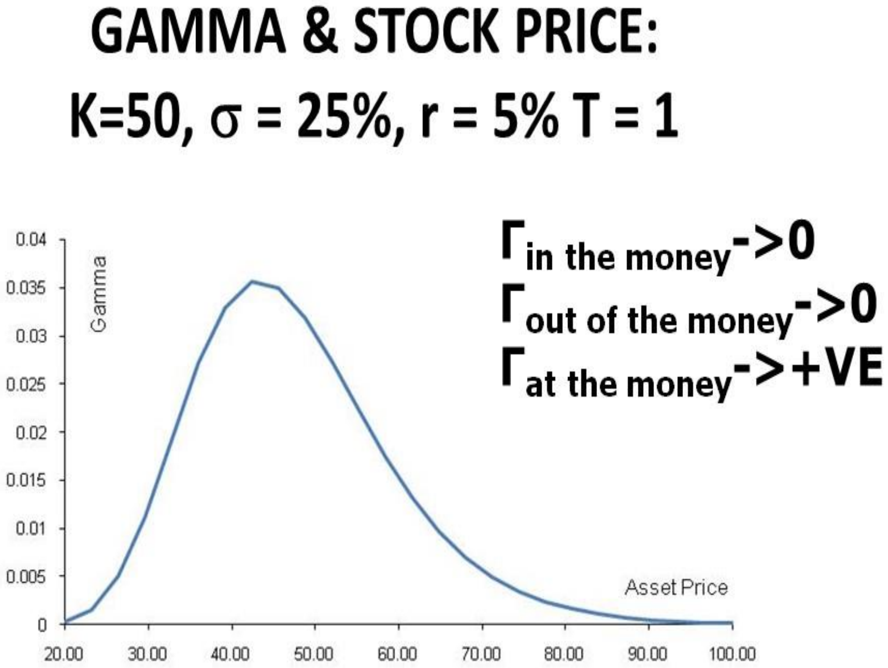
$$d_1=\frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\text{ so that }\frac{\partial d_1}{\partial S}=\frac{1}{\sigma S\sqrt{T-t}}.$$

It is clear from the above expression that Γ→0, as S→+∞ and S→0 and Γ>0 for all intermediate values of S. In other words, Γ→0 for both OTM & ITM calls, remaining positive throughout and reaching a maxima for ATM calls.

Long calls always carry a positive gamma; short ones have a negative gamma.

In the Black–Scholes model, we also have:

$$\Gamma_c = \frac{\partial^2 c}{\partial S^2} = \frac{\partial \Delta_c}{\partial S};\ \Gamma_p = \frac{\partial^2 p}{\partial S^2} = \frac{\partial \Delta_p}{\partial S};\ \Gamma_c = \Gamma_p = \frac{\mathbf{N}'(d_1)}{S\sigma\sqrt{(T-t)}} > 0$$



(i) As calls become further in-the-money, they act increasingly like the stock itself so that delta approaches one and gamma approaches zero.

(ii) For out-of-the-money options, option prices are much less sensitive to changes in the underlying stock so that delta and gamma both approach zero.

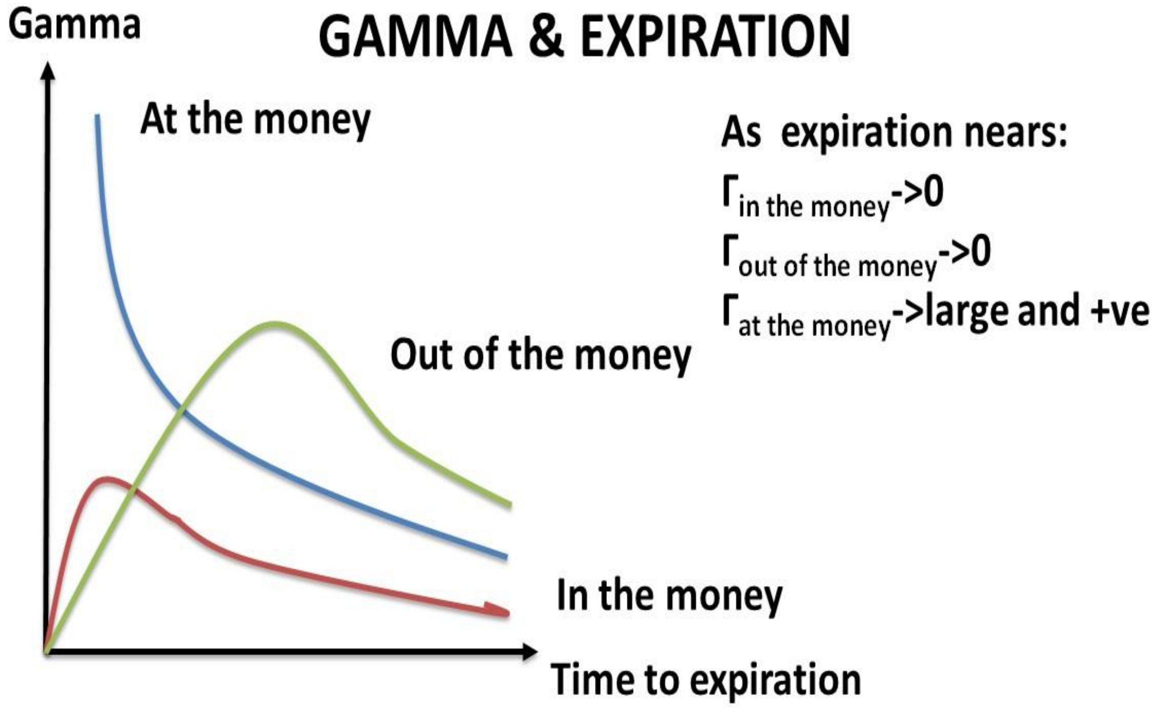
(iii) Thus, gamma generally at its peak value when the stock price is near the strike of the option and decreases as the option goes deeper into or out of the money.

(iv) For a given strike price and expiration, the call gamma equals the put gamma.

As calls go deeper in the money, they become certain to be exercised. Hence, a given change in stock price produces an equivalent change in expected payoff of the call. Δ settles down rigidly infinitesimally close to one and does not change much for a small change in stock price, whence Γ approaches zero.

With calls going deep out of the money a similar situation arises, with the call becoming certain not to be exercised. A small change in stock price does not significantly alter this "certainty" and so the call price does not change, Δ settles near zero. Further, the Δ also does not respond to small changes in stock price so that $\Gamma \to 0$.

<u>Gamma & expiration</u>

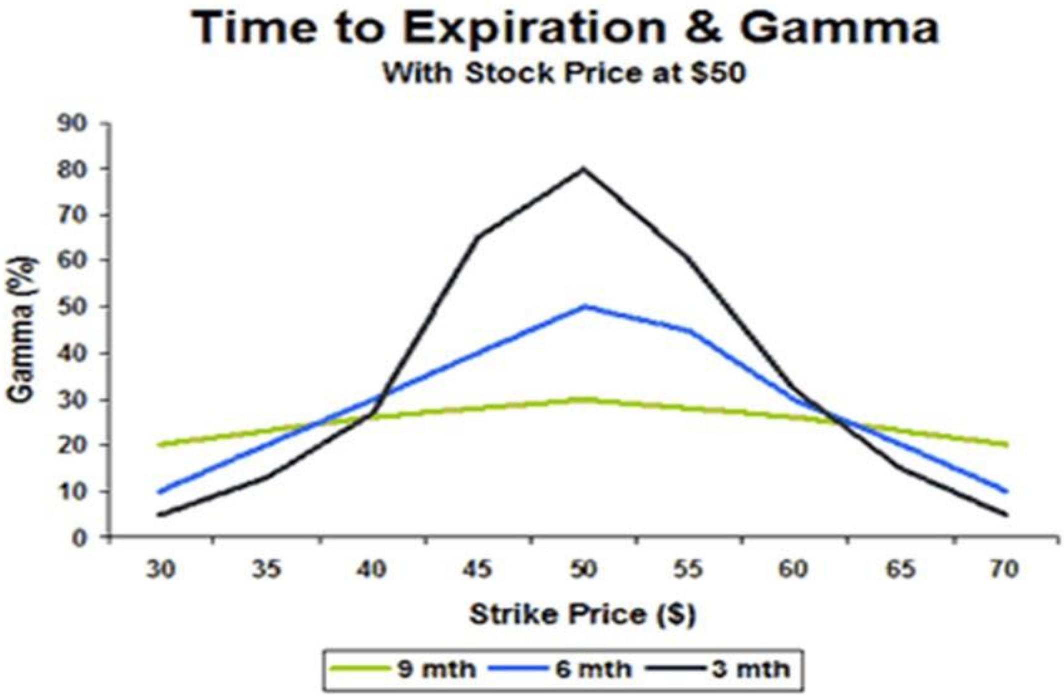


For an at-the-money option, gamma RAPIDLY increases as the time to maturity decreases. Short-life at-the-money options have very high gammas, which means that the value of the option holder's position is highly sensitive to jumps in the stock price.

As the time to expiration draws nearer, the gamma of In-The-Money and Out-of-The-Money options decreases.

As expiration approaches, because the realizability of the payoff becomes more and more certain, Γ also tends to decrease and approaches 0 at expiration. Both OTM & ITM calls' $\Gamma \to 0$ as expiration approaches.

As ATM calls approach expiration, Γ increases very rapidly. ATM calls have very high Γ making their Δ extremely sensitive to the stock price. A large and positive Γ means that an investor takes a long position in short-maturity ATM calls. But if an investor has a portfolio with large positive Γ, it follows that the portfolio Δ changes rapidly due to even small change in stock price. But the important thing is that a positive Γ operates in favour of the investor, as will be shown later.



In the above diagram, we have calls with varying maturities, the yellow line is the 9-month maturity call, the blue line 6-month and the black line 3-month maturity call. The following features are apparent:

(i) As the calls move from ATM towards ITM or OTM, Γ approaches zero. This holds for all the three calls and, is therefore, independent of maturity;

(ii) The maximum Γ of all the calls occurs when they are at the money;

(iii) The near-maturity (black) call shows the largest Γ of the three options in the region when the calls are ATM, although when the calls are ITM or OTM all of them have $\Gamma \rightarrow 0$.

Volatility & gamma

When volatility is low, the gamma of ATM options is high while the gamma for deeply ITM or OTM options approaches 0. The reason is that when volatility is low, the time value of deep ITM & OTM options is low but it goes up dramatically as the underlying stock price approaches the strike price.

When volatility is high, gamma tends to be stable across all strike prices. This is due to the fact that when volatility is high, the time value of deeply ITM/OTM options are already quite substantial. Thus, the increase in the time value of these options as they go nearer the money will be less dramatic and hence the low and stable gamma.

If the volatility is low, then an option which is ITM or OTM is likely to remain so, because the stock price does not fluctuate too much (low volatility). There is little chance of an OTM option bouncing ATM/ITM or vice versa. In other words, the moneyness is more or less crystallized and that being the case, the time value factor is small & not significant. If the time value factor of the option is not significant, Γ remains stable close to zero for ITM/OTM options.

Time value is most for ATM options, so in the case of such options, because this time value erodes rapidly as the option approaches maturity Γ will be large and increasing at the maturity of the option approaches.

However, if the volatility is high i.e. the stock price is showing significant fluctuations, then even deep ITM/OTM options have likelihood of bouncing into ATM or even OTM/ITM respectively. Thus, if volatility is high, Γ tends to be relatively stable across the spectrum of stock prices. If volatility is high, then ITM, ATM & OTM options carry the possibility of changing their moneyness character. The impact of stock price change is more equally distributed. All the money type options have significant time values and this time value must erode in each case to zero at maturity (time value of any option must necessarily be zero at maturity). Thus, Γ of OTM/ATM/ITM on high volatility stocks tends to be high and uniform on approaching maturity across the entire spectrum of moneyness.