Financial Derivatives & Risk Management
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Lecture 49 - Solution of BS PDE; Option Greeks

Solution of the Black Scholes equation

Although several approaches exist for solving the Black Scholes equation, we shall be doing so by transforming the equation to the diffusion equation followed by solving the emanating diffusion equation by constructing the appropriate Green's function for the problem.

(i) **Transformation of variables**

The Black Scholes equation is: $\frac{\partial C}{\partial t}+\frac{1}{2}(\sigma S)^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0$ (1)

subject to the boundary conditions:

$$t=T:C(S,t)=\max\left[S_T-K,0\right],\ S=0:C(0,t)=0,\ S\to\infty:C(S,t)\sim S \quad (2)$$

The boundary conditions are

(i) at t=T, i.e. the maturity of the call (Recall that we are valuing a European call), the value of the call will equal its payoff i.e. max(S_T-K,0);

(ii) if stock price touches 0 at any time during (0,T), then it remains 0 thereafter throughout, including at t=T i.e. S_T=0 with certainty whence, there exists no probability of exercise of the call and the realization of any payoff thereon. Hence, the value of the call at any earlier time after S touches 0, also becomes 0 because it is, now, (i.e. after S touches 0) known with certainty that the call will yield a payoff of 0 on maturity.

(iii) if the stock price rises unboundedly high at any point in (0,T), then it is certainly not going to fall below the exercise price K at t=T, so that the option will definitely be exercised in this case and the payoff of S_T-K will certainly materialize at t=T. Since K is a constant independent of S and t, this payoff is simply the payoff from the holding of a stock displaced by a fixed amount K. Thus, in this case, the call mimics the stock for all practical purposes. Hence, the value of the call at any earlier time in (0,T) after the stock has risen to a sufficiently high value, will be the value of the stock less the present value of K at that point. Further, if S is large enough compared to K such that the present value of K be ignored compared to S, then then the call price will approximate the stock price S, at that point in time.

We make the following transformation of variables:

$$x=\ln\left(\frac{S}{K}\right) \quad (3a)$$

$$t=T-\frac{\tau}{\left(\sigma^2/2\right)} \quad (3b)$$

$$C=Kf(x,\tau) \quad (3c)$$

Transformation (3a) i.e. $x = \ln\left(\frac{S}{K}\right)$ serves three purposes (i) it converts our independent variable S into a dimensionless variable (S/K) because S and K are in the same dimensions, both being in money units; (ii) by introducing the new variable as a logarithm i.e. $x = \ln\left(\frac{S}{K}\right)$, we eliminates the S dependency of the coefficients of the Black Scholes PDE i.e. the drift and diffusion coefficients become independent of S; and (iii) since S is assumed to be log-normally distributed in the model, it immediately follows that ln S and therefore ln(S/K), with K being a constant, is normally distributed. Hence, x is normally distributed and we can work in the environment of the normal distribution.

Transformation (3b) i.e. $t = T - \frac{\tau}{\left(\sigma^2/2\right)}$ converts this terminal value problem to an initial value problem since t=T gives τ=0.

Transformation (3c) i.e. $C = Kf\left(x,\tau\right)$ makes the dependent variable C dimensionless because C and K are in the same dimensions, both being in money units. Therefore, f(x,τ) which is our new dependent variable is dimensionless.

Making these substitutions yields:

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial x^2} + \left(k - 1\right)\frac{\partial f}{\partial x} - kf \qquad (4)$$

With the boundary conditions:

$$\tau = 0: f\left(x,0\right) = \max\left(e^x - 1,0\right),\ x \to -\infty: f\left(x,\tau\right) \to 0,\ x \to +\infty: f\left(x,\tau\right) \sim e^x \qquad (5)$$

<u>Transformation to the diffusion equation</u>

A diffusion equation has the form: $\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2}$ (6)

To convert eq (4) to the form of eq (6), we make a second set of substitutions given by the following:

$$f\left(x,\tau\right) = e^{ax+b\tau} g\left(x,\tau\right) \qquad (7)$$

where a,b are free real numbers to be determined in such manner as to convert eq (4) with the substitution of eq (7) into the form of eq (6).

When we substitute eq (7) into eq (4), we get:

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2} + \left[2a + \left(k - 1\right)\right]\frac{\partial g}{\partial x} + \left[a^2 + \left(k - 1\right)a - k - b\right]g \qquad (8)$$

In order that eq (8) takes the form of eq (6), a, b need to be so chosen that the coefficients of $\frac{\partial g}{\partial x}$ and g, both vanish. Thus, we need to set:

$$a = -\frac{1}{2}(k-1) \text{ and } b = a^2 + (k-1)a - k = -\frac{1}{4}(k+1)^2 \qquad (9)$$

On setting these values, we get $\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2}$. (10)

The boundary conditions on g take the form:

$$\tau = 0: g(x,0) = \max\left[e^{(k+1)\frac{x}{2}} - e^{(k-1)\frac{x}{2}}, 0\right] \Rightarrow g(x,0)e^{-\alpha x^2} \xrightarrow{|x|\to\infty} 0(\alpha > 0)$$

$$\tau > 0: g(x,\tau) \xrightarrow{|x|\to\infty} e^{(k+1)\frac{x}{2} - (k+1)^2\frac{\tau}{4}} \Rightarrow g(x,\tau)e^{-\alpha x^2} \xrightarrow{|x|\to\infty} 0(\alpha > 0) \qquad (11)$$

where α is any real positive constant.

Solution of the diffusion equation

We need to solve eq (10) subject to the boundary conditions (11).

From the substitution $t = T - \frac{\tau}{(\sigma^2/2)}$, it is clear that $\tau = t = \frac{\sigma^2}{2}(T-t)$ cannot be negative. In other words, $g(x,\tau)$, is defined only for positive τ. Therefore, in order to extend this definition of this function over the entire real line i.e. $-\infty < \tau < +\infty$, we invoke the Heaviside step function defined by $\Theta(\tau) = \begin{cases} 0 & for\ \tau < 0 \\ 1 & for\ \tau \geq 0 \end{cases}$ and write $\bar{g}(x,\tau) := \Theta(\tau) g(x,\tau)$ whence $\bar{g}(x,\tau)$ is clearly defined over the entire real line. Now,

$$\left(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2}\right)\bar{g}(x,\tau) = \left(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2}\right)\Theta(\tau) g(x,\tau)$$

$$= \delta(\tau) g(x,\tau) + \Theta(\tau)\frac{\partial g(x,\tau)}{\partial \tau} - \Theta(\tau)\frac{\partial^2 g(x,\tau)}{\partial x^2} \text{ since } \frac{\partial \Theta(\tau)}{\partial \tau} = \delta(\tau)$$

$$= \delta(\tau) g(x,\tau) + \Theta(\tau)\left[\frac{\partial g(x,\tau)}{\partial \tau} - \frac{\partial^2 g(x,\tau)}{\partial x^2}\right] = g(x,\tau)\delta(\tau) = \bar{g}(x,0)\delta(\tau)$$

where the last step follows from $\bar{g}(x,0)\delta(\tau) = \Theta(0) g(x,0)\delta(\tau)$ $= g(x,0)\delta(\tau) = g(x,\tau)\delta(\tau)$. Thus, we have

$$\left(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2}\right)\bar{g}(x,\tau) = \bar{g}(x,0)\delta(\tau) \qquad (12)$$

Eq (12) has the solution

$$\bar{g}(x,\tau) = \int_{-\infty}^{+\infty} dy \bar{g}(y,0)\, p(x,\tau | y,0) \qquad (13)$$

where $p\left(x,\tau|y,0\right)$ is the Green's function for eq (12) representing the transition probability function and the integration kernel. It satisfies:

$$\left(\frac{\partial}{\partial\ \tau}-\frac{\partial^2}{\partial x^2}\right)p\left(x,\tau|y,0\right)=\delta\left(x-y\right)\delta\left(\tau\right) \tag{14}$$

with the solution:

$$p\left(x,\tau|y,0\right)=\frac{1}{\sqrt{4\pi\ \tau}}\mathbf{exp}\left[-\frac{\left(x-y\right)^2}{4\tau}\right] \tag{15}$$

whence

$$\bar{g}\left(x,\tau\right)=\int_{-\infty}^{+\infty}dy\bar{g}\left(y,0\right)p\left(x,\tau|y,0\right)=\frac{1}{\sqrt{4\pi\ \tau}}\int_{-\infty}^{+\infty}dy\bar{g}\left(y,0\right)\mathbf{exp}\left[-\frac{\left(x-y\right)^2}{4\tau}\right] \tag{16}$$

and so

$$g\left(x,\tau\right)=\int_{-\infty}^{+\infty}dyg\left(y,0\right)p\left(x,\tau|y,0\right)=\frac{1}{\sqrt{4\pi\ \tau}}\int_{-\infty}^{+\infty}dyg\left(y,0\right)\mathbf{exp}\left[-\frac{\left(x-y\right)^2}{4\tau}\right] \tag{17}$$

Now, the boundary conditions. We are given the boundary condition for g(x,0) as:

$$\tau=0:g\left(x,0\right)=\max\left[e^{(k+1)\frac{x}{2}}-e^{(k-1)\frac{x}{2}},0\right]\Rightarrow g\left(x,0\right)e^{-\alpha x^2}\xrightarrow{|x|\to\infty}0\left(\alpha>0\right)$$

Applying this condition to g(y,0) in the integral (17), we get:

$$\tau=0:g\left(y,0\right)=\mathbf{max}\left(e^{(k+1)\frac{y}{2}}-e^{(k-1)\frac{y}{2}},0\right)=\mathbf{max}\left[e^{(k+1)\frac{y}{2}}\left(1-e^{-y}\right),0\right]$$

But, $\left(1-e^{-y}\right)<0, e^{(k+1)\frac{y}{2}}>0\ \ for\ \ y<0\ so\ that\ e^{(k+1)\frac{y}{2}}\left(1-e^{-y}\right)<0\ \ for\ \ y<0$

$whence\ \mathbf{max}\left[e^{(k+1)\frac{y}{2}}\left(1-e^{-y}\right),0\right]=0\ \ for\ \ y<0$. Thus, this boundary condition limits the integration region to positive y. Also, we need only retain $g\left(y,0\right)=e^{(k+1)\frac{y}{2}}-e^{(k-1)\frac{y}{2}}$ which is positive for all y>0 whence $\mathbf{max}\left(e^{(k+1)\frac{y}{2}}-e^{(k-1)\frac{y}{2}},0\right)=e^{(k+1)\frac{y}{2}}-e^{(k-1)\frac{y}{2}}\ \forall y>0\ as\ k>0$. Hence,

$$g\left(x,\tau\right)=\frac{1}{\sqrt{4\pi\ \tau}}\int_{0}^{\infty}dy\left(e^{(k+1)\frac{y}{2}}-e^{(k-1)\frac{y}{2}}\right)\mathbf{exp}\left[-\frac{\left(x-y\right)^2}{4\tau}\right] \tag{18}$$

$$\textit{Setting } z=\frac{y-x}{\sqrt{2\tau}}, \textit{ we get } g(x,\tau)=\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}\frac{dz}{\sqrt{2\pi}}\left(e^{(k+1)\frac{(z\sqrt{2\tau}+x)}{2}}-e^{(k-1)\frac{(z\sqrt{2\tau}+x)}{2}}\right)e^{-\frac{z^2}{2}}$$

$$=e^{(k+1)\frac{x}{2}+(k+1)^2\frac{\tau}{4}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(z-\frac{1}{2}(k+1)\sqrt{2\tau}\right)^2}-e^{(k-1)\frac{x}{2}+(k-1)^2\frac{\tau}{4}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(z-\frac{1}{2}(k-1)\sqrt{2\tau}\right)^2}$$

$$=e^{(k+1)\frac{x}{2}+(k+1)^2\frac{\tau}{4}}\mathbf{N}(d_1)-e^{(k-1)\frac{x}{2}+(k-1)^2\frac{\tau}{4}}\mathbf{N}(d_2) \text{ where} \tag{19}$$

$$d_1=\frac{x}{\sqrt{2\tau}}+\frac{1}{2}(k+1)\sqrt{2\tau}=\frac{\mathbf{ln}\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2=\frac{\mathbf{ln}\left(\frac{S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}=d_1-\sigma\sqrt{T-t}$$

Returning to the original variables, we have $C=Ke^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}g(x,\tau)$

$$C(S,t)=S\mathbf{N}(d_1)-Ke^{-r(T-t)}\mathbf{N}(d_2) \tag{20}$$

Appendix

A note on Green's functions

Let

$$Lg(x) = f(x) \quad \text{(A1)}$$

be a given differential equation. We need to solve for g(x).

Let $G(x;y)$ represent the Green's function for eq (A1). Then, $G(x;y)$ satisfies

$$LG(x;y) = \delta(x-y) \quad \text{(A2)}$$

so that

$$Lg(x) = f(x) = \int dy\delta(x-y) f(y) = \int_{-\infty}^{+\infty} dyLG(x;y) f(y) = L\int_{-\infty}^{+\infty} dyG(x;y) f(y) \quad \text{(A3)}$$

which gives $g(x) = \int_{-\infty}^{+\infty} dyG(x;y) f(y) \; where \; LG(x;y) = \delta(x-y)$ (A4)