Financial Derivatives & Risk Management
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Lecture 48 - Features of BS Model

We continue with our analysis of the Black Scholes Equation.

Condition expectation of stock price subject to call exercise in risk neutral world

The quantity that we need to work out here can symbolically be represented by $E_Q(S_T|S_T>K)$ i.e. the expectation of the stock price at maturity S_T subject to the condition that the call is exercised i.e. that maturity stock price finishes greater than the exercise price. The important thing here is that while calculating expectation we need to consider only those possible values of S_T that exceed K, possible values of S_T <K are not to be included while calculating expectation.

Now, we cannot write $E_Q(S_T|S_T>K)= S_T*P_Q(S_T>K)$ like we wrote $E_Q(K|S_T>K)= K*P_Q(S_T>K)$ because, while in the latter case, K was a constant so that, each probability that was involved in the expectation $P_Q(S_T)$ was multiplied by the same factor K to get the expectation of K. However, the case of $E_Q(S_T|S_T>K)$ is different. The probability of each S_T, $P_Q(S_T)$ needs to be multiplied by the corresponding S_T. Thus, each term in the expectation contains a different factor S_T, which can, therefore, not be separated out.

Further, the expectation is to be worked out with reference to risk-neutral probabilities.

We, now, proceed to the computations:

$$E\left[S_T\left|\left(S_T \geq K\right)\right.\right] = E\left[e^{\ln S_T}\left|\left(e^{\ln S_T} \geq e^{\ln K}\right)\right.\right] = E\left[e^{\xi}\left|\left(e^{\xi} \geq e^{\ln K}\right)\right.\right]$$

$$where\, \xi = \ln\, S_T \;\; is\;\; N\left[\ln\, S_0 + \left(r - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right] = N\left(\lambda, \theta^2\right)$$

$$E\left[e^{\xi}\left|\left(e^{\xi} \geq e^{\ln K}\right)\right.\right] = \frac{1}{\sqrt{2\pi\theta^2}}\int_{\ln K}^{\infty} e^{\xi} e^{-\frac{(\xi-\lambda)^2}{2\theta^2}} d\xi = \frac{e^{\left(\lambda+\frac{1}{2}\theta^2\right)}}{\sqrt{2\pi\theta^2}}\int_{\ln K}^{\infty} e^{-\frac{\left[\xi-\left(\lambda+\theta^2\right)\right]^2}{2\theta^2}} d\xi$$

$$\sin ce\;\; \xi - \frac{(\xi-\lambda)^2}{2\theta^2} = \frac{1}{2\theta^2}\left(2\theta^2\xi - \xi^2 - \lambda^2 + 2\lambda\xi\right)$$

$$= -\frac{1}{2\theta^2}\left[\xi^2 - 2\xi\left(\lambda+\theta^2\right) + \left(\lambda+\theta^2\right)^2 - 2\lambda\theta^2 - \theta^4\right] = \left(\lambda + \frac{1}{2}\theta^2\right) - \frac{1}{2\theta^2}\left[\xi - \left(\lambda+\theta^2\right)\right]$$

$$Now,\; e^{\lambda+\frac{1}{2}\theta^2} = e^{\left[\ln S_0 + \left(r-\frac{1}{2}\sigma^2\right)T + \frac{1}{2}\sigma^2 T\right]} = e^{\left(\ln S_0 + rT\right)} = S_0 e^{rT} \text{ so that}$$

$$E\left[e^{\xi}\left|\left(e^{\xi} \geq e^{\ln K}\right)\right.\right] = \frac{e^{\left(\lambda+\frac{1}{2}\theta^2\right)}}{\sqrt{2\pi\theta^2}}\int_{\ln K}^{\infty} e^{-\frac{\left[\xi-\left(\lambda+\theta^2\right)\right]^2}{2\theta^2}} d\xi = \frac{S_0 e^{rT}}{\sqrt{2\pi\theta^2}}\int_{\ln K}^{\infty} e^{-\frac{\left[\xi-\left(\lambda+\theta^2\right)\right]^2}{2\theta^2}} d\xi$$

$$Writing\ z = \frac{\xi - \left(\lambda + \theta^2\right)}{\theta},\ we\ have\ \theta dz = d\xi$$

$$E\left[e^{\xi} \left| \left(e^{\xi} \geq e^{\ln K}\right)\right.\right] = \frac{S_0 e^{rT}}{\sqrt{2\pi\theta^2}} \int_{\ln K}^{\infty} e^{-\frac{\left[\xi - \left(\lambda + \theta^2\right)\right]^2}{2\theta^2}} d\xi = \frac{S_0 e^{rT}}{\sqrt{2\pi}} \int_{\frac{\ln K - \left(\lambda + \theta^2\right)}{\theta}}^{\infty} e^{-\frac{1}{2}z^2} dz$$

$$= S_0 e^{rT} \left[1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln K - \left(\lambda + \theta^2\right)}{\theta}} e^{-\frac{1}{2}z^2} dz \right] = S_0 e^{rT} \left[1 - \mathbf{N}\left(\frac{\ln K - \left(\lambda + \theta^2\right)}{\theta}\right)\right]$$

$$= S_0 e^{rT} \mathbf{N}\left(-\frac{\ln K - \left(\lambda + \theta^2\right)}{\theta}\right) = S_0 e^{rT} \mathbf{N}\left(\frac{-\ln K + \left[\ln\ S_0 + \left(r - \frac{1}{2}\sigma^2\right)T + \sigma^2 T\right]}{\sigma\sqrt{T}}\right)$$

$$= S_0 e^{rT} \mathbf{N}\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) = S_0 e^{rT} \mathbf{N}\left(d_1\right)$$

Interpretation of the Black Scholes solution

Now let us assume that an investor writes a call option, then he creates an obligation to honor the call if it is exercised. However, if he has written a call, he takes the risk that if the stock finishes above the exercise price, he will have to honor the call.

But he can protect himself, that is the underlying philosophy of this entire Black Scholes model, that one can create a riskless portfolio by taking opposite positions in the derivative and the stock. In the current case, since the investor is short in the derivative, he can create a riskless portfolio by taking a long position in the stock. He can take a long position in $\Delta = \frac{\partial c}{\partial S} = \mathbf{N}\left(d_1\right)$ units of the stock and the combination of the short call and Δ units of the long stock will result in a neutralization of the volatility of the call and as a result of which his portfolio would be riskless.

Therefore by writing a call and by taking a long position in Δ units of the stock, the investor is really exposing himself to no additional risk. Now, to buy Δ units, the investor will have to make an initial cash outflow at t=0 of $\Delta S_0 = \mathbf{N}(d_1) * S_0$. This cashflow occurs at t=0.

Against this, the investor has the possibility of receiving the exercise price K at t=T, the maturity of the call option. Hence, its present value is relevant to the investor who is doing the analysis at t=0. This present value is Ke^{-rT}. But he will receive the exercise price only if the call is exercised and the probability of the call being exercised is given by $\mathbf{N}(d_2)$. Since this inflow on account of exercise price will only materialize if the option holder decides to exercise the call i.e. with probability **N**(d_2). Therefore, the expected value of present value of exercise price is Ke^{-rT}***N**(d_2) + 0*[1-N(d_2)]= Ke^{-rT}***N**(d_2).

Therefore, the value of the short call, given by c_{short}=Expected value of cash inflow-Cash outflow = Ke^{-rT}***N**(d_2)- **N**(d_1)*S_0 which is precisely the Black Scholes formula since

c_{long}=-c_{short} = S_0**N**(d_1)- Ke^{-rT}**N**(d_2)

Interpretation of N(d₁) & N(d₂)

We have, $c = e^{-rT} E_Q\left[f(S_T)\right] = e^{-rT} E_Q\left[\mathbf{max}(S_T - K, 0)\right]$. In the Black Scholes framework, $c = e^{-rT}\left[e^{rT} S_0 \mathbf{N}(d_1) - K\mathbf{N}(d_2)\right]$. Now, the first component of the payoff is –K, with the negative sign showing that it is a cash outflow from the perspective of a long call. However, it will only materialize if the option is exercised i.e. if S_T finishes >K.

Hence, we can write this component as a contingent payoff: $C_T^1 = \begin{bmatrix} -K \ \ if \ \ S_T > K \\ 0 \ \ otherwise \end{bmatrix}$ with the expected value $E(C_T^1) = -K \times P(S_T \geq K) + 0 \times P(S_T < K) = -K \times P(S_T \geq K) = -K\mathbf{N}(d_2)$ so that $PV\left[E(C_T^1)\right] = -e^{-rT} K\mathbf{N}(d_2) = E\left[PV(C_T^1)\right]$.

The second component of the payoff is the stock, since an option, if exercised results in the receipt of the stock by the long party by paying the exercise price to the short party. But the transfer will occur only if the option is exercised. Hence, $C_T^2 = \begin{bmatrix} +S_T \ \ if \ \ S_T \geq K \\ 0 \ \ otherwise \end{bmatrix}$. This component has the expected value $E(C_T^2) = E\left[S_T \middle| (S_T \geq K)\right] + 0 \times P(S_T < K) = E\left[S_T \middle| (S_T \geq K)\right] = e^{rT} S_0 \mathbf{N}(d_1)$ so that $PV\left[E(C_T^2)\right] = S_0\mathbf{N}(d_1) = E\left[PV(C_T^2)\right]$.
The point to be noted is that if the price finishes below the threshold of K then it does not contribute to the expected value of the stock price. The expectation is calculated only with reference to those values of the stock price that exceed the exercise price because it is only in these circumstances that the option will be exercised and the transfer of stock will actually take place.

The Black Scholes equation: $c = S_0\mathbf{N}(d_1) - Ke^{-rT}\mathbf{N}(d_2) = E\left[PV(C_T^2)\right] - E\left[PV(C_T^1)\right]$ $= E\left[PV(C_T^2) - PV(C_T^1)\right] = E\left[PV(C_T^2 - C_T^1)\right] = PV\left[E(C_T^2 - C_T^1)\right]$
is then equal to the expected present value of the net cash flow to the option holder subject to option exercise or equivalently, present value of the net expected cash flow subject to option exercise. This gives the current price of the option which is obviously in tandem with the principles of finance.

Risk neutral derivation of Black Scholes formula

We have, $c = e^{-rT} E_Q\left[\max\left(S_T - K, 0\right)\right] = e^{-rT} E_Q\left[\left(S_T - K\right) \middle| S_T > K\right]$

$$= e^{-rT}\left\{E_Q\left[S_T \middle| S_T > K\right] - E_Q\left[K \middle| S_T > K\right]\right\} = e^{-rT}\left\{E_Q\left[S_T \middle| S_T > K\right] - KP_Q\left(S_T > K\right)\right\}$$

$$= e^{-rT}\left[e^{rT} S_0 N\left(d_1\right) - KN\left(d_2\right)\right] = S_0 \mathbf{N}\left(d_1\right) - Ke^{-rT}\mathbf{N}\left(d_2\right)$$

<u>Example 1</u>

A stock price follows. Geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is 38. What is the probability that a European call option on the stock with an exercise price of 40 and a maturity date in 6 months will be exercised?

<u>Solution</u>

The required probability is the probability of the stock price exceeding 40 at t=6 months from now.

We know that $\mathbf{ln}\, S_T \sim N\left(\mathbf{ln}\, S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$. Thus, in this problem,

$$\mathbf{ln}\, S_{0.50} \sim N\left(\mathbf{ln}\, 38 + \left(0.16 - \frac{0.35^2}{2}\right)\times 0.5, 0.35^2 \times 0.5\right) = N\left(3.687, 0.247^2\right).$$

Now,P($S_{0.50}$>40)=P(ln$S_{0.50}$>ln40)=P(ln $S_{0.50}$>3.689)=$1 - \mathbf{N}\left(\frac{3.689 - 3.687}{0.247}\right) = 1 - \mathbf{N}\left(0.008\right)$=0.4968.

<u>Example 2</u>

Using put-call parity, obtain the Black Scholes formula for the price of a put option.

<u>Solution</u>

$$p = c + Ke^{-rT} - S_0 = S_0 \mathbf{N}\left(d_1\right) - Ke^{-rT}\mathbf{N}\left(d_2\right) + Ke^{-rT} - S_0$$

$$= -S_0\left[1 - \mathbf{N}\left(d_1\right)\right] + Ke^{-rT}\left[1 - \mathbf{N}\left(d_2\right)\right] = -S_0 \mathbf{N}\left(-d_1\right) + Ke^{-rT}\mathbf{N}\left(-d_2\right)$$

<u>Example 3</u>

What is the price of a European put option on a non-dividend-paying stock when the stock price is 69, the strike price is 70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is 6 months?

<u>Solution</u>

$$\textit{In this case } S_0 = 69, K = 70, r = 0.05, \sigma = 0.35 \textit{ and } T = 0.5$$

$$d_1 = \frac{\mathbf{ln}\left(\frac{69}{70}\right) + \left(0.05 + \frac{0.35^2}{2}\right)\times 0.5}{0.35\sqrt{0.5}} = 0.1666; \; d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

$$p = 70e^{-0.05\times0.5}\mathbf{N}\left(0.0809\right) - 69\mathbf{N}\left(-0.1666\right) = 70e^{-0.05} \times 0.5323 - 69 \times 0.4338 = 6.40$$